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INFINITELY MANY SOLUTIONS FOR A *P*-SUPERLINEAR *P*-LAPLACIAN PROBLEMS*

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Abstract We are concerned with the existence of infinitely many solutions for p-Laplacian problem

$$\begin{cases} -(\varphi_p(u'))' = g(u) + h(x, u, u'), & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{P}$$

where $\varphi_p(s) := |s|^{p-2} \cdot s$, $p > 1$, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function and satisfies p-superlinear growth at infinity, $h : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function satisfying $|h(x, \xi, \xi_1)| \leqslant C + \frac{1}{2}|\varphi_p(\xi)|$. Based on global bifurcation techniques, we obtain infinitely many solutions of (P) having specified nodal properties.

Keywords Solutions, p-Laplacian, p-superlinear, bifurcation.

MSC(2010) 34B05, 35B32.

1. Introduction

We are concerned with the existence of infinitely many solutions for p-Laplacian problem

$$\begin{cases} -(\varphi_p(u'))' = g(u) + h(x, u, u'), & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where $\varphi_p(s) := |s|^{p-2}s$, $p > 1$, nonlinear functions g and h satisfy

(H1) $g : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$\lim_{|\xi| \to \infty} \frac{g(\xi)}{\varphi_p(\xi)} = \infty; \tag{1.2}$$

(H2) $h : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and there exists $C > 0$ such that

$$|h(x, \xi, \xi_1)| \leqslant C + \frac{1}{2}|\varphi_p(\xi)|. \tag{1.3}$$

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Problems involving the p-Laplacian have been attracting the attention of many researchers in the last few years due to its applicability in several areas of science, such as electrorheological fluids and imagine processing, see [4]. Hence, problems similar as (1.1) have been treated by many authors, the related researches can be founded in Ambrosetti [3], del Pino and Manásevich [8], Naito and Tanaka [14], Dai and Ma [6,7], and [2,5,9–12,14,15,19].

When $h = 0$ and g is of the form $a(x)f(u)$, problem (1.1) were investigated by Wang [20] under the condition:

$$f_0 := \lim_{s\to 0^+} \frac{f(s)}{\varphi_p(s)} = 0, \quad \text{and} \quad f_\infty := \lim_{s\to\infty} \frac{f(s)}{\varphi_p(s)} = \infty.$$

By using the fixed point theorem in cones, he obtained a positive solution of the problem, and then, Dai and Ma [6] studied the existence of nodal solutions for p-Laplacian problem

$$\begin{cases} -(\varphi_p(u'))' = f(x,u),\ x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.4}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and satisfies

(F1) $\lambda_k(p) \leqslant a(x) \equiv \lim_{|s|\to\infty} \frac{f(x,s)}{\varphi_p(s)}$ uniformly on [0,1], and the inequality is strict on some subset of positive measure in (0,1);

(F2) $0 \leqslant \lim_{|s|\to 0} \frac{f(x,s)}{\varphi_p(s)} \equiv c(x) \leqslant \lambda_k(p)$ uniformly on [0,1], and the inequality is strict on some subset of positive measure in (0,1);

(F3) $f(x,s)\varphi_p(s) > 0$ for a.e. $x \in (0,1)$ and $s \neq 0$,

where $\lambda_k(p)$ is the k-th eigenvalue of eigenvalue problem of (1.4). They obtained the result as follows:

Theorem A [6, Theorem 4.1] *Suppose that $f(x,u)$ satisfies (F1), (F2) and (F3), then problem (1.4) possesses two solutions u_k^+ and u_k^- such that u_k^+ has exactly $k-1$ zeros in $(0,1)$ and is positive near 0, and u_k^- has exactly $k-1$ zeros in $(0,1)$ and is negative near 0.*

Noting that in the works we mentioned above, the asymptotic behavior of nonlinearity near 0 is crucial. The assumption $f_0 = 0$ in [20] can ensure that f satisfies the conditions of fixed point theorem in cones near 0. As for (F3) in [6], which ensures that $f(0) = 0$ and f have constant sign when $s > 0$ or $s < 0$, then combining the other assumptions and bifurcation technology, there is a component of solutions of (1.4), which bifurcates from trivial solution.

Going back to problem (1.1) and letting $F(s) = g(s) + h(x,s,v)$, we observe that there is a possibility of $F(0) \neq 0$. This means one could not treat the case in the same way with (1.4). Now it motivates the question: how do we deal with the case $F(0) \neq 0$?

With respect to the case $F(0) < 0$, finding solutions to the problem similar as (1.1) has been actively studied for a long time, and it is not an easy task, we can refer to Alotaibi et al. [2], Agarwal et al. [1], Chu et al. [5] and Morres et al. [13], but they only obtained the existence of positive solutions, and no one got the result about nodal solutions. The likely reason is that due to $F(0) < 0$, it is difficult to give the nodal properties and bifurcation behavior of solution near 0.

Incidentally, let us mention that an important work of Rynne [17], he established a result of nodal solutions about fourth-order problem when $F(0) \neq 0$. Motivated by [6] and [17], we attempt to investigate the nodal solutions for (1.1). But it is well-known that there are many differences between fourth-order problem and p-Laplacian problem, for example, the Green functions of (1.1) do not exist and the p-Laplacian operator is not self-adjoint, then the techniques in [17] are not applicable to (1.1), such as the proof of Lemmas 2.1-2.3 and 3.8.

For any integer $r \geqslant 0$, let $C^r[0,1]$ denote the standard Banach space of real valued, r-times continuously differentiable functions defined on [0, 1], with the norm

$$\|u\|_r = \sum_{i=0}^{r} \|u^{(i)}\|_0,$$

where $\|\cdot\|_0$ denotes by $\|u\|_0 = \max_{x\in[0,1]} |u(x)|$.

Let

$$E = \{u \in C^1[0,1] : u(0) = u(1) = 0\},\ X = E \cap \{u : \varphi_p(u') \in C^1[0,1]\},\ Y = C[0,1].$$

In what follows, we use the terminology of Rabinowitz [16]. Let $S_{k,+}$ denote the set of functions in E which have exactly $k-1$ interior nodal (i.e. non-degenerate) zeros in (0,1) and are positive near $t = 0$, and set $S_{k,-} = -S_{k,+}$, and $S_k = S_{k,+} \cup S_{k,-}$. It is clear that $S_{k,+}$ and $S_{k,-}$ are disjoint and open in E. A solution of (1.1) is a function $u \in X$ satisfying (1.1).

Theorem 1.1. *There exists an integer $k_0 \geqslant 1$, such that for all integers $k \geqslant k_0$, the problem (1.1) have the solutions $u_{k,+} \in S_{k,+}$ and $u_{k,-} \in S_{k,-}$.*

Remark 1.2. Observed that Theorem A shows all solutions of (1.4) having nodal properties for all $k \in \mathbb{N}^+$. Compared with the case of $F(0) = 0$, the effect of putting no restrictions on F near 0 is that the solutions of (1.1) have nodal properties only when $k \geqslant k_0$ and $\|u\|_0 \geqslant \zeta_3(\lambda)$ (defined in (2.8)).

2. Auxiliary results

For any $u \in X$ we define $e(u)(x) : [0,1] \to \mathbb{R}$ by

$$e(u)(x) = h(x, u(x), u'(x)), \qquad x \in [0,1].$$

It follows from (1.3) that

$$|e(u)(x)| \leqslant C + \frac{1}{2}|\varphi_p(u(x))|, \qquad x \in [0,1]. \tag{2.1}$$

Define continuous functions γ, Φ, G as follows:

$$\gamma(s) = \max\{|g(\xi)| : |\xi| \leqslant s,\ s \geqslant 0\}, \tag{2.2}$$

$$\Phi(u(x)) = \int_0^x |\varphi_p(u(t))| \mathrm{d}t, \qquad t \in [0,1], \tag{2.3}$$

$$G(u(x)) = \int_0^x |g(u(t))| \mathrm{d}t, \qquad t \in [0,1]. \tag{2.4}$$

We now consider the boundary value problem

$$\begin{cases} -(\varphi_p(u'))' = \lambda\varphi_p(u) + \alpha(g(u) + e(u)), & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{2.5}$$

where $\alpha \in [0,1]$ is an arbitrary fixed number and $\lambda \in \mathbb{R}$. Obviously, u is a solution of (1.1) if and only if u is a solution of (2.5) when $\lambda = 0$ and $\alpha = 1$. In the following lemmas, $(\lambda, u) \in \mathbb{R} \times X$ will be supposed an arbitrary solution of (2.5) and $R \geqslant 0$ will be an constant. Also, η_1, η_2 will be positive constants and $\zeta_i : [0,\infty) \to [0,\infty)$ $(i = 1, 2, \cdots)$ will be continuous functions.

By the boundary condition of (1.1) and Role's theorem, for any $u \in X$, there exists a $\tau \in (0,1)$ such that $u'(\tau) = 0$, then

$$|u(x)| = |\int_0^x u'(t)\mathrm{d}t| \leqslant \int_0^1 |u'(t)|\mathrm{d}t \leqslant \|u'\|_0,$$
$$|u'(x)| = |\int_\tau^x u''(t)\mathrm{d}t| \leqslant \int_0^1 |u''(t)|\mathrm{d}t \leqslant \|u''\|_0.$$

Hence

$$\|u\|_0 \leqslant \|u'\|_0 \leqslant \|u''\|_0. \tag{2.6}$$

Lemma 2.1. *If*

$$0 \leqslant \lambda \leqslant R, \qquad \|u\|_0 \leqslant R,$$

then there exists a ζ_1 such that

$$\|u\|_1 \leqslant \zeta_1(R).$$

Proof. Integrate (2.5) over $[\tau, x]$ we get

$$-\varphi_p(u'(x)) = \int_\tau^x [\lambda\varphi_p(u(t)) + \alpha g(u(t)) + \alpha e(u)(t)]\mathrm{d}t,$$

Combining with (2.2) we get

$$\begin{aligned} |-\varphi_p(u'(x))| &= |\varphi_p(u'(x))| \\ &\leqslant \left|\int_\tau^x [\lambda|\varphi_p(u(t))| + \alpha|g(u(t))| + \alpha|e(u)(t)|]\mathrm{d}t\right| \\ &\leqslant \left|\int_0^1 [\lambda|\varphi_p(u(t))| + \alpha|g(u(t))| + \alpha|e(u)(t)|]\mathrm{d}t\right| \\ &\leqslant \left|\int_0^1 [R \cdot R^{p-1} + \gamma(R) + C + \frac{1}{2}R^{p-1}]\mathrm{d}t\right| \\ &= R^p + \gamma(R) + C + \frac{1}{2}R^{p-1}. \end{aligned}$$

Moreover,

$$|\varphi_p(u'(x))| = \varphi_p(|u'(x)|) \leqslant R^p + \gamma(R) + C + \frac{1}{2}R^{p-1},$$

then we get

$$|u'(x)| \leqslant \varphi_p^{-1}(R^p + \gamma(R) + C + \frac{1}{2}R^{p-1}).$$

Define $\zeta_1(R)/2 := \varphi_p^{-1}(R^p + \gamma(R) + C + R^{p-1}/2)$, this shows $\|u\|_1 \leqslant \zeta_1(R)$. □

Lemma 2.2. *For any $x_0, x_1 \in [0,1]$ and $x_0 \geqslant x_1$, then there exists an increasing function ζ_2 such that if $0 \leqslant \lambda \leqslant R$, and $|u(x_0)| + |u'(x_0)| \leqslant R$, then $\|u\|_0 \leqslant \zeta_2(R)$.*

Proof. Choose $x_1 \in [0,1]$ such that $|u'(x_1)| = \|u'\|_0$, then we get

$$\varphi_p(u'(x_1)) = \lambda \int_{x_1}^{x_0} \varphi_p(u)\mathrm{d}x + \alpha \int_{x_1}^{x_0} g(u(x))\mathrm{d}x + \alpha \int_{x_1}^{x_0} e(u(x))\mathrm{d}x + \varphi_p(u'(x_0)).$$

We can choose $x_0 = x_1$ when $x_1 = 1$ (the conclusion clearly holds) and $x_0 > x_1$ when $x_1 \in [0,1)$, then by (2.1), (2.3), (2.4) and (2.6), we obtain

$$\begin{aligned}
&|\varphi_p(u'(x_1))| \\
&= |u'(x_1)|^{p-1} \\
&\leqslant \lambda \int_{x_1}^{x_0} |\varphi_p(u(t))|\mathrm{d}t + \alpha \int_{x_1}^{x_0} |g(u(t))|\mathrm{d}t + \alpha \int_{x_1}^{x_0} |e(u)(t)|\mathrm{d}t + |\varphi_p(u'(x_0))| \\
&= \lambda\Phi(u(x_0)) - \lambda\Phi(u(x_1)) + \alpha G(u(x_0)) - \alpha G(u(x_1)) \\
&\quad + \alpha \int_{x_1}^{x_0} |e(u)(t)|\mathrm{d}t + |\varphi_p(u'(x_0))|,
\end{aligned}$$

that is,

$$\begin{aligned}
&|u'(x_1)|^{p-1} + \lambda\Phi(u(x_1)) + \alpha G(u(x_1)) \\
\leqslant&\lambda\Phi(u(x_0)) + \alpha G(u(x_0)) + \alpha \int_{x_1}^{x_0} |e(u)(t)|\mathrm{d}t + |\varphi_p(u'(x_0))|,
\end{aligned}$$

then we get

$$\begin{aligned}
|u'(x_1)|^{p-1} &\leqslant \lambda\Phi(u(x_0)) + G(u(x_0)) + \int_{x_1}^{x_0} |e(u)(t)|\mathrm{d}x + |\varphi_p(u'(x_0))| \\
&\leqslant R\Phi(R) + R^{p-1} + G(R) + C + \frac{1}{2}\|u\|_0^{p-1} \\
&\leqslant R\Phi(R) + R^{p-1} + G(R) + C + \frac{1}{2}\|u'\|_0^{p-1}.
\end{aligned}$$

Let

$$K(R) := R\Phi(R) + R^{p-1} + G(R) + C.$$

If $\|u'\|_0 \leqslant 1$, then $\|u\|_0 \leqslant 1$, and if $\|u'\|_0 > 1$, then

$$|u'(x_1)|^{p-1} = \|u'\|_0^{p-1} \leqslant K(R) + \frac{1}{2}\|u'\|_0^{p-1},$$

then

$$\|u\|_0 \leqslant \|u'\|_0 \leqslant (2K(R))^{\frac{1}{p-1}}.$$

Define $\zeta_2(R) := \max\{1, (2K(R))^{\frac{1}{p-1}}\}$, then $\|u\|_0 \leqslant \zeta_2(R)$. □

By (1.2) we can choose $\eta_1 \geqslant 1$ such that if $|\xi| \geqslant \eta_1$, then

$$|g(\xi)| \geqslant C + \frac{1}{2}|\varphi_p(\xi)|. \tag{2.7}$$

We also define function $\zeta_3 : \mathbb{R} \to [\eta_1, \infty)$ by

$$\zeta_3(\xi) = \begin{cases} \zeta_2(\xi + \xi^2) + \eta_1, & \xi \geqslant \eta_1, \\ \zeta_3(\eta_1), & \xi < \eta_1. \end{cases} \tag{2.8}$$

Clearly, ζ_3 is increasing.

Lemma 2.3. *If $R \geqslant \eta_1$, $0 \leqslant \lambda \leqslant R$ and $\|u\|_0 \geqslant \zeta_3(R)$, then for any $x_0 \in [0,1]$ with $|u(x_0)| \leqslant R$, we have $|u'(x_0)| \geqslant R^2$.*

Proof. Suppose that for some $R \geqslant \eta_1$ there exists $x_0 \in [0,1]$ such that $|u(x_0)| \leqslant R$ and $|u'(x_0)| < R^2$. Then Lemma 2.2 shows

$$\|u\|_0 \leqslant \zeta_2(R + R^2).$$

This is a contradiction! □

3. Proof of main result

We now consider the problem

$$-(\varphi_p(u'))' = \lambda\varphi_p(u) + \rho(\frac{\|u\|_0}{\zeta_3(\lambda)})(g(u) + e(u)), \quad u \in X, \tag{3.1}$$

where $\rho : \mathbb{R} \to \mathbb{R}$ is an increasing C^∞ function with $\rho(s) = 0$, $s \leqslant 1$ and $\rho(s) = 1$, $s \geqslant 2$. Note that we have replaced α in (2.5) with the function $\rho(\frac{\|u\|_0}{\zeta_3(\lambda)})$. The nonlinearity in (3.1) is a continuous function of $(\lambda, u) \in \mathbb{R} \times X$, and the nonlinearity becomes zero if $\|u\|_0 \leqslant \zeta_3(\lambda)$. So (3.1) becomes a linear eigenvalue problem, and we can regard (3.1) as a bifurcation problem bifurcating from $u \equiv 0$.

Consider the eigenvalue problem

$$\begin{cases} -(\varphi_p(u'))' = \mu\varphi_p(u), \; x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{3.2}$$

Letting $h = 1$ in [11] we can obtain the following lemma:

Lemma 3.1. [11, Proposition 2.6] *(i) The set of all eigenvalues of (3.2) is a countable set $\{\mu_k \mid k \in \mathbb{N}\}$ satisfying*

$$0 < \mu_1 < \mu_2 < \cdots < \mu_k < \cdots \to \infty;$$

(ii) Let ϕ_k be a corresponding eigenfunction to μ_k, then the number of interior zeros of ϕ_k is $k - 1$.

Definition $\nu \in \{+,-\}$, which is used throughout the rest of the paper. The following lemma follows directly from the above statement.

Lemma 3.2. *The set of solutions (λ, u) of (3.1) with $\|u\|_0 \leqslant \zeta_3(\lambda)$ is*

$$\{(\lambda, 0)\} \cup \{(\mu_k, t\phi_k) : k \geqslant 1,\ |t| \leqslant \zeta_3(\lambda)/\|\phi_k\|_0\}.$$

Lemma 3.3. *For each $k \geqslant 1$ and $\nu \in \{+,-\}$, there exists a connected set $C_{k,\nu} \subset \mathbb{R}\times E$ of non-trivial solutions of (3.1) such that $C_{k,\nu}\cup(\mu_k, 0)$ is closed and connected. Moreover,*

(i) there exists a neighbourhood U_k of $(\mu_k, 0)$ in $\mathbb{R} \times E$ such that $U_k \cup C_{k,\nu} \subset \mathbb{R} \times S_{k,\nu}$;

(ii) either $C_{k,\nu} \cap C_{k',\nu'} \neq \emptyset$, for some $(k,\nu) \neq (k',\nu')$, or $C_{k,\nu}$ meets infinity in $\mathbb{R} \times E$, that is, there exists a sequence $(\lambda_n, u_n) \in C_{k,\nu}$, $n = 1, 2, \cdots$ such that $|\lambda_n| + \|u_n\|_1 \to \infty$ as $n \to \infty$.

Proof. Since $\varphi_p^{-1} : Y \to X$ exists and is bounded, (3.1) can be rewritten of the form

$$u(x) = \int_0^x \varphi_p^{-1}\left(\int_t^\tau \lambda u(s) + \rho(\|u\|_0/\zeta_3(\lambda))[g(u(s)) + e(u)(s)]\mathrm{d}s\right)\mathrm{d}t := Fu(x). \tag{3.3}$$

It is easy to verify $F : Y \to X$ is a compact operator, and a solution of (3.1) is equivalent to a solution of (3.3). Applying the Rabinowitz global bifurcation theorem [16] to operator equation (3.3), by a similar argument with [6], one has that there exists a component $C_{k,\nu}$ of $S_{k,\nu}$ and either $C_{k,\nu}$ is unbounded or passes through $(\mu_k', 0)(\mu_k' \neq \mu_k)$. [16, Lemma 1.24] implies $U_k \cup C_{k,\nu} \subset \mathbb{R} \times S_{k,\nu}$. □

Lemma 3.4. [6, Lemma 2.3] *The first alternative in part (ii) of Lemma 3.3 is impossible.*

Lemma 3.4 shows $C_{k,\nu}$ must be unbounded. By a same argument with [16, Lemma 2.7] we obtain the following lemma:

Lemma 3.5. *If (λ, u) is a solution of (3.1) with $\lambda \geqslant 0$ and $\|u\|_0 \geqslant \zeta_3(\lambda)$ then $u \in S_{k,\nu}$ for some $k \geqslant 1$ and ν.*

In view of Lemmas 3.2 and 3.5, in the following lemmas we suppose that (λ, u) is an arbitrary non-trivial solution of (3.1) with $\lambda \geqslant 0$ and $u \in S_{k,\nu}$, for some $k \geqslant 1$ and ν. And if $\|u\|_0 \geqslant \zeta_3(\lambda)$, then $(0, u)$ is a solution of (1.1).

Lemma 3.6. *There exists an integer $k_0 \geqslant 1$ (depending only on $\zeta_3(0)$) such that if $\lambda = 0$ and $\zeta_3(0) \leqslant \|u\|_0 \leqslant 2\zeta_3(0)$ then $k < k_0$.*

Proof. Let $x_{i_0}, x_{i_0+1} \in (0,1)$, $i_0 \in \{0, 1, 2, \cdots, k-1\}$, be consecutive zeros of u. Then there exists $x_j \in (x_{i_0}, x_{i_0+1})$ such that $u'(x_j) = 0$, and hence, by Lemma 2.3 (with $R = \eta_1$), $|u(x_j)| \geqslant R \geqslant 1$. Hence,

$$\begin{aligned}\|u'\|_0(x_{i_0+1} - x_{i_0}) &= \|u'\|_0(x_j - x_{i_0}) + \|u'\|_0(x_{i_0+1} - x_j)\\ &\geqslant |u(x_j) - u(x_{i_0})| + |u(x_{i_0+1}) - u(x_j)|\\ &\geqslant 2,\end{aligned}$$

that is

$$(x_{i_0+1} - x_{i_0}) \geqslant \frac{2}{\|u'\|_0}. \tag{3.4}$$

Since $\|u\|_0 \leqslant 2\zeta_3(0)$, then from Lemma 2.1,

$$\|u\|_1 = \|u\|_0 + \|u'\|_0 \leqslant \zeta_1(2\zeta_3(0)),$$

then

$$\|u'\|_0 \leqslant \zeta_1(2\zeta_3(0)) - \|u\|_0 \leqslant \zeta_1(2\zeta_3(0)) - \zeta_3(0).$$

From (3.4)

$$(x_{i_0+1} - x_{i_0}) \geqslant \frac{2}{\|u'\|_0} \geqslant \frac{2}{\zeta_1(2\zeta_3(0)) - \zeta_3(0)}.$$

And let

$$1 = (1 - x_{k-1}) + (x_{k-1} - x_{k-2}) + \cdots + (x_{i_0+1} - x_{i_0}) + \cdots + (x_2 - x_1) + (x_1 - 0),$$

then we get

$$1 \geqslant \frac{2k}{\zeta_1(2\zeta_3(0)) - \zeta_3(0)} > 1$$

if we take $k \geqslant \frac{\zeta_1(2\zeta_3(0))-\zeta_3(0)}{2} + 1$. This is a contradiction! Hence $k < k_0 := \frac{\zeta_1(2\zeta_3(0))-\zeta_3(0)}{2} + 1$. □

Now let

$$V_R(u) = \{x \in [0,1] : |u(x)| \geqslant R\},$$
$$W_R(u) = \{x \in [0,1] : |u(x)| < R\}.$$

Lemma 3.7. *Suppose that $R \geqslant \eta_1$, $0 \leqslant \lambda \leqslant R$ and $\|u\|_0 \geqslant \zeta_3(R)$. Then $W_R(u)$ consists of exactly $k+1$ intervals, each of length less than $2/R$, and $V_R(u)$ consists of exactly k intervals.*

Proof. Lemma 2.3 implies that $|u'(x)| \geqslant R^2$ for all $x \in W_R(u)$, then for any $x_1, x_2 \in W_R(u)$ and $x_1 < x_2$,

$$|u(x_2)| - |u(x_1)| = \int_{x_1}^{x_2} |u'(t)|\mathrm{d}x \geqslant \int_{x_1}^{x_2} R^2 \mathrm{d}x = R^2(x_2 - x_1),$$

hence

$$x_2 - x_1 \leqslant \frac{|u(x_2)| - |u(x_1)|}{R^2} \leqslant \frac{|u(x_2)| + |u(x_1)|}{R^2} < \frac{2R}{R^2} = \frac{2}{R}.$$

This completes the proof. □

Lemma 3.8. *There exists ζ_4 satisfying $\lim\limits_{R\to\infty} \zeta_4(R) = 0$, and $\eta_2 \geqslant \eta_1$ such that, for any $R \geqslant \eta_2$, if either* *(a)* $0 \leqslant \lambda \leqslant R$ *and* $\|u\|_0 = 2\zeta_3(R)$, *or*

(b) $\lambda = R$ *and* $\zeta_3(R) \leqslant \|u\|_0 \leqslant 2\zeta_3(R)$,
then the length of each interval of $V_R(u)$ *is less than* $\zeta_4(R)$.

Proof. Define $H = H(R)$ by

$$H(R) := \min\left\{R, \min\left\{\frac{g(\xi)}{\varphi_p(\xi)} : |\xi| \geqslant R\right\} - \left(\frac{C}{\varphi_p(R)} + \frac{1}{2}\right)\right\}.$$

We may choose $\eta_2 \geqslant \eta_1$ sufficiently large that $H(R) > 0$ for all $R \geqslant \eta_2$.

Define $[x_0, x_2] \subset (0,1)$ such that $u(x_0) = u(x_2) = R$ and $u(x) > 0$ in $[x_0, x_2]$ (the case $u(x) < 0$ is similar), then $[x_0, x_2] \subset V_R(u)$. By (3.1) and the construction of H, if either (a) or (b) holds, then

$$-(\varphi_p(u'))' \geqslant H\varphi_p(u) > 0$$

for $x \in [x_0, x_2]$.

Consider boundary value problem

$$\begin{cases} -(\varphi_p(u'))' \geqslant H(R)\varphi_p(u),\ x \in (0,1), \\ u(x_0) = u(x_2) = R, \end{cases} \tag{3.5}$$

let $y(x) = u(x) - R$, then (3.5) is equivalent to

$$\begin{cases} -(\varphi_p(y'))' \geqslant H(R)\varphi_p(y+R),\ x \in (0,1), \\ y(x_0) = y(x_2) = 0. \end{cases} \tag{3.6}$$

Take

$$\zeta_4(R) := \frac{8q}{[H(R)]^{q-1}},$$

where $q > 1$ satisfies $1/p + 1/q = 1$. Then $\zeta_4(R) \to 0$ as $R \to \infty$.

Define $T : C[0,1] \to X$ by

$$Ty(x) = \begin{cases} \displaystyle\int_{x_0}^{x} \varphi_p^{-1}\left(\int_s^{\sigma} H(R)\varphi_p(y(\tau)+R)\mathrm{d}\tau\right)\mathrm{d}s, & x_0 \leqslant x \leqslant \sigma, \\ \displaystyle\int_{x}^{x_2} \varphi_p^{-1}\left(\int_{\sigma}^{s} H(R)\varphi_p(y(\tau)+R)\mathrm{d}\tau\right)\mathrm{d}s, & \sigma \leqslant x \leqslant x_2, \end{cases}$$

where $\sigma \in (x_0, x_2)$. By [18, Lemma 3.1], if there exists a y satisfies (3.6), then

$$\frac{1}{2}\|y\|_0 \leqslant y(x) \leqslant \|y\|_0, \qquad x \in [\frac{3x_0+x_2}{4}, \frac{3x_2+x_0}{4}].$$

We have two cases: either $\frac{x_0+x_2}{2} < \sigma$ or $\frac{x_0+x_2}{2} \geqslant \sigma$. If $\frac{x_0+x_2}{2} < \sigma$, then

$$\|y\|_0 \geqslant y(\frac{x_0+x_2}{2}) \geqslant \int_{\frac{3x_0+x_2}{4}}^{\frac{x_0+x_2}{2}} \varphi_p^{-1}\left(\int_s^{\sigma} H(R)\cdot\varphi_p(\frac{1}{2}\|y\|_0 + R)\mathrm{d}\tau\right)\mathrm{d}s$$

$$
\begin{aligned}
&\geqslant \int_{\frac{3x_0+x_2}{4}}^{\frac{x_0+x_2}{2}} \varphi_p^{-1}\left(\int_s^{\frac{x_0+x_2}{2}} H(R)\cdot\varphi_p(\frac{1}{2}\|y\|_0+R)\mathrm{d}\tau\right)\mathrm{d}s\\
&\geqslant \int_{\frac{3x_0+x_2}{4}}^{\frac{x_0+x_2}{2}} \varphi_p^{-1}\left(\int_s^{\frac{x_0+x_2}{2}} H(R)\cdot\varphi_p(\frac{1}{2}\|y\|_0)\mathrm{d}\tau\right)\mathrm{d}s\\
&= \frac{1}{2}[H(R)]^{q-1}\|y\|_0 \int_{\frac{3x_0+x_2}{4}}^{\frac{x_0+x_2}{2}} \varphi_p^{-1}\left(\int_s^{\frac{x_0+x_2}{2}} \mathrm{d}\tau\right)\mathrm{d}s\\
&= \frac{x_2-x_0}{8}\cdot\frac{[H(R)]^{q-1}\|y\|_0}{q}.
\end{aligned}
$$

For the case $\frac{x_0+x_2}{2}\geqslant\sigma$, we get

$$
\begin{aligned}
\|y\|_0 &\geqslant y(\frac{x_0+x_2}{2})\\
&\geqslant \int_{\frac{x_0+x_2}{2}}^{\frac{x_0+3x_2}{4}} \varphi_p^{-1}\left(\int_\sigma^s H(R)\cdot\varphi_p(\frac{1}{2}\|y\|_0+R)\mathrm{d}\tau\right)\mathrm{d}s\\
&\geqslant \int_{\frac{x_0+x_2}{2}}^{\frac{x_0+3x_2}{4}} \varphi_p^{-1}\left(\int_{\frac{x_0+x_2}{2}}^s H(R)\cdot\varphi_p(\frac{1}{2}\|y\|_0+R)\mathrm{d}\tau\right)\mathrm{d}s\\
&\geqslant \int_{\frac{x_0+x_2}{2}}^{\frac{x_0+3x_2}{4}} \varphi_p^{-1}\left(\int_{\frac{x_0+x_2}{2}}^s H(R)\cdot\varphi_p(\frac{1}{2}\|y\|_0)\mathrm{d}\tau\right)\mathrm{d}s\\
&= \frac{1}{2}[H(R)]^{q-1}\|y\|_0 \int_{\frac{x_0+x_2}{2}}^{\frac{x_0+3x_2}{4}} \varphi_p^{-1}\left(\int_{\frac{x_0+x_2}{2}}^s \mathrm{d}\tau\right)\mathrm{d}s\\
&= \frac{x_2-x_0}{8}\cdot\frac{[H(R)]^{q-1}\|y\|_0}{q}.
\end{aligned}
$$

This implies

$$
x_2-x_0\leqslant\frac{8q}{[H(R)]^{q-1}}=\zeta_4(R),
$$

and we completes the proof. $\square$

Now choose an arbitrary integer $k\geqslant k_0$ and ν, and choose $\Lambda>\max\{\eta_2,\mu_k\}$ such that

$$
\frac{2(k+1)}{\Lambda}+k\zeta_4(\Lambda)<1. \tag{3.7}
$$

Let

$$
\begin{aligned}
&B:=\{(\lambda,u):0\leqslant\lambda\leqslant\Lambda,\ \zeta_3(\lambda)\leqslant\|u\|_0\leqslant2\zeta_3(\Lambda)\},\\
&D_1=\{(\lambda,u):0\leqslant\lambda\leqslant\Lambda,\ \|u\|_0=\zeta_3(\lambda)\},\\
&D_2=\{(0,u):2\zeta_3(0)\leqslant\|u\|_0\leqslant\zeta_3(\Lambda)\},\\
&D_3=\{(\lambda,u):0\leqslant\lambda\leqslant\Lambda,\ \|u\|_0=2\zeta_3(\lambda)\},\\
&D_4=\{(\Lambda,u):\zeta_3(\Lambda)\leqslant\|u\|_0\leqslant2\zeta_3(\Lambda)\},\\
&D_5=\{(0,u):\zeta_3(0)\leqslant\|u\|_0\leqslant2\zeta_3(0)\}.
\end{aligned}
$$

It follows from Lemma 3.2 that $C_{k,\nu}$ "enters" B through the set D_1, while from Lemma 3.5, $C_{k,\nu} \cap B \subset \mathbb{R} \times S_{k,\nu}$. Thus, by Lemmas 2.1, 3.3 and 3.4, $C_{k,\nu}$ is unbounded and must "leave" B, and since $C_{k,\nu}$ is connected it must intersect ∂B. However, (3.7) and Lemma 3.7-3.8 show $C_{k,\nu}$ can not intersect with D_3 and D_4 (if $u \in C_{k,\nu}$, then the sum of the lengths of intervals in $W_R(u)$ and $V_R(u)$ is 1). And Lemma 3.6 shows $C_{k,\nu}$ can not intersect with D_5 when $k \geqslant k_0$. Then the only portion of ∂B (other than D_1) which $C_{k,\nu}$ can intersect is D_2. Thus there exists a point $(0, u_{k,\nu}) \in C_{k,\nu} \cap D_2$, and clearly $u_{k,\nu}$ provides the desired solution of (1.1) when $\|u\|_0 \geqslant \zeta_3(\lambda)$, which completes the proof of the theorem.

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References

[1] R. Agarwal, D. Cao and H. Lu, *Existence and multiplicity of positive solutions for singular semipositone p-Laplacian equations*, Can. J. Math., 2006, 58(3), 449–475.

[2] T. Alotaibi, D. D. Hai and R. Shivaji, *Existence and nonexistence of positive radial solutions for a class of p-laplacian superlinear problems with nonlinear boundary conditions*, Commun. Pure Appl. Anal., 2020, 19(9), 4655–4666.

[3] A. Ambrosetti, J. G. Azorero and I. Peral, *Quasilinear equations with a multiple bifurcation*, Differential Integral Equations, 1997, 10(1), 37–50.

[4] G. Bognar, *Numerical and Numerical and Analytic Investigation of some Nonlinear Problems in Fluid Mechanics*, Computer and Simulation in Modern Science II, WSEAS Press, 2008.

[5] K. D. Chu, D. D. Hai and R. Shivaji, *A uniqueness result for infinite semipositone p-Laplacian problems in a ball*, Complex Var. Elliptic Equ., 2022, 67(6), 1496–1503.

[6] G. Dai and R. Ma, *Unilateral global bifurcation phenomena and nodal solutions for p-Laplacian*, J. Differential Equations, 2012, 252(3), 2448–2468.

[7] G. Dai, R. Ma and J. Xu, *Global bifurcation and nodal solutions of N-dimensional p-Laplacian in unit ball*, Appl. Anal., 2013, 92(7), 1345–1356.

[8] M. A. Del Pino and R. F. Manásevich, *Global bifurcation from the eigenvalues of the p-Laplacian*, J. Differential Equations, 1991, 92(2), 226–251.

[9] R. Kajikiya, Y.-H. Lee and I. Sim, *Bifurcation of sign-changing solutions for one-dimensional p-Laplacian with a strong singular weight; p-sublinear at ∞*, Nonlinear Anal., 2009, 71(3–4), 1235–1249.

[10] K. Lan, X. Yang and G. Yang, *Positive solutions of one-dimensional p-Laplacian equations and applications to population models of one species*, Topol. Methods Nonlinear Anal., 2015, 46(1), 431–445.

[11] Y.-H. Lee and I. Sim, *Global bifurcation phenomena for singular one-dimensional p-Laplacian*, J. Differential Equations, 2006, 229(1), 229–256.

[12] R. Ma, *Connected component of positive solutions for singular superlinear semipositone problems*, Topol. Methods Nonlinear Anal., 2020, 55(1), 51–62.

[13] Q. Morres, R. Shivajia and I. Sim, *Existence of positive radial solutions for a superlinear semipositone p-Laplacian problem on the exterior of a ball*, Proc. Roy. Soc. Edinburgh, 2018, 148A(2), 409–428.

[14] Y. Naito and S. Tanaka, *Sharp conditions for the existence of sign-changing solutions to equations involving the one-dimensional p-Laplacian*, Nonlinear Anal., 2008, 69(9), 3070–3083.

[15] L. V. Nasirova, *Global bifurcation from intervals in nonlinear Sturm-Liouville problem with indefinite weight function*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb., 2021, 47(2), 346–356.

[16] P. H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal., 1971, 7, 487–513.

[17] B. P. Rynne, *Infinitely many solutions of superlinear fourth order boundary value problems*. Topol. Methods Nonlinear Anal., 2002, 19(2), 303–312.

[18] I. Sim and S. Tanaka, *Three positive solutions for one-dimensional p-Laplacian problem with sign-changing weight*, Appl. Math. Lett., 2015, 49(42–50), 42–50.

[19] B. Son and P. Wang, *Analysis of positive radial solutions for singular superlinear p-Laplacian systems on the exterior of a ball*, Nonlinear Anal., 2020, 192, 1–15.

[20] J. Wang, *The existence of positive solutions for the one-dimensional p-Laplacian*, Proc. Amer. Math. Soc., 1997, 125(8), 2275–2283.