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VARIATIONAL METHODS FOR A FRACTIONAL ADVECTION-DISPERSION EQUATION WITH INSTANTANEOUS AND NON-INSTANTANEOUS IMPULSES AND NONLINEAR STURM-LIOUVILLE CONDITIONS*

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Abstract In this paper, a class of fractional advection-dispersion equations with instantaneous and non-instantaneous impulses and nonlinear Sturm-Liouville boundary conditions is considered. Firstly, based on the problem, we define an appropriate function space and construct corresponding variational structures. Then, under weaker conditions than the Ambrosetti-Rabinowitz condition, the existence and multiplicity of solutions to the equation are proven through the Mountain Pass Lemma and genus properties. Finally, an example is provided to illustrate the results obtained in this paper.

Keywords Fractional advection-dispersion equations, instantaneous and non-instantaneous impulses, nonlinear Sturm-Liouville conditions, variational methods.

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1. Introduction

Compared to integer order differential equations, fractional order differential equations have advantages and higher accuracy in describing practical problems with non-local or genetic properties and memory. Therefore, fractional order differential equations are widely used in fields such as physics, cybernetics, biology, economics, and dynamical systems, such as macroeconomic models with dynamic memory and dynamic analysis of infectious disease dynamical systems with memory (see [11]). They got the attention of many researchers and considerable work has been done in this regard in recent years. Various theories and methods are used to study it in the

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literatures, such as critical point theory, fixed point theory and topological degree theory (see [3, 7, 18, 19, 24] and references therein). As is well known, the general idea of methods for studying boundary value problems, such as fixed point theorems, upper and lower solution methods, fixed point index theorems, and degree theory, is to first find the integral equation corresponding to the solution of the equation under consideration. It is precisely this limitation of these methods that reflects the superiority of variational methods. Some interesting results are obtained by using the variational method and the critical point theory (see [2, 4, 14] and references therein). For example, in [10], the authors used the variational method for the first time to solve the boundary value problem of the following fractional differential equations

$$\begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) = \nabla F(t,u(t)),\ a.e.\ t\in[0,T], \\ u(0)=u(T)=0, \end{cases}$$

where $0<\alpha\le 1$, ${}_0D_t^{\alpha}$ and ${}_tD_T^{\alpha}$ denote the left and right Riemann-Liouville fractional derivatives of order α, respectively. Under the Ambrosetti-Rabinowitz condition, they obtained the existence result of the solution to the problem.

Speaking of fractional advection-dispersion equations, as a typical class of fractional order differential equations, play a crucial role in simulating anomalous diffusion phenomena. Due to reasons such as uneven transmission media or complex flow fields, abnormal diffusion problems such as groundwater and soil pollution, porous media, and biohydrodynamics commonly exist in real life. Therefore, it is necessary to promote the traditional second-order advection-diffusion equation. The fractional advection-dispersion equation can not only simulate anomalous diffusion, but also describe physical phenomena such as chaotic dynamics in turbulent classical conservative systems. In recent years, there has also been some research work on fractional advection-dispersion equations (see [5, 6, 23] and references therein). In the bargain, the Sturm-Liouville problem plays an important role in the heat conduction of finite length uniform thin tubes, axial and torsional vibrations of rods, and microwave transmission (see [9,12] and references therein). For instance, in [20], Tian and Nieto studied the following fractional advection-dispersion equation with homogeneous Sturm-Liouville boundary value conditions

$$\begin{cases} -\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\frac{1}{2}\ {}_0D_t^{-\beta}(u'(t))+\frac{1}{2}\ {}_tD_T^{-\beta}(u'(t))\bigg)=\lambda f(t,u(t)),\ a.e.\ t\in[0,T], \\ -au(0)-b\bigg(\frac{1}{2}\ {}_0D_t^{-\beta}(u'(0))+\frac{1}{2}\ {}_tD_T^{-\beta}(u'(0))\bigg)=0, \\ cu(T)+d\bigg(\frac{1}{2}\ {}_0D_t^{-\beta}(u'(T))+\frac{1}{2}\ {}_tD_T^{-\beta}(u'(T))\bigg)=0, \end{cases}$$

where a, $c>0$ and b, $d\ge 0$ are constants. $0\le\beta<1$, ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ denote the left and right Riemann-Liouville fractional integrals of order β, respectively. The variational structure was established and some existence results for solutions were obtained.

Additionally, instantaneous impulses have been widely considered by many scholars due to their ability to reflect the changing patterns of things more deeply and accurately, as well as their ability to fully consider the impact of homeopathic mutations on states (see [1, 15]). However, considering only fractional differential equations with instantaneous impulses cannot simulate the evolution process of phenomena such as dynamics. In this case, the addition of non-instantaneous impulses ([8]) can effectively solve this difficulty. Non-instantaneous impulses can start at a certain moment and last for a certain period of time, and this feature is particularly useful in applications in clinical medicine, bioengineering, chemistry, and physics (see [21,25] and references therein). Cite a case to you, [13] were concerned with a class of instantaneous and non-instantaneous impulsive fractional differential equations involving a ψ-Caputo fractional derivative

$$\begin{cases} {}^cD_{T^-}^{\alpha,\psi}({}^cD_{0^+}^{\alpha,\psi}u(t)) = \lambda f_i(t,u(t)),\ t\in(s_i,t_{i+1}],\ i=0,1,\cdots,m, \\ \Delta({}^cD_{T^-}^{\alpha,\psi}(I_{0^+}^{1-\alpha,\psi}u(t_i))) = I_i(u(t_i)),\ i=1,\cdots,m, \\ {}^cD_{T^-}^{\alpha,\psi}(I_{0^+}^{1-\alpha,\psi}u(t)) = {}^cD_{T^-}^{\alpha,\psi}(I_{0^+}^{1-\alpha,\psi}u(t_i^+)),\ t\in(t_i,s_i],\ i=1,\cdots,m, \\ {}^cD_{T^-}^{\alpha,\psi}(I_{0^+}^{1-\alpha,\psi}u(s_i^-)) = {}^cD_{T^-}^{\alpha,\psi}(I_{0^+}^{1-\alpha,\psi}u(s_i^+)),\ i=1,\cdots,m, \\ u(0)=u(T)=0, \end{cases}$$

where $\lambda>0$, $0<\alpha\le 1$, ${}^cD_{T^-}^{\alpha,\psi}$ and ${}^cD_{0^+}^{\alpha,\psi}$ are the right and left ψ-Caputo fractional derivatives, $I_{0^+}^{1-\alpha,\psi}$ is the left ψ-Riemann-Liouville fractional integral with order $1-\alpha$. The authors proved the multiplicity of solutions to this problem through critical point theory.

Inspired by the above research works and their significances, this article considers the following class of fractional advection-dispersion equation with instantaneous and non-instantaneous impulses and nonlinear Sturm-Liouville boundary conditions

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \dfrac{1}{2}\,{}_tD_T^{-\beta}(u'(t))\right) + g(t)u(t) = \lambda f_i(t,u(t)),\ t\in(s_i,t_{i+1}], \\ i=0,1,\cdots,m, \\ a\left(\dfrac{1}{2}\,{}_0D_t^{-\beta}(u'(0)) + \dfrac{1}{2}\,{}_tD_T^{-\beta}(u'(0))\right) - bh(u(0)) = A, \\ c\left(\dfrac{1}{2}\,{}_0D_t^{-\beta}(u'(T)) + \dfrac{1}{2}\,{}_tD_T^{-\beta}(u'(T))\right) + dh(u(T)) = B, \\ \Delta\left(\dfrac{1}{2}\,{}_0D_t^{-\beta}(u'(t_i)) + \dfrac{1}{2}\,{}_tD_T^{-\beta}(u'(t_i))\right) = \mu I_i(u(t_i)),\ i=1,\cdots,m, \\ {}_0D_t^{-\beta}(u'(t)) + {}_tD_T^{-\beta}(u'(t)) = {}_0D_t^{-\beta}(u'(t_i^+)) + {}_tD_T^{-\beta}(u'(t_i^+)),\ t\in(t_i,s_i], \\ i=1,\cdots,m, \\ {}_0D_t^{-\beta}(u'(s_i^-)) + {}_tD_T^{-\beta}(u'(s_i^-)) = {}_0D_t^{-\beta}(u'(s_i^+)) + {}_tD_T^{-\beta}(u'(s_i^+)),\ i=1,\cdots,m, \end{cases} \tag{1.1}$$

where $0\le\beta<1$, $\lambda>0$, a, b, c, $d>0$, A and B are constants. ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ denote the left and right Riemann-Liouville fractional integrals of order β, respectively. And $0=s_0<t_1<s_1<t_2<s_2<\cdots<t_m<s_m<t_{m+1}=T$. The function $g\in C^1([0,T],\mathbb{R})$ and there are g_1 and g_2 such that $0<g_1\le g(t)\le g_2$. $f_i\in C^1([s_i,t_{i+1}]\times\mathbb{R},\mathbb{R})$ and $F_i(t,y)=\int_{s_i}^{t_{i+1}} f_i(t,s)\mathrm{d}s$ for $i=0,1,\cdots,m$. The function $h\in C^1(\mathbb{R},\mathbb{R})$ and $H(y)=\int_0^y h(s)\mathrm{d}s$. The instantaneous impulses

$I_i \in C^1(\mathbb{R},\mathbb{R})$ start to change suddenly at the points t_i and the non-instantaneous impulses continue during the finite intervals $(t_i, s_i]$, for $i = 1, \cdots, m$. Besides,

$$\begin{aligned}&\Delta\left(\frac{1}{2}\,{}_0D_t^{-\beta}(u'(t_i)) + \frac{1}{2}\,{}_tD_T^{-\beta}(u'(t_i))\right)\\ =&\left(\frac{1}{2}\,{}_0D_t^{-\beta}(u'(t_i^+)) + \frac{1}{2}\,{}_tD_T^{-\beta}(u'(t_i^+))\right)\\ &-\left(\frac{1}{2}\,{}_0D_t^{-\beta}(u'(t_i^-)) + \frac{1}{2}\,{}_tD_T^{-\beta}(u'(t_i^-))\right),\end{aligned}$$

where

$$\begin{aligned}{}_0D_t^{-\beta}(u'(s_i^\pm)) + {}_tD_T^{-\beta}(u'(s_i^\pm)) &= \lim_{t\to s_i^\pm}\left({}_0D_t^{-\beta}(u'(t)) + {}_tD_T^{-\beta}(u'(t))\right),\\ {}_0D_t^{-\beta}(u'(t_i^\pm)) + {}_tD_T^{-\beta}(u'(t_i^\pm)) &= \lim_{t\to t_i^\pm}\left({}_0D_t^{-\beta}(u'(t)) + {}_tD_T^{-\beta}(u'(t))\right).\end{aligned}$$

In this article, we attempt to get the existence and multiplicity of weak solutions to boundary value problem (abbreviated as BVP) (1.1) through variational methods and critical point theory. Specifically, after constructing the variational structure, assuming that the nonlinear term satisfies weaker conditions than the Ambrosetti-Rabinowitz condition, the existence of the solution to BVP (1.1) is proven via the Mountain Pass Lemma. In addition, we divide the nonlinear term into two parts, one satisfying the superlinear condition and the other satisfying the sublinear condition. Then, we prove the multiplicity of solutions for BVP (1.1) by the genus property. As far as we know, very few articles have studied the existence and multiplicity of solutions for fractional order differential equations such as BVP (1.1). And the conditions set in this article are weaker than the famous Ambrosetti-Rabinowitz condition.

2. Preliminaries

For convenience, we will recall some necessary definitions and lemmas of fractional calculus.

Definition 2.1. [11] Let u be a function defined on $[a, b]$. The left and right Riemann-Liouville fractional integrals of order $0 < \gamma \le 1$ for u denoted by ${}_aD_t^{-\gamma}u(t)$ and ${}_tD_b^{-\gamma}u(t)$, respectively, are defined by

$${}_aD_t^{-\gamma}u(t) = \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1}u(s)\mathrm{d}s, \quad {}_tD_b^{-\gamma}u(t) = \frac{1}{\Gamma(\gamma)}\int_t^b (s-t)^{\gamma-1}u(s)\mathrm{d}s.$$

Definition 2.2. [11] Let u be a function defined on $[a, b]$. The left and right Riemann-Liouville fractional derivatives of order $0 < \gamma \le 1$ for u denoted by ${}_aD_t^{\gamma}u(t)$ and ${}_tD_b^{\gamma}u(t)$, respectively, are defined by

$$\begin{aligned}{}_aD_t^{\gamma}u(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\,{}_aD_t^{\gamma-1}u(t) = \frac{1}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_a^t (t-s)^{-\gamma}u(s)\mathrm{d}s\right),\\ {}_tD_b^{\gamma}u(t) &= -\frac{\mathrm{d}}{\mathrm{d}t}\,{}_tD_b^{\gamma-1}u(t) = \frac{-1}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_t^b (s-t)^{-\gamma}u(s)\mathrm{d}s\right).\end{aligned}$$

Definition 2.3. [11] Let $u(t) \in AC([a,b],\mathbb{R}^N)$. Then the left and right Caputo fractional derivatives of order $0<\gamma\le 1$ for the function u denoted by ${}_a^cD_t^\gamma u(t)$ and ${}_t^cD_b^\gamma u(t)$, respectively, are defined by

$$
{}_a^cD_t^\gamma u(t) = {}_aD_t^{\gamma-1}u'(t) = \frac{1}{\Gamma(1-\gamma)}\int_a^t (t-s)^{-\gamma}u'(s)\mathrm{d}s,
$$

$$
{}_t^cD_b^\gamma u(t) = -\,{}_tD_b^{\gamma-1}u'(t) = \frac{-1}{\Gamma(1-\gamma)}\int_t^b (s-t)^{-\gamma}u'(s)\mathrm{d}s.
$$

Proposition 2.1. [11] *Let u is continuous for a.e. $t\in[a,b]$, γ_1, $\gamma_2>0$. The left and right Riemann-Liouville fractional integral operators have the following properties*

$$
{}_aD_t^{-\gamma_1}({}_aD_t^{-\gamma_2}u(t)) = {}_aD_t^{-\gamma_1-\gamma_2}u(t), \quad {}_tD_b^{-\gamma_1}({}_tD_b^{-\gamma_2}u(t)) = {}_tD_b^{-\gamma_1-\gamma_2}u(t).
$$

Proposition 2.2. [11] *If $u\in L^p([a,b],\mathbb{R}^N)$, $v\in L^q([a,b],\mathbb{R}^N)$ and $p\ge 1$, $q\ge 1$, $\frac{1}{p}+\frac{1}{q}\le 1+\gamma$ with $p\ne 1$, $q\ne 1$ when $\frac{1}{p}+\frac{1}{q}=1+\gamma$, then*

$$
\int_a^b [{}_aD_t^{-\gamma}u(t)]v(t)\mathrm{d}t = \int_a^b [{}_tD_b^{-\gamma}v(t)]u(t)\mathrm{d}t.
$$

According to Definition 2.3 and Proposition 2.1, we know that

$$
\frac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}\,{}_tD_T^{-\beta}(u'(t)) = \frac{1}{2}\,{}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - \frac{1}{2}\,{}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t)), \tag{2.1}
$$

where $\alpha = 1-\frac{\beta}{2}$ and $\frac{1}{2}<\alpha\le 1$.

Let $L^p([0,T],\mathbb{R})(1\le p<+\infty)$ and $C([0,T],\mathbb{R})$ be the $p-$Lebesgue space and continuous function space, respectively, with the norms

$$
\|u\|_{L^p} = \left(\int_0^T |u(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}, \quad u\in L^p([0,T],\mathbb{R})\ (1\le p<+\infty),
$$

$$
\|u\|_\infty = \max_{t\in[0,T]}|u(t)|, \quad u\in C([0,T],\mathbb{R}).
$$

Definition 2.4. Let $\frac{1}{2}<\alpha\le 1$ and $1\le p<+\infty$. The fractional derivative space $E^{\alpha,p}$ is defined as the closure of $C^\infty([0,T],\mathbb{R})$, that is, $E^{\alpha,p}=\overline{C^\infty([0,T],\mathbb{R})}$ with the norm

$$
\|u\|_{\alpha,p} = \left(\int_0^T |{}_0^cD_t^\alpha u(t)|^p\mathrm{d}t + \int_0^T |u(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}.
$$

Reference [20] shows that the space $E^{\alpha,p}$ with $p\in(1,+\infty)$ is a reflexive and separable Banach space. When $p=2$, we denote the Hilbert space $E^{\alpha,2}$ as X.

Lemma 2.1. [20] *For any $u\in X$, we have*

$$
\begin{aligned}
-\cos(\pi\alpha)\int_0^T |{}_0^cD_t^\alpha u(t)|^2\mathrm{d}t &\le -\int_0^T ({}_0^cD_t^\alpha u(t))({}_t^cD_T^\alpha u(t))\mathrm{d}t \\
&\le -\frac{1}{\cos(\pi\alpha)}\int_0^T |{}_0^cD_t^\alpha u(t)|^2\mathrm{d}t.
\end{aligned}
$$

Lemma 2.2. *The norm $\|u\|_{\alpha,2}$ in X is equivalent to*

$$\|u\| = \left(-\int_0^T ({}_0^cD_t^\alpha u(t))({}_t^cD_T^\alpha u(t))\mathrm{d}t + \sum_{i=0}^m \int_{s_i}^{t_{i+1}} g(t)(u(t))^2\mathrm{d}t \right)^{\frac{1}{2}}.$$

Similar to Lemma 5.4 and Proposition 5.5 in [20], the following lemma can be proven.

Lemma 2.3. *There is a continuous and compact embedding $X \hookrightarrow C([0,T],\mathbb{R})$. And for $u \in X$, there exists constant $\Lambda > 0$ such that $\|u\|_\infty \le \Lambda\|u\|$.*

Lemma 2.4. [10] *Assume that $\frac{1}{2} < \alpha \le 1$ and the sequence $\{u_n\}$ weakly converges to u in X, that is, $u_n \rightharpoonup u$ in X. Then $u_n \to u$ in $C([0,T],\mathbb{R})$, that is, $\|u_n - u\|_\infty \to 0$ as $n \to \infty$.*

We take $v \in C^\infty([0,T],\mathbb{R})$ and multiply $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\ {}_0D_t^{-\beta}u'(t) + \frac{1}{2}\ {}_tD_T^{-\beta}u'(t)\right)$ and integrate between 0 and T. Furthermore, it follows from (2.1), the boundary conditions and Proposition 2.2 that

$$\begin{aligned}
&\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\ {}_0D_t^{-\beta}u'(t) + \frac{1}{2}\ {}_tD_T^{-\beta}u'(t)\right)v(t)\mathrm{d}t \\
=&\sum_{i=0}^m \int_{s_i}^{t_{i+1}} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\ {}_0D_t^{-\beta}u'(t) + \frac{1}{2}\ {}_tD_T^{-\beta}u'(t)\right)v(t)\mathrm{d}t \\
&+\sum_{i=1}^m \int_{t_i}^{s_i} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\ {}_0D_t^{-\beta}u'(t) + \frac{1}{2}\ {}_tD_T^{-\beta}u'(t)\right)v(t)\mathrm{d}t \\
=&\sum_{i=0}^m \left(\frac{1}{2}\ {}_0D_t^{-\beta}u'(t) + \frac{1}{2}\ {}_tD_T^{-\beta}u'(t)\right)v(t)\Big|_{s_i}^{t_{i+1}} \\
&-\sum_{i=0}^m \int_{s_i}^{t_{i+1}} \left(\frac{1}{2}\ {}_0D_t^{-\beta}u'(t) + \frac{1}{2}\ {}_tD_T^{-\beta}u'(t)\right)v'(t)\mathrm{d}t \\
&+\sum_{i=1}^m \left(\frac{1}{2}\ {}_0D_t^{-\beta}u'(t) + \frac{1}{2}\ {}_tD_T^{-\beta}u'(t)\right)v(t)\Big|_{t_i}^{s_i} \\
&-\sum_{i=1}^m \int_{t_i}^{s_i} \left(\frac{1}{2}\ {}_0D_t^{-\beta}u'(t) + \frac{1}{2}\ {}_tD_T^{-\beta}u'(t)\right)v'(t)\mathrm{d}t \\
=&-\frac{A}{a}v(0) + \frac{B}{c}v(T) - \frac{b}{a}h(u(0))v(0) - \frac{d}{c}h(u(T))v(T) - \mu\sum_{i=0}^m I_i(u(t_i))v(t_i) \\
&+\frac{1}{2}\int_0^T \left(({}_0^cD_t^\alpha u(t))({}_t^cD_T^\alpha v(t)) + ({}_t^cD_T^\alpha u(t))({}_0^cD_t^\alpha v(t))\right)\mathrm{d}t.
\end{aligned}$$

Therefore, we can define the weak solution u of the BVP (1.1) as follows.

Definition 2.5. A function $u \in X$ is called the weak solution of the BVP (1.1) if u satisfying the following equation

$$-\frac{1}{2}\int_0^T \left[({}_0^cD_t^\alpha u(t))({}_t^cD_T^\alpha v(t)) + ({}_t^cD_T^\alpha u(t))({}_0^cD_t^\alpha v(t))\right]\mathrm{d}t$$

$$
\begin{aligned}
&+\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} g(t)u(t)v(t)\mathrm{d}t+\mu\sum_{i=1}^{m} I_i(u(t_i))v(t_i)+\frac{b}{a}h(u(0))v(0)\\
&+\frac{d}{c}h(u(T))v(T)+\frac{A}{a}v(0)-\frac{B}{c}v(T)\\
=&\lambda\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} f_i(t,u(t))v(t)\mathrm{d}t,\ \forall v\in X.
\end{aligned}
$$

For $\forall u \in X$, we consider the functional $J : X \to \mathbb{R}$ as follows

$$
\begin{aligned}
J(u)=&-\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha u(t))({}_t^cD_T^\alpha u(t))\mathrm{d}t+\frac{1}{2}\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} g(t)(u(t))^2\mathrm{d}t\\
&+\mu\sum_{i=1}^{m}\int_0^{u(t_i)} I_i(s)\mathrm{d}s+\frac{b}{a}H(u(0))+\frac{d}{c}H(u(T))+\frac{A}{a}u(0)-\frac{B}{c}u(T)\\
&-\lambda\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} F_i(t,u(t))\mathrm{d}t\\
=&\frac{1}{2}\|u\|^2+\mu\sum_{i=1}^{m}\int_0^{u(t_i)} I_i(s)\mathrm{d}s+\frac{b}{a}H(u(0))+\frac{d}{c}H(u(T))+\frac{A}{a}u(0)-\frac{B}{c}u(T)\\
&-\lambda\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} F_i(t,u(t))\mathrm{d}t,
\end{aligned}
$$

where $F_i(t, u) = \int_0^u f_i(t, s)\mathrm{d}s$ and $H(u) = \int_0^u h(s)\mathrm{d}s$. Then, for all $v \in X$, we can get

$$
\begin{aligned}
< J'(u),v>=&-\frac{1}{2}\int_0^T\Big[({}_0^cD_t^\alpha u(t))({}_t^cD_T^\alpha v(t))+({}_t^cD_T^\alpha u(t))({}_0^cD_t^\alpha v(t))\Big]\mathrm{d}t\\
&+\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} g(t)u(t)v(t)\mathrm{d}t+\mu\sum_{i=1}^{m} I_i(u(t_i))v(t_i)+\frac{b}{a}h(u(0))v(0)\\
&+\frac{d}{c}h(u(T))v(T)+\frac{A}{a}v(0)-\frac{B}{c}v(T)-\lambda\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} f_i(t,u(t))v(t)\mathrm{d}t.
\end{aligned}
$$

It can obtain that the functional J is differentiable on X and $u \in X$ is a solution of $< J'(u), v >= 0$ for all $v \in X$, then u is a weak solution of the BVP (1.1).

Lemma 2.5. [22] *For the functional $\varphi : \Theta \subseteq E \to [-\infty, +\infty]$ with $\Theta \neq \emptyset$, $\min_{u\in\Theta} \varphi(u) = \varepsilon$ has a solution when the following hold*
(1) *E is a real reflexive Banach space.*
(2) *Θ is bounded and weak sequentially closed, i.e., by definition, for each sequence $\{u_n\}$ in Θ such that $u_n \rightharpoonup u$ as $n \to +\infty$, we always have $u \in \Theta$.*
(3) *φ is sequentially weakly lower semi-continuous on Θ.*

Lemma 2.6. [16] *Let E be a real Banach space and $\varphi \in C^1(E, \mathbb{R})$ satisfy Palais-Smale (for short (PS)) -condition. Suppose that φ satisfies the following conditions*
(1) *$\varphi(0) = 0$.*
(2) *There exist $\rho, \sigma > 0$ such that $\varphi(u_0) \geq \sigma$ for every $u_0 \in E$ with $\|u_0\| = \rho$.*
(3) *There exists $u_1 \in E$ with $\|u_1\| \geq \rho$ such that $\varphi(u_1) < \sigma$.*

Then, φ possesses a critical value $\varepsilon \geq \sigma$. Moreover, ε can be characterized as $\varepsilon = \inf_{k\in\Upsilon} \max_{s\in[0,1]} \varphi(k(s))$, where $\Upsilon = \{k \in C^1([0,1], E) : k(0) = u_0, k(1) = u_1\}$.

Immediately after, we give the signs, definitions and lemmas of genus properties. Let X be a real Banach space, $J \in C^1(X, \mathbb{R})$ and $z \in \mathbb{R}$. Set

$$\Sigma = \{U \subset X - \{0\} : U \text{ is closed in } X \text{ and symmetric with respect to } 0\},$$
$$K_z = \{u \in X : J(u) = z, J'(u) = 0\},$$
$$J^z = \{u \in X : J(u) \leq z\}.$$

Definition 2.6. [16,17] For $U \in \Sigma$, we say genus of U is n (denoted by $\gamma(U) = n$) if there is an odd map $\Psi \in C^1(U, \mathbb{R}^n \setminus \{0\})$, and n is the smallest integer with this property.

Lemma 2.7. [16] *Let J be an even C^1 functional on X and satisfy the (PS)-condition. For any $n \in \mathbb{N}$, set*

$$\Sigma_n = \{U \in \Sigma : \gamma(U) \geq n\}, \quad z_n = \inf_{U\in\Sigma_n} \sup_{u\in U} J(u).$$

(1) *If $\Sigma_n \neq \emptyset$; and $z_n \in \mathbb{R}$, then z_n is a critical value of J.*
(2) *If there exists $l \in \mathbb{N}$ such that $z_n = z_{n+1} = z_{n+2} = \cdots = z_{n+l} = z \in \mathbb{R}$, and $z \neq J(0)$, then $\gamma(K_z) \geq l + 1$.*

Remark 2.1. From Remark 7.3 in [16] , we know that if $K_z \in \Sigma$ and $\gamma(K_z) > 1$, then K_z contains infinitely many distinct points, that is, J has infinitely many distinct critical points in X.

3. Assumptions and main results

Firstly, we make the following assumptions which are met when needed in the rest of the article.
(H_1) There exist constants $1 \leq \tau < 2$, $Q > 0$ and $L > 0$ such that

$$|h(y)| \leq Q|y|^{\tau-1} + L, \quad y \in \mathbb{R}.$$

(H_2) For $i = 1, \cdots, m$, there exist $0 \leq \tau_i < 1$, $L_i > 0$ and $M_i > 0$ such that

$$|I_i(y)| \leq L_i|y|^{\tau_i} + M_i, \quad y \in \mathbb{R}.$$

(H_3) For $i = 0, 1, \cdots, m$, $\limsup_{|y|\to 0} \frac{F_i(t,y)}{|y|^2} = 0$ uniformly for a.e. $t \in (s_i, t_{i+1}]$.
(H_4) For $i = 0, 1, \cdots, m$, there is $\theta > 2$, such that

$$\limsup_{|y|\to\infty} \frac{\theta F_i(t, y) - f_i(t, y)y}{|y|^2} \leq 0$$

uniformly for a.e. $t \in (s_i, t_{i+1}]$.
(H_5) For $i = 0, 1, \cdots, m$, there exist subsets Ω_i of $(s_i, t_{i+1}]$ with $meas(\Omega_i) > 0$, such that

$$\liminf_{|y|\to\infty} \frac{F_i(t, y)}{|y|^2} > 0$$

uniformly for a.e. $t \in \Omega_i$.

Lemma 3.1. *Assume that (H_1) and (H_4) hold, then J satisfies the (PS)-condition.*

Proof. Assume that $\{u_n\}_{n\in\mathbb{N}} \subset X$ is a sequence such that $\{J(u_n)\}_{n\in\mathbb{N}}$ is bounded and $\lim_{n\to\infty} J'(u_n) = 0$. Then there exists $D > 0$ such that for $n \in \mathbb{N}$,

$$|J(u_n)| \le D, \quad \|J'(u_n)\|_* \le D, \tag{3.1}$$

where $\|\cdot\|_*$ is the norm of the dual space of X. Proof by contradiction. Suppose $\{u_n\}$ is unbounded. We assume that $\|u_n\| \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. By Lemma 2.4, passing to a subsequence, we suppose that $v_n \rightharpoonup v_0$ in X, then $v_n \to v_0$ in $C^1([0,T],\mathbb{R})$ as $n \to \infty$. By (3.1) there exists a constant $D_1 > 0$ such that

$$\begin{aligned}
(\frac{\theta}{2} - 1)\|u_n\|^2 =& \theta J(u_n) - J'(u_n)u_n - \theta\mu\sum_{i=1}^{m}\int_0^{u_n(t_i)} I_i(s)\mathrm{d}s + \mu\sum_{i=1}^{m} I_i(u_n(t_i))u_n(t_i) \\
&- \frac{\theta b}{a}H(u_n(0)) + \frac{b}{a}h(u_n(0))u_n(0) - \frac{\theta d}{c}H(u_n(T)) \\
&+ \frac{d}{c}h(u_n(T))u_n(T) - \frac{A}{a}(\theta-1)u_n(0) + \frac{B}{c}(\theta-1)u_n(T) \\
&+ \lambda\sum_{i=1}^{m}\int_{s_i}^{t_{i+1}}\Big(\theta F_i(t,u_n(t)) - f_i(t,u_n(t))u_n(t)\Big)\mathrm{d}t \\
\le& D_1(1 + \|u_n\|) + \theta\mu\sum_{i=1}^{m}(\frac{L_i\Lambda^{\tau_i+1}\|u_n\|^{\tau_i+1}}{\tau_i+1} + M_i\Lambda\|u_n\|) + \mu\sum_{i=1}^{m}(L_i \\
&\times \Lambda^{\tau_i+1}\|u_n\|^{\tau_i+1} + M_i\Lambda\|u_n\|) + \theta(\frac{b}{a} + \frac{d}{c})(\frac{Q\Lambda^{\tau}\|u_n\|^{\tau}}{\tau} + L\Lambda\|u_n\|) \\
&+ (\frac{b}{a} + \frac{d}{c})(Q\Lambda^{\tau}\|u_n\|^{\tau} + L\Lambda\|u_n\|) + (\theta-1)(\frac{|A|}{a} + \frac{|B|}{c})\Lambda\|u_n\| \\
&+ \lambda\sum_{i=1}^{m}\int_{s_i}^{t_{i+1}}\Big(\theta F_i(t,u_n(t)) - f_i(t,u_n(t))u_n(t)\Big)\mathrm{d}t.
\end{aligned}$$

Notice that $\|u_n\| \to \infty$, we have

$$\begin{aligned}
(\frac{\theta}{2} - 1)\|v_n\|^2 \le& \frac{D_1(1+\|u_n\|)}{\|u_n\|^2} + (\theta-1)(\frac{|A|}{a} + \frac{|B|}{c})\frac{\Lambda\|u_n\|}{\|u_n\|^2} \\
&+ \theta\mu\sum_{i=1}^{m}\left(\frac{L_i\Lambda^{\tau_i+1}\|u_n\|^{\tau_i+1}}{(\tau_i+1)\|u_n\|^2} + \frac{M_i\Lambda\|u_n\|}{\|u_n\|^2}\right) \\
&+ \mu\sum_{i=1}^{m}\left(\frac{L_i\Lambda^{\tau_i+1}\|u_n\|^{\tau_i+1}}{\|u_n\|^2} + \frac{M_i\Lambda\|u_n\|}{\|u_n\|^2}\right) \\
&+ \theta(\frac{b}{a} + \frac{d}{c})\left(\frac{Q\Lambda^{\tau}\|u_n\|^{\tau}}{\tau\|u_n\|^2} + \frac{L\Lambda\|u_n\|}{\|u_n\|^2}\right) \\
&+ (\frac{b}{a} + \frac{d}{c})\left(\frac{Q\Lambda^{\tau}\|u_n\|^{\tau}}{\|u_n\|^2} + \frac{L\Lambda\|u_n\|}{\|u_n\|^2}\right) \\
&+ \lambda\frac{\sum_{i=1}^{m}\int_{s_i}^{t_{i+1}}\Big(\theta F_i(t,u_n(t)) - f_i(t,u_n(t))u_n(t)\Big)\mathrm{d}t}{\|u_n\|^2}.
\end{aligned} \tag{3.2}$$

By (H_4), there exists $\Omega_{i0} \subset (s_i, t_{i+1}]$ with $meas(\Omega_{i0}) = 0$ such that

$$\limsup_{|u|\to\infty} \frac{\theta F_i(t,u) - f_i(t,u)u}{|u|^2} \le 0$$

uniformly for $t \in (s_i, t_{i+1}] \setminus \Omega_{i0}$. We claim that

$$\limsup_{n\to\infty} \frac{\theta F_i(t,u_n(t)) - f_i(t,u_n(t))u_n(t)}{\|u_n\|^2} \le 0 \tag{3.3}$$

for $t \in (s_i, t_{i+1}] \setminus \Omega_{i0}$. If not, there exists $t_0 \in (s_i, t_{i+1}] \setminus \Omega_{i0}$ and a subsequence of $\{u_n\}$ such that

$$\limsup_{n\to\infty} \frac{\theta F_i(t_0,u_n(t_0)) - f_i(t_0,u_n(t_0))u_n(t_0)}{\|u_n\|^2} > 0. \tag{3.4}$$

If $\{u_n(t_0)\}$ is bounded, then there exists $D_2 > 0$ such that $|u_n(t_0)| \le D_2$ for all $n \in \mathbb{N}$. Based on the continuity of f_i on Ω_{i0}, we have

$$\frac{\theta F_i(t_0,u_n(t_0)) - f_i(t_0,u_n(t_0))u_n(t_0)}{\|u_n\|^2} \le \frac{(\theta+1)D_2}{\|u_n\|^2} \to 0$$

as $n \to \infty$, which contradicts (3.4). So, there is a subsequence of $\{u_n(t_0)\}$ such that $|u_n(t_0)| \to \infty$ as $n \to \infty$.

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{\theta F_i(t_0,u_n(t_0)) - f_i(t_0,u_n(t_0))u_n(t_0)}{\|u_n\|^2}\\
=&\limsup_{n\to\infty} \frac{\theta F_i(t_0,u_n(t_0)) - f_i(t_0,u_n(t_0))u_n(t_0)}{|u_n(t_0)|^2}|v_n(t_0)|^2\\
=&\limsup_{n\to\infty} \frac{\theta F_i(t_0,u_n(t_0)) - f_i(t_0,u_n(t_0))u_n(t_0)}{|u_n(t_0)|^2}\lim_{n\to\infty}|v_n(t_0)|^2\\
\le&0.
\end{aligned}$$

This contradicts (3.4). Thus, (3.3) holds.

From (3.2) and (3.3) we obtain $\limsup_{n\to\infty}(\frac{\theta}{2}-1)\|v_n\|^2 \le 0$. Since $\theta > 2$, we get $\|v_n\|^2 \to 0$ as $n \to \infty$, but $\|v_n\| = 1$. This is a contradiction. Hence, $\{u_n\}$ is a bounded sequence in X.

In what follows, we claim that $u_n \to u$ in X. In fact, since X is a reflexive space, there exists a weakly convergent subsequence such that $u_n \rightharpoonup u$ in X. Hence, we have $u_n \to u$ in $C^1([0,T],\mathbb{R})$. And further, we get

$$\begin{aligned}
&< J'(u_n) - J'(u), u_n - u > \to 0,\\
&\Big(I_i(u_n(t)) - I_i(u(t))\Big)(u_n(t) - u(t)) \to 0,\\
&\Big(h(u_n(t)) - h(u(t))\Big)(u_n(t) - u(t)) \to 0,\\
&\int_{s_i}^{t_{i+1}} \Big(f_i(t,u_n(t)) - f_i(t,u(t))\Big)(u_n(t) - u(t))\mathrm{d}t \to 0
\end{aligned}$$

as $n \to \infty$. Since

$$\begin{aligned}
&\|u_n - u\|^2 \\
=&< J'(u_n) - J'(u), u_n - u > -\mu \sum_{i=1}^{m} \Big(I_i(u_n(t)) - I_i(u(t)) \Big)(u_n(t) - u(t)) \\
&- \frac{b}{a}\Big(h(u_n(0)) - h(u(0)) \Big)(u_n(0) - u(0)) - \frac{d}{c}\Big(h((u_n(T)) - h(u(T)) \Big) \\
&(u_n(T) - u(T)) - \lambda \sum_{i=0}^{m} \int_{s_j}^{t_{j+1}} \Big(f_j(t, u_n(t)) - f_j(t, u(t)) \Big)(u_n(t) - u(t))\mathrm{d}t \\
\to& 0
\end{aligned}$$

as $n \to \infty$. This implies that $u_n \to u$ in X. □

Theorem 3.1. *If (H_1)-(H_5) hold, and constant $P = \frac{\delta^2}{2\Lambda^2} - \mu \sum_{i=1}^{m}(\frac{L_i}{\tau_i+1}\delta^{\tau_i+1} + M_i\delta) - (\frac{b}{a} + \frac{d}{c})(\frac{Q}{\tau}\delta^{\tau} + L\delta) - (\frac{|A|}{a} + \frac{|B|}{c})\delta > 0$, then the BVP (1.1) admits at least two different weak solution for each $\lambda \in (0, \frac{P}{\chi T \delta^2})$.*

Proof. Let B_r denotes the open ball in X with radius r and centered at 0 and let ∂B_r and $\bar{B}_r$ represent the boundary and closure of B_r, respectively. It is easy to get $\bar{B}_{\frac{\delta}{\Lambda}}$ is a bounded weak closed set. Direct verification yields that $J(u)$ is sequentially weakly lower semi-continuous in X, based on Lemma 2.5, we get $J(u)$ has a local minimum point u_0 in $\bar{B}_{\frac{\delta}{\Lambda}}$, in other word, $J(u_0) \le J(0) = 0$.

It follows from (H_3) that for $i = 0, 1, \cdots, m$ and $\chi > 0$, there exists $\delta \in (0, \chi)$ such that

$$|F_i(t, u)| \le \chi |u|^2 \tag{3.5}$$

for a.e. $t \in (s_i, t_{i+1}]$ and $u \in \mathbb{R}$ with $|u| \le \delta$.

Let $\|u\| \le \frac{\delta}{\Lambda}$, it follows from Lemma 2.3 that $\|u\|_\infty \le \delta$. Then for $u \in \partial B_r$ $(r \le \frac{\delta}{\Lambda})$, according to (H_1), (H_2) and (3.5), we know

$$\begin{aligned}
J(u) \ge& \frac{1}{2}\|u\|^2 - \mu \sum_{i=1}^{m}(\frac{L_i \Lambda^{\tau_i+1}}{\tau_i+1}\|u\|^{\tau_i+1} + M_i \Lambda \|u\|) - (\frac{b}{a} + \frac{d}{c})L\Lambda\|u\| - (\frac{b}{a} + \frac{d}{c})\frac{Q}{\tau} \\
&\times \Lambda^{\tau}\|u\|^{\tau} - (\frac{|A|}{a} + \frac{|B|}{c})\Lambda\|u\| - \lambda \chi T \Lambda^2 \|u\|^2 \\
=& \Big(\frac{1}{2} - \lambda \chi T \Lambda^2\Big) r^2 - \mu \sum_{i=1}^{m}(\frac{L_i \Lambda^{\tau_i+1}}{\tau_i+1} r^{\tau_i+1} + M_i \Lambda r) - (\frac{b}{a} + \frac{d}{c})L\Lambda r \\
&- (\frac{b}{a} + \frac{d}{c})\frac{Q}{\tau}\Lambda^{\tau} r^{\tau} - (\frac{|A|}{a} + \frac{|B|}{c})\Lambda r.
\end{aligned}$$

And then for all $u \in \partial B_{\frac{\delta}{\Lambda}}$, one has

$$\begin{aligned}
J(u) \ge& \Big(\frac{1}{2} - \lambda \chi T \Lambda^2\Big)\frac{\delta^2}{\Lambda^2} - \mu \sum_{i=1}^{m}(\frac{L_i}{\tau_i+1}\delta^{\tau_i+1} + M_i\delta) - (\frac{b}{a} + \frac{d}{c})\frac{Q}{\tau}\delta^{\tau} \\
&- (\frac{|A| + bL}{a} + \frac{|B| + dL}{c})\delta \\
=& M_\lambda.
\end{aligned}$$

Since $\lambda \in (0, \frac{P}{\chi T \delta^2})$, we have $J(u) = M_\lambda > 0 \geq J(u_0)$ for $u \in \partial B_{\frac{\delta}{\Lambda}}$. Therefore, $\inf_{u \in \partial B_{\frac{\delta}{\Lambda}}} J(u) > J(u_0)$.

For $i = 0, 1, \cdots, m$, combining (H_4), (H_5) and the continuities of $f_i(t,u)$, it can be inferred that there are $\varrho > \frac{(\theta-2)\Lambda^2}{4\lambda}$, $K > 0$ and a subset of Ω_i, still denote as Ω_i, with $meas(\Omega_i) > 0$, such that

$$|F_i(t,u)| \geq \frac{2\varrho}{\theta - 2}|u|^2 - K \tag{3.6}$$

for $t \in \Omega_i$ and $u \in \mathbb{R}$. Choosing $u_0(t) \in X$ with $\|u_0\| \leq \Lambda$ and $\int_{\Omega_i} |u_0(t)|^2 \mathrm{d}t = 1$. And for $\varsigma > 0$, based on (3.6), we have

$$\begin{aligned}
J(\varsigma u_0) \leq& \frac{1}{2}\|\varsigma u_0\|^2 + \mu \sum_{i=1}^{m} \int_0^{\varsigma u_0(t_i)} I_i(s)\mathrm{d}s + \frac{b}{a}H(\varsigma u_0(0)) + \frac{d}{c}H(\varsigma u_0(T)) + \frac{A}{a}\varsigma u_0(0) \\
&- \frac{B}{c}\varsigma u_0(T) - \lambda \sum_{i=0}^{m} \int_{\Omega_i} F_i(t, \varsigma u_0(t))\mathrm{d}t \\
\leq& \frac{\varsigma^2}{2}\|u_0\|^2 + \mu \sum_{i=1}^{m} \Big(\frac{L_i \Lambda^{\tau_i+1}\varsigma^{\tau_i+1}}{\tau_i+1}\|u_0\|^{\tau_i+1} + M_i \varsigma \Lambda \|u_0\|\Big) + \Big(\frac{b}{a} + \frac{d}{c}\Big)L\Lambda\varsigma\|u_0\| \\
&+ \Big(\frac{b}{a} + \frac{d}{c}\Big)\frac{Q}{\tau}\Lambda^\tau \varsigma^\tau \|u_0\|^\tau + \Big(\frac{|A|}{a} + \frac{|B|}{c}\Big)\Lambda\varsigma\|u_0\| - \lambda(m+1)\frac{2\varrho\varsigma^2}{\theta-2} \\
&+ \lambda \sum_{i=0}^{m} K meas(\Omega_i) \\
\leq& \left(\frac{\Lambda^2}{2} - \lambda(m+1)\frac{2\varrho}{\theta-2}\right)\varsigma^2 + \mu \sum_{i=1}^{m} \Big(\frac{L_i \Lambda^{2(\tau_i+1)}\varsigma^{\tau_i+1}}{\tau_i+1} + M_i \Lambda^2 \varsigma\Big) \\
&+ \Big(\frac{b}{a} + \frac{d}{c}\Big)L\Lambda^2\varsigma + \Big(\frac{b}{a} + \frac{d}{c}\Big)\frac{Q}{\tau}\Lambda^{2\tau}\varsigma^\tau + \Big(\frac{|A|}{a} + \frac{|B|}{c}\Big)\Lambda^2\varsigma \\
&+ \lambda \sum_{i=0}^{m} K meas(\Omega_i).
\end{aligned}$$

Notice that $\varrho > \frac{(\theta-2)\Lambda^2}{4\lambda(m+1)}$, we can get that $I(\varsigma u_0) \to -\infty$ as $\varsigma \to \infty$. Thus, there exists $u_1 > 0$ with $\|u_1\| > \frac{\delta}{\Lambda}$ such that $\inf_{u \in \partial B_{\frac{\delta}{\Lambda}}} J(u) > J(u_1)$. Based on Lemma 2.6 and Lemma 3.1, it is easy to know that there is $u_2 \in X$ such that $J'(u_2) = 0$ and $J(u_2) > \max\{J(u_0), J(u_1)\}$. In summary, u_0 and u_2 are two different weak solutions of the BVP (1.1). □

In the remaining part of the article, we consider the case where $A = B = 0$ in BVP (1.1).

For $i = 0, 1, \cdots, m$, let $f_i(t,u) = f_{i_1}(t,u) + f_{i_2}(t,u)$, where $f_{i_1}(t,u)$ is superlinear as $|u| \to \infty$ and $f_{i_2}(t,u)$ is sublinear at infinity. $F_{i_1}(t,u) = \int_0^u f_{i_1}(t,s)\mathrm{d}s$ and $F_{i_2}(t,u) = \int_0^u f_{i_2}(t,s)\mathrm{d}s$. The specific assumptions are as below.
(H_3') For $i = 0, 1, \cdots, m$, there is $\theta > 2$, such that

$$\lim_{|y| \to \infty} \frac{F_{i_1}(t,y)}{|y|^\theta} = \infty$$

uniformly for $t \in (s_i, t_{i+1}]$.
(H_4') For $i = 0, 1, \cdots, m$, there exist $\eta_0 > 0$ and $R > 0$, such that

$$f_{i_1}(t, y)y - \theta F_{i_1}(t, y) \geq -\eta_0 y^2$$

for $t \in (s_i, t_{i+1}]$ and $|y| \geq R$.
(H_5') For $i = 0, 1, \cdots, m$, there exist $1 < \kappa < 2$ and $K_1 \in L^1((s_i, t_{i+1}], \mathbb{R}^+)$, such that

$$|f_{i_2}(t, y)| \leq K_1(t)|y|^{\kappa-1}, \quad (t, y) \in (s_i, t_{i+1}] \times \mathbb{R}.$$

(H_6) For $i = 0, 1, \cdots, m$, there exists $K_2 \in C^1((s_i, t_{i+1}], \mathbb{R}^+)$, such that

$$F_{i_2}(t, y) \geq K_2(t)|y|^\kappa, \quad (t, y) \in (s_i, t_{i+1}] \times \mathbb{R},$$

where $\mathbb{R}^+ = (0, \infty)$.

Lemma 3.2. *When $A = B = 0$, if (H_1), (H_2) and (H_3')-(H_5') hold, then J satisfies the (PS)-condition.*

Proof. At first, we prove $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X. We choose to use proof by contradiction. Suppose $\|u_n\| \to \infty$ as $n \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. By Lemma 2.4, passing to a subsequence, we can suppose that $v_n \rightharpoonup v_0$ in X, then $v_n \to v_0$ in $C^1([0, T], \mathbb{R})$.

It follows from (H_5') that

$$|f_{i_2}(t, u)u| \leq K_1(t)|u|^\kappa, \quad |F_{i_2}(t, u)| \leq \frac{1}{\kappa}K_1(t)|u|^\kappa. \tag{3.7}$$

The following is divided into two cases: $v_0 \equiv 0$ and $v_0 \neq 0$.

Case 1. $v_0 \equiv 0$. Note that the continuity of f_i and (H_4'), then there exists $\eta_1 > 0$, such that

$$f_{i_1}(t, u)u - \theta F_{i_1}(t, u) \geq -\eta_0 u^2 - \eta_1, \quad t \in (s_i, t_{i+1}], \quad u \in \mathbb{R}. \tag{3.8}$$

Thus, by (3.1), (3.7) and (3.8), we get

$$\begin{aligned}
o(1) =& \frac{\theta D + D\|u_n\|}{\|u_n\|^2} \\
\geq& \frac{\theta J(u_n) - J'(u_n)u_n}{\|u_n\|^2} \\
=& (\frac{\theta}{2} - 1) + \frac{\mu}{\|u_n\|^2}\sum_{i=1}^m \left(\theta\int_0^{u_n(t_i)} I_i(s)\mathrm{d}s - I_i(u_n(t_i))u_n(t_i)\right) \\
&+ \frac{b}{a\|u_n\|^2}\left(\theta H(u_n(0)) - h(u_n(0))u_n(0)\right) \\
&+ \frac{d}{c\|u_n\|^2}\left(\theta H(u_n(T)) - h(u_n(T))u_n(T)\right) \\
&+ \frac{\lambda}{\|u_n\|^2}\sum_{i=0}^m \int_{s_i}^{t_{i+1}} \left(f_i(t, u_n(t))u_n(t) - \theta F_i(t, u_n(t))\right)\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
\geq&(\frac{\theta}{2}-1)-\frac{\mu}{\|u_n\|^2}\sum_{i=1}^{m}\left((\frac{\theta L_i}{\tau_i+1}+L_i)\|u_n\|_\infty^{\tau_i+1}+(\theta+1)M_i\|u_n\|_\infty\right)\\
&-\left(\frac{b}{a\|u_n\|^2}+\frac{d}{c\|u_n\|^2}\right)\left((\frac{\theta Q}{\tau}+Q)\|u_n\|_\infty^{\tau}+(\theta L+L)\|u_n\|_\infty\right)\\
&-\frac{\lambda}{\|u_n\|^2}\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\left(\eta_0|u_n|^2+\eta_1+(1+\frac{\theta}{\kappa})K_1(t)|u_n|^\kappa\right)\mathrm{d}t\\
\geq&(\frac{\theta}{2}-1)-\frac{\mu}{\|u_n\|^2}\sum_{i=1}^{m}\left((\frac{\theta L_i}{\tau_i+1}+L_i)\Lambda^{\tau_i+1}\|u_n\|^{\tau_i+1}+(\theta+1)M_i\Lambda\|u_n\|\right)\\
&-\left(\frac{b}{a\|u_n\|^2}+\frac{d}{c\|u_n\|^2}\right)\left((\frac{\theta Q}{\tau}+Q)\Lambda^{\tau}\|u_n\|^{\tau}+(\theta L+L)\Lambda\|u_n\|\right)\\
&-\frac{\lambda T}{\|u_n\|^2}(\eta_0\Lambda^2\|u_n\|^2+\eta_1)-\frac{\lambda}{\|u_n\|^2}(1+\frac{\theta}{\kappa})\|K_1\|_{L^1}\Lambda^\kappa\|u_n\|^\kappa\\
\geq&(\frac{\theta}{2}-1)-\lambda T\eta_0\Lambda^2
\end{aligned}
$$

as $n\to\infty$. which indicates that $\{u_n\}$ is bounded in X.

Case 2. $v_0\neq 0$. For $i=0,1,\cdots,m$, let $\Omega_i'=\{t\in(s_i,t_{i+1}]:|v_0(t)|>0\}$, then $meas(\Omega_i')>0$. In addition, $\|u_n\|\to\infty$ as $n\to\infty$ and $|u_n(t)|=|v_n(t)|\cdot\|u_n\|$ indicates that for $t\in\Omega_i'$, $|u_n(t)|\to\infty$ as $n\to\infty$. So, it follows from (3.1) and (3.7) that

$$
\begin{aligned}
&\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}F_{i_1}(t,u_n(t))\mathrm{d}t\\
=&-\frac{1}{\lambda}J(u_n)+\frac{1}{2\lambda}\|u_n\|^2+\frac{\mu}{\lambda}\sum_{i=1}^{m}\int_0^{u_n(t_i)}I_i(s)\mathrm{d}s+\frac{b}{\lambda a}H(u_n(0))+\frac{d}{\lambda c}H(u_n(T))\\
&-\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}F_{i_2}(t,u_n(t))\mathrm{d}t\\
\leq&\frac{1}{2\lambda}\|u_n\|^2+\frac{\mu}{\lambda}\sum_{i=1}^{m}\left(\frac{L_i}{\tau_i+1}\|u_n\|_\infty^{\tau_i+1}+M_i\|u_n\|_\infty\right)+(\frac{b}{\lambda a}+\frac{d}{\lambda c})(\frac{Q}{\tau}\|u_n\|_\infty^{\tau}\\
&+L\|u_n\|_\infty)+\frac{1}{\kappa}\int_0^T K_1(t)|u_n(t)|^\kappa\mathrm{d}t+\frac{D}{\lambda}\\
\leq&\frac{1}{2\lambda}\|u_n\|^2+\frac{\mu}{\lambda}\sum_{i=1}^{m}\left(\frac{L_i\Lambda^{\tau_i+1}}{\tau_i+1}\|u_n\|^{\tau_i+1}+M_i\Lambda\|u_n\|\right)+(\frac{b}{\lambda a}+\frac{d}{\lambda c})(\frac{Q}{\tau}\Lambda^\tau\|u_n\|^{\tau}\\
&+L\Lambda\|u_n\|)+\frac{\Lambda^\kappa}{\kappa}\|K_1\|_{L^1}\|u_n\|^\kappa+\frac{D}{\lambda}.
\end{aligned}
$$

Owing to $\theta > 2$, $1 \le \tau < 2$, $1 \le \tau_i + 1 < 2$ and $1 < \kappa < 2$, we have

$$\sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} \frac{F_{i_1}(t, u_n(t))}{\|u_n\|^{\theta}} \mathrm{d}t \le o(1) \tag{3.9}$$

as $n \to \infty$. However, Fatou's lemma and (H_3') derive that

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} \frac{F_{i_1}(t, u_n(t))}{\|u_n\|^{\theta}} \mathrm{d}t \ge& \lim_{n\to\infty} \sum_{i=0}^{m} \int_{\Omega_i'} \frac{F_{i_1}(t, u_n(t))}{\|u_n\|^{\theta}} \mathrm{d}t \\ =& \lim_{n\to\infty} \sum_{i=0}^{m} \int_{\Omega_i'} \frac{F_{i_1}(t, u_n(t))}{|u_n(t)|^{\theta}} |v_n(t)|^{\theta} \mathrm{d}t \\ =& \infty,\end{aligned}$$

which contradicts (3.9). In conclusion, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X. And then, like the proof of Lemma 3.1, we know that $\|u_n - u\| \to 0$ as $n \to \infty$ in X. □

Theorem 3.2. *Suppose $A = B = 0$, if (H_1), (H_2), (H_3')-(H_5') and (H_6) hold, moreover, $h(u)$, $f_i(t, u)$ $(i = 0, 1, \cdots, m)$ and $I_i(u)$ $(i = 1, \cdots, m)$ are odd in u, then the BVP* (1.1) *possesses infinitely many nontrivial weak solutions.*

Proof. It is obviously to know that J is even and $J(0) = 0$. Let $\{e_n\}_{n=1}^{\infty}$ be the standard orthogonal basis of X, that is,

$$\|e_q\| = 1 \quad \text{and} \quad < e_q, e_{q'} >= 0, \quad 1 \le q \ne q'.$$

For any $n \in \mathbb{N}$, define $E_n = span\{e_1, \cdots, e_n\}$, $S_n = \{u \in E_n : \|u\| = 1\}$, then for any $u \in E_n$, there exist $\omega_j \in \mathbb{R}$ $(j = 1, \cdots, n)$, such that

$$u(t) = \sum_{j=1}^{n} \omega_j e_j(t), \quad t \in [0, T], \tag{3.10}$$

which means that

$$\begin{aligned}\|u\|^2 =& -\int_0^T ({}_0^cD_t^{\alpha} u(t))({}_t^cD_T^{\alpha} u(t))\mathrm{d}t + \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} g(t)(u(t))^2 \mathrm{d}t \\ =& \sum_{j=0}^{n} \omega_j^2 \Big(-\int_0^T ({}_0^cD_t^{\alpha} e_j(t))({}_t^cD_T^{\alpha} e_j(t))\mathrm{d}t + \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} g(t)(e_j(t))^2 \mathrm{d}t \Big) \\ =& \sum_{j=0}^{n} \omega_j^2 \|e_j\|^2 \\ =& \sum_{j=0}^{n} \omega_j^2.\end{aligned} \tag{3.11}$$

On the other side, in view of (H_6), for any bounded open set $\Pi_i \subset (s_i, t_{i+1}]$ $(i = 0, 1, \cdots, m)$, there exists $K_3 > 0$ such that

$$F_{i_2}(t, u) \ge K_2(t)|u|^{\kappa} \ge K_3 |u|^{\kappa}, \quad (t, u) \in \Pi_i \times \mathbb{R}. \tag{3.12}$$

For the above Π_i $(i=0,1,\cdots,m)$, (H_3') indicates that there exist $K_4>0$ and $K_5>0$, such that

$$F_{i_1}(t,u)\geq K_4|u|^\theta - K_5,\quad (t,u)\in \Pi_i\times\mathbb{R}. \tag{3.13}$$

Consequently, from (3.11)-(3.13), for any $u\in S_n$,

$$\begin{aligned}
J(\vartheta u) =&\frac{1}{2}\|\vartheta u\|^2+\mu\sum_{i=1}^{m}\int_0^{\vartheta u(t_i)} I_i(s)\mathrm{d}s+\frac{b}{a}H(\vartheta u(0))+\frac{d}{c}H(\vartheta u(T))\\
&-\lambda\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}F_i(t,\vartheta u(t))\mathrm{d}t\\
\leq&\frac{1}{2}\|\vartheta u\|^2+\mu\sum_{i=1}^{m}\int_0^{\vartheta u(t_i)} I_i(s)\mathrm{d}s+\frac{b}{a}H(\vartheta u(0))+\frac{d}{c}H(\vartheta u(T))\\
&-\lambda\sum_{i=0}^{m}\int_{\Pi_i}F_i(t,\vartheta u(t))\mathrm{d}t\\
\leq&-\lambda\vartheta^\theta K_4\sum_{i=0}^{m}\int_{\Pi_i}\left|\sum_{j=1}^{n}\omega_j e_j(t)\right|^\theta\mathrm{d}t-\lambda\vartheta^\kappa K_3\sum_{i=0}^{m}\int_{\Pi_i}\left|\sum_{j=1}^{n}\omega_j e_j(t)\right|^\kappa\mathrm{d}t\\
&+\frac{\vartheta^2}{2}\|u\|^2+\mu\sum_{i=1}^{m}(\frac{L_i\Lambda^{\tau_i+1}\vartheta^{\tau_i+1}}{\tau_i+1}\|u\|^{\tau_i+1}+M_i\Lambda\vartheta\|u\|)+(\frac{b}{a}+\frac{d}{c})L\Lambda\vartheta\|u\|\\
&+(\frac{b}{a}+\frac{d}{c})\frac{Q}{\tau}\Lambda^\tau\vartheta^\tau\|u\|^\tau+\lambda K_5 T\\
=&-\lambda\vartheta^\theta K_4\sum_{i=0}^{m}\int_{\Pi_i}\left|\sum_{j=1}^{n}\omega_j e_j(t)\right|^\theta\mathrm{d}t-\lambda\vartheta^\kappa K_3\sum_{i=0}^{m}\int_{\Pi_i}\left|\sum_{j=1}^{n}\omega_j e_j(t)\right|^\kappa\mathrm{d}t+\frac{\vartheta^2}{2}\\
&+\mu\sum_{i=1}^{m}(\frac{L_i\Lambda^{\tau_i+1}\vartheta^{\tau_i+1}}{\tau_i+1}+M_i\Lambda\vartheta)+(\frac{b}{a}+\frac{d}{c})L\Lambda\vartheta+(\frac{b}{a}+\frac{d}{c})\frac{Q}{\tau}\Lambda^\tau\vartheta^\tau\\
&+\lambda K_5 T.
\end{aligned}$$

Besides, it's easy to prove that $\sum_{i=0}^{m}\int_{\Pi_i}|\sum_{j=1}^{n}\omega_j e_j(t)|^\theta\mathrm{d}t>0$. Pay attention to $\theta>2$, $1\leq\tau<2$, $1<\kappa<2$ and $1\leq\tau_i+1<2$ $(i=1,\cdots,m)$, so there exist $\zeta>0$ and $\varpi>0$ such that

$$J(\varpi u)<-\zeta,\quad u\in S_n. \tag{3.14}$$

Let $S_n^\varpi=\{\varpi u: u\in S_n\}$, $\Theta=\{(\omega_1,\cdots,\omega_n)\in\mathbb{R}^n:\sum_{j=1}^{n}\omega_j^2<\varpi^2\}$. Then, by (3.14), we can get

$$J(u)<-\zeta,\quad u\in S_n^\varpi,$$

which together with the fact that even functional $J\in C^1(X,\mathbb{R})$, yields that $S_n^\varpi\subset J^{-\zeta}\in\Sigma$.

In addition, from (3.10) and (3.11), there exists an odd homeomorphism mapping $\Psi\in C^1(\partial\Theta,S_n^\varpi)$. By some properties of the genus, we obtain

$$\gamma(J^{-\zeta})\geq\gamma(S_n^\varpi)=n, \tag{3.15}$$

so $J^{-\zeta} \in \Sigma_n$, that is, $\Sigma_n \neq \emptyset$. Let $z_n = \inf_{U \in \Sigma_n} \sup_{u \in U} J(u)$, then, from (3.15) and the fact that J is bounded from below on X, we have $-\infty \leq z_n < -\zeta < 0$, that is, for any $n \in \mathbb{N}$, z_n is a negative real number. So then by Lemma 2.7 and Remark 2.1, J has infinitely many nontrivial critical points, in other words, BVP (1.1) admits infinitely many negative energy solutions. □

4. Examples

Example 4.1. Let $\beta = \frac{1}{2}$, $T = 1$, $m = 1$, $\lambda = \frac{1}{4}$, $a = 3$, $b = c = 1$, $d = 2$ and $\mu = A = B = 2$. Consider the following fractional boundary value problem

$$\begin{cases} -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\ {}_0D_t^{-\frac{1}{2}}(u'(t)) + \frac{1}{2}\ {}_tD_1^{-\frac{1}{2}}(u'(t))\right) + g(t)u(t) = \frac{1}{4}f_i(t,u(t)),\ t \in (s_i, t_{i+1}], \\ \ i = 0, 1, \\ 3\left(\frac{1}{2}\ {}_0D_t^{-\frac{1}{2}}(u'(0)) + \frac{1}{2}\ {}_tD_1^{-\frac{1}{2}}(u'(0))\right) - h(u(0)) = 2, \\ \left(\frac{1}{2}\ {}_0D_t^{-\frac{1}{2}}(u'(1)) + \frac{1}{2}\ {}_tD_1^{-\frac{1}{2}}(u'(1))\right) + 2h(u(1)) = 2, \\ \Delta\left(\frac{1}{2}\ {}_0D_t^{-\frac{1}{2}}(u'(t_1)) + \frac{1}{2}\ {}_tD_1^{-\frac{1}{2}}(u'(t_1))\right) = 2I_1(u(t_1)), \\ {}_0D_t^{-\frac{1}{2}}(u'(t)) + {}_tD_1^{-\frac{1}{2}}(u'(t)) = {}_0D_t^{-\frac{1}{2}}(u'(t_1^+)) + {}_tD_1^{-\frac{1}{2}}(u'(t_1^+)),\ t \in (t_1, s_1], \\ {}_0D_t^{-\frac{1}{2}}(u'(s_1^-)) + {}_tD_1^{-\frac{1}{2}}(u'(s_1^-)) = {}_0D_t^{-\frac{1}{2}}(u'(s_1^+)) + {}_tD_1^{-\frac{1}{2}}(u'(s_1^+)), \end{cases} \tag{4.1}$$

where $0 = s_0 < t_1 = \frac{1}{3} < s_1 = \frac{2}{3} < t_2 = 1$. Let $g(t) = \ln(1+t^2)$, then, for $t \in [0,1]$, we can get $g_1 = 0 \leq g(t) \leq g_2 = \ln 2$. Let $h(u) = \frac{1}{2}\sin u$, then, for $Q = L = 1$ and $\tau = \frac{3}{2}$, (H_1) holds. Choose $I_1(u) = \sin u$, there exist $L_1 = M_1 = 2$ and $\tau_1 = \frac{1}{2}$, such that (H_2) holds. $f_i(t,u) = f_{i1}(t,u) + f_{i2}(t,u)$, where $f_{i1}(t,u) = (1+\sin t)u^5$ and $f_{i2}(t,u) = (2+\cos t)u^{\frac{1}{3}}$, then, $F_{i1}(t,u) = \frac{1}{6}(1+\sin t)u^6$ and $F_{i2}(t,u) = \frac{3}{4}(2+\cos t)u^{\frac{4}{3}}$. According to the calculation, when $\eta_0 = 2$, $R = 3$ and $\theta = 4$, (H_3') and (H_4') hold. What's more, setting $\kappa = \frac{4}{3}$, $K_1(t) = \frac{1}{3}t + 5$ and $K_2(t) = \frac{1}{4}(1+t)$ can make (H_5') and (H_6) true. Of course, $h(u)$, $I_1(u)$ and $f_i(t,u)$ are all odd functions related to u. Therefore, all conditions of Theorem 3.2 are satisfied, that is, the BVP (4.1) admits infinitely many nontrivial weak solutions.

5. Conclusion

In this work, we study a class of fractional advection-dispersion equation with instantaneous and non-instantaneous impulses and nonlinear Sturm-Liouville boundary conditions. In order to carry out the study, we defined firstly a suitable fractional derivative space and the weak solution for BVP (1.1) in this space. And then we establish the variational structure of BVP (1.1). Then we use the Mountain Pass Lemma and genus properties to consider the existence and multiplicity of critical points of energy functional in this space. It is worth mentioning that the problem considered in this article are rarely addressed, and the Sturm-Liouville boundary conditions are more general, including traditional Sturm-Liouville condition cases. Last but not least, the assumptions set in this article are weaker.

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