Journal of Applied Analysis and Computation
Volume 14, Number 3, June 2024, 1790–1819
Website:http://www.jaac-online.com
DOI:10.11948/20230418

AN EXTENSIONAL CONFORMABLE FRACTIONAL DERIVATIVE AND ITS EFFECTS ON SOLUTIONS AND DYNAMICAL PROPERTIES OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS*

Weijun He[1], Weiguo Rui[1] and Xiaochun Hong[2,†]

Abstract Investigations have shown that the conformable fractional derivative is very different from the classical fractional derivatives, it does not have function of memory like the classical fractional derivatives, so it is more appropriate to be called as cognate derivative of the classical integer-order derivative. In this paper, following the idea of constructing the conformable fractional derivative, an extensional conformable fractional derivative named sech-fractional derivative is proposed. The effects of the new conformable fractional differential operator on dynamical properties of nonlinear partial differential equations (PDEs) are discussed. As example, by using the dynamical system method, traveling wave solutions and their dynamical properties of a nonlinear fractional Schrödinger equation are investigated under the sech-fractional differential operator. The solutions and their dynamical properties of the nonlinear Schrödinger equation are compared under three kinds of differential operators, their distinction and connection are revealed. Some interesting phenomena are found and deserve attentions further.

Keywords Extensional conformable fractional derivative, dynamical system method, nonlinear Schrödinger equation, exact solution, dynamical property.

MSC(2010) 26A33, 34A05, 34K18, 35D05.

1. Introduction

Over 320 years have passed since the concept of the fractional calculus was born in 1695. However, the theory of fractional calculus is not perfect enough compared with the integer-order calculus, and the development of the theory to fractional calculus still continues. Following the definitions of earlier fractional derivatives such as Riemann-Liouville fractional derivative, Grünwald-Letnikov fractional derivative,

[†]The corresponding author.
[1]School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China
[2]School of Statistics and Mathematics, Yunnan University of Finance and Economics, Kunming, Yunnan 650221, China
[*]The authors were supported by National Natural Science Foundation of China (Grant No. 11761075) and Research Project of Chongqing Education Commission (Grant No. CXQT21014).
Email: heweijun2022@163.com(W. He), wgruihhu@163.com(W. Rui), xchong@ynufe.edu.cn(X. Hong)

Caputo fractional derivative, Riesz fractional derivative, Weyl fractional derivative, Weyl-Marchaud fractional derivative, Hadamard fractional derivative, Capto-Hadamard fractional derivative and so forth, some new definitions of fractional derivatives such as Jumarie fractional derivative [16–18], conformable fractional derivative [19], truncated M-fractional derivative and M-fractional derivative [33] have been proposed in recent years. Unfortunately, the fractional chain rule and fractional Leibniz rule of Jumarie fractional derivative were proved to be wrong by some authors in [10,11,23,28,36]. Therefore, the Jumarie fractional derivative loses the prospect of its application. In contrast, the other two new derivatives named conformable fractional derivative and truncated M-fractional derivative have been used by many authors [1–5,7–9,12–15,24–27,31,32,34,35,37–43] to investigate various exact traveling wave solutions of some nonlinear partial differential equations due to their correctness and convenience in operation.

In these definitions of fractional derivatives mentioned above, the definition of Riemann-Liouville fractional derivative and the definition of Grünwald-Letnikov fractional derivative are equivalent, they are only different in form. Indeed, the definition of Grünwald-Letnikov fractional derivative can be regarded as a limit form of discretization to Riemann-Liouville fractional derivative that is often used in the field of numerical calculation for corresponding fractional models. The definition of Caputo fractional derivative is a direct modification of the definition of Riemann-Liouville fractional derivative, just the sequence (precedence) of their integral and derivative are different. Undoubtedly, the Caputo fractional derivative is much simpler and more convenient than the Riemann-Liouville fractional derivative in terms of operation (computation) and determination of initial value conditions for mathematical models. Other definitions of earlier fractional derivatives contain integral operators, most of them are combinations of integral and derivative. Their construction ideas are all derived from the Riemann-Liouville fractional derivative. Therefore, more classical and widely used fractional differential operators are still Riemann-Liouville fractional differential operator and Caputo fractional differential operator. However, the two kinds of fractional differential operators have neither corresponding Leibniz rule nor chain rule as in the integer-order calculus. The absence of the two rules causes very great difficulties in solving nonlinear fractional differential equations defined by the Riemann-Liouville fractional differential operator and Caputo fractional differential operator. Many classical and effective methods in the field of integer-order differential equations cannot be directly applied to solve fractional differential equations defined by Riemann-Liouville fractional differential operator or Caputo fractional differential operator. It is for this reason that a new definition of fractional derivative named conformable fractional derivative was proposed by Khalil et al in [19]. Further, the conformable fractional derivative was improved into the truncated M-fractional derivative and M-fractional derivative later (see reference [33] and cited therein). Indeed, only one constant factor $\frac{1}{\Gamma(\gamma+1)}$ varies between the truncated M-fractional derivative and the conformable fractional derivative, it's only a very small difference between them. Just the opposite, there is a very big difference between the Riemann-Liouville fractional derivative and the conformable fractional derivative, and their construction ideas are completely different. Next, we make a simple comparison for the definitions of the two kinds of fractional derivatives.

The Riemann-Liouville fractional derivative of order α is defined by

$$ {}_{a}^{RL}D_t^{\alpha} f(t) = {}_{a}^{RL}D_t^{n}[{}_{a}^{RL}I_t^{n-\alpha} f(t)] = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-1-\alpha} f(\tau) d\tau, \quad (1.1) $$

where $f(t)$ is a continuous function in the interval $t \in [a, +\infty)$ and $n-1 \leq \alpha < n$. We only use the left Riemann-Liouville fractional derivative as example at here.

The conformable fractional derivative of $f(t)$ of order α is defined by

$$ {}_{0}^{K}D_t^{\alpha} f(t) \equiv f^{(\alpha)}(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} = t^{1-\alpha} \frac{df(t)}{dt}, \quad (1.2) $$

where $t > 0,\ 0 < \alpha < 1$ and $f(t)$ is smooth.

From (1.1), it is easy to know that

$$ {}_{a}^{RL}D_t^{\alpha} f(t) = {}_{a}^{RL}D_t^{n}[{}_{a}^{RL}I_t^{n-\alpha} f(t)] = {}_{a}^{RL}D_t^{n}[{}_{a}^{RL}D_t^{\alpha-n} f(t)]. $$

This implies that Def. (1.1) contains arbitrary $n-\alpha$ order integral (i.e. arbitrary $\alpha - n$ order derivative). Therefore, Riemann-Liouville fractional derivative is a really arbitrary-order derivative, it's just that people already habitually call it a fractional-order derivative (or fractional derivative). In contrast, Def. (1.2) only contains components of the integer-order derivative $\frac{df(t)}{dt}$ and power function $t^{1-\alpha}$, no implication of fractional-order or arbitrary-order derivative. Therefore, the conformable fractional derivative is not a fractional derivative in the true sense, but more like a twin of the classical integer-order derivative. Moreover, according to the standard way to define the fractional derivatives given in [20], the conformable fractional derivative should not be a real fractional derivative yet, it should be a cognate derivative (an isogenesis derivative) of the classical integer-order derivative. On the other hand, the Riemann-Liouville fractional differential operator defined by (1.1) has function (characteristic) of memory, but the conformable fractional differential operator defined by (1.2) has not. Indeed, the conformable fractional derivative is a direct modification of the classical integer-order derivative, its properties are very close to those of the integer-order derivative. So, all operational rules of the conformable derivatives can be directly converted into operational rules of the integer-order derivative. It is for this reason that all solving methods in the field of integer-order differential equations can be directly applied to the field of conformable fractional differential equations.

Although the conformable fractional derivative is not a fractional derivative in the true sense, but people have habitually called it conformable fractional derivative and it is paid attention to many researchers due to it has many excellent properties like integer-order derivative. Although the definition and operational rules of conformable fractional derivative is very similar as those of the classical integer-order derivative, their effects on the dynamical properties of differential equations are very different. So what impact will these differences have on physics or other applications? This is exactly the concern of our institute.

In this paper, following the idea of constructing the conformable fractional derivative, we will introduced an extensional conformable fractional derivative named sech-fractional derivative. And then, we will discuss effects of this new conformable fractional differential operator on solutions of nonlinear PDEs. As example, we will investigate traveling wave solutions and their dynamical properties of a nonlinear fractional Schrödinger equation under the sech-fractional differential operator. By the way, the sech-fractional derivative is completely different from the truncated M-fractional derivative [33].

The organization of this paper is as follows: In Sec. 2, we will introduc an extensional conformable fractional derivative named sech-fractional derivative. Further, we will discuss effects of this new conformable fractional differential operator on solutions of linear or nonlinear PDEs. Several theorems of replacement solutions are proposed and proved. In Sec. 3, we will investigate traveling wave solutions and their dynamical properties of a nonlinear fractional Schrödinger equation under the sech-fractional differential operator. Under three kinds of differential operators, we will compare the dynamical properties of the solutions of the nonlinear Schrödinger equation. In Sec. 4, a concise conclusion is given. In Appendix, under sech-fractional differential operator and integer-order differential operator, differences of derivatives to the six kinds of basic elementary functions are shown.

2. An extensional conformable fractional derivative and its effects on dynamical properties of fractional PDEs

In this section, we will introduce an extensional conformable fractional derivative named sech-fractional derivative based on the the conformable fractional derivative. And then, we will discuss effects of this new conformable fractional differential operator on solutions and dynamical properties of sech-fractional PDEs.

2.1. Definition of an extensional conformable fractional derivative

First, we introduce an extensional conformable fractional derivative named sech-fractional derivative as follows:

Definition 2.1. Suppose that the function $f(t)$ is differential in $\mathbb{R}$, then the sech-fractional derivative for $f(t)$ of α-order is defined by

$$ {}^{\mathrm{s}}D_t^{\alpha} f(t) = \frac{d^{\alpha} f(t)}{dt^{\alpha}} = \lim_{\varepsilon \to 0} \frac{f\left[t + \varepsilon(\operatorname{sech}(1-\alpha)t)\right] - f(t)}{\varepsilon}, \tag{2.1} $$

where $t \in \mathbb{R},\ 0 < \alpha < 1,\ \operatorname{sech}(1-\alpha)t \in (0, 1]$.

Letting $\varepsilon(\operatorname{sech}(1-\alpha)t) = \Delta t$, (2.1) is reduced to

$$ {}^{\mathrm{s}}D_t^{\alpha} f(t) = [\operatorname{sech}(1-\alpha)t] \lim_{\Delta t \to 0} \frac{f\left(t + \Delta t\right) - f(t)}{\Delta t} = [\operatorname{sech}(1-\alpha)t] \frac{df(t)}{dt}, \tag{2.2} $$

where $0 < \alpha \leq 1$. In particular, ${}^{\mathrm{s}}D_t^{\alpha} f(t) = f'(t)$ when $\alpha = 1$. Obviously, the definition domain of the function $f(t)$ is significantly improved compared to the conformable fractional derivative; in the definition of conformable fractional derivative (1.2), the independent variable $t \in [0, +\infty)$, in the definition of sech-fractional derivative (2.1), the independent variable $t \in (-\infty, +\infty)$. Although the derivative defined by (2.1) is not a fractional derivative in the true sense yet, habitually we still call it sech-fractional derivative.

According to $\varepsilon(\operatorname{sech}(1-\alpha)t) = \Delta t$, we find that sech-fractional differentiate and integer-order differentiate have the following relationship:

$$dt^\alpha = \cosh(1-\alpha)t\ dt, \tag{2.3}$$

where $\varepsilon \sim dt^\alpha$ and $\Delta t \sim dt$. If $\frac{dG(t)}{dt} = g(t)$, then we obtain a sech-fractional derivative formula and an integral formula as follows:

$$\begin{aligned}
{}^sD_t^\alpha G\left(\frac{\sinh(1-\alpha)t}{1-\alpha}\right) &= [\operatorname{sech}(1-\alpha)t]g\left(\frac{\sinh(1-\alpha)t}{1-\alpha}\right)[\cosh(1-\alpha)t] \\
&= g\left(\frac{\sinh(1-\alpha)t}{1-\alpha}\right), \\
{}^sI_t^\alpha g\left(\frac{\sinh(1-\alpha)t}{1-\alpha}\right) &= \int g\left(\frac{\sinh(1-\alpha)t}{1-\alpha}\right)\ dt^\alpha \\
&= \int g\left(\frac{\sinh(1-\alpha)t}{1-\alpha}\right)\ \cosh(1-\alpha)t\ dt \\
&= \int g\left(\frac{\sinh(1-\alpha)t}{1-\alpha}\right) d\left(\frac{\sinh(1-\alpha)t}{1-\alpha}\right) \\
&= G\left(\frac{\sinh(1-\alpha)t}{1-\alpha}\right) + C.
\end{aligned}$$

Generally, we have

$${}^sI_t^\alpha g(t) = \int g(t)dt^\alpha = \int g(t)[\cosh(1-\alpha)t]\ dt. \tag{2.4}$$

When $n < \alpha \le n+1$, the higher order derivative of $f(t)$ is defined as follows:

Definition 2.2. Suppose that the function $f(t)$ is n-order differential in $\mathbb{R}$. If $n < \alpha \le n+1$, then the sech-fractional derivative for $f(t)$ of higher order is defined by

$${}^sD_t^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \lim_{\varepsilon\to 0}\frac{f^{(n)}[t+\varepsilon(\operatorname{sech}(n+1-\alpha)t)] - f^{(n)}(t)}{\varepsilon}, \tag{2.5}$$

where $t \in \mathbb{R},\ n < \alpha \le n+1$.

Similarly, letting $\varepsilon(\operatorname{sech}(1-\alpha)t) = \Delta t$, (2.5) is reduced to

$${}^sD_t^\alpha f(t) = \operatorname{sech}[(n+1-\alpha)t]f^{(n+1)}(t), \quad (n < \alpha \le n+1) \tag{2.6}$$

and ${}^sD_t^\alpha f(t) = f^{(n+1)}(t)$ when $\alpha = n+1$. Obviously, ${}^sD_t^\alpha[{}^sD_t^\alpha f(t)] \ne {}^sD_t^{2\alpha} f(t)$, that is, the fractional differential operator does not satisfy the superposition principle ${}^sD_t^\alpha \cdot {}^sD_t^\sigma \ne {}^sD_t^{\alpha+\sigma}$.

Regarding the sech-fractional chain rule and Leibniz rule, we have two lemmas as follows:

Lemma 2.1. *Supposes that $f(t)$, $g(t)$ and $f[g(t)]$ are smooth functions in $t \in (-\infty,+\infty)$. For the sech-fractional derivative of the compound function $f[g(t)]$, the following chain rule holds*

$${}^sD_t^\alpha f[g(t)] = [\operatorname{sech}(1-\alpha)t]\frac{df[g(t)]}{dt} = [\operatorname{sech}(1-\alpha)t]f'(\tau)g'(t)\big|_{\tau=g(t)}. \tag{2.7}$$

Proof. Based on the Def. 2.1, we have

$$ {}^{\mathrm{s}}D_t^{\alpha} f[g(t)] = \lim_{\varepsilon \to 0} \frac{f\left[g\left(t+\varepsilon(\operatorname{sech}(1-\alpha)t)\right] - f[g(t)]\right.}{\varepsilon}. \tag{2.8} $$

Letting $\varepsilon(\operatorname{sech}(1-\alpha)t) = \Delta t$, Eq. (2.8) is reduced to

$$ \begin{aligned} {}^{\mathrm{s}}D_t^{\alpha} f[g(t)] &= [\operatorname{sech}(1-\alpha)t] \lim_{\Delta t \to 0} \frac{f\left[g\left(t+\Delta t\right)\right] - f[g(t)]}{\Delta t} \\ &= [\operatorname{sech}(1-\alpha)t] \frac{df[g(t)]}{dt} \\ &= [\operatorname{sech}(1-\alpha)t] f'(\varphi) g'(t)\big|_{\varphi=g(t)}. \end{aligned} $$

Indeed, this is an indirect form of chain rule need to be assisted by the chain rule of the integer-order derivative. It is important to note that its direct form of chain rule does not hold, that is, ${}^{\mathrm{s}}D_t^{\alpha} f[g(t)] \neq {}^{\mathrm{s}}D_{\varphi}^{\alpha} f(\varphi) \cdot {}^{\mathrm{s}}D_t^{\alpha} g(t)\big|_{\varphi=g(t)}$ because

$$ \begin{aligned} {}^{\mathrm{s}}D_{\varphi}^{\alpha} f(\varphi) \cdot {}^{\mathrm{s}}D_t^{\alpha} g(t)\big|_{\varphi=g(t)} &= [\operatorname{sech}(1-\alpha)\varphi] f'(\varphi) \cdot [\operatorname{sech}(1-\alpha)t] g'(t)\big|_{\varphi=g(t)} \\ &= [\operatorname{sech}(1-\alpha)g(t)][\operatorname{sech}(1-\alpha)t] f'(\varphi) g'(t)\big|_{\varphi=g(t)} \\ &\neq [\operatorname{sech}(1-\alpha)t] f'(\varphi) g'(t)\big|_{\varphi=g(t)}. \end{aligned} $$

□

Lemma 2.2. *Supposes that $f(t)$, $g(t)$ are two smooth functions in $t \in (-\infty, +\infty)$. For the sech-fractional derivative of the function $f(t)g(t)$, the following sech-fractional Leibniz's rule holds*

$$ \begin{aligned} {}^{s}D_t^{\alpha}[f(t)g(t)] &= [\,{}^{s}D_t^{\alpha} f(t)]g(t) + f(t)[\,{}^{s}D_t^{\alpha} g(t)] \\ &= [\operatorname{sech}(1-\alpha)t][f'(t)g(t) + f(t)g'(t)]. \end{aligned} \tag{2.9} $$

Proof. As in Lemma 2.1, letting $\varepsilon(\operatorname{sech}(1-\alpha)t) = \Delta t$, we can easily obtain

$$ {}^{s}D_t^{\alpha}[f(t)g(t)] = [\operatorname{sech}(1-\alpha)t] \frac{d[f(t)g(t)]}{dt} = [\operatorname{sech}(1-\alpha)t][f'(t)g(t) + f(t)g'(t)]. $$

On the other hand, by using (2.2), we get

$$ \begin{aligned} &[\,{}^{\mathrm{s}}D_t^{\alpha} f(t)]g(t) + f(t)[\,{}^{\mathrm{s}}D_t^{\alpha} g(t)] \\ =&[\operatorname{sech}(1-\alpha)t] \frac{df(t)}{dt} g(t) + f(t)[\operatorname{sech}(1-\alpha)t] \frac{dg(t)}{dt} \\ =&[\operatorname{sech}(1-\alpha)t][f'(t)g(t) + f(t)g'(t)]. \end{aligned} $$

So, the conclusion is proved. Obviously, direct sech-fractional Leibniz's rule holds.

□

It is not difficult to see that the sech-fractional derivative has one more functional factor formed as $\operatorname{sech}(1-\alpha)t$ than the integer-order derivative when $0 < \alpha < 1$. Also, we find that the sech-fractional derivative of higher order has one more functional factor formed as $\operatorname{sech}(n+1-\alpha)t$ than the integer-order derivative $f^{(n+1)}(t)$ of order $n+1$ when $n < \alpha < n+1$. Although the sech-fractional derivative has similar definition and operational rules as in classical integer derivative, they have very different effects on the dynamical properties of differential equations. Then, what are their essential difference and correlation? Next, we will answer this question.

2.2. Effects of the sech-fractional differential operator on solutions and dynamical properties of nonlinear PDEs

Affected by the sech-fractional differential operator, the solutions of the sech-fractional PDEs are different from those of the integer-order PDEs. Of course, there are distinction and connection between the solutions of the two types of linear or nonlinear PDEs due to there are difference and connection between these two differential operators. In the next, we give their association and difference by theorems.

Theorem 2.1. *If $u = \varphi(kx+\omega t)$ is a traveling wave solution of a linear or nonlinear integer-order PDE with constant coefficients formed as*

$$F\left(u,\ \frac{\partial u}{\partial t},\ \frac{\partial u}{\partial x},\ \frac{\partial^2 u}{\partial x \partial t}\ \frac{\partial^2 u}{\partial x^2},\ \cdots,\ \frac{\partial^n u}{\partial x^n}\right) = 0, \tag{2.10}$$

then $u = \varphi\left(\frac{k\sinh(1-\beta)x}{1-\beta} + \frac{\omega \sinh(1-\alpha)t}{1-\alpha}\right)$ must be a traveling wave solution of a linear or nonlinear sech-fractional PDE with constant coefficients formed as

$$F\left[u,\ {}^sD_t^\alpha u,\ {}^sD_x^\beta u,\ {}^sD_x^\beta({}^sD_t^\alpha u),\ ({}^sD_x^\alpha)^2 u,\ \cdots,\ ({}^sD_x^\beta)^n u\right] = 0, \tag{2.11}$$

where $0 < \alpha < 1$, ${}^sD_t^\alpha$ is sech-fractional differential operator and $({}^sD_x^\beta)^2 u = {}^sD_x^\beta({}^sD_x^\beta u) \neq {}^sD_x^{2\beta} u$.

Proof. Making traveling wave transformation as follows:

$$u(x,t) = \phi(\xi), \qquad \xi = kx + \omega t, \tag{2.12}$$

where k and ω are two nonzero constant, and then substituting the transformation (2.12) into PDE (2.10), we obtain

$$F\left(\varphi,\ \omega\frac{d\varphi}{d\xi},\ k\frac{d\varphi}{d\xi},\ \omega k\frac{d^2\varphi}{d\xi^2},\ k^2\frac{d^2\varphi}{d\xi^2},\ \cdots,\ k^n\frac{d^n\varphi}{d\xi^n}\right) \equiv 0. \tag{2.13}$$

Similarly, we make another traveling wave transformation as follows:

$$u(x,t) = \varphi(\zeta), \qquad \zeta = \frac{k\sinh(1-\beta)x}{1-\beta} + \frac{\omega\sinh(1-\alpha)t}{1-\alpha}. \tag{2.14}$$

Obviously, $\frac{d\zeta}{dt} = \omega\cosh[(1-\alpha)t]$, $\frac{d\zeta}{dx} = k\cosh[(1-\beta)x]$. Thus, by using the sech-fractional derivative formula (2.2) and chain rule, we easily obtain the following expressions

$${}^sD_t^\alpha u = \operatorname{sech}[(1-\alpha)t]\frac{\partial u}{\partial t} = \operatorname{sech}[(1-\alpha)t]\frac{d\varphi}{d\zeta}\frac{d\zeta}{dt} = \omega\frac{d\varphi}{d\zeta}, \tag{2.15}$$

$${}^sD_x^\beta u = \operatorname{sech}[(1-\beta)x]\frac{\partial u}{\partial x} = \operatorname{sech}[(1-\beta)x]\frac{d\varphi}{d\zeta}\frac{d\zeta}{dx} = k\frac{d\varphi}{d\zeta}, \tag{2.16}$$

$${}^sD_x^\beta({}^sD_t^\alpha u) = {}^sD_x^\beta\left[\omega\frac{d\varphi}{d\zeta}\right] = \operatorname{sech}[(1-\beta)x]\frac{d}{d\zeta}\left[\omega\frac{d\varphi}{d\zeta}\right]\frac{d\zeta}{dx} = \omega k\frac{d^2\varphi}{d\zeta^2}, \tag{2.17}$$

$$({}^sD_x^\beta)^2 u = D_x^\beta(D_x^\beta u) = {}^sD_x^\beta\left[k\frac{d\varphi}{d\zeta}\right] = \operatorname{sech}[(1-\beta)x]\frac{d}{d\zeta}\left[k\frac{d\varphi}{d\zeta}\right]\frac{d\zeta}{dx} = k^2\frac{d^2\varphi}{d\zeta^2}, \tag{2.18}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdot\cdot$$

$$({}^{\mathrm{s}}D_x^{\beta})^n u = {}^{\mathrm{s}}D_x^{\beta}\left[k^{n-1}\frac{d^{n-1}\varphi}{d\zeta^{n-1}}\right] = \mathrm{sech}[(1-\beta)x]\frac{d}{d\zeta}\left[k^{n-1}\frac{d^{n-1}\varphi}{d\zeta^{n-1}}\right]\frac{d\zeta}{dx} = k^n\frac{d^n\varphi}{d\zeta^n}. \tag{2.19}$$

Plugging (2.14)-(2.19) into fractional PDE (2.11), we get

$$F\left(\varphi,\ \omega\frac{d\varphi}{d\zeta},\ k\frac{d\varphi}{d\zeta},\ \omega k\frac{d^2\varphi}{d\zeta^2},\ k^2\frac{d^2\varphi}{d\zeta^2},\ \cdots,\ k^n\frac{d^n\varphi}{d\zeta^n}\right) \equiv 0. \tag{2.20}$$

Obviously, the forms of Eqs. (2.13) and (2.20) are same except their independent variables ξ and ζ are different. So, the forms of their solutions are also same except the independent variables of solutions are different. In other words, the structures of their solutions are very similar, only that the independent variables of solutions are different. If $\varphi(\xi)$ is a solution of ODE (2.10), then $\varphi(\zeta)$ must be a solution of ODE (2.11). Thus, when $u = \varphi(kx + \omega t)$ is a traveling wave solution of the PDE (2.10), then $u = \varphi\left(\frac{k\sinh(1-\beta)x}{1-\beta} + \frac{\omega\sinh(1-\alpha)t}{1-\alpha}\right)$ must be a traveling wave solution of a sech-fractional PDE (2.11). □

Corollary 2.1. *If $u = \varphi(kx + \omega t)$ is a traveling wave solution of a linear or nonlinear integer-order PDE (2.10) formed as*

$$F\left(u,\ \frac{\partial u}{\partial t},\ \frac{\partial u}{\partial x},\ \frac{\partial^2 u}{\partial x\partial t}\ \frac{\partial^2 u}{\partial x^2},\ \cdots,\ \frac{\partial^n u}{\partial x^n}\right) = 0,$$

then $u = \varphi\left(\frac{kx^\beta}{\beta} + \frac{\omega t^\alpha}{\alpha}\right)$ must be a traveling wave solution of a linear or nonlinear conformable fractional PDE with constant coefficients formed as

$$F\left[u,\ {}_0^K D_t^\alpha u,\ {}_0^K D_x^\beta u,\ {}_0^K D_x^\beta({}_0^K D_t^\alpha u),\ ({}_0^K D_x^\alpha)^2 u,\ \cdots,\ ({}_0^K D_x^\beta)^n u\right] = 0, \tag{2.21}$$

where $0 < \alpha < 1$, ${}_0^K D_t^\alpha$ is conformable fractional differential operator and $({}_0^K D_x^\beta)^2 u = {}_0^K D_x^\beta({}_0^K D_x^\beta u) \neq {}_0^K D_x^{2\beta} u$.

Theorem 2.2. *If $u = \phi(kx + \omega t)$ is a periodic wave solution of the integer-order PDE (2.10), then $u = \phi\left(\frac{k\sinh(1-\beta)x}{1-\beta} + \frac{\omega\sinh(1-\alpha)t}{1-\alpha}\right)$ must be a non-periodic traveling wave solution of the sech-fractional PDE (2.11).*

Proof. Let

$$\xi = kx + \omega t, \qquad \zeta = \frac{k\sinh(1-\beta)x}{1-\beta} + \frac{\omega\sinh(1-\alpha)t}{1-\alpha}.$$

According to Theorem 2.1, the $u = \phi\left(\frac{k\sinh(1-\beta)x}{1-\beta} + \frac{\omega\sinh(1-\alpha)t}{1-\alpha}\right)$ must be a traveling wave solution of the sech-fractional PDE (2.11). Since $u = \phi(kx + \omega t)$ is a periodic solution, then there must be two nonzero constants X and T that make the following equations hold

$$\phi[k(x + X) + \omega t] = \phi(kx + \omega t) \tag{2.22}$$

and

$$\phi[kx + \omega(t + T)] = \phi(kx + \omega t). \tag{2.23}$$

So

$$k(x+X)+\omega t = kx+\omega t+kX = \xi + kX, \tag{2.24}$$
$$kx+\omega(t+T) = kx+\omega t+\omega T = \xi+\omega T, \tag{2.25}$$

thereby

$$\phi(\xi+kX) = \phi(\xi) = \phi(\xi+\omega T),$$

this indicates that kX and ωT are the periods of the function $\phi(\xi)$ about the independent variables x and t, respectively.

Next, we will prove that $u = \phi\left(\frac{k\sinh(1-\beta)x}{1-\beta} + \frac{\omega\sinh(1-\alpha)t}{1-\alpha}\right)$ is not periodic. By using anti-proof method, we assume that $\phi\left(\frac{k\sinh(1-\beta)x}{1-\beta} + \frac{\omega\sinh(1-\alpha)t}{1-\alpha}\right)$ is also a periodic function, then there must be two nonzero constants X and T that make the following equations hold

$$\begin{aligned}&\phi\left(\frac{k\sinh(1-\beta)(x+X)}{1-\beta} + \frac{\omega\sinh(1-\alpha)t}{1-\alpha}\right)\\ =&\phi\left(\frac{k\sinh(1-\beta)x}{1-\beta} + \frac{\omega\sinh(1-\alpha)t}{1-\alpha}\right)\end{aligned} \tag{2.26}$$

and

$$\begin{aligned}&\phi\left(\frac{k\sinh(1-\beta)x}{1-\beta} + \frac{\omega\sinh(1-\alpha)(t+T)}{1-\alpha}\right)\\ =&\phi\left(\frac{k\sinh(1-\beta)x}{1-\beta} + \frac{\omega\sinh(1-\alpha)t}{1-\alpha}\right).\end{aligned} \tag{2.27}$$

From above assumption that the $\phi(\zeta)$ is periodic function, it is clear that the following equation holds

$$\phi(\zeta+C_1X) = \phi(\zeta) = \phi(\zeta+C_2T), \tag{2.28}$$

where C_1 and C_2 are two nonzero constants which can be determined, the C_1X and C_2T are two periods of the function $\phi(\zeta)$ about the independent variables x and t, respectively. By using (2.26), (2.27) and (2.28), we directly obtain two equations as follows:

$$\begin{aligned}&\frac{k\sinh(1-\beta)(x+X)}{1-\beta} + \frac{\omega\sinh(1-\alpha)t}{1-\alpha}\\ =&\frac{k\sinh(1-\beta)x}{1-\beta} + \frac{\omega\sinh(1-\alpha)t}{1-\alpha} + C_1X,\end{aligned} \tag{2.29}$$

$$\begin{aligned}&\frac{k\sinh(1-\beta)x}{1-\beta} + \frac{\omega\sinh(1-\alpha)(t+T)}{1-\alpha}\\ =&\frac{k\sinh(1-\beta)x}{1-\beta} + \frac{\omega\sinh(1-\alpha)t}{1-\alpha} + C_2T,\end{aligned} \tag{2.30}$$

that is,

$$\frac{k\sinh(1-\beta)(x+X)}{1-\beta} + \frac{\omega\sinh(1-\alpha)t}{1-\alpha} = \zeta + C_1X, \tag{2.31}$$

$$\frac{k\sinh(1-\beta)x}{1-\beta}+\frac{\omega\sinh(1-\alpha)(t+T)}{1-\alpha}=\zeta+C_2T. \tag{2.32}$$

However,

$$\begin{aligned}
&\frac{k\sinh(1-\beta)(x+X)}{1-\beta}+\frac{\omega\sinh(1-\alpha)t}{1-\alpha}\\
=&\frac{k[\sinh(1-\beta)x\ \cosh(1-\beta)X+\cosh(1-\beta)x\ \sinh(1-\beta)X]}{1-\beta}\\
&+\frac{\omega\sinh(1-\alpha)t}{1-\alpha}\\
\neq&\zeta+C_1X, \qquad (2.33)\\
&\frac{k\sinh(1-\beta)x}{1-\beta}+\frac{\omega\sinh(1-\alpha)(t+T)}{1-\alpha}\\
=&\frac{k\sinh(1-\beta)x}{1-\beta}\\
&+\frac{\omega[\sinh(1-\alpha)t\ \cosh(1-\alpha)T+\cosh(1-\alpha)t\ \sinh(1-\alpha)T]}{1-\alpha}\\
\neq&\zeta+C_2T. \qquad (2.34)
\end{aligned}$$

Obviously, the above results are contradictory, and therefore the previous assumption is not true. This proves that the $u=\phi\left(\frac{k\sinh(1-\beta)x}{1-\beta}+\frac{\omega\sinh(1-\alpha)t}{1-\alpha}\right)$ must be a non-periodic traveling wave solution. □

For example, it is easy to verify that

$$u=A\sin(kx+kct) \tag{2.35}$$

is a periodic traveling wave solution to the following linear wave equation

$$\frac{\partial^2u}{\partial t^2}=c^2\frac{\partial^2u}{\partial x^2}. \tag{2.36}$$

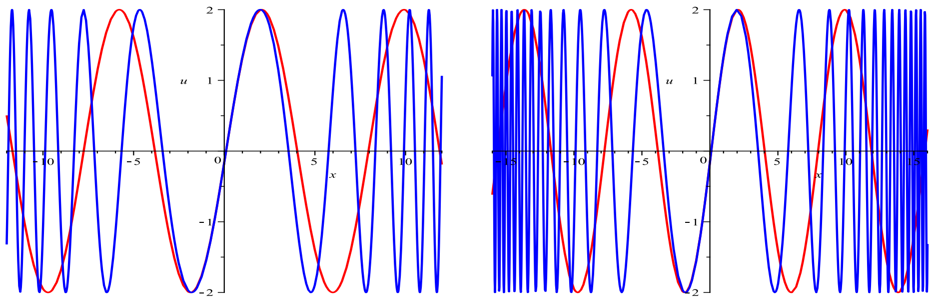
According to Theorem (2.1),

$$u=A\sin\left(\frac{k\sinh(1-\beta)x}{1-\beta}+\frac{kc\sinh(1-\alpha)t}{1-\alpha}\right) \tag{2.37}$$

must be a non-periodic traveling wave solution of the linear sech-fractional wave equation formed as follows:

$$({}^{\mathrm{s}}D_t^{\alpha})^2u=c^2({}^{\mathrm{s}}D_x^{\beta})^2u. \tag{2.38}$$

In order to compare the properties of the solutions (2.35) and (2.37), we draw their curve graphs in the same coordinate system, see Figure 1. In Figure 1, the curve graph of the solution (2.35) is marked in red, the curve graph of the solution (2.37) is marked in blue.



(a) $x \in [-12, 12]$ (b) $x \in [-16, 16]$

Figure 1. Graphs of the (2.35) and (2.37) under the $A = 2,\ k = 0.8,\ c = 0.5,\ \alpha = 0.25,\ \beta = 0.75,\ t = 1.$

It is not difficult to see from Figure 1 that the solution (2.37) is not a periodic function and does not satisfy the property of the periodic function. Although the solution (2.37) has property of similar periodic function (the maximum and minimum values appeared repeatedly), its oscillation frequency will become faster and faster. As the interval expands, the distance between two maximum value points (or two minimum value points) will become smaller and smaller. Therefore, the solution (2.37) does not exist a nonzero constant called period as in periodic function.

Corollary 2.2. *If $u = \phi(kx + \omega t)$ is a periodic wave solution of the integer-order PDE (2.10), then $u = \phi\left(\frac{kx^\beta}{\beta} + \frac{\omega t^\alpha}{\alpha}\right)$ must be a non-periodic traveling wave solution of the conformable fractional nonlinear PDE (2.21).*

Moreover, under Def. 2.1 of the extensional conformable fractional derivative, the fractional derivatives of six kinds of basic elementary functions are completely different from those of their integer-order derivatives, their differences are detailed in the Appendix of Section 5.

To show the effects of this new conformable fractional derivative on nonlinear fractional partial differential equations, as example, we will investigate traveling wave solutions and their dynamical properties of a nonlinear fractional Schrödinger equation by using the dynamical system method. The dynamical properties of the solutions of the nonlinear Schrödinger equation will be compared under the sech-fractional differential operator, integer-order differential operator and conformable fractional differential operator, see the next section.

3. Comparisons of solutions of nonlinear Schrödinger equation under three kinds of differential operators

In the previous section, we compared the dynamical properties of the solutions of a linear wave equation under two different differential operators. It was found that structure of the periodic wave solution of integer-order wave equation is very similar to structure of the non-periodic traveling wave solution of sech-fractional wave equation (with only different independent variables), but their dynamical properties are very different. In this section, by using separation method of variables combined with dynamical system method [29,30], we will investigate the difference and connection between the solutions under the sech-fractional differential operator (extensional conformable fractional differential operator) and integer-order differential operator.

The nonlinear sech-fractional Schrödinger equation is defined by

$$i\ ({}^{\mathrm{s}}D_t^{\alpha}\psi) + ({}^{\mathrm{s}}D_x^{\beta})^2\psi + \rho|\psi|^2\psi + \kappa\psi = 0, \tag{3.1}$$

where $\psi = \psi(x,t)$, $0 < \alpha < 1$, $0 < \beta < 1$, $i = \sqrt{-1}$, the parameters ρ, κ are two nonzero constants, the sign ${}^{\mathrm{s}}D_t^{\alpha}$ denotes sech-fractional differential operator and $({}^{\mathrm{s}}D_x^{\beta})^2\psi = {}^{\mathrm{s}}D_x^{\beta}({}^{\mathrm{s}}D_x^{\beta}\psi) \neq {}^{\mathrm{s}}D_x^{2\beta}\psi$. When $\alpha = \beta = 1$, Eq. (3.1) becomes the classical nonlinear Schrödinger equation as follows:

$$i\ \psi_t + \psi_{xx} + \rho|\psi|^2\psi + \kappa\psi = 0. \tag{3.2}$$

When the differential operator ${}^{\mathrm{s}}D_t^{\alpha}$ becomes ${}_0^K D_t^{\alpha}$, Eq. (3.1) becomes the nonlinear conformable fractional Schrödinger equation [6] as follows:

$$i\ ({}_0^K D_t^{\alpha}\psi) + ({}_0^K D_x^{\beta})^2\psi + \rho|\psi|^2\psi + \kappa\psi = 0, \tag{3.3}$$

where $\psi = \psi(x,t)$, $0 < \alpha < 1$, $0 < \beta < 1$, $i = \sqrt{-1}$, the sign ${}_0^K D_t^{\alpha}$ denotes conformable fractional differential operator and $({}_0^K D_x^{\beta})^2\psi = {}_0^K D_x^{\beta}({}_0^K D_x^{\beta}\psi) \neq {}_0^K D_x^{2\beta}\psi$.

According to Theorem 2.1 and Corollary 2.1, we know that the structures of solutions for the above three kinds of nonlinear Schrödinger equations are very similar, just that their independent variables are different. Therefore, we only need to solve one of the above three equations is enough. Thus, we only solve the equation (3.1) at here. The traveling wave solutions of the other two equations (3.2) and (3.3) can be replaced by the solutions of the equation (3.1), that is, the solutions of Eqs. (3.2) and (3.3) can be directly obtained by variable replacement such as $\frac{\sinh(1-\beta)x}{1-\beta} \sim x$, $\frac{\sinh(1-\alpha)t}{1-\alpha} \sim t$ and $x \sim \frac{x^{\beta}}{\beta}$, $t \sim \frac{t^{\alpha}}{\alpha}$ from the solutions of Eq. (3.1).

Making a traveling wave transformation

$$\psi = u(\zeta)e^{i\theta}, \tag{3.4}$$

where

$$\zeta = \frac{k\sinh(1-\beta)x}{1-\beta} - \frac{c\sinh(1-\alpha)t}{1-\alpha}, \quad \theta = \frac{p\sinh(1-\beta)x}{1-\beta} - \frac{\omega\sinh(1-\alpha)t}{1-\alpha}, \tag{3.5}$$

then substituting (3.4) and (3.5) into (3.1) and letting both real part and imaginary part in the equation be zero, Eq. (3.1) is reduce to the following two equations:

$$\rho u^3 + k^2\frac{d^2u}{d\zeta^2} + (\kappa + \omega - p^2)u = 0 \tag{3.6}$$

and

$$(2kp - c)\frac{du}{d\zeta} = 0. \tag{3.7}$$

Provisionally regarding $u(\zeta)$ is arbitrary function, solving (3.7) it yields

$$c = 2kp. \tag{3.8}$$

Of cause, when $c \neq 2kp$, the solution of (3.7) is a trivial solution (constant solution) formed as $u = C$, but this kind of constant solution makes no significance. So, we only consider the case $c = 2kp$. Plugging (3.8) into (3.5), we get

$$\zeta = \frac{k\sinh(1-\beta)x}{1-\beta} - \frac{2kp\sinh(1-\alpha)t}{1-\alpha}, \quad \theta = \frac{p\sinh(1-\beta)x}{1-\beta} - \frac{\omega\sinh(1-\alpha)t}{1-\alpha}. \tag{3.9}$$

Thereby, the solutions of Eq. (3.1) is determined by the solutions of ODE (3.6) and the transformations (3.4) and (3.9). Next, we will solve Eq. (3.6).

Letting $\frac{du}{d\xi} = v$, Eq. (3.6) is reduced to a nonlinear planar system as follows:

$$\begin{cases} \dfrac{du}{d\zeta} = v, \\ \dfrac{dv}{d\zeta} = \dfrac{(p^2-\kappa-\omega)u - \rho u^3}{k^2}. \end{cases} \tag{3.10}$$

The first integral of the system (3.10) is defined by

$$v^2 = \frac{p^2-\kappa-\omega}{k^2}u^2 - \frac{\rho}{2k^2}u^4 + h, \tag{3.11}$$

where h is an integral constant. For convenience on discussion, we rewrite Eq. (3.11) as following:

$$H(u,v) \equiv v^2 + \frac{\rho}{2k^2}u^4 - \frac{p^2-\kappa-\omega}{k^2}u^2 = h. \tag{3.12}$$

When $\frac{p^2-\kappa-\omega}{\rho} \leq 0$, the system (3.10) has only one equilibrium point $O(0,0)$. When $\frac{p^2-\kappa-\omega}{\rho} > 0$, the system (3.10) has three equilibrium points $O(0,0)$ and $A_{1,2}\left(\pm\sqrt{\frac{p^2-\kappa-\omega}{\rho}}, 0\right)$. Substituting these points into Eq. (3.12), it yields

$$h_0 = H(0,0) = 0, \qquad h_1 = H\left(\pm\sqrt{\frac{p^2-\kappa-\omega}{\rho}}, 0\right) = -\frac{(p^2-\kappa-\omega)^2}{2\rho k^2}. \tag{3.13}$$

From (3.10) and (3.12), it is easily verify that $\frac{\partial H}{\partial v} \neq -\frac{du}{d\zeta}$ and $\frac{\partial H}{\partial u} \neq \frac{dv}{d\zeta}$, so system (3.10) isn't a Hamiltonian system.

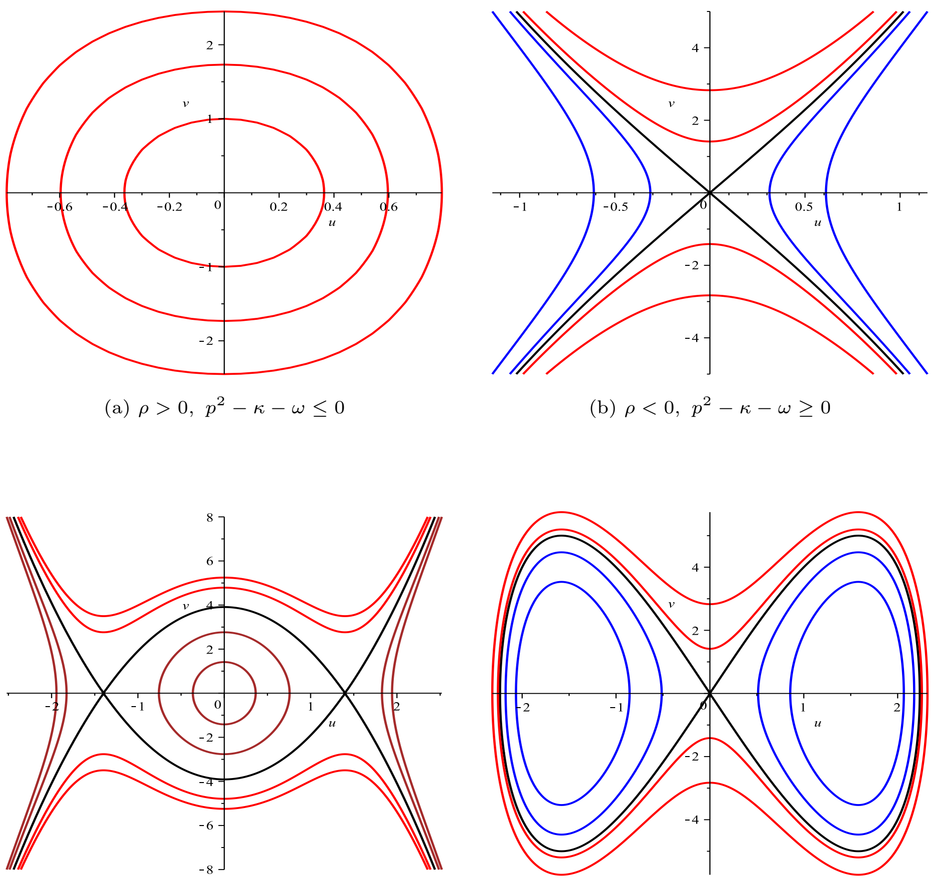
Letting $P = v$, $Q = \frac{(p^2-\kappa-\omega)u-\rho u^3}{k^2}$ in the system (3.10), we write Jacobian matrix and Jacobian determinant of the system (3.10) as follows:

$$\begin{aligned} M(u,v) &= \begin{bmatrix} \frac{\partial P}{\partial u} & \frac{\partial P}{\partial v} \\ \frac{\partial Q}{\partial u} & \frac{\partial Q}{\partial v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{(p^2-\kappa-\omega)-3\rho u^2}{k^2} & 0 \end{bmatrix}, \\ J(u,v) &= \det M(u,v). \end{aligned}$$

Obviously,

$$J(0,0) = -\frac{p^2-\kappa-\omega}{k^2}, \qquad J\left(\pm\sqrt{\frac{p^2-\kappa-\omega}{\rho}}, 0\right) = \frac{2(p^2-\kappa-\omega)}{k^2}. \tag{3.14}$$

According to discriminant method of the types of equilibrium points for nonlinear dynamical system given in [21,22], we know that the point $O(0,0)$ is a saddle point and the points $A_{1,2}\left(\pm\sqrt{\frac{p^2-\kappa-\omega}{\rho}},0\right)$ are center points when $p^2-\kappa-\omega>0$. On the contrary, the point $O(0,0)$ is a center point and the points $A_{1,2}\left(\pm\sqrt{\frac{p^2-\kappa-\omega}{\rho}},0\right)$ are saddle points when $p^2-\kappa-\omega<0$. According to above information, the graphs of phase portraits of the system (3.10) are illustrated in Fig. 2.



(a) $\rho>0,\ p^2-\kappa-\omega\le 0$ (b) $\rho<0,\ p^2-\kappa-\omega\ge 0$

(c) $\rho<0,\ p^2-\kappa-\omega<0$ (d) $\rho>0,\ p^2-\kappa-\omega>0$

Figure 2. Graphs of phase portraits of the system (3.10).

Through the distribution of orbits in each phase portraits in Figure 2, we will discuss the solutions of the ODE (3.6) in the corresponding parametric conditions, and further search different kinds of exact traveling wave solutions of Eqs. (3.1), (3.2) and (3.3).

Case 1. When $p^2-\kappa-\omega\le 0,\ \rho>0$ and $h>0$, system (3.10) has a family of closed orbits (infinity many closed orbits) surround the center point $O(0,0)$, which is shown in Fig.2a. This indicates that the ODE (3.6) exists a family of periodic solutions. In particular, when $p^2-\kappa-\omega=0$ (i.e. $p=\sqrt{\kappa+\omega}$) and $\rho>0,\ h>0$, the Eq. (3.11) is reduced to

$$v = \pm\frac{\sqrt{2\rho}}{2k}\sqrt{(a^2+u^2)(a^2-u^2)}, \tag{3.15}$$

where $a^2 = \sqrt{\frac{2k^2h}{\rho}}$. Substituting (3.15) into $\frac{du}{d\zeta} = v$ in (3.10) and then integrating it, we get

$$\int_u^a \frac{du}{\sqrt{(a^2+u^2)(a^2-u^2)}} = \pm\frac{\sqrt{2\rho}}{2k}\int_\zeta^0 d\zeta. \tag{3.16}$$

Solving (3.16), we obtain an exact periodic solution of ODE (3.6) as follows:

$$u = \sqrt[4]{\frac{2k^2h}{\rho}}\mathrm{cn}\left(\sqrt[4]{\frac{2\rho h}{k^2}}\zeta, \frac{\sqrt{2}}{2}\right). \tag{3.17}$$

Plugging (3.17), (3.9) and $p = \sqrt{\kappa+\omega}$ into (3.4), we obtain a non-periodic traveling wave solution of Eq. (3.1) as follows:

$$\psi = \sqrt[4]{\frac{2k^2h}{\rho}}\ \mathrm{cn}\left(\sqrt[4]{\frac{2\rho h}{k^2}}\left(\frac{k\sinh(1-\beta)x}{1-\beta} - \frac{2k\sqrt{\kappa+\omega}\sinh(1-\alpha)t}{1-\alpha}\right), \frac{\sqrt{2}}{2}\right)e^{i\theta}, \tag{3.18}$$

where $\theta = \frac{\sqrt{\kappa+\omega}\sinh(1-\beta)x}{1-\beta} - \frac{\omega\sinh(1-\alpha)t}{1-\alpha}$. According to Theorem 2.1 and using the replacements of independent variables $\frac{\sinh(1-\beta)x}{1-\beta} \sim x$ and $\frac{\sinh(1-\alpha)t}{1-\alpha} \sim t$, the solution (3.18) is converted into a periodic wave solution of the classical nonlinear Schrödinger equation (3.2) as follows:

$$\psi = \sqrt[4]{\frac{2k^2h}{\rho}}\ \mathrm{cn}\left(\sqrt[4]{\frac{2\rho h}{k^2}}\big(kx - 2k\sqrt{\kappa+\omega}\ t\big), \frac{\sqrt{2}}{2}\right)e^{i\left(\sqrt{\kappa+\omega}x-\omega t\right)}. \tag{3.19}$$

According to Corollary 2.1 and using the replacements of independent variables $x \sim \frac{x^\beta}{\beta}$ and $t \sim \frac{t^\alpha}{\alpha}$, the solution (3.19) is converted into a non-periodic traveling wave solution of the nonlinear conformable fractional Schrödinger equation (3.3) as follows:

$$\psi = \sqrt[4]{\frac{2k^2h}{\rho}}\ \mathrm{cn}\left(\sqrt[4]{\frac{2\rho h}{k^2}}\left(\frac{kx^\beta}{\beta} - \frac{2k\sqrt{\kappa+\omega}\ t^\alpha}{\alpha}\right), \frac{\sqrt{2}}{2}\right)e^{i\left(\frac{\sqrt{\kappa+\omega}x^\beta}{\beta}-\frac{\omega t^\alpha}{\alpha}\right)}. \tag{3.20}$$

In order to compare dynamical properties of the solutions (3.18), (3.19) and (3.20), we plot the comparison graph of modules $|\psi|$ to the solutions (3.18) and (3.19), which is shown in Figure 3. Also, we plot the comparison graph of modules $|\psi|$ to the solutions (3.20) and (3.19), which is shown in Figure 4. In Figs. 3 and 4, the values of parameters are taken as $\rho = 2,\ \kappa = 0.5,\ \omega = 0.8,\ h = 4,\ \alpha = 0.25,\ \beta = 0.75,\ k = 1,\ t = 2$. In comparison graphs of modules of solutions, the curve to solution of the sech-fractional Schrödinger equation is marked in blue, the curve to solution of the classical integer-order Schrödinger equation is marked in red and the curve to solution of the conformable fractional Schrödinger equation is marked in black. All the curves in the below graphs are colored by this convention and will not be repeated.

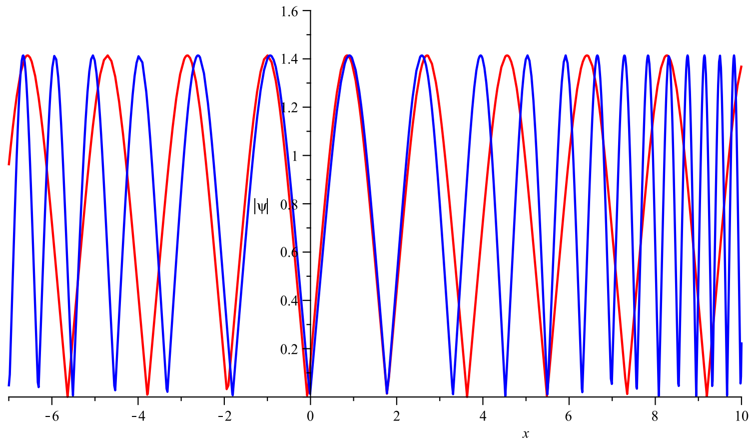


Figure 3. Comparison graph of modules of the solutions (3.18) and (3.19).

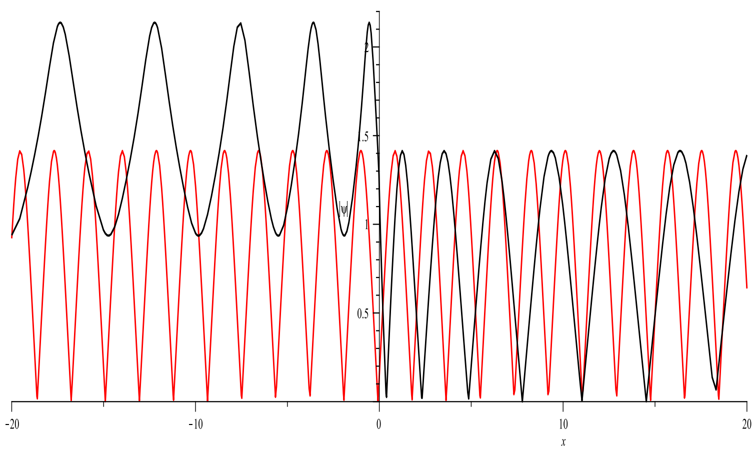


Figure 4. Comparison graph of modules of solutions (3.20) and (3.19).

As can be seen from Figures 3 and 4, the oscillation frequency of the blue curve will be faster and faster with increasing distance in the sech-fractional system, the oscillation frequency of the black curve will be slower and slower with increasing distance in the conformable fractional system, the oscillation frequency of the red curve is always kept constant in the integer-order system.

Case 2. When $p^2 - \kappa - \omega \geq 0,\ \rho < 0$ and $h = h_0 = 0$, system (3.10) has two line orbits (marked in black) cross over the saddle point $O(0,0)$, which is shown in Fig.2b. This indicates that the ODE (3.6) exists two unbounded solution.

(i) When $p^2 - \kappa - \omega > 0,\ \rho < 0$ and $h = h_0 = 0$, the Eq. (3.11) is reduced to

$$v = \pm \frac{\sqrt{-2\rho}}{2k} u \sqrt{\frac{2(\kappa + \omega - p^2)}{\rho} + u^2}. \tag{3.21}$$

Substituting (3.21) into $\frac{du}{d\zeta} = v$ in (3.10) and then integrating it, we get

$$\int \frac{du}{u\sqrt{\frac{2(\kappa+\omega-p^2)}{\rho} + u^2}} = \pm\frac{\sqrt{-2\rho}}{2k}\int d\zeta. \tag{3.22}$$

Solving (3.22) and letting the integral constant as zero, we obtain two bounded solutions of ODE (3.6) as follows:

$$u = \mp\sqrt{\frac{2(\kappa+\omega-p^2)}{\rho}}\operatorname{csch}\left(\frac{\sqrt{p^2-\kappa-\omega}}{k}\zeta\right). \tag{3.23}$$

Plugging (3.23) and (3.9) into (3.4), we obtain two non-periodic traveling wave solutions of Eq. (3.1) as follows:

$$\begin{aligned}\psi = \mp\sqrt{\frac{2(\kappa+\omega-p^2)}{\rho}}\operatorname{csch}\Bigg(&\frac{\sqrt{p^2-\kappa-\omega}}{k}\\ &\times\left(\frac{k\sinh(1-\beta)x}{1-\beta} - \frac{2kp\sinh(1-\alpha)t}{1-\alpha}\right)\Bigg)e^{i\theta},\end{aligned} \tag{3.24}$$

where $\theta = \frac{p\sinh(1-\beta)x}{1-\beta} - \frac{\omega\sinh(1-\alpha)t}{1-\alpha}$. Similarly, according to Theorem 2.1 and Corollary 2.1, we obtain the following solutions of Eqs. (3.2) and (3.3), respectively.

$$\psi = \mp\sqrt{\frac{2(\kappa+\omega-p^2)}{\rho}}\operatorname{csch}\left(\frac{\sqrt{p^2-\kappa-\omega}}{k}(kx-2kpt)\right)e^{i(px-\omega t)}, \tag{3.25}$$

$$\psi = \mp\sqrt{\frac{2(\kappa+\omega-p^2)}{\rho}}\operatorname{csch}\left(\frac{\sqrt{p^2-\kappa-\omega}}{k}\left(\frac{kx^\beta}{\beta} - \frac{2kpt^\alpha}{\alpha}\right)\right)e^{i\left(\frac{px^\beta}{\beta}-\frac{\omega t^\alpha}{\alpha}\right)}. \tag{3.26}$$

(ii) When $p^2-\kappa-\omega=0$ (i.e. $p=\sqrt{\kappa+\omega}$) and $\rho<0,\ h=0$, the Eq. (3.11) is reduced to

$$v = \pm\frac{\sqrt{-2\rho}}{2k}u^2. \tag{3.27}$$

By using the same method as before, we can obtain two unbounded solutions of ODE (3.6) as follows:

$$u = \mp\frac{2k}{\sqrt{-2\rho}}\zeta^{-1}. \tag{3.28}$$

Thus, Eq. (3.1) has two exact traveling wave solutions as follows:

$$\begin{aligned}\psi = \mp\frac{2k}{\sqrt{-2\rho}}&\left(\frac{k\sinh(1-\beta)x}{1-\beta} - \frac{2k\sqrt{\kappa+\omega}\sinh(1-\alpha)t}{1-\alpha}\right)^{-1}\\ &\times e^{i\left(\frac{\sqrt{\kappa+\omega}\sinh(1-\beta)x}{1-\beta}-\frac{\omega\sinh(1-\alpha)t}{1-\alpha}\right)}.\end{aligned} \tag{3.29}$$

Similarly, according to Theorem 2.1 and Corollary 2.1, we obtain the following solutions of Eqs. (3.2) and (3.3), respectively.

$$\psi = \mp\frac{2ke^{i\left(\sqrt{\kappa+\omega}x-\omega t\right)}}{\sqrt{-2\rho}\left(kx-2k\sqrt{\kappa+\omega}t\right)}, \tag{3.30}$$

$$\psi = \mp \frac{2k}{\sqrt{-2\rho}} \left(\frac{kx^\beta}{\beta} - \frac{2k\sqrt{\kappa+\omega}t^\alpha}{\alpha} \right)^{-1} e^{i\left(\frac{\sqrt{\kappa+\omega}x^\beta}{\beta} - \frac{\omega t^\alpha}{\alpha}\right)}. \tag{3.31}$$

Case 3. When $p^2-\kappa-\omega<0,\ \rho<0$ and $h=h_1$, system (3.10) has two heteroclinic orbits (marked in black) pass through the saddle points A_1 and A_2, which is shown in Fig.2c; this indicates that the ODE (3.6) exists two heteroclinic solutions shaped as kink wave and anti-kink wave. When $p^2-\kappa-\omega<0,\ \rho<0$ and $0<h<h_1$, system (3.10) has a family of closed orbits (marked in brown) around the center point $O(0,0)$ which is shown in Fig.2c; this indicates that the ODE (3.6) exists a family of periodic wave solutions.

(i) When $p^2-\kappa-\omega<0,\ \rho<0$ and $h=h_1=-\frac{(p^2-\kappa-\omega)^2}{2\rho k^2}$, Eq. (3.11) is reduced to

$$v = \pm \frac{\sqrt{-2\rho}}{2k} \left(u^2 - \frac{p^2-\kappa-\omega}{\rho} \right). \tag{3.32}$$

Substituting (3.32) into $\frac{du}{d\zeta}=v$ in (3.10) and then integrating it, we get

$$\int \frac{du}{u^2 - \frac{p^2-\kappa-\omega}{\rho}} = \pm \frac{\sqrt{-2\rho}}{2k} \int d\zeta. \tag{3.33}$$

Solving (3.33), we obtain two heteroclinic solutions of ODE (3.6) as follows:

$$u = \pm \sqrt{\frac{p^2-\kappa-\omega}{\rho}} \tanh\left(\frac{\sqrt{2(\kappa+\omega-p^2)}}{2k}\zeta \right). \tag{3.34}$$

Plugging (3.34) and (3.9) into (3.4), we obtain two soliton solutions of Eq. (3.1) as follows:

$$\psi = \pm \sqrt{\frac{p^2-\kappa-\omega}{\rho}} \tanh\left(\frac{\sqrt{2(\kappa+\omega-p^2)}}{2k} \left(\frac{k\sinh(1-\beta)x}{1-\beta} - \frac{2kp\sinh(1-\alpha)t}{1-\alpha} \right) \right) e^{i\theta}, \tag{3.35}$$

where $\theta = \frac{p\sinh(1-\beta)x}{1-\beta} - \frac{\omega\sinh(1-\alpha)t}{1-\alpha}$. Similarly, according to Theorem 2.1 and Corollary 2.1, we obtain the following soliton solutions of Eqs. (3.2) and (3.3), respectively.

$$\psi = \pm \sqrt{\frac{p^2-\kappa-\omega}{\rho}} \tanh\left(\frac{\sqrt{2(\kappa+\omega-p^2)}}{2k} (kx-2kpt) \right) e^{i(px-\omega t)}, \tag{3.36}$$

$$\psi = \pm \sqrt{\frac{p^2-\kappa-\omega}{\rho}} \tanh\left(\frac{\sqrt{2(\kappa+\omega-p^2)}}{2k} \left(\frac{kx^\beta}{\beta} - \frac{2kpt^\alpha}{\alpha} \right) \right) e^{i\left(\frac{px^\beta}{\beta} - \frac{\omega t^\alpha}{\alpha}\right)}. \tag{3.37}$$

(ii) When $p^2-\kappa-\omega<0,\ \rho<0$ and $0<h<h_1$, Eq. (3.11) is reduced to

$$v = \pm \frac{\sqrt{-2\rho}}{2k} \sqrt{(u_1-u)(u_3-u)(u-u_4)(u-u_2)}, \tag{3.38}$$

where

$$u_{1,2} = \pm\sqrt{\frac{(p^2-\kappa-\omega)+\sqrt{p^2(p^2-2\kappa-2\omega)+(\kappa+\omega)^2+2\rho k^2 h}}{\rho}},$$
$$u_{3,4} = \pm\sqrt{\frac{(p^2-\kappa-\omega)-\sqrt{p^2(p^2-2\kappa-2\omega)+(\kappa+\omega)^2+2\rho k^2 h}}{\rho}}.$$

Taking $(u_3, 0)$ as initial value point and then substituting (3.38) into $\frac{du}{d\zeta} = v$ in (3.10) to integrate, we get

$$\int_u^{u_3} \frac{du}{\sqrt{(u_1-u)(u_3-u)(u-u_4)(u-u_2)}} = \pm\frac{\sqrt{-2\rho}}{2k}\int_\zeta^0 d\zeta. \tag{3.39}$$

Solving (3.39), we obtain a periodic solution of ODE (3.6) as follows:

$$u = \frac{u_3 - u_1\lambda_1 \mathrm{sn}^2\left(\frac{\sqrt{-2\rho(u_1-u_2)(u_3-u_4)}}{4k}\zeta, m_1\right)}{1-\lambda_1\mathrm{sn}^2\left(\frac{\sqrt{-2\rho(u_1-u_2)(u_3-u_4)}}{4k}\zeta, m_1\right)}, \tag{3.40}$$

where $\lambda_1 = \frac{u_3-u_4}{u_1-u_4}$, $m_1 = \sqrt{\frac{(u_3-u_4)(u_1-u_2)}{(u_1-u_4)(u_3-u_2)}}$, the $\mathrm{sn}(*, m_1)$ is Jacobian elliptic function and m_1 is its module. Plugging (3.40) and (3.9) into (3.4), we obtain an exact traveling wave solution of Eq. (3.1) as follows:

$$\psi = \frac{u_3 - u_1\lambda_1\mathrm{sn}^2\left(\frac{\sqrt{-2\rho(u_1-u_2)(u_3-u_4)}}{4k}\left(\frac{k\sinh(1-\beta)x}{1-\beta} - \frac{2kp\sinh(1-\alpha)t}{1-\alpha}\right), m_1\right)}{1-\lambda_1\mathrm{sn}^2\left(\frac{\sqrt{-2\rho(u_1-u_2)(u_3-u_4)}}{4k}\left(\frac{k\sinh(1-\beta)x}{1-\beta} - \frac{2kp\sinh(1-\alpha)t}{1-\alpha}\right), m_1\right)} e^{i\theta}, \tag{3.41}$$

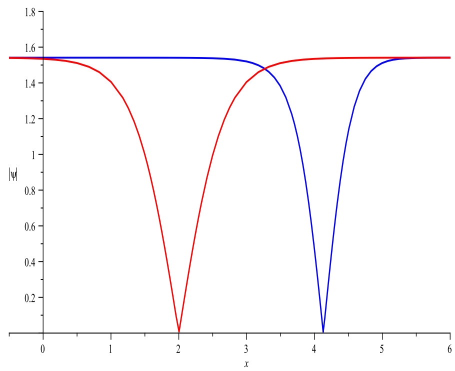
where $\theta = \frac{p\sinh(1-\beta)x}{1-\beta} - \frac{\omega\sinh(1-\alpha)t}{1-\alpha}$. According to Theorem 2.1 and Corollary 2.1, we obtain the following traveling wave solutions of Eqs. (3.2) and (3.3), respectively.

$$\psi = \frac{u_3 - u_1\lambda_1\mathrm{sn}^2\left(\frac{\sqrt{-2\rho(u_1-u_2)(u_3-u_4)}}{4k}(kx-2kpt), m_1\right)}{1-\lambda_1\mathrm{sn}^2\left(\frac{\sqrt{-2\rho(u_1-u_2)(u_3-u_4)}}{4k}(kx-2kpt), m_1\right)} e^{i(px-\omega t)}, \tag{3.42}$$

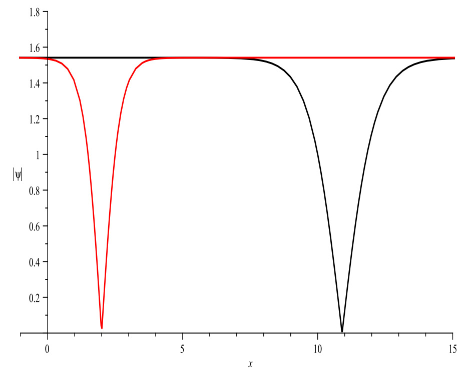
$$\psi = \frac{u_3 - u_1\lambda_1\mathrm{sn}^2\left(\frac{\sqrt{-2\rho(u_1-u_2)(u_3-u_4)}}{4k}\left(\frac{kx^\beta}{\beta} - \frac{2kpt^\alpha}{\alpha}\right), m_1\right)}{1-\lambda_1\mathrm{sn}^2\left(\frac{\sqrt{-2\rho(u_1-u_2)(u_3-u_4)}}{4k}\left(\frac{kx^\beta}{\beta} - \frac{2kpt^\alpha}{\alpha}\right), m_1\right)} e^{i\left(\frac{px^\beta}{\beta} - \frac{\omega t^\alpha}{\alpha}\right)}, \tag{3.43}$$

where the expressions of the parameters u_1, u_2, u_3, u_4, λ and m_1 have been given above.

In order to show difference of dynamical properties to soliton solutions, as examples, we plot the comparison graph of modules $|\psi|$ to the solutions (3.35) and (3.36), which is shown in Fig.5a. Also, we plot the comparison graph of modules $|\psi|$ to the solutions (3.37) and (3.36), which is shown in Fig.5b. In graphs, the values of parameters are taken as $\rho=-2,\ p=0.5,\ \kappa=3,\ \omega=2,\ h=4,\ \alpha=0.25,\ \beta=0.75,\ k=1,\ t=1$.



(a) graph of solutions (3.35) and (3.36)



(b) graph of solutions (3.37) and (3.36)

Figure 5. Comparison graphs of modules of solutions (3.35), (3.36) and (3.37).

Case 4. When $p^2-\kappa-\omega>0,\ \rho>0$ and $h=h_0=0$, system (3.10) has two homoclinic orbits (marked in black) pass through the saddle point $O(0,0)$, which is shown in Fig.2d; this indicates that the ODE (3.6) exists two homoclinic solutions shaped as bright soliton and dark soliton. When $p^2-\kappa-\omega>0,\ \rho>0$ and $h_1<h<0$, system (3.10) has two family of closed orbits (marked in blue) respectively around the center points A_1 and A_2, which is shown in Fig.2d; this indicates that the ODE (3.6) exists two family of periodic solutions. When $p^2-\kappa-\omega>0,\ \rho>0$ and $h>0$, system (3.10) has a family of closed orbits (marked in red) around the two homoclinic orbits, which is shown in Fig.2d; this indicates that the ODE (3.6) exists a family of periodic solutions.

(i) When $p^2-\kappa-\omega>0,\ \rho>0$ and $h=h_0=0$, Eq. (3.11) is reduced to

$$v=\pm\frac{\sqrt{2\rho}}{2k}u\sqrt{\frac{2(p^2-\kappa-\omega)}{\rho}-u^2}. \tag{3.44}$$

Taking $\left(\pm\sqrt{\frac{2(p^2-\kappa-\omega)}{\rho}},0\right)$ as initial value points, substituting (3.44) into $\frac{du}{d\zeta}=v$ in (3.10) and then integrating it, we get

$$\int_u^{\sqrt{\frac{2(p^2-\kappa-\omega)}{\rho}}}\frac{du}{u\sqrt{\frac{2(p^2-\kappa-\omega)}{\rho}-u^2}}=\pm\frac{\sqrt{2\rho}}{2k}\int_\zeta^0 d\zeta, \tag{3.45}$$

$$\int_{-\sqrt{\frac{2(p^2-\kappa-\omega)}{\rho}}}^u\frac{du}{u\sqrt{\frac{2(p^2-\kappa-\omega)}{\rho}-u^2}}=\pm\frac{\sqrt{2\rho}}{2k}\int_0^\zeta d\zeta. \tag{3.46}$$

Respectively solving (3.45) and (3.46), we obtain two exact traveling wave solutions of ODE (3.6) as follows:

$$u=\pm\sqrt{\frac{2(p^2-\kappa-\omega)}{\rho}}\operatorname{sech}\left(\frac{\sqrt{p^2-\kappa-\omega}}{k}\zeta\right). \tag{3.47}$$

Thus, Eq. (3.1) has two soliton solutions formed as

$$\psi = \pm\sqrt{\frac{2(p^2-\kappa-\omega)}{\rho}}\operatorname{sech}\left(\frac{\sqrt{p^2-\kappa-\omega}}{k}\right.$$
$$\left.\left(\frac{k\sinh(1-\beta)x}{1-\beta}-\frac{2kp\sinh(1-\alpha)t}{1-\alpha}\right)\right)e^{i\theta}, \tag{3.48}$$

where $\theta = \frac{p\sinh(1-\beta)x}{1-\beta} - \frac{\omega\sinh(1-\alpha)t}{1-\alpha}$. According to Theorem 2.1 and Corollary 2.1, we obtain the following soliton solutions of Eqs. (3.2) and (3.3), respectively.

$$\psi = \pm\sqrt{\frac{2(p^2-\kappa-\omega)}{\rho}}\operatorname{sech}\left(\frac{\sqrt{p^2-\kappa-\omega}}{k}(kx-2kpt)\right)e^{i(px-\omega t)}, \tag{3.49}$$

$$\psi = \pm\sqrt{\frac{2(p^2-\kappa-\omega)}{\rho}}\operatorname{sech}\left(\frac{\sqrt{p^2-\kappa-\omega}}{k}\left(\frac{kx^\beta}{\beta}-\frac{2kpt^\alpha}{\alpha}\right)\right)e^{i\left(\frac{px^\beta}{\beta}-\frac{\omega t^\alpha}{\alpha}\right)}. \tag{3.50}$$

(ii) When $p^2-\kappa-\omega>0$, $\rho>0$ and $h_1<h<0$, Eq. (3.11) is reduced to

$$v = \pm\frac{\sqrt{2\rho}}{2k}\sqrt{(u_1-u)(u-u_3)(u-u_4)(u-u_2)} \tag{3.51}$$

or

$$v = \pm\frac{\sqrt{2\rho}}{2k}\sqrt{(u_1-u)(u_3-u)(u_4-u)(u-u_2)}, \tag{3.52}$$

where

$$u_{1,2} = \pm\sqrt{\frac{(p^2-\kappa-\omega)+\sqrt{p^2(p^2-2\kappa-2\omega)+(\kappa+\omega)^2+2\rho k^2 h}}{\rho}},$$
$$u_{3,4} = \pm\sqrt{\frac{(p^2-\kappa-\omega)-\sqrt{p^2(p^2-2\kappa-2\omega)+(\kappa+\omega)^2+2\rho k^2 h}}{\rho}}.$$

Respectively substituting (3.51) and (3.52) into $\frac{du}{d\zeta}=v$ in (3.10) and then integrating it, we get

$$\int_u^{u_1}\frac{du}{\sqrt{(u_1-u)(u-u_3)(u-u_4)(u-u_2)}} = \pm\frac{\sqrt{2\rho}}{2k}\int_\zeta^0 d\zeta, \tag{3.53}$$

$$\int_{u_2}^{u}\frac{du}{\sqrt{(u_1-u)(u_3-u)(u_4-u)(u-u_2)}} = \pm\frac{\sqrt{2\rho}}{2k}\int_\zeta^0 d\zeta. \tag{3.54}$$

Respectively solving (3.53) and (3.54), we obtain two periodic solutions of ODE (3.6) as follows:

$$u = \frac{u_1+u_2\lambda_2\operatorname{sn}^2\left(\frac{\sqrt{2\rho(u_1-u_4)(u_3-u_2)}}{4k}\zeta, m_2\right)}{1+\lambda_2\operatorname{sn}^2\left(\frac{\sqrt{2\rho(u_1-u_4)(u_3-u_2)}}{4k}\zeta, m_2\right)}, \tag{3.55}$$

$$u=\frac{u_2+u_1\lambda_3\mathrm{sn}^2\left(\frac{\sqrt{2\rho(u_1-u_4)(u_3-u_2)}}{4k}\zeta,m_2\right)}{1+\lambda_3\mathrm{sn}^2\left(\frac{\sqrt{2\rho(u_1-u_4)(u_3-u_2)}}{4k}\zeta,m_2\right)}, \tag{3.56}$$

where $\lambda_2=\frac{u_1-u_3}{u_3-u_2},\ \lambda_3=\frac{u_4-u_2}{u_1-u_4},\ m_2=\sqrt{\frac{(u_1-u_3)(u_4-u_2)}{(u_1-u_4)(u_3-u_2)}}$. Thereby, Eq. (3.1) has two non-periodic traveling wave solutions formed as

$$\psi=\frac{u_1+u_2\lambda_2\mathrm{sn}^2\left(\frac{\sqrt{2\rho(u_1-u_4)(u_3-u_2)}}{4k}\left(\frac{k\sinh(1-\beta)x}{1-\beta}-\frac{2kp\sinh(1-\alpha)t}{1-\alpha}\right),m_2\right)}{1+\lambda_2\mathrm{sn}^2\left(\frac{\sqrt{2\rho(u_1-u_4)(u_3-u_2)}}{4k}\left(\frac{k\sinh(1-\beta)x}{1-\beta}-\frac{2kp\sinh(1-\alpha)t}{1-\alpha}\right),m_2\right)}\ e^{i\theta}, \tag{3.57}$$

$$\psi=\frac{u_2+u_1\lambda_3\mathrm{sn}^2\left(\frac{\sqrt{2\rho(u_1-u_4)(u_3-u_2)}}{4k}\left(\frac{k\sinh(1-\beta)x}{1-\beta}-\frac{2kp\sinh(1-\alpha)t}{1-\alpha}\right),m_2\right)}{1+\lambda_3\mathrm{sn}^2\left(\frac{\sqrt{2\rho(u_1-u_4)(u_3-u_2)}}{4k}\left(\frac{k\sinh(1-\beta)x}{1-\beta}-\frac{2kp\sinh(1-\alpha)t}{1-\alpha}\right),m_2\right)}\ e^{i\theta}, \tag{3.58}$$

where $\theta=\frac{p\sinh(1-\beta)x}{1-\beta}-\frac{\omega\sinh(1-\alpha)t}{1-\alpha}$. According to Theorem 2.1 and Corollary 2.1, we obtain two kinds of traveling wave solutions of Eqs. (3.2) and (3.3) as follows:

$$\psi=\frac{u_1+u_2\lambda_2\mathrm{sn}^2\left(\frac{\sqrt{2\rho(u_1-u_4)(u_3-u_2)}}{4k}(kx-2kpt),m_2\right)}{1+\lambda_2\mathrm{sn}^2\left(\frac{\sqrt{2\rho(u_1-u_4)(u_3-u_2)}}{4k}(kx-2kpt),m_2\right)}\ e^{i(px-\omega t)}, \tag{3.59}$$

$$\psi=\frac{u_2+u_1\lambda_3\mathrm{sn}^2\left(\frac{\sqrt{2\rho(u_1-u_4)(u_3-u_2)}}{4k}(kx-2kpt),m_2\right)}{1+\lambda_3\mathrm{sn}^2\left(\frac{\sqrt{2\rho(u_1-u_4)(u_3-u_2)}}{4k}(kx-2kpt),m_2\right)}\ e^{i(px-\omega t)}, \tag{3.60}$$

$$\psi=\frac{u_1+u_2\lambda_2\mathrm{sn}^2\left(\frac{\sqrt{2\rho(u_1-u_4)(u_3-u_2)}}{4k}\left(\frac{kx^\beta}{\beta}-\frac{2kpt^\alpha}{\alpha}\right),m_2\right)}{1+\lambda_2\mathrm{sn}^2\left(\frac{\sqrt{2\rho(u_1-u_4)(u_3-u_2)}}{4k}\left(\frac{kx^\beta}{\beta}-\frac{2kpt^\alpha}{\alpha}\right),m_2\right)}\ e^{i\left(\frac{px^\beta}{\beta}-\frac{\omega t^\alpha}{\alpha}\right)}, \tag{3.61}$$

$$\psi=\frac{u_2+u_1\lambda_3\mathrm{sn}^2\left(\frac{\sqrt{2\rho(u_1-u_4)(u_3-u_2)}}{4k}\left(\frac{kx^\beta}{\beta}-\frac{2kpt^\alpha}{\alpha}\right),m_2\right)}{1+\lambda_3\mathrm{sn}^2\left(\frac{\sqrt{2\rho(u_1-u_4)(u_3-u_2)}}{4k}\left(\frac{kx^\beta}{\beta}-\frac{2kpt^\alpha}{\alpha}\right),m_2\right)}\ e^{i\left(\frac{px^\beta}{\beta}-\frac{\omega t^\alpha}{\alpha}\right)}. \tag{3.62}$$

(iii) When $p^2-\kappa-\omega>0,\ \rho>0$ and $h>0$, Eq. (3.11) is reduced to

$$v=\pm\frac{\sqrt{2\rho}}{2k}\sqrt{(u_1^2-u^2)(u^2+u_2^2)}, \tag{3.63}$$

where

$$u_1^2=\frac{(p^2-\kappa-\omega)+\sqrt{p^2(p^2-2\kappa-2\omega)+(\kappa+\omega)^2+2\rho k^2h}}{\rho},$$

$$u_2^2 = \frac{(p^2 - \kappa - \omega) - \sqrt{p^2(p^2 - 2\kappa - 2\omega) + (\kappa + \omega)^2 + 2\rho k^2 h}}{\rho}.$$

Taking $(u_1, 0)$ as initial value conditions, substituting (3.63) into $\frac{du}{d\zeta} = v$ in (3.10) to integrate, it yields

$$\int_u^{u_1} \frac{du}{\sqrt{(u_1^2 - u^2)(u^2 + u_2^2)}} = \pm\frac{\sqrt{2\rho}}{2k}\int_\zeta^0 d\zeta. \tag{3.64}$$

Solving (3.64), we get

$$u = u_1 \mathrm{cn}\left(\frac{\sqrt{p^2 - \kappa - \omega}}{k}\ \zeta, m_3\right), \tag{3.65}$$

where

$$u_1 = \sqrt{\frac{(p^2 - \kappa - \omega) + \sqrt{p^2(p^2 - 2\kappa - 2\omega) + (\kappa + \omega)^2 + 2\rho k^2 h}}{\rho}},$$

$$m_3 = \sqrt{\frac{1}{2}\left(1 + \frac{\sqrt{p^2(p^2 - 2\kappa - 2\omega) + (\kappa + \omega)^2 + 2\rho k^2 h}}{p^2 - \kappa - \omega}\right)}.$$

Plugging (3.65) and (3.9) into (3.4), we obtain a non-periodic traveling wave solution of Eq. (3.1) as follows:

$$\psi = u_1 \mathrm{cn}\left(\frac{\sqrt{p^2 - \kappa - \omega}}{k}\left(\frac{k\sinh(1-\beta)x}{1-\beta} - \frac{2kp\sinh(1-\alpha)t}{1-\alpha}\right), m_3\right)e^{i\theta}, \tag{3.66}$$

where $\theta = \frac{p\sinh(1-\beta)x}{1-\beta} - \frac{\omega\sinh(1-\alpha)t}{1-\alpha}$ and m_3 is given above. According to Theorem 2.1 and using the replacements of independent variables $\frac{\sinh(1-\beta)x}{1-\beta} \sim x$ and $\frac{\sinh(1-\alpha)t}{1-\alpha} \sim t$, the solution (3.66) is converted into a periodic wave solution of the classical nonlinear Schrödinger equation (3.2) as follows:

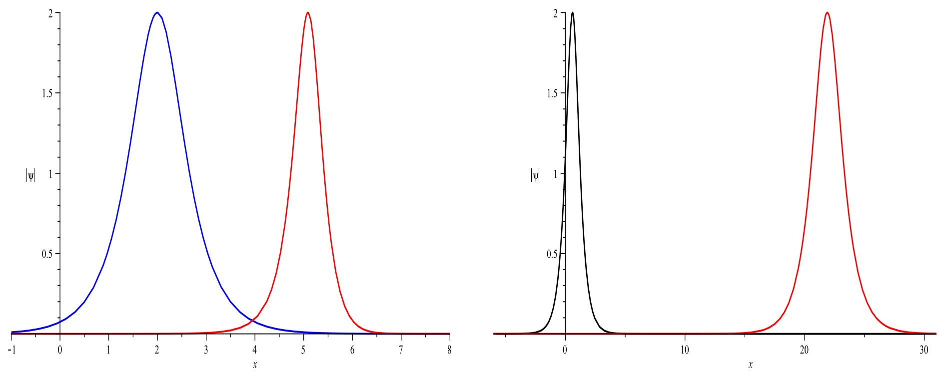
$$\psi = u_1 \mathrm{cn}\left(\frac{\sqrt{p^2 - \kappa - \omega}}{k}\,(kx - 2kpt)\,, m_3\right)e^{i(px - \omega t)}. \tag{3.67}$$

According to Corollary 2.1 and using the replacements of independent variables $x \sim \frac{x^\beta}{\beta}$ and $t \sim \frac{t^\alpha}{\alpha}$, the solution (3.67) is converted into a non-periodic traveling wave solution of the nonlinear conformable fractional Schrödinger equation (3.3) as follows:

$$\psi = u_1 \mathrm{cn}\left(\frac{\sqrt{p^2 - \kappa - \omega}}{k}\left(\frac{kx^\beta}{\beta} - \frac{2kpt^\alpha}{\alpha}\right), m_3\right)e^{i\left(\frac{px^\beta}{\beta} - \frac{\omega t^\alpha}{\alpha}\right)}. \tag{3.68}$$

In order to show difference of dynamical properties to soliton solutions, as examples, we plot the comparison graph of modules $|\psi|$ to the solutions (3.48) and (3.49), which is shown in Fig.6a. Also, we plot the comparison graph of modules $|\psi|$ to the solutions (3.50) and (3.49), which is shown in Fig.6b. In these two graphs, the values of parameters are taken as $\rho = 2,\ p = 3,\ \kappa = 3,\ \omega = 2,\ h = 4,\ \alpha = 0.25,\ \beta = 0.75,\ k = 1$.

As can be seen from Figure 6, the wave amplitudes of the three kinds of solitary waves have not changed, but their fat and thin shapes have been changed. In other words, their wave amplitudes are always the same during the propagation process, but the soliton of the sech-fractional system (3.1) is significantly fatter than that of the integer-order system (3.2), and the soliton of the conformable fractional system (3.3) is significantly thinner than that of the integer-order system (3.2).



(a) graph of (3.48) and (3.49) at $t = 1$

(b) graph of (3.50) and (3.49) at $t = 0.1$

Figure 6. Comparison graphs of modules of solutions (3.48), (3.49) and (3.50).

4. Conclusions

The facts show that the conformable fractional derivative has not any property of fractional-order derivative, it should belong to a cognate derivative of the classical integer-order derivative. Of course, habitually we still call it conformable fractional derivative. In this work, following the idea of constructing the conformable fractional derivative, an extensional conformable fractional derivative named sech-fractional derivative was proposed. And then, we discussed the distinctions and connections between sech-fractional derivative and integer-order derivative. In addition, the investigations shew that structure to solutions of an integer-order PDE and corresponding fractional PDE is same except difference of their independent variables but their dynamical properties are very different. Thereby, we gave two theorems about replacement solution and their corresponding Corollaries, see Theorems 2.1 and 2.2, Corollaries 2.1 and 2.2.

The effects of sech-fractional differential operator on solutions of PDEs were also discussed in this work. As example, traveling wave solutions and their dynamical properties of a nonlinear fractional Schrödinger equation were investigated under the sech-fractional differential operator. Some interesting phenomena have been found; the investigations shew that when an integer-order PDE becomes a conformable fractional PDE, the periodic solution of the clasical integer-order PDE becomes a non-periodic solution of the conformable fractional PDE and their oscillation frequencies must change. In the extensional fractional system, the oscillation frequency will be faster and faster. In the conformable fractional system, the oscillation frequency will be slower and slower. In the integer-order system, the oscillation frequency always keep constant. But the soliton solutions including their amplitudes among three kinds of nonlinear PDEs (such as sech-fractional Schrödinger equation, integer-order Schrödinger equation and conformable fractional Schrödinger equation) do not change, only their waveforms change somewhat in being fat and thin.

Can these interesting and strange phenomena be found in the field of natural science? Or, what natural science problems can be modeled by the sech-fractional derivative and the conformable fractional derivative? Expect this issue to be addressed in future research works. Meanwhile, I hope that the mathematical and physical experts will pay more attention to this problem.

Acknowledgements

The authors are grateful to the anonymous referees for their useful comments and suggestions.

CRediT authorship contribution statement

Weijun He: Calculations and drawing.
Weiguo Rui: Theme conception and method.
Xiaochun Hong: Formal analysis and proof, Writing.

Appendix

In terms of geometry, the $f'(t)$ represents the slope of the tangent line passed through an arbitrary point (t, y) in the curve $y = f(t)$. So, the ${}^{\mathrm{s}}D_t^{\alpha} f(t)$ can be essentially regard as a weighted slope of the tangent line passed through an arbitrary point (t, y) in the curve $y = f(t)$ when $0 < \alpha < 1$. In terms of physics, the $f'(t)$ is a change rate of function $f(t)$ at certain instant t. Similarly, the ${}^{\mathrm{s}}D_t^{\alpha} f(t)$ can be essentially regard as a weighted change rate of function $f(t)$ at certain instant t when $0 < \alpha < 1$. Next, we compare the derivative functions of the six basic elementary functions such as constant function, power function, exponential function, logarithmic function, trigonometric function, anti-trigonometric function and their curve graphs in the definitions of the integer-order derivative and sech-fractional derivative.

$$(C)' = 0, \qquad {}^{\mathrm{s}}D_x^{\beta} C = 0, \quad (C \text{ is constant}), \tag{4.1}$$

$$(x^m)' = mx^{m-1}, \qquad {}^{\mathrm{s}}D_x^{\beta} x^m = mx^{m-1} \operatorname{sech}(1-\beta)x, \tag{4.2}$$

$$(e^x)' = e^x, \qquad {}^{\mathrm{s}}D_x^{\beta} e^x = e^x \operatorname{sech}(1-\beta)x, \tag{4.3}$$

$$(\ln x)' = \frac{1}{x}, \qquad {}^{\mathrm{s}}D_x^{\beta} \ln x = \frac{\operatorname{sech}(1-\beta)x}{x}, \tag{4.4}$$

$$(\sin x)' = \cos x, \qquad {}^{\mathrm{s}}D_x^{\beta} \sin x = \cos x \operatorname{sech}(1-\beta)x, \tag{4.5}$$

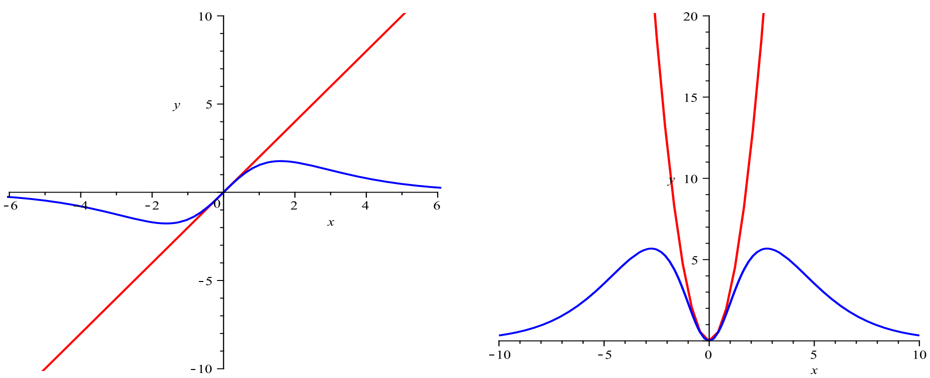
$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \qquad {}^{\mathrm{s}}D_x^{\beta} \arcsin x = \frac{\operatorname{sech}(1-\beta)x}{\sqrt{1-x^2}}, \tag{4.6}$$

$$(\tan x)' = \sec^2 x, \qquad {}^{\mathrm{s}}D_x^{\beta} \tan x = \sec^2 x \operatorname{sech}(1-\beta)x, \tag{4.7}$$

$$(\arctan x)' = \frac{1}{1+x^2}, \qquad {}^{\mathrm{s}}D_x^{\beta} \arctan x = \frac{\operatorname{sech}(1-\beta)x}{1+x^2}, \tag{4.8}$$

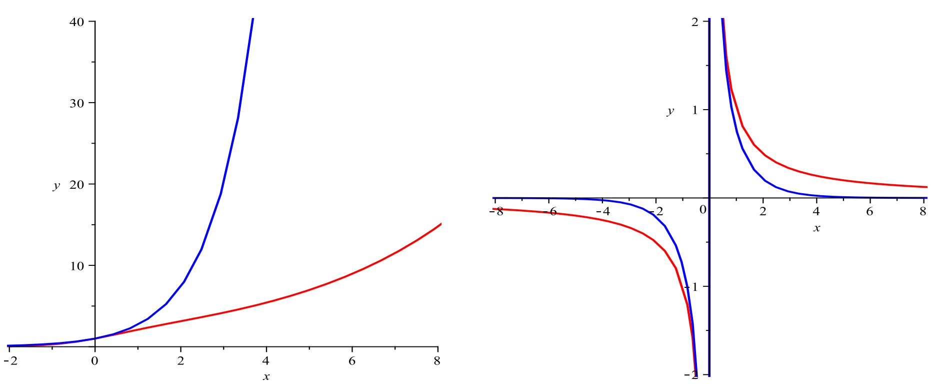
where $f'(x) = \frac{df(x)}{dx}$, $0 < \beta < 1$. Clearly, the derivative of the constant function is always equal to zero either under the integer-order differential operator or under the sech-fractional differential operator. The derivatives of other functions are different under the integer-order and sech-fractional differential operators, all the sech-fractional derivatives have one more functional factor $\operatorname{sech}(1-\beta)x$ than corresponding integer-order derivatives.

To intuitively show the geometric properties to derivative functions of the above several elementary functions under two different kinds of differential operators, taking $\beta = 0.25$ as an example, we plot their graphs, see Figures 7-10. In Figures 7-10, the curve graphs of integer-order derivative functions are shown in red curves, and the curve graphs of sech-fractional derivative functions are shown in blue curves.



(a) $m = 2$ (b) $m = 3$

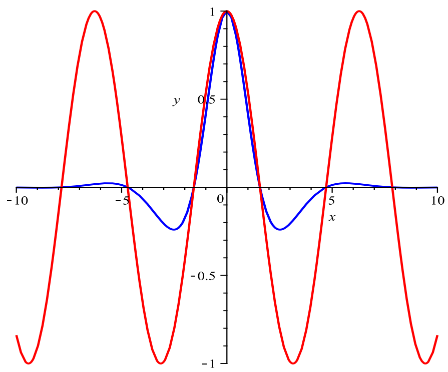
Figure 7. Graphs of derivatives defined by (5.2) under two operators.



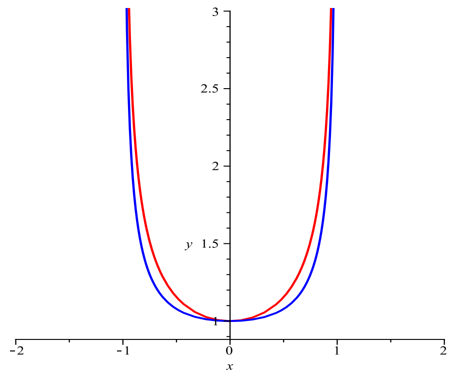
(a) Two kinds of derivatives of e^x (b) Two kinds of derivatives of $\ln x$

Figure 8. Graphs of functions defined by (5.3) and (5.4) under two operators.

From Figure 7-10 and the expressions (5.2)-(5.8), it is not difficult to find that the geometric properties of the derivative functions of the $\ln x$ and all anti-triangle

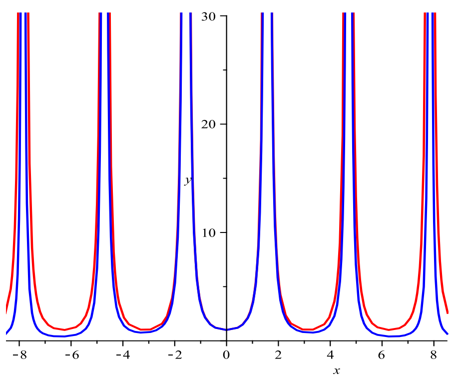


(a) Two kinds of derivatives of $\sin x$

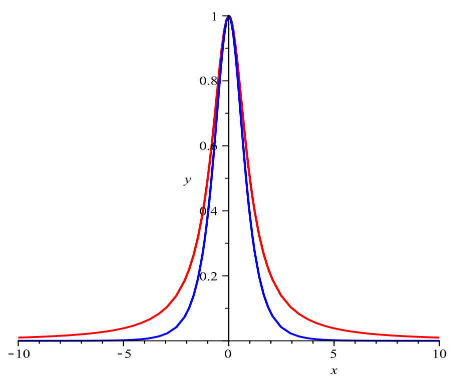


(b) Two kinds of derivatives of $\arcsin x$

Figure 9. Graphs of functions defined by (5.5) and (5.6) under two operators.



(a) Two kinds of derivatives of $\tan x$



(b) Two kinds of derivatives of $\arctan x$

Figure 10. Graphs of functions defined by (5.7) and (5.8) under two operators.

functions are very similar (close) under the integer-order and sech-fractional differential operators, but the geometric properties of other derivative functions are very different. In particular, the sech-fractional derivative functions of the periodic functions such as $\sin x$ and $\cos x$ are no longer periodic functions, these are very different from integer-order derivative functions of the periodic functions.

References

[1] T. Abdeljawad, *On conformable fractional calculus*, Journal of Computational and Applied Mathematics, 2015, 279, 57–66.

[2] B. Acay, E. Bas and T. Abdeljawad, *Non-local fractional calculus from different viewpoint generated by truncated M-derivative*, Journal of Computational and Applied Mathematics, 2020, 366, 112410.

[3] E. Balci, İ. Öztürk and S. Kartal, *Dynamical behaviour of fractional order tumor model with Caputo and conformable fractional derivative*, Chaos, Solitons and Fractals, 2019, 123, 43–51.

[4] Y. Cenesiz and K. Ali, *The new solution of time fractional wave equation with conformable fractional derivative definition*, Journal of New Theory, 2015, 7, 79–85.

[5] W. S. Chung, *Fractional Newton mechanics with conformable fractional derivative*, Journal of Computational and Applied Mathematics, 2015, 290, 150–158.

[6] M. T. Darvishi, M. Najafi and A. R. Seadawy, *Dispersive bright, dark and singular optical soliton solutions in conformable fractional optical fiber Schrödinger models and its applications*, Optical and Quantum Electronics, 2018, 50(4), 1–16.

[7] M. Ekici, M. Mirzazadeh, M. Eslami, et al., *Optical soliton perturbation with fractional-temporal evolution by first integral method with conformable fractional derivatives*, Optik, 2016, 127(22), 10659–10669.

[8] A. El-Ajou, M. N. Oqielat, Z. Al-Zhour, et al, *Solitary solutions for time-fractional nonlinear dispersive PDEs in the sense of conformable fractional derivative*, Chaos, 2019, 29(9), 093102.

[9] M. Eslami and H. Rezazadeh, *The first integral method for Wu-Zhang system with conformable time-fractional derivative*, Calcolo, 2016, 53(3), 475–485.

[10] J. H. He, S. K. Elagan and Z. B. Li, *Geometrical explanation of the fractional complex trnsform and derivative chain rule for fractional calculus*, Physica Letters A, 2012, 376, 257–259.

[11] Y. He and Y. Zhao, *Applications of separation variables approach in solving time-fractional PDEs*, Mathematical Problems in Engineering, 2018, 2018.

[12] X. Hong, A. G. Davodi, S. M. Mirhosseini-Alizamini, et al., *New explicit solitons for the general modified fractional Degasperis-Procesi-Camassa-Holm equation with a truncated M-fractional derivative*, Modern Physics Letters B, 2021, 35(33), 2150496.

[13] A. Hussain, A. Jhangeer, N. Abbas, et al., *Optical solitons of fractional complex Ginzburg-Landau equation with conformable, beta, and M-truncated derivatives: A comparative study*, Advances in Difference Equations, 2020, 2020(1), 1–19.

[14] E. İhan and İ O. Kiymaz, *A generalization of truncated M-fractional derivative and applications to fractional differential equations*, Applied Mathematics and Nonlinear Sciences, 2020, 5(1), 171–188.

[15] O. S. Iyiola and E. R. Nwaeze, *Some new results on the new conformable fractional calculus with application using D'Alambert approach*, Progress in Fractional Differentiation and Applications, 2016, 2(2), 115–122.

[16] G. Jumarie, *Modified Riemann-Liouville derivative and fractional Talor series of non-differentiable functions further results*, Computers and Mathematics with Applications, 2006, 51(9–10), 1367–1376.

[17] G. Jumarie, *Fractional partial differential equations and modified Riemann-Liouville derivative new methods for solution*, Journal of Applied Mathematics and Computing, 2007, 24(1–2), 31–48.

[18] G. Jumarie, *Cauchy's integral formula via the modified Riemann-Liouville derivative for analytic functions of fractional order*, Applied Mathematics Letters, 2010, 23(12), 1444–1450.

[19] R. Khalil, M. Al Horani, A. Yousef, et al., *A new definition of fractional derivative*, Journal of Computational and Applied Mathematics, 2014, 264, 65–70.

[20] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, 2006.

[21] J. Li and Z. Liu, *Smooth and non-smooth traveling waves in a nonlinearly dispersive equation*, Applied Mathematical Modelling, 2000, 25(1), 41–56.

[22] J. Li and L. Zhang, *Bifurcations of traveling wave solutions in generalized Pochhammer-Chree equation*, Chaos, Solitons and Fractals, 2002, 14(4), 581–593.

[23] X. Li, *Comment on "Applications of homogenous balanced principle on investigating exact solutions to a series of time fractional nonlinear PDEs"*, Communications in Nonlinear Science and Numerical Simulation, 2018, 59, 606–607.

[24] N. G. N'Gbo and Y. Xia, *Traveling wave solution of bad and good modified Boussinesq equations with conformable fractional-order derivative*, Qualitative Theory of Dynamical Systems, 2022, 21(1), 1–21.

[25] M. S. Osman, A. Zafar, K. K. Ali, et al., *Novel optical solitons to the perturbed Gerdjikov-Ivanov equation with truncated M-fractional conformable derivative*, Optik, 2020, 222, 165418.

[26] H. Rezazadeh, S. Mirhosseini-Alizamin and R. Attia, *New exact optical soliton solutions for the General Modified fractional Degasperis-Procesi-Camassa-Holm equations with a truncated M-fractional derivative*, Modern Physics Letters B, 2021, 35(33), 17 pp.

[27] M. B. Riaz, A. Jhangeer, J. Awrejcewicz, et al., *Fractional propagation of short light pulses in monomode optical fibers: Comparison of Beta derivative and truncated M-fractional derivative*, Journal of Computational and Nonlinear Dynamics, 2022, 17(3), 031002, 9 pp.

[28] W. Rui, *Applications of integral bifurcation method together with homogeneous balanced principle on investigating exact solutions of time fractional nonlinear PDEs*, Nonlinear Dynamics, 2018, (91), 697-712.

[29] W. Rui, *Dynamical system method for investigating existence and dynamical property of solution of nonlinear time-fractional PDEs*, Nonlinear Dynamics, 2020, 99(3), 2421–2440.

[30] W. Rui, *Separation method of semi-fixed variables together with dynamical system method for solving nonlinear time-fractional PDEs with higher-order terms*, Nonlinear Dynamics, 2022, 109, 943–961.

[31] S. Salahshour, A. Ahmadian, S. Abbasbandy, et al., *M-fractional derivative under interval uncertainty: Theory, properties and applications*, Chaos, Solitons and Fractals, 2018, 117, 84–93.

[32] J. E. Solís-Pérez and J. F. Gómez-Aguilar, *Novel fractional operators with three orders and power-law, exponential decay and Mittag-Leffler memories involving the truncated M-derivative*, Symmetry, 2020, 12(4), 626.

[33] J. Sousa, E. C. de Oliveira, *A new truncated M-fractional derivative type unifying some fractional derivative types with classical properties*, arXiv preprint, arXiv:1704.08187, 2017.

[34] T. A. Sulaiman, G. Yel and H. Bulut, *M-fractional solitons and periodic wave solutions to the Hirota-Maccari system*, Modern Physics Letters B, 2019, 33(05), 1950052.

[35] A. S. Tagne, J. M. Ema'a Ema'a, G. H. Ben-Bolie, et al., *A new truncated M-fractional derivative for air pollutant dispersion*, Indian Journal of Physics 2020, 94(11), 1777-1784.

[36] V. E. Tarasov, *On chain rule for fractional derivatives*, Communications in Nonlinear Science and Numerical Simulation, 2016, 30(1), 1–4.

[37] K. U. Tariq, M. Younis, S. T. R. Rizvi, et al., *M-truncated fractional optical solitons and other periodic wave structures with Schrödinger-Hirota equation*, Modern Physics Letters B, 2020, 34(supp01), 2050427.

[38] U. Younas, M. Younis, A. R. Seadawy, et al., *Diverse exact solutions for modified nonlinear Schrödinger equation with conformable fractional derivative*, Results in Physics, 2021, 20, 103766.

[39] A. Zafar, A. Bekir, M. Raheel, et al., *Optical soliton solutions to Biswas-Arshed model with truncated M-fractional derivative*, Optik, 2020, 222, 165355.

[40] B. Zhang, W. Zhu, Y. Xia, et al., *A unified analysis of exact traveling wave solutions for the fractional-order and integer-order Biswas-Milovic equation: via bifurcation theory of dynamical system*, Qualitative Theory of Dynamical Systems, 2020, 19(1), 1–28.

[41] D. Zhao and M. Luo, *General conformable fractional derivative and its physical interpretation*, Calcolo, 2017, 54(3), 903–917.

[42] H. Zheng, Y. Xia, Y. Bai, et al., *Travelling wave solutions of the general regularized long wave equation*, Qualitative Theory of Dynamical Systems, 2021, 20(1), 1–21.

[43] W. Zhu, Y. Xia, B. Zhang, et al., *Exact traveling wave solutions and bifurcations of the time-fractional differential equations with applications*, International Journal of Bifurcation and Chaos, 2019, 29(03), 1950041.