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MONOTONE ITERATIVE TECHNIQUE FOR IMPULSIVE EVOLUTION EQUATIONS WITH INFINITE DELAY*

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Abstract In this paper, we use a monotone iterative technique in the presence of lower and upper solutions to discuss the existence of solutions for the initial value problem of impulsive evolution equations with infinite delay in an ordered Banach space X. Finally, we give an example to illustrate our main results.

Keywords Infinite delay, impulsive evolution equations, measure of noncompactness, lower and upper solutions.

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1. Introduction

Impulsive differential equations arising from real world problems are used to describe the dynamics of processes in which sudden, discontinuous jumps occur and such processes occur in biology, physics, engineering, etc. Due to their significance, they have attracted much attention in the last decade, we refer the reader to [2, 11, 14, 21, 29] and the references therein for more details. However, many papers on impulsive differential equations do not consider the influence of delay, see [6, 8, 9, 17, 28] and the references therein. A large number of theoretical and practical studies show that the simultaneous introduction of impulsive and delay into a system can better describe the interaction and influence of many factors inside the system, and better depict the real world, see [22, 25, 33]. Of course, such systems are more complex and generally more difficult to study theoretically. When the length of the delay is close to infinity, the delay evolution equation transfers to the infinite delay evolution equation. In [13], the author studied the existence and regularity of mild solutions for a class of abstract neutral functional differential equations with infinite delay by using fraction power theory and fixed point theorem. In 2022, by using the theory of

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resolvent operators for linear neutral integro-differential evolution systems, Huang and Fu [20] investigated optimal control and time optimal control for a neutral integro-differential evolution system with infinite delay. The theoretical methods of infinite delay and finite delay are very different. The idea of studying finite delay in the space with the supremum norm is no longer applicable to infinite delay. Therefore, we introduce to the study of impulsive differential equations with infinite delay an abstract admissible phase space which was initiated by Hale and Kato [18]. For the general theory and applications of such equations with infinite delay we refer the interested reader to the papers [1,4,5,12,15,26,31,32,35,36] and the references therein.

Inspired by the above mentioned aspects, in this paper we will use the monotone iterative technique to consider the existence of mild solutions for impulsive evolution equations with infinite delay:

$$\begin{cases} u'(t) + Au(t) = f(t, u(t), u_t), \quad t \in I = [0, b], \ \ t \neq t_k, \\ \Delta u|_{t=t_k} = J_k(u(t_k)), \quad k = 1, 2, \cdots, m, \\ u_0 = \varphi \in \mathcal{B}, \end{cases} \tag{1.1}$$

where $A : D(A) \subset X \to X$ is a closed linear operator and $-A$ generates a C_0-semigroup $T(t)(t \geq 0)$ on X, $b > 0$ is a constant, $f : [0, b] \times X \times \mathcal{B} \to X$ is a Carathéodory continuous function, $0 < t_1 < t_2 < \cdots < t_m < b$, $J_k \in C(X, X)$, $k = 1, 2, \cdots, m$. $u_t : (-\infty, 0] \to X$, $u_t(\tau) = u(t + \tau)$ belongs to an abstract phase space $\mathcal{B}$, $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively.

The monotone iterative technique in the presence of upper and lower solutions for nonlinear differential equations has received a lot of attention. Li and Liu [24], Guo and Liu [17] investigated the existence of extremal solutions for the initial value problem of the integro-differential equation when the nonlinear term satisfies a monotonicity condition and a noncompactness measure condition. The iterative method has been extended to evolution equations; see Chen and Li [8,9], Li and Gou [23] for evolution equations with impulsive, and Zhang, Chen and Li [34] for retarded evolution equations with nonlocal and impulsive conditions in Banach spaces. We should mention that Chaudhary and Dwijendar [7] investigated neutral fractional differential equations with infinite delay without impulsive effects by using the monotone iterative method. However we have not seen any relevant papers that study infinite delay evolution equations with impulsive condition applying the monotone iterative method. In this paper the nonlinear term f satisfies a monotone condition and a noncompactness measure condition and we use a monotone iterative method to discuss the existence of solutions for the impulsive evolution equations with infinite delay (1.1).

This paper is organized as follows. In section 2, we define the admissible phase space $\mathcal{B}$ and recall some basic definitions and lemmas. In section 3, we investigate the existence of extremal solutions for the initial value problem of impulsive evolution equation with infinite delay (1.1) with a compact semigroup. In section 4, we investigate the existence and uniqueness of solutions for the initial value problem of impulsive evolution equation with infinite delay (1.1) with a noncompact semigroup. Lastly, in section 5, we present an example to illustrate the main theorem.

2. Preliminaries

In this paper, we assume that X is an ordered Banach space with norm $\|\cdot\|$ and partial order " $\leq$ ", whose positive cone $P=\{u\in X \mid u\geq\theta\}$ (θ is the zero element of X) is normal with normal constant N.

For impulsive differential equations with infinite delay, we will adopt an axiomatic definition of the phase space introduced in [18].

Definition 2.1. The phase space $\mathcal{B}$ is a linear space of functions $(-\infty,0]$ into X endowed with a norm $\|\cdot\|_{\mathcal{B}}$. We will assume that $\mathcal{B}$ satisfies the following axioms:

(A) for $b>0$, if $u:(-\infty,\sigma+b]\to X$ is continuous on $[\sigma,\sigma+b]$ and $u_\sigma\in\mathcal{B}$, then for every $t\in[\sigma,\sigma+b]$ the following conditions hold:

(i) $u_t\in\mathcal{B}$;

(ii) $\|u(t)\|\leq H\|u_t\|_{\mathcal{B}}$;

(iii) $\|u_t\|_{\mathcal{B}}\leq K(t-\sigma)\sup\{\|u(s)\|:\sigma\leq s\leq t\}+P(t-\sigma)\|u_\sigma\|_{\mathcal{B}}$,

where $H>0$ is a constant, $K,P:[0,+\infty)\to[0,+\infty)$, $K(\cdot)$ is continuous, $P(\cdot)$ is locally bounded and $K(\cdot),P(\cdot)$ are independent of $u(\cdot)$.

(A_1) For the function $u(\cdot)$ in (A), the function $t\to u_t$ is continuous from $[\sigma,\sigma+b]$ into $\mathcal{B}$.

(B) The space $\mathcal{B}$ is complete.

Consider the space

$$\begin{aligned}\mathcal{B}_b=&\{u:(-\infty,b]\to X \mid u \text{ is continuous at } t\neq t_k,\\ &u(t_k^-)=u(t_k),u(t_k^+)\text{ exists, } k=1,2,...,m,\text{and } u_0=\varphi\}\end{aligned}$$

endowed with the semi-norm

$$\|u\|_{\mathcal{B}_b}=\|u_0\|_{\mathcal{B}}+\sup_{t\in I}\|u(t)\|.$$

Now, $\mathcal{B}_b$ is also an ordered Banach space with the partial order " $\leq$ " induced by the positive cone $K_{\mathcal{B}_b}=\{u\in\mathcal{B}_b \mid u(t)\geq\theta,\ t\in(-\infty,b]\}$. $K_{\mathcal{B}_b}$ is also normal with the same normal constant N. For $v,\ w\in\mathcal{B}_b$ with $v\leq w$, we use $[v,w]$ to denote the order interval

$$\{u\in\mathcal{B}_b \mid v\leq u\leq w\}$$

in $\mathcal{B}_b$, and $[v(t),w(t)]$ to denote the order interval

$$\{x\in X \mid v(t)\leq x\leq w(t),\ t\in(-\infty,b]\}$$

in X.

Let $I_0=[0,t_1]$, $I_k=(t_k,t_{k+1}]$, $I'_k=[t_k,t_{k+1}]$, $k=1,2,\cdots,m$, $t_{m+1}=b$, $I'=(-\infty,b]\backslash\{t_1,t_2,\cdots,t_m\}$, $I''=(-\infty,b]\backslash\{0,t_1,t_2,\cdots,t_m\}$. An abstract function $u\in\mathcal{B}_b\cap C^1(I'',X)\cap C(I',X_1)$ (X_1 is the Banach space endowed with the norm $\|\cdot\|_1=\|\cdot\|+\|A\cdot\|$) is called the solution of the initial value problem of the impulsive evolution equation with infinite delay (1.1), if $u(t)$ satisfies(1.1).

Definition 2.2. If the abstract function $u \in \mathcal{B}_b \cap C^1(I'', X) \cap C(I', X_1)$ satisfies

$$\begin{cases} u'(t) + Au(t) \leq f(t, u(t), u_t), \quad t \in I = [0, b], \quad t \neq t_k, \\ \Delta u|_{t=t_k} \leq J_k(u(t_k)), \quad k = 1, 2, \cdots, m, \\ u_0 \leq \varphi \in \mathcal{B}, \end{cases} \tag{2.1}$$

we call it a lower solution of the initial value problem of the impulsive evolution equation with infinite delay (1.1); if all the inequalities in (2.1) are reversed, we call it an upper solution of the initial value problem of the impulsive evolution equation with infinite delay (1.1).

Definition 2.3. A function $u : (-\infty, b] \to X$ is said to be a mild solution of the initial value problem of the impulsive evolution equation with infinite delay (1.1) if $u_0 = \varphi \in \mathcal{B}$ and

$$u(t) = T(t)\varphi(0) + \int_0^t T(t-s) f(s, u(s), u_s) ds + \sum_{0<t_k<t} T(t-t_k) J_k(u(t_k)), \quad t \in I. \tag{2.2}$$

Definition 2.4. If a function $f : [0, b] \times X \times \mathcal{B} \to X$ satisfies

(i) for all $(u, v) \in X \times \mathcal{B} \to X, f(\cdot, u, v) : [0, b] \to X$ is measurable;

(ii) for a.e. $t \in [0, b], f(t, \cdot, \cdot) : X \times \mathcal{B} \to X$ is continuous, then we say f is a Carathéodory continuous function.

In this paper, let $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generates a positive C_0-semigroup $T(t)(t \geq 0)$ on X. Therefore, there exist constants M_1 and $\delta \in \mathbb{R}$ such that

$$\|T(t)\| \leq M_1 e^{\delta t}, \quad t \geq 0. \tag{2.3}$$

From (2.3) we see that

$$M := \sup_{t \in I} \|T(t)\|_{\mathcal{L}(X)} \geq 1 \tag{2.4}$$

is a finite number, where $\mathcal{L}(X)$ is the Banach space of all bounded linear operators from X to X.

Therefore, we see that for any contant $C > 0$, $-(A + CI)$ generates a positive C_0-semigroup $S(t) = e^{-Ct}T(t)(t \geq 0)$ on X and

$$\sup_{t \in I} \|S(t)\|_{\mathcal{L}(X)} = \sup_{t \in I} \|e^{-Ct}T(t)\|_{\mathcal{L}(X)} = M \geq 1. \tag{2.5}$$

Hence, $S(t)(t \geq 0)$ is a positive C_0-semigroup in X if $T(t)(t \geq 0)$ is a positive C_0-semigroup in X; $S(t)(t \geq 0)$ is a compact C_0-semigroup in X if $T(t)(t \geq 0)$ is a compact C_0-semigroup in X; $S(t)(t \geq 0)$ is a equicontinuous C_0-semigroup in X if $T(t)(t \geq 0)$ is a equicontinuous C_0-semigroup in X; for more details concerning the properties of the operator and the C_0-semigroup, we refer the reader to Pazy [26] and Vrabie [29].

Next, we present the definitions and properties copncerning the Kuratowski measure of non-compactness. In the paper, we use $\alpha(\cdot)$ and $\alpha_{\mathcal{B}}(\cdot)$ to denote the Kuratowski measure of non-compactness on the bounded set of X and $\mathcal{B}$, respectively.

Definition 2.5 ([10]). The Kuratowski measure of noncompactness $\alpha(\cdot)$ defined on a bounded set S of the Banach space X is

$$\alpha(S) = \inf\{\delta > 0 : S = \cup_{i=1}^{m} S_i \text{ with } diam(S_i) \leq \delta \ for\ i = 1, 2, \cdots, m\}. \qquad (2.6)$$

Lemma 2.1 ([19]). *Let $B = \{u_n\} \subset \mathcal{B}_b$ be a bounded and countable set. Then $\alpha(B(t))$ is Lebesgue integrable on I, and*

$$\alpha\Big(\Big\{\int_I u_n(t)dt\Big\}\Big) \leq 2\int_I \alpha(B(t))dt.$$

Lemma 2.2 ([3]). *Let $\lambda > 0$. If $g(t)$ and $\beta(t)$ are nonnegative continuous functions satisfying*

$$g(t) = \lambda + \int_0^t \beta(s)g(s)ds, \quad t \in I,$$

then

$$g(t) \leq \lambda e^{\int_0^t \beta(s)ds}, \quad t \in I.$$

Lemma 2.3 ([16]). *Let P be a normal cone of the ordered Banach space X and $v_0, w_0 \in X$ with $v_0 \leq w_0$. Suppose that $F : [v_0, w_0] \to X$ is a nondecreasing strict set-contraction such that $v_0 \leq Fv_0$ and $Fw_0 \leq w_0$. Then F has a minimal fixed point $\underline{u}$ and a maximal point $\overline{u}$ in $[v_0, w_0]$; Moreover*

$$v_n \to \underline{u}, \quad w_n \to \overline{u} \quad as \quad n \to \infty,$$

where $v_n = Fv_{n-1}$ and $w_n = Fw_{n-1}$ $(n = 1, 2, \cdots)$ satisfy

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq \cdots \leq \underline{u} \leq \overline{u} \leq \cdots \leq w_n \leq \cdots \leq w_1 \leq w_0.$$

3. $T(t)(t \geq 0)$ is a compact C_0-semigroup

In this section, we study the existence of extremal solutions for the initial value problem of the impulsive evolution equation with infinite delay (1.1) with a compact C_0-semigroup condition.

Theorem 3.1. *Let X be an ordered Banach space, whose positive cone P is normal with a normal constant N, $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generates a compact positive C_0-semigroup $T(t)(t \geq 0)$ on X. Assume that the nonlinear function $f : I \times X \times \mathcal{B} \to X$ is Carathéodory continuous, and $J_k \in C(X, X)$, $k = 1, 2, \cdots, m$. Suppose the initial value problem of the impulsive evolution equation with infinite delay (1.1) has a lower solution v_0 and an upper solution w_0 with $v_0 \leq w_0$ and assume the following conditions is satisfied:*

(H1) There exists a positive constant C such that

$$f(t, u_2, v_2) - f(t, u_1, v_1) \geq -C(u_2 - u_1),$$

for any $t \in I, u_1, u_2 \in X$ and $v_1, v_2 \in \mathcal{B}$ with $v_0(t) \leq u_1 \leq u_2 \leq w_0(t)$ and $(v_0)_t \leq v_1 \leq v_2 \leq (w_0)_t$;

(H2) For any $u_1, u_2 \in X$ with $v_0(t) \leq u_1 \leq u_2 \leq w_0(t)$, $t \in I$, we have

$$J_k(u_1) \leq J_k(u_2), \quad k = 1, 2, \cdots, m.$$

Then the initial value problem of the impulsive evolution equation with infinite delay (1.1) has a minimal mild solution and a maximal mild solution between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.

Proof. The initial value problem of the impulsive evolution equation with infinite delay (1.1) is equivalent to the following impulsive evolution equation with infinite delay

$$\begin{cases} u'(t) + Au(t) + Cu(t) = f(t, u(t), u_t) + Cu(t), \quad t \in I, \ t \neq t_k, \\ \Delta u|_{t=t_k} = J_k(u(t_k)), \quad k = 1, 2, \cdots, m, \\ u_0 = \varphi \in \mathcal{B}, \end{cases} \tag{3.1}$$

where C is the same constant as in (H1).We consider the operator $F : \mathcal{B}_b \to \mathcal{B}_b$ defined by

$$(Fu)(t) = \begin{cases} S(t)\varphi(0) + \displaystyle\int_0^t S(t-s)[f(s, u(s), u_s) + Cu(s)]ds \\ + \displaystyle\sum_{0<t_k<t} S(t - t_k)J_k(u(t_k)), \quad t \in I, \\ \varphi(t), \quad t \in (-\infty, 0], \end{cases} \tag{3.2}$$

where $S(t) = e^{-Ct}T(t)(t \geq 0)$ is the positive C_0-semigroup generated by $-(A+CI)$. By Definition 2.3, we see that the mild solution of the initial value problem of the impulsive evolution equation with infinite delay (1.1) is equivalent to the fixed point of the operator F defined by (3.2). For any $\varphi \in \mathcal{B}$, let

$$\psi(t) = \begin{cases} S(t)\varphi(0), \quad t \in I, \\ \varphi(t), \quad t \in (-\infty, 0], \end{cases} \tag{3.3}$$

then $\psi \in \mathcal{B}_b$. Further, for any $t \in (-\infty, b]$, let $u(t) = z(t) + \psi(t)$. Now, $u(\cdot)$ satisfies (3.1) if and only if z satisfies $z_0 = 0$, and

$$\begin{aligned} z(t) = &\int_0^t S(t-s)[f(s, z(s) + \psi(s), z_s + \psi_s) + C(z(s) + \psi(s))]ds \\ &+ \sum_{0<t_k<t} S(t - t_k)J_k(z(t_k) + \psi(t_k)), \quad t \in I. \end{aligned} \tag{3.4}$$

Consider the space $\mathcal{B}_b' = \{z : (-\infty, b] \to X \mid z \in \mathcal{B}_b \text{ and } z_0 = 0\}$ endowed with the norm $\|z\|_b = \|z_0\|_{\mathcal{B}} + \sup\limits_{t\in I} \|z(t)\| = \sup\limits_{t\in I} \|z(t)\|$. Note $(\mathcal{B}_b', \|\cdot\|_b)$ is a Banach space.

We define the mapping $\mathcal{F} : [\widetilde{v}_0, \widetilde{w}_0] \to \mathcal{B}_b'$ by

$$(\mathcal{F}z)(t) = \begin{cases} \displaystyle\int_0^t S(t-s)[f(s, z(s) + \psi(s), z_s + \psi_s) + C(z(s) + \psi(s))]ds \\ + \displaystyle\sum_{0<t_k<t} S(t - t_k)J_k(z(t_k) + \psi(t_k)), \quad t \in I, \\ 0, \quad t \in (-\infty, 0], \end{cases} \tag{3.5}$$

where $\widetilde{v}_0,\ \widetilde{w}_0 \in \mathcal{B}'_b \cap C^1(I'',X) \cap C(I',X_1)$ with $v_0(t) = \widetilde{v}_0(t) + \psi(t),\ w_0(t) = \widetilde{w}_0(t) + \psi(t), t \in I$. The operator F has a fixed point if the operator $\mathcal{F}$ has a fixed point.

First, we prove that the operator $\mathcal{F} : [\widetilde{v}_0, \widetilde{w}_0] \to \mathcal{B}'_b$ defined by (3.5) is continuous. For this purpose, let $\{z_n\}_{n=1}^{\infty}$ be a sequence such that $\lim\limits_{n\to\infty} z_n = z$ in $[\widetilde{v}_0, \widetilde{w}_0]$. Then $\lim\limits_{n\to\infty} (z_n)_t = z_t,\ t \in I$. If $t \in I$, by the Carathéodory continuity of the nonlinear function f, and the continuity of the impulsive function J_k for $k = 1, 2, \cdots, m$, we have

$$\lim_{n\to\infty} \|J_k(z_n(t_k) + \psi(t_k)) - J_k(z(t_k) + \psi(t_k))\| = 0, \quad k = 1, 2, \cdots, m, \tag{3.6}$$

and

$$\begin{aligned}&\lim_{n\to\infty} \|f(s, z_n(s) + \psi(s), (z_n)_s + \psi_s) - f(s, z(s) + \psi(s), z_s + \psi_s) \\ &+ C(z_n(s) + \psi(s)) - C(z(s) + \psi(s))\| = 0, \quad s \in [0, t].\end{aligned} \tag{3.7}$$

From condition (H1), we see that for any $z \in [\widetilde{v}_0, \widetilde{w}_0]$ and $s \in [0, t], t \in I$, we have

$$\begin{aligned}&f(s, \widetilde{v}_0(s) + \psi(s), (\widetilde{v}_0)_s + \psi_s) + C(\widetilde{v}_0(s) + \psi(s)) \\ \le& f(s, z(s) + \psi(s), z_s + \psi_s) + C(z(s) + \psi(s)) \\ \le& f(s, \widetilde{w}_0(s) + \psi(s), (\widetilde{w}_0)_s + \psi_s) + C(\widetilde{w}_0(s) + \psi(s)).\end{aligned}$$

The above inequality combined with the normality of the positive cone P, guarantees that there exists a constant $R_1 > 0$, such that

$$\|f(s, z(s) + \psi(s), z_s + \psi_s) + C(z(s) + \psi(s))\| \le R_1, \quad s \in [0, t], \quad t \in I. \tag{3.8}$$

Combining with (2.5), (3.4)-(3.8) and Lebesgue's dominated convergence theorem, for any $t \in I$, we have

$$\begin{aligned}\|(\mathcal{F}z_n)(t) - (\mathcal{F}z)(t)\| \le& M \int_0^t \|[f(s, z_n(s) + \psi(s), (z_n)_s + \psi_s) + C(z_n(s) + \psi(s))] \\ &- [f(s, z(s) + \psi(s), z_s + \psi_s) + C(z(s) + \psi(s))]\|ds \\ &+ M \sum_{0<t_k<t} \|J_k(z_n(t_k) + \psi(t_k)) - J_k(z(t_k) + \psi(t_k))\| \\ \to& 0\ (n \to \infty).\end{aligned} \tag{3.9}$$

Hence, by (3.9) we have

$$\|(\mathcal{F}z_n)(t) - (\mathcal{F}z)(t)\|_b = \sup\{\|(\mathcal{F}z_n)(t) - (\mathcal{F}z)(t)\| : t \in I\} \to 0 \quad (n \to \infty),$$

which means that $\mathcal{F}$ is a continuous operator.

Next, we prove that the operator $\mathcal{F}$ maps $[\widetilde{v}_0, \widetilde{w}_0]$ to $[\widetilde{v}_0, \widetilde{w}_0]$ and is monotonic increasing. Let $z_1, z_2 \in [\widetilde{v}_0, \widetilde{w}_0]$ and $z_1 \le z_2$, then $z_1(t) \le z_2(t)$ for $t \in (-\infty, b]$ and $(z_1)_t \le (z_2)_t$ for $t \in I$. By assumptions (H1), (H2) and the properties of the C_0-semigroup, we see that

$$\mathcal{F}z_1 \le \mathcal{F}z_2, \tag{3.10}$$

which means that $\mathcal{F}$ is an increasing operator in $[\widetilde{v}_0, \widetilde{w}_0]$. Next, we show that $\widetilde{v}_0 \leq \mathcal{F}\widetilde{v}_0, \mathcal{F}\widetilde{w}_0 \leq \widetilde{w}_0$. Let

$$h(t) = v_0'(t) + Av_0(t) + Cv_0(t),\ t \in I,\ t \neq t_k,\ k = 1, 2, \cdots, m.$$

By the definition of the lower solution, we see that

$$h(t) \leq f(t, v_0(t), (v_0)_t) + Cv_0(t), \qquad t \in I.$$

Therefore, by Definitions 2.2 and 2.3 and (3.4), we have

$$\begin{aligned}
&\widetilde{v}_0(t) + \psi(t)\\
=&v_0(t)\\
=&S(t)\varphi(0) + \int_0^t S(t-s)h(s)ds + \sum_{0<t_k<t} S(t-t_k)[v_0({t_k}^+) - v_0({t_k}^-)]\\
\leq&S(t)\varphi(0) + \int_0^t S(t-s)[f(s, v_0(s), (v_0)_s) + Cv_0(s)]ds\\
&+ \sum_{0<t_k<t} S(t-t_k)J_k(v_0(t_k))\\
\leq&\psi(t) + \int_0^t S(t-s)[f(s, \widetilde{v}_0(s) + \psi(s), (\widetilde{v}_0)_s + \psi_s) + C(\widetilde{v}_0(s) + \psi(s))]ds\\
&+ \sum_{0<t_k<t} S(t-t_k)J_k(\widetilde{v}_0(t_k) + \psi(t_k)), \qquad t \in I.
\end{aligned}$$

By the above inequality, we have

$$\begin{aligned}
\widetilde{v}_0(t) \leq &\int_0^t S(t-s)[f(s, \widetilde{v}_0(s) + \psi(s), (\widetilde{v}_0)_s + \psi_s) + C(\widetilde{v}_0(s) + \psi(s))]ds\\
&+ \sum_{0<t_k<t} S(t-t_k)J_k(\widetilde{v}_0(t_k) + \psi(t_k))\\
=&(\mathcal{F}\widetilde{v}_0)(t), \qquad t \in I,
\end{aligned} \tag{3.11}$$

namely, $\widetilde{v}_0 \leq \mathcal{F}\widetilde{v}_0$. Similarly, it can be shown that $\mathcal{F}\widetilde{w}_0 \leq \widetilde{w}_0$. Therefore, $\mathcal{F} : [\widetilde{v}_0, \widetilde{w}_0] \to [\widetilde{v}_0, \widetilde{w}_0]$ is a continuously increasing operator.

Now, we define two sequences $\{\widetilde{v}_n\}$ and $\{\widetilde{w}_n\}$ in $[\widetilde{v}_0, \widetilde{w}_0]$ by the following iterative scheme:

$$\widetilde{v}_n = \mathcal{F}\widetilde{v}_{n-1}, \quad \widetilde{w}_n = \mathcal{F}\widetilde{w}_{n-1}, \quad n = 1, 2, \cdots. \tag{3.12}$$

Then from the monotonicity of $\mathcal{F}$, it follows that

$$\widetilde{v}_0 \leq \widetilde{v}_1 < \widetilde{v}_2 \leq \cdots \leq \widetilde{v}_n \leq \cdots \leq \widetilde{w}_n \leq \cdots \leq \widetilde{w}_2 \leq \widetilde{w}_1 \leq \widetilde{w}_0. \tag{3.13}$$

In the following, we show that the sequence $\{\widetilde{v}_n\}$ and $\{\widetilde{w}_n\}$ converge on I.

For convenience, let $B = \{\widetilde{v}_n \mid n \in \mathbb{N}\}$ and $B_0 = \{\widetilde{v}_{n-1} \mid n \in \mathbb{N}\}$, then $B = \mathcal{F}(B_0)$. For any $t \in I$ and $\widetilde{v}_{n-1} \in B_0$, let

$$(\mathcal{F}_1\widetilde{v}_{n-1})(t) = \int_0^t S(t-s)[f(s, \widetilde{v}_{n-1}(s) + \psi(s), (\widetilde{v}_{n-1})_s + \psi_s)$$

$$+ C(\widetilde{v}_{n-1}(s) + \psi(s))]ds, \tag{3.14}$$

$$(\mathcal{F}_2\widetilde{v}_{n-1})(t) = \sum_{0<t_k<t} S(t-t_k)J_k(\widetilde{v}_{n-1}(t_k) + \psi(t_k)). \tag{3.15}$$

For any $\widetilde{v}_{n-1} \in B_0$, $s \in [0,t]$, and $t \in (0,b]$, by assumption (H1), we have

$$\begin{aligned}
&f(s, \widetilde{v}_0(s) + \psi(s), (\widetilde{v}_0)_s + \psi_s) + C(\widetilde{v}_0(s) + \psi(s)) \\
\le &f(s, \widetilde{v}_{n-1}(s) + \psi(s), (\widetilde{v}_{n-1})_s + \psi_s) + C((\widetilde{v}_{n-1}(s) + \psi(s)) \\
\le &f(s, \widetilde{w}_0(s) + \psi(s), (\widetilde{w}_0)_s + \psi_s) + C(\widetilde{w}_0(s) + \psi(s)).
\end{aligned}$$

Combining with the above inequality and the normality of the positive cone P, we see that there exists a constant $R_2 > 0$, such that for any $\widetilde{v}_{n-1} \in B_0$ and $s \in [0,t]$, $t \in (0,b]$, we have

$$\|f(s, \widetilde{v}_{n-1}(s) + \psi(s), (\widetilde{v}_{n-1})_s + \psi_s) + C(\widetilde{v}_{n-1}(s) + \psi(s))\| \le R_2. \tag{3.16}$$

Hence, for $t \in (0,b]$, take ϵ sufficiently small such that $t - \epsilon \in (0,b]$. Let

$$\begin{aligned}
(\mathcal{F}_1^{\epsilon}\widetilde{v}_{n-1})(t) = &\int_0^{t-\epsilon} S(t-s)[f(s, \widetilde{v}_{n-1}(s) + \psi(s), (\widetilde{v}_{n-1})_s + \psi_s) \\
&+ C(\widetilde{v}_{n-1}(s) + \psi(s))]ds \\
= &S(\epsilon) \int_0^{t-\epsilon} S(t-s-\varepsilon)[f(s, \widetilde{v}_{n-1}(s) + \psi(s), (\widetilde{v}_{n-1})_s + \psi_s) \\
&+ C(\widetilde{v}_{n-1}(s) + \psi(s))]ds.
\end{aligned} \tag{3.17}$$

Since for any $t > 0$, $S(t)$ is a compact operator in X, $\{(\mathcal{F}_1^{\epsilon}\widetilde{v}_{n-1})(t) \mid \widetilde{v}_{n-1} \in B_0\}$ is precompact in X. From (3.14), (3.16) and (3.17), we get

$$\begin{aligned}
\|(\mathcal{F}_1\widetilde{v}_{n-1})(t) - (\mathcal{F}_1^{\epsilon}\widetilde{v}_{n-1})(t)\| = &\int_{t-\epsilon}^{t} \|S(t-s)[f(s, \widetilde{v}_{n-1}(s) + \psi(s), (\widetilde{v}_{n-1})_s + \psi_s) \\
&+ C(\widetilde{v}_{n-1}(s) + \psi(s))]\|ds \\
\le &MR_2\epsilon.
\end{aligned} \tag{3.18}$$

Therefore

$$\|(\mathcal{F}_1\widetilde{v}_{n-1})(t) - (\mathcal{F}_1^{\epsilon}\widetilde{v}_{n-1})(t)\|_b \le MR_2\epsilon. \tag{3.19}$$

This means that there exists a precompact set $\{(\mathcal{F}_1^{\epsilon}\widetilde{v}_{n-1})(t) \mid \widetilde{v}_{n-1} \in B_0\}$ sufficiently close to the set $\{(\mathcal{F}_1\widetilde{v}_{n-1})(t) \mid \widetilde{v}_{n-1} \in B_0\}$ for every $t \in (0,b]$. Therefore, for $t \in (0,b]$, the set $\{(\mathcal{F}_1\widetilde{v}_{n-1})(t) \mid \widetilde{v}_{n-1} \in B_0\}$ is precompact in X.

On the other hand, for any $\widetilde{v}_{n-1} \in B_0$ and $k = 1, 2, \cdots, m$, by assumption (H2), we have

$$J_k(\widetilde{v}_0(t_k) + \psi(t_k)) \le J_k(\widetilde{v}_{n-1}(t_k) + \psi(t_k)) \le J_k(\widetilde{w}_0(t_k) + \psi(t_k)).$$

By the above inequality and the normality of the positive cone P, we see that there exists a constant $R_3 > 0$, such that for any $\widetilde{v}_{n-1} \in B_0$ and $k = 1, 2, \cdots, m$,

$$\|J_k(\widetilde{v}_{n-1}(t_k) + \psi(t_k))\| \le R_3. \tag{3.20}$$

Since for any $t > 0$, $S(t)$ is a compact operator in X, $\{(\mathcal{F}_2\widetilde{v}_{n-1})(t) \mid \widetilde{v}_{n-1} \in B_0\}$ is precompact in X by (3.16) and (3.20). Therefore, for any $t \in (-\infty, b]$, $\{(\mathcal{F}\widetilde{v}_{n-1})(t) \mid \widetilde{v}_{n-1} \in B_0\}$ is precompact in X, which means that $\{\widetilde{v}_n(t)\}$ has a convergent subsequence. Combining with the monotonicity (3.13), we see that $\{\widetilde{v}_n(t)\}$ is convergent for every $t \in (-\infty, b]$, that is

$$\lim_{n\to\infty} \widetilde{v}_n(t) = \underline{z}(t), \quad t \in (-\infty, b].$$

Similarly, we can prove that

$$\lim_{n\to\infty} \widetilde{w}_n(t) = \overline{z}(t), \quad t \in (-\infty, b].$$

Note, $\{\widetilde{v}_n(t)\} \subset \mathcal{B}'_b$. Therefore, for any $t \in I$, by the definition of the operator $\mathcal{F}$, we have

$$\begin{aligned}\widetilde{v}_n(t) =& (\mathcal{F}\widetilde{v}_{n-1})(t)\\ =& \int_0^t S(t-s)[f(s, \widetilde{v}_{n-1}(s) + \psi(s), (\widetilde{v}_{n-1})_s + \psi_s) + C(\widetilde{v}_{n-1}(s) + \psi(s))]ds\\ &+ \sum_{0<t_k<t} S(t-t_k)J_k(\widetilde{v}_{n-1}(t_k) + \psi(t_k)).\end{aligned} \tag{3.21}$$

Letting $n \to \infty$ in the above inequality (3.21), then by the Lebesgue dominated convergence theorem, for $t \in I$, we have

$$\begin{aligned}\underline{z}(t) =& (\mathcal{F}\underline{z})(t)\\ =& \int_0^t S(t-s)[f(s, \underline{z}(s) + \psi(s), (\underline{z})_s + \psi_s) + C(\underline{z}(s) + \psi(s))]ds\\ &+ \sum_{0<t_k<t} S(t-t_k)J_k(\underline{z}(t_k) + \psi(t_k)).\end{aligned} \tag{3.22}$$

Therefore, $\underline{z} \in \mathcal{B}'_b$ and $\underline{z} = \mathcal{F}\underline{z}$. Similarly, we can prove that $\overline{z} \in \mathcal{B}'_b$ and $\overline{z} = \mathcal{F}\overline{z}$. Combining the above conclusion with the monotonicity condition (3.13), we see that $\widetilde{v}_0 \le \underline{z} \le \overline{z} \le \widetilde{w}_0$.

Finally, we prove that $\underline{z}$ and $\overline{z}$ are the minimal and maximal fixed points of $\mathcal{F}$ in $[\widetilde{v}_0, \widetilde{w}_0]$, respectively. In fact, for any $z \in [\widetilde{v}_0, \widetilde{w}_0]$, we have $\mathcal{F}z = z$ and $\widetilde{v}_0 \le z \le \widetilde{w}_0$. Combining this fact with the monotonicity of the operator $\mathcal{F}$, we see $\widetilde{v}_1 = \mathcal{F}\widetilde{v}_0 \le \mathcal{F}z = z \le \mathcal{F}\widetilde{w}_0 = \widetilde{w}_1$. Continuing such a progress, we get $\widetilde{v}_n \le z \le \widetilde{w}_n$. Letting $n \to \infty$, we get $\underline{z} \le z \le \overline{z}$. This means that $\underline{z}$ and $\overline{z}$ are the minimal and the maximal fixed points of the operator $\mathcal{F}$, respectively. Therefore, the operator F has a minimum fixed point $\underline{u}$ and a maximum fixed point $\overline{u}$ between v_0 and w_0, that is, $\underline{u}$ and $\overline{u}$ are the minimum and maximum solution of the initial value problem of the impulsive evolution equation with infinite delay (1.1) on $(-\infty, b]$. □

4. $T(t)(t \ge 0)$ is a non-compact C_0-semigroup

In this section, we study the existence and uniqueness of solutions for the initial value problem of the impulsive evolution equation with infinite delay (1.1) with the noncompact C_0-semigroup condition using the properties of the non-compactness measure and the monotone iterative technique.

Theorem 4.1. *Let X be an ordered Banach space, whose positive cone P is normal with a normal constant N, $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generates the positive C_0-semigroup $T(t)(t \geq 0)$ on X. Assume that the nonlinear function $f : I \times X \times \mathcal{B} \to X$ is Carathéodory continuous, and $J_k \in C(X, X)$, $k = 1, 2, \cdots, m$. Suppose the initial value problem of the impulsive evolution equation with infinite delay* (1.1)*) has a lower solution v_0 and an upper solution w_0 with $v_0 \leq w_0$, and conditions (H1)-(H2) and the following condition are satisfied:*

(H3) For any bounded set $V_1 \subset X$, $V_2 \subset \mathcal{B}$, there exist a continuous function $\mu : I \to \mathbb{R}^+$ such that

$$\alpha(f(t, V_1, V_2)) \leq \mu(t)[\alpha(V_1) + \alpha_{\mathcal{B}}(V_2)], \quad t \in I.$$

Then the initial value problem of the impulsive evolution equation with infinite delay (1.1) *has a minimal mild solution and a maximal mild solution between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Proof. According to Theorem 3.1, if the assumptions (H1) and (H2) hold, then the operator $\mathcal{F}$ defined by (3.4) is continuous and monotonically increasing. The sequence defined by (3.12) satisfies (3.13). In the following, we show that $\{\widetilde{v}_n\}$ and $\{\widetilde{w}_n\}$ are uniformly convergent on I.

For convenience, let $B = \{\widetilde{v}_n \mid n \in \mathbb{N}\}$ and $B_0 = \{\widetilde{v}_{n-1} \mid n \in \mathbb{N}\}$, then $B = \mathcal{F}(B_0)$. From $B = B_0 \cup \{\widetilde{v}_0\}$ it follows that $\alpha(B(t)) = \alpha(B_0(t))$ for any $t \in I$.

For $t \in I_0 = [0, t_1]$, from (3.3),using Lemma 2.1 and assumption (H3), we have

$$\begin{aligned}
&\alpha(B(t)) \\
=&\alpha(\mathcal{F}(B_0)(t)) \\
=&\alpha(\{\int_0^t S(t-s)[f(s, \widetilde{v}_{n-1}(s) + \psi(s), (\widetilde{v}_{n-1})_s + \psi_s) + C(\widetilde{v}_{n-1}(s) + \psi(s))]ds\}) \\
\leq&2M \int_0^t \alpha(\{f(s, \widetilde{v}_{n-1}(s) + \psi(s), (\widetilde{v}_{n-1})_s + \psi_s) + C(\widetilde{v}_{n-1}(s) + \psi(s))\})ds \\
\leq&2M \int_0^t \{\mu(s)[\alpha(B_0(s)) + \alpha_{\mathcal{B}}((B_0)_s)] + C\alpha(B_0(s))\}ds \\
\leq&2M \int_0^t \{\mu(s)[\alpha(B(s)) + \alpha_{\mathcal{B}}(B_s)] + C\alpha(B(s))\}ds \\
\leq&2M \int_0^t \{\mu(s)[\alpha(B(s)) + K \sup_{0\leq\tau\leq s} \alpha(B(\tau))] + C\alpha(B(s))\}ds \\
\leq&2M \int_0^t (C + \mu(s))\alpha(B(s)) + K\mu(s) \sup_{0\leq\tau\leq s} \alpha(B(\tau))ds, \qquad (4.1)
\end{aligned}$$

where $K = \max_{t\in I} K(t)$. Let $p_1(t) = \sup_{0\leq s\leq t} \alpha(B(s))$, $t \in I_0$, then

$$p_1(t) \leq 2M \int_0^t [C + (K + 1)\mu(s)]p_1(s)ds, \quad t \in I_0.$$

By Lemma 2.2, we obtain that $p_1(t) \equiv 0$ on I_0. In particular, $\alpha(B_0(t_1)) = \alpha(B(t_1)) = 0$, and this means that $B(t_1)$ and $B_0(t_1)$ are precompact in X. Thus $J_1(B_0(t_1))$ is precompact in X, and $\alpha(J_1(B_0(t_1))) = 0$.

For $t \in I_1 = (t_1, t_2]$, from the defininition $\mathcal{F}$ and the above discussion of $t \in I_0$, we have

$$\begin{aligned}
&\alpha(B(t)) = \alpha(\mathcal{F}(B_0)(t)) \\
\leq&\alpha(\{\int_0^t S(t-s)[f(s, \widetilde{v}_{n-1}(s) + \psi(s), (\widetilde{v}_{n-1})_s + \psi_s) + C(\widetilde{v}_{n-1}(s) + \psi(s))]ds\}) \\
&+ \alpha(J_1(\widetilde{v}_{n-1}(t_1) + \psi(t_1))) \\
\leq&2M \int_0^t \alpha(\{f(s, \widetilde{v}_{n-1}(s) + \psi(s), (\widetilde{v}_{n-1})_s + \psi_s) + C(\widetilde{v}_{n-1}(s) + \psi(s))\})ds \\
&+ \alpha(J_1(\widetilde{v}_{n-1}(t_1) + \psi(t_1))) \\
\leq&2M \int_0^t \{\mu(s)[\alpha(B_0(s)) + \alpha_{\mathcal{B}}((B_0)_s)] + C\alpha(B_0(s))\}ds \\
\leq&2M \int_0^t \{\mu(s)[\alpha(B(s)) + \alpha_{\mathcal{B}}(B_s)] + C\alpha(B(s))\}ds \\
\leq&2M \int_{t_1}^t \{\mu(s)[\alpha(B(s)) + K \sup_{0 \leq \tau \leq s} \alpha(B(\tau))] + C\alpha(B(s))\}ds \\
\leq&2M \int_{t_1}^t (C + \mu(s))\alpha(B(s)) + K\mu(s) \sup_{t_1 \leq \tau \leq s} \alpha(B(\tau))ds, \qquad (4.2)
\end{aligned}$$

where $K = \max_{t \in I} K(t)$. Let $p_2(t) = \sup_{t_1 \leq s \leq t} \alpha(B(s))$, $t \in I_1$, then

$$p_2(t) \leq 2M \int_{t_1}^t [C + (K+1)\mu(s)]p_2(s)ds, \quad t \in I_1.$$

Again by Lemma 2.2, we obtain that $p_2(t) \equiv 0$ on I_1. Therefore, $\alpha(B_0(t_2)) = \alpha(B(t_2)) = 0$ and $\alpha(J_2(B_0(t_2))) = 0$.

Continuing such a process interval by interval up to I_m, we prove that $\alpha(B(t)) = \alpha(B_0(t)) \equiv 0$ on every I_k, $k = 0, 1, 2, \cdots, m$.

For any I'_k, if we modify the value of v_n at $t = t_k$ via $\widetilde{v}_n(t_{k-1}) = \widetilde{v}_n(t^+_{k-1})$, $n \in \mathbb{N}$, then $\{\widetilde{v}_n\} \subset C(I'_k, X)$ and it is equicontinuous. Since $\alpha(\{\widetilde{v}_n(t)\}) \equiv 0$, $\{\widetilde{v}_n(t)\}$ is precompact in X for every $t \in I'_k$. By the Arzela-Ascoli theorem, $\{\widetilde{v}_n\}$ is precompact in $C(I'_k, X)$. Hence, $\{\widetilde{v}_n\}$ has a convergent subsequence in $C(I'_k, X)$. Combining this with the monotonicity (3.13), we can easily prove $\{\widetilde{v}_n\}$ itself is convergent in $C(I'_k, X)$. In particular, $\{\widetilde{v}_n(t)\}$ is uniformly convergent in I_k. Consequently, $\{\widetilde{v}_n(t)\}$ is uniformly convergent over the whole of I. Hence, $\{\widetilde{v}_n(t)\}$ is convergent in $\mathcal{B}'_b$, that is

$$\lim_{n \to \infty} \widetilde{v}_n(t) = \underline{z}(t), \quad t \in (-\infty, b].$$

Similarly, we can prove that

$$\lim_{n \to \infty} \widetilde{w}_n(t) = \overline{z}(t), \quad t \in (-\infty, b].$$

Using a proof method similar to that in Theorem 3.1, we can prove that $\underline{z}$ and $\overline{z}$ are the minimal and maximal fixed points of $\mathcal{F}$, respectively. Therefore, the operator F also has a minimal fixed point $\underline{u}$ and a maximal fixed point $\overline{u}$, that is, $\underline{u}$ and $\overline{u}$ are the minimum mild solution and the maximum mild solution of the initial value problem of the impulsive evolution equation with infinite delay (1.1) on $(-\infty, b]$, respectively. $\square$

Suppose we replace the assumption (H3) by the following assumption:

(H4) there exist continuous functions $\mu_1,\ \mu_2 \in I \to \mathbb{R}^+$, such that for any $u, v \in [v_0, w_0]$ and $t \in I$, $u_t, v_t \in \mathcal{B}$, we have

$$f(t, u(t), u_t) - f(t, v(t), v_t) \leq \mu_1(t)(u(t) - v(t)) + \mu_2(t)(u_t - v_t), \quad \forall t \in I.$$

Then we have the following unique result.

Theorem 4.2. *Let X be an ordered Banach space, whose positive cone P is normal with a normal constant N, $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generates a positive C_0-semigroup $T(t)(t \geq 0)$ on X. Assume that the nonlinear function $f : I \times X \times \mathcal{B} \to X$ is Carathéodory continuous, and $J_k \in C(X, X)$, $k = 1, 2, \cdots, m$. Suppose the initial value problem of the impulsive evolution equation with infinite delay* (1.1) *has a lower solution v_0 and an upper solution w_0 with $v_0 \leq w_0$, and assume conditions (H1), (H2) and (H4) hold. Then the initial value problem of the impulsive evolution equation with infinite delay* (1.1) *has a unique solution between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Proof. First, we prove that (H1) and (H4) imply (H3). For $t \in I$, let $\{u_n\} \subset [v_0(t), w_0(t)]$ be an increasing sequence. For any $m, n \in \mathbb{N}$ and $m > n$, from conditions (H1) and (H4), we have

$$\begin{aligned} \theta \leq & f(t, u_m(t), (u_m)_t) - f(t, u_n, (u_n)_t) + C(u_m(t) - u_n(t)) \\ \leq & (\mu_1(t) + C)(u_m(t) - u_n(t)) + \mu_2(t)((u_m)_t - (u_n)_t), \end{aligned}$$

by the above inequality and normality of the positive cone P, we have

$$\begin{aligned} & \|f(t, u_m(t), (u_m)_t) - f(t, u_n(t), (u_n)_t)\| \\ \leq & N(C + \mu_1(t))\|u_m(t) - u_n(t)\| + N\mu_2(t)\|(u_m)_t - (u_n)_t\|_{\mathcal{B}} + C\|u_m(t) - u_n(t)\| \\ \leq & [N(C + \mu_1(t)) + C]\|u_m(t) - u_n(t)\| + N\mu_2(t)\|(u_m)_t - (u_n)_t\|_{\mathcal{B}}. \end{aligned}$$

From the above inequality and the definition of the measure of non-compactness, we see that there exist a bounded set $V_1 \subset X,\ V_2 \subset \mathcal{B}$ such that for any $t \in I$, we have

$$\begin{aligned} \alpha(\{f(t, V_1, V_2)\}) \leq & (N\mu_1(t) + NC + C)\alpha(V_1) + N\mu_2(t)\alpha_{\mathcal{B}}(V_2) \\ \leq & (N \max_{t \in I} \mu_1(t) + NC + C)\alpha(V_1) + N \max_{t \in I} \mu_2(t)\alpha_{\mathcal{B}}(V_2) \\ \leq & H(\alpha(V_1) + \alpha_{\mathcal{B}}(V_2)), \end{aligned} \tag{4.3}$$

where $H = \max\{N \max_{t \in I} \mu_1(t) + NC + C,\ N \max_{t \in I} \mu_2(t)\}$. If $\{u_n\}$ is a decreasing sequence, the above inequality is also valid. Hence (H3) holds.

Therefore, by Theorem 3.1, the initial value problem of the impulsive evolution equation with infinite delay (1.1) has the minimal solution $\underline{u}$ and the maximal solution $\overline{u}$ in $[v_0, w_0]$. Going from I_0 and I_m interval by interval we show that $\overline{u} = \underline{u}$ in every I_k.

Clearly, for $t \in (-\infty, 0]$, $\overline{u} = \underline{u}$. For $t \in I_0$, by (3.2), (3.12) and assumption (H4), we get

$$\begin{aligned}
\theta \leq & \overline{u}(t) - \underline{u}(t) \\
= & (F\overline{u})(t) - (F\underline{u})(t) \\
\leq & \int_0^t S(t-s)\big[f(s, \overline{u}(s), (\overline{u})_s) + C\overline{u}(s) - f(s, \underline{u}(s), (\underline{u})_s) - C\underline{u}(s)\big] ds \\
\leq & \int_0^t S(t-s)\big[\mu_1(s)(\overline{u}(s) - \underline{u}(s)) + \mu_2(s)((\overline{u})_s - (\underline{u})_s) + C(\overline{u}(s) - \underline{u}(s))\big] ds \\
\leq & \int_0^t S(t-s)[(\mu_1(s) + C)(\overline{u}(s) - \underline{u}(s)) + K\mu_2(s) \sup_{0 \leq \tau \leq s} (\overline{u}(\tau) - \underline{u}(\tau))] ds. \quad (4.4)
\end{aligned}$$

By (4.4) and the normality of the cone P, we have

$$\begin{aligned}
\|\overline{u}(t) - \underline{u}(t)\| \leq & NM \int_0^t (\mu_1(s) + C)\|\overline{u}(s) - \underline{u}(s)\| + K\mu_2(s) \sup_{0 \leq \tau \leq s} \|\overline{u}(\tau) - \underline{u}(\tau)\| ds \\
\leq & NM \int_0^t \big(\mu_1(s) + C + K\mu_2(s)\big)\|\overline{u}(s) - \underline{u}(s)\|_{\mathcal{B}_b} ds. \quad (4.5)
\end{aligned}$$

Therefore

$$\|\overline{u}(t) - \underline{u}(t)\|_{\mathcal{B}_b} \leq NM \int_0^t \big(\mu_1(s) + C + K\mu_2(s)\big)\|\overline{u}(s) - \underline{u}(s)\|_{\mathcal{B}_b} ds.$$

From Lemma 2.2, we obtain that $\overline{u}(t) \equiv \underline{u}(t)$ in I_0.

For $t \in I_1$, since $J_1(\overline{u}(t_1)) = J_1(\underline{u}(t_1))$, use (3.2) and complete the same argument as above for $t \in I_0$, we get

$$\begin{aligned}
\|\overline{u}(t) - \underline{u}(t)\|_{\mathcal{B}_b} \leq & NM \int_0^t \big(\mu_1(s) + C + K\mu_2(s)\big)\|\overline{u}(s) - \underline{u}(s)\|_{\mathcal{B}_b} ds \\
\leq & NM \int_{t_1}^t \big(\mu_1(s) + C + K\mu_2(s)\big)\|\overline{u}(s) - \underline{u}(s)\|_{\mathcal{B}_b} ds.
\end{aligned}$$

Again by Lemma 2.2, we obtain that $\overline{u}(t) \equiv \underline{u}(t)$ in I_1.

Continuing such a process interval by interval up to I_m, we can prove that $\overline{u}(t) \equiv \underline{u}(t)$ over the whole of $(-\infty, b]$. Hence, $\widetilde{u} = \overline{u} = \underline{u}$ is the unique mild solution of the initial value problem of the impulsive evolution equation with infinite delay (1.1) in $[v_0, w_0]$, which can be obtained by the monotone iterative procedure (3.12) starting from v_0 and w_0. □

5. Application

In this section, the application of the abstract results obtained in this paper in specific problems is illustrated by an example of the initial value problem of the nonlinear heat equation with infinite delay and impulse.

We consider the initial value problem of nonlinear heat equation with infinite delay and impulse:

$$\begin{aligned}
&\frac{\partial}{\partial t}w(x,t)-\iota\frac{\partial^2}{\partial x^2}w(x,t)=L\left(\frac{|w(x,t)|}{1+|w(x,t)|}\right)+\int_{-\infty}^{0}G(s)w(x,t+s)ds,\\
&\qquad\qquad x\in[c,d],\ t\in[0,b],\ t\neq t_k,\\
&w(x,t_k^+)=w(x,t_k^-)+\frac{\sqrt{|w(x,t)|}}{1+|w(x,t)|},\ k=1,2,\cdots,m,\\
&w(x,s)=\phi(x,s),\ x\in[c,d],\ s\in(-\infty,0],
\end{aligned}\tag{5.1}$$

where $\iota>0$ is the thermal conductivity; $b,L>0$ is constant; $0<t_1<t_2<\cdots<t_m<b$, $G\in L((-\infty,b],\mathbb{R}^+)$, $\phi\in C([c,d]\times(-\infty,0],\mathbb{R}^+)$.

Let $X=L^2([c,d],\mathbb{R})$, and its norm is $\|\cdot\|_2$, $P=\{w\in L^2\mid w(x)\geq 0,\ x\in[c,d]\}$, then X is a Banach space, P is a normal cone in X, and the normal constant is $N=1$. Define the operator in X as follows

$$D(A)=H^2(c,d)\cap H_0^1(c,d),\quad Aw=\iota\frac{\partial^2}{\partial x^2}w,$$

then $-A$ generates the positive C_0-semigroup $T(t)(t\geq 0)$.

Let

$$\begin{aligned}
&u(t)=w(\cdot,t),\quad t\in(-\infty,b],\\
&f(t,u(t),u_t)=L\left(\frac{|w(x,t)|}{1+|w(x,t)|}\right)+\int_{-\infty}^{0}G(s)w(x,t+s)ds,\ t\in[0,b],\\
&J_k(u(t_k))=\frac{\sqrt{|w(x,t)|}}{1+|w(x,t)|},\quad k=1,2,\cdots,m,\\
&\varphi(t)=\phi(\cdot,t),\quad t\in(-\infty,0].
\end{aligned}$$

Then the initial value problem of the nonlinear heat equation with infinite delay and impulse (5.1) is transformed into the initial value problem of the impulsive evolution equation with infinite delay (1.1)) in the Banach space X.

By the properties and assumptions of the nonlinear term f and the impulsive term J_k, $k=1,2,\cdots,m$, we can easily verify that $v_0=0$ and $w_0=w(x,t)$ are the lower and upper solutions of problem (5.1), respectively, and there exists a constant $C>0$ such that assumptions (H1) and (H3) hold.

For any $t\in[0,b]$, $u_1,u_2\in X$ satisfy $\theta\leq u_1\leq u_2$, we have that

$$\begin{aligned}
\theta\leq& f\big(t,u_2(t),(u_2)_t\big)-f\big(t,u_1(t),(u_1)_t\big)\\
=&L\left(\frac{u_2(t)}{1+u_2(t)}-\frac{u_1(t)}{1+u_1(t)}\right)+\int_{-\infty}^{0}G(s)\big((u_2)_s-(u_1)_s\big)ds\\
\leq&\frac{L}{1+u_1(t)}\big(|u_2(t)|-|u_1(t)|\big)+\int_{-\infty}^{0}G(s)\big((u_2)_t-(u_1)_t\big)ds.
\end{aligned}$$

Combining with the above inequality and the normality of the positive cone P, we get

$$\|f(t,u_2(t),(u_2)_t)-f(t,u_1(t),(u_1)_t)\|_2$$

$$\leq \frac{L}{1+u_1(t)}\|u_2(t)-u_1(t)\|_2 + \int_{-\infty}^{0} G(s)\|(u_2)_s-(u_1)_s\|_{\mathcal{B}}ds$$
$$\leq \frac{L}{1+u_1(t)}\|u_2(t)-u_1(t)\|_2 + \int_{-\infty}^{0} G(s)ds\|(u_2)_t-(u_1)_t\|_{\mathcal{B}}.$$

Therefore, for any bounded set $V_1 \subset X$, $V_2 \subset \mathcal{B}$, we have

$$\alpha\big(f(t,V_1,V_2)\big) \leq H_1[\alpha(V_1)+\alpha(V_2)], \quad t\in[0,b],$$

where $H_1 = \max\Big\{\max\limits_{t\in[0,b]} \frac{L}{1+u_1(t)}, \int_{-\infty}^{0} G(t)dt\Big\}$. Thus assumption (H3) is established. Therefore, from Theorem 3.1, we see that the initial value problem (1.1) has a minimum mild solution and a maximum mild solution, which can be obtained by the monotone iteration method from v_0 and w_0, respectively. That is to say, the initial value problem of the heat equation with infinite delay and impulse (5.1) has the minimum mild solution and the maximum mild solution between 0 and $w(x,t)$, which can be obtained by the monotone iteration method from v_0 and w_0, respectively.

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