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EXISTENCE OF OSCILLATORY SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS*

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Abstract In this paper, we use Schauder-Tychonoff theorem to obtain a new sufficient condition for the global existence of oscillatory solutions for forced fractional delay differential equations.

Keywords Fractional, Liouville derivative, oscillatory solutions, global existence, Schauder-Tychonoff theorem.

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1. Introduction

Fractional differential equations have garnered significant attention owing to their widespread utility across diverse domains, including but not limited to fluid mechanics, chemical physics, electronic networks, dynamic system control theory, fluid dynamics, economics, and various other fields with broad and multifaceted applications, see Book [3, 10, 12].

Scholars have shown a preference for the oscillation theory of integral order functional differential equations, recognizing its vital theoretical importance and practical significance, which has led to substantial advancements in this field, see classic Book [1, 2, 4, 6].

In recent studies, Duan et al. [7], Harikrishnan et al. [9], Raheem et al. [13], Zhou et al. [17] and Feng et al. [5] have investigated oscillation and forced oscillation characteristics for fractional-order delay differential equations. The nonoscillatory theory for fractional differential equations has been further discussed by scholars including Zhou et al. [16], Sun et al. [14] and Grace et al. [8]. Nonetheless, there has been limited exploration into the existence of oscillatory solutions in fractional functional differential equations that involve distributed delays. We will now address this issue in the following discussion.

In this paper, we study the existence of oscillatory solutions for the fractional

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delay differential equations with forced term

$$D_t^{\alpha}\left[r(t)\Phi(x'(t))\right] + \sum_{i=1}^{m} f_i(t, x(g_i(t))) = q(t), \quad t \geq t_0, \tag{1.1}$$

where D_t^{α} is Liouville fractional derivative of order $\alpha \geq 0$ on the half-axis, $r \in C\big([t_0,\infty), R^+\big), q, g_i \in C\big([t_0,\infty), R\big), f_i \in C\big([t_0,\infty)\times R, R\big)$, and $g_i(t) \leq t, \lim_{t\to\infty} g_i(t) = \infty, i = 1, 2, \cdots, m, \Phi(u)$ is continuously increasing real function with respect to u defined on R, and $\Phi^{-1}(u)$ satisfies the local Lipschitz condition.

2. Preliminaries

In this section, we introduce preliminary details that will be used throughout this paper.

Definition 2.1. A solution of Equation 1.1 is a function $x(t)$ defined on $[T,\infty)$ such that $x(t)$ and $r(t)\Phi(x'(t))$ exist on $[t_1,\infty)$ and equation 1.1 holds for all $t_1 > T$. Such a solution is said to be oscillatory if it has a sequence of zeros tending to infinity. Otherwise, it is said to be nonoscillatory.

Definition 2.2. [10] (Liouville fractional integrals on the half-axis) The Liouville fractional derivative on the half-axis is defined by

$$D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^{\infty} (s-t)^{\alpha-1} f(s)ds,$$

where $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} t^{\alpha-1}e^{-t}dt, t \in R$ and$\alpha \in [0,\infty)$.

Definition 2.3. [10] (Liouville fractional derivatives on the half-axis) The Liouville fractional derivative on the half-axis is defined by

$$D_t^{\alpha} f(t) = \frac{d^n}{dt^n}(D_t^{-(n-\alpha)} f(t)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^{\infty} (s-t)^{n-\alpha-1} f(s)ds,$$

where $n = [\alpha]+1, \alpha \in [0,\infty), [\alpha]$ denotes the integer part of α and $t \in R$. In particular, if $\alpha = n \in N$, then$D_t^n f(t) = f^{(n)}(t)$, where $f^{(n)}(t)$ is the usual derivative of $f(t)$ of order n.

Property 2.1. [10] For $\alpha > 0$, we have

$$D_t^{\alpha}(D_t^{-\alpha} f)(t) = f(t).$$

We will prove a general result about equation 1.1 on the existence of oscillatory solutions.

Here, for any $\sigma \geq t_0$, let $T = \min_{1\leq i\leq m} \inf_{t\geq\sigma} g_i(t)$. For a constant $\gamma > 0, p_i(t)_\gamma = \max_{|x|\leq\gamma} \frac{1}{\gamma}|f_i(t,x)|, t \geq t_0, i = 1, 2, \cdots, m$, L_γ denote the local *lipischitz* constants of functions $\Phi^{-1}(u)$.

3. The main results

Lemma 3.1 ([15]). *Let S be a locally convex topological space. For any nonempty compact convex set $K \subset S$, any continuous map $F : K \to K$ has a fixed point.*

Theorem 3.1. *Assume there exists $\eta, \gamma > 0$ such that $r(t) > \eta$,*

$$\frac{1}{r(t)} \int_t^\infty s^{\alpha-1} q(s) ds \text{ is integrable on } [t_0, \infty), \tag{3.1}$$

$$\frac{1}{r(t)} \int_t^\infty s^{\alpha-1} \sum_{i=1}^m p_i(s)_\gamma ds \text{ is integrable on } [t_0, \infty), \tag{3.2}$$

moreover, there exist two increasing divergent sequences $\{t_n\}$ and $\{s_n\}$, such that

$$\int_{t_n}^\infty \Phi^{-1}\left(\frac{1}{\Gamma(\alpha) r(s)} \int_s^\infty (u-t)^{\alpha-1} (q(u) + \gamma \sum_{i=1}^m p_i(u)_\gamma) du\right) ds < 0, \tag{3.3}$$

$$\int_{s_n}^\infty \Phi^{-1}\left(\frac{1}{\Gamma(\alpha) r(s)} \int_s^\infty (u-t)^{\alpha-1} (q(u) - \gamma \sum_{i=1}^m p_i(u)_\gamma) du\right) ds > 0, \tag{3.4}$$

then equation (1.1) has an oscillatory solution $x(t)$ defined on $[t_0, \infty)$ with $|x| \le \gamma$, and $\lim_{t\to\infty} x(t) = 0$.

Proof. The proof is based on an application of the well known Schauder-Tychonoff fixed point theorem.

From (3.1) and (3.2), for any $\gamma > 0$ we can choose a large number T_γ, such that for all $t \ge T_\gamma \ge T$,

$$\int_t^\infty \Phi^{-1}\left(\frac{1}{\Gamma(\alpha) r(s)} \int_s^\infty (u-t)^{\alpha-1} (q(u) + \gamma \sum_{i=1}^m p_i(u)_\gamma) du\right) ds \le \gamma, \tag{3.5}$$

$$\int_t^\infty \Phi^{-1}\left(\frac{1}{\Gamma(\alpha) r(s)} \int_s^\infty (u-t)^{\alpha-1} (q(u) - \gamma \sum_{i=1}^m p_i(u)_\gamma) du\right) ds \ge -\gamma. \tag{3.6}$$

Let $C[T, \infty)$ denote the locally convex space of all continuous functions with topology of uniform convergence on compact subsets of $[T, \infty)$, let $S = \{x \in C[T, \infty),\ |x(t)| \le \gamma\}$, clearly, S is a close convex subset of $C[T, \infty)$.

Introduce an operator F by the following formula,

$$(Fx)(t) = \begin{cases} \displaystyle\int_t^\infty \Phi^{-1}\left(\frac{1}{\Gamma(\alpha) r(t)} \int_s^\infty (u-t)^{\alpha-1} (q(u) - \sum_{i=1}^m f_i(u, x(g_i(u)))) du\right) ds, & t > T_\gamma, \\ (Fx)(T_\gamma), & T \le t \le T_\gamma. \end{cases}$$

It is easy to see that for any $x \in S$, $(Fx)(t)$ is well defined on $[T, \infty)$ continuously. From (3.5) and (3.6) we obtain

$$(Fx)(t) \le \int_t^\infty \Phi^{-1}\left(\frac{1}{\Gamma(\alpha) r(t)} \int_s^\infty (u-t)^{\alpha-1} (q(u) + \sum_{i=1}^m p_i(u)_\gamma) du\right) ds$$

$$\leq \gamma, \quad t \geq T,$$

and

$$(Fx)(t) \geq \int_t^\infty \Phi^{-1}\left(\frac{1}{\Gamma(\alpha)r(t)}\int_s^\infty (u-t)^{\alpha-1}(q(u)-\sum_{i=1}^m p_i(u)_\gamma)du\right)ds$$
$$\geq -\gamma, \quad t \geq T.$$

Hence $|(Fx)(t)| \leq \gamma$, thus we have $FS \subset S$ and Fx is uniformly bounded on S.

Let$\{x_n\}_{n=1}^\infty \in S$ be any sequence and $x_0 \in S$ with $\lim\limits_{n\to\infty} x_n = x_0$. Let T_1 be a large constant with $T_1 > T$, for any $\epsilon > 0$ that

$$\int_{T_1}^\infty \frac{1}{\Gamma(\alpha)r(s)}\int_s^\infty \left((u-t)^{\alpha-1}\sum_{i=1}^m p_i(u)_\gamma du\right)ds < \frac{\epsilon}{3\gamma L_\gamma}. \tag{3.7}$$

From the compactness of the domain of f_i, there exists a large $N(\epsilon) > 0$ and a constant $\delta(\epsilon) > 0$, let $t \in [T, T_1]$ and $n \geq N$ when $|x_n - x_0| < \delta(\epsilon)$,

$$\max_{1\leq i\leq m} |f_i(t, x_n(g_i(t))) - f_i(t, x_0(g_i(t)))| \leq \frac{\epsilon}{3L_\gamma mM}, \tag{3.8}$$

where $M = \int_T^{T_1} \frac{(s-T)^{\alpha-1}}{\Gamma(\alpha)r(s)}ds$. By the virtue of (3.1)-(3.8), we have that for any $t \geq T$ and $|x_n - x_0| < \delta$,

$$\begin{aligned}
&|(Fx_n)(t) - (Fx_0)(t)|\\
=&|\int_t^\infty \Phi^{-1}\left(\frac{1}{\Gamma(\alpha)r(s)}\int_s^\infty (u-t)^{\alpha-1}(q(u)-\sum_{i=1}^m f_i(u, x_n(g_i(u)))du\right)ds\\
&-\int_t^\infty \Phi^{-1}\left(\frac{1}{\Gamma(\alpha)r(s)}\int_s^\infty (u-t)^{\alpha-1}(q(u)-\sum_{i=1}^m f_i(u, x_0(g_i(u))))du\right)ds|\\
\leq&\int_t^\infty \left(\frac{L_\gamma}{\Gamma(\alpha)r(s)}\int_s^\infty (u-t)^{\alpha-1}\sum_{i=1}^m |f_i(u, x_n(g_i(u))) - f_i(u, x_0(g_i(u)))|du\right)ds\\
\leq&\int_T^{T_1} \left(\frac{L_\gamma}{\Gamma(\alpha)r(s)}\int_s^\infty (u-t)^{\alpha-1}\sum_{i=1}^m |f_i(u, x_n(g_i(u))) - f_i(u, x_0(g_i(u)))|du\right)ds\\
&+\int_{T_1}^\infty \left(\frac{L_\gamma}{\Gamma(\alpha)r(s)}\int_s^\infty (u-t)^{\alpha-1}\sum_{i=1}^m |f_i(u, x_n(g_i(u))) - f_i(u, x_0(g_i(u)))|du\right)ds\\
<&\frac{\epsilon}{3} + \frac{2\epsilon}{3}\\
=&\epsilon,
\end{aligned}$$

the continuity of F on S is proved.

Moreover, for all $t_2, t_1 > T$,

$$\begin{aligned}
&(Fx)(t_2) - (Fx)(t_1)\\
=&\int_{t_1}^{t_2} \Phi^{-1}\left(\frac{1}{\Gamma(\alpha)r(s)}\int_s^\infty (u-t)^{\alpha-1}(q(u)-\sum_{i=1}^m f_i(u, x(g_i(u))))du\right)ds
\end{aligned}$$

$$\leq \int_{t_1}^{t_2} \Phi^{-1}\left(\frac{1}{\Gamma(\alpha)r(s)}\int_s^\infty (u-t)^{\alpha-1}(q(u)+\sum_{i=1}^m p_i(u)_\gamma)du\right) ds$$
$$\leq K_1(t_2-t_1),$$

where $K_1 = \sup_{t\geq t_0} \Phi^{-1}(\frac{1}{\Gamma(\alpha)r(s)}\int_s^\infty (u-t)^{\alpha-1}(q(u)+\sum_{i=1}^m p_i(u)_\gamma)du)$.

$$(Fx)(t_2)-(Fx)(t_1)$$
$$=\int_{t_1}^{t_2} \Phi^{-1}\left(\frac{1}{\Gamma(\alpha)r(s)}\int_s^\infty (u-t)^{\alpha-1}(q(u)-\sum_{i=1}^m f_i(u,x(g_i(u))))du\right) ds$$
$$\geq \int_{t_1}^{t_2} \Phi^{-1}\left(\frac{1}{\Gamma(\alpha)r(s)}\int_s^\infty (u-t)^{\alpha-1}(q(u)-\sum_{i=1}^m p_i(u)_\gamma)du\right) ds$$
$$\geq K_2(t_2-t_1),$$

where $K_2 = \inf_{t\geq t_0} \Phi^{-1}\left(\frac{1}{\Gamma(\alpha)r(s)}\int_s^\infty (u-t)^{\alpha-1}(q(u)-\sum_{i=1}^m p_i(u)_\gamma)du\right)$, thus,

$$|(Fx)(t_2)-(Fx)(t_1)| \leq K|t_2-t_1|,$$

where $K = \max\{|K_1|,|K_2|\}$, this implies Fx is equicontinuous, hence by the Ascoli-Arzela Theorem the operator is a completely continous on S. By Lemma, there exists $\tilde{x} \in S$ satisfying $\widetilde{x}(t) = (F\widetilde{x})(t)$,

$$\widetilde{x}(t) = \int_t^\infty \Phi^{-1}\left(\frac{1}{\Gamma(\alpha)r(t)}\int_s^\infty (u-t)^{\alpha-1}(q(s)-\sum_{i=1}^m f_i(s,\widetilde{x}(g_i(s)))ds\right),$$
$$r(t)\Phi(\widetilde{x}'(t)) = \frac{1}{\Gamma(\alpha)}\int_s^\infty (u-t)^{\alpha-1}(q(s)-\sum_{i=1}^m f_i(s,\widetilde{x}(g_i(s)))ds,$$

from Property 2.1, it is easy to see that $\tilde{x}(t)$ is a solution of Equation 1.1.

On the other hand, from (3.3) and (3.4), we find

$$\widetilde{x}(t_n) \leq \int_{t_n}^\infty \Phi^{-1}\left(\frac{1}{\Gamma(\alpha)r(s)}\int_s^\infty (u-t)^{\alpha-1}(q(u)+\gamma\sum_{i=1}^m p_i(u)_\gamma)du\right) ds < 0,$$

and

$$\widetilde{x}(s_n) \geq \int_{s_n}^\infty \Phi^{-1}\left(\frac{1}{\Gamma(\alpha)r(s)}\int_s^\infty (u-t)^{\alpha-1}(q(u)-\gamma\sum_{i=1}^m p_i(u)_\gamma)du\right) ds > 0,$$

which implies that $\tilde{x}(t)$ is a bounded oscillatory solution of 1.1 and $\lim_{t\to\infty}\widetilde{x}(t) = 0$. The proof is complete. □

Corollary 3.1. *Assume (3.1) and (3.2) of Theorem hold, and specially,* $\Phi(u) = u^\lambda, \lambda \geq 1$ *is the ratio of two positive odd integers, there exist two increasing divergent sequences* $\{t_n\}$ *and* $\{s_n\}$*, such that*

$$\int_{t_n}^\infty \left(\frac{1}{\Gamma(\alpha)r(s)}\int_s^\infty (u-t)^{\alpha-1}(q(u)+\gamma\sum_{i=1}^m p_i(u)_\gamma)du\right)^{\frac{1}{\lambda}} ds < 0,$$

and

$$\int_{s_n}^{\infty}\left(\frac{1}{\Gamma(\alpha)r(s)}\int_{s}^{\infty}(u-t)^{\alpha-1}(q(u)-\gamma\sum_{i=1}^{m}p_i(u)_\gamma)du\right)^{\frac{1}{\lambda}}ds>0.$$

Then (1.1) *has an oscillatory solution* $x(t)$ *defined on* $[t_0,\infty)$ *with* $|x|\le\gamma$*, and* $\lim_{t\to\infty}x(t)=0$.

4. Remark

We consider the existence of oscillatory solutions of equation 1.1 for any order $\alpha>0$. In particular, for $\alpha=1$, equation 1.1 reduces to equation 1.1 of reference [11].

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