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EXISTENCE OF SOLUTIONS TO A GENERALIZED KADOMTSEV-PETVIASHVILI EQUATION WITH A POTENTIAL AND CONCAVE-CONVEX NONLINEARITY

Jiaoping Chen[1] and Jianqing Chen[1,†]

Abstract In this paper, we firstly prove the existence of infinitely many solutions with positive energy to a class of generalized Kadomtsev-Petviashvili equation with a potential and concave-convex nonlinearity. Secondly, with the help of *genus*, we are able to prove the existence of infinitely many solutions with negative energy for a suitable parameter λ. Our results can be looked on as a generalization to previous works in the literature.

Keywords Anisotropic Sobolev embedding, generalized Kadomtsev-Petviashvili equation, concave-convex nonlinearity, variational method.

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1. Introduction and main results

In describing water waves that propagate in straits or rivers rather than unbounded surfaces, a generalized Kadomtsev-Petviashvili (GKP) equation with variable coefficients has been proposed by David et. al. [5, 6]. From then on, there are some works studying the existence of solitary waves or soliton solutions of the GKP with variable coefficients, see, for instance [11] and the references therein. More precisely, in $\mathbb{R}^2$, a class of GKP with the form

$$\begin{aligned}&(u(t)+r(t)uu_x+q(t)u_{xxx})_x+\sigma(y,t)u_{yy}+a(y,t)u_y+b(y,t)u_{xy}\\&+c(y,t)u_{xx}+e(y,t)u_x+f(y,t)u+\rho(y,t)=0\end{aligned}\tag{1.1}$$

has been considered by Güngör and Winternitz [11], where r, q, σ, a, b, c, e, f and ρ are functions satisfying some technical conditions.

A lot of mathematicians have studied the existence of solitary waves. A pioneering work has been achieved by De Bouard and Saut [7], where the authors have studied the existence of solitary waves to the following

$$\begin{cases} u_t+f'(u)u_x+u_{xxx}+\beta v_y=0, \\ v_x=u_y, \end{cases} \quad (x,y)\in\mathbb{R}^2 \tag{1.2}$$

†The corresponding author.
[1]School of Mathematics and Statistics, Fujian Normal University, Fuzhou 350117, China
Email: 1131643863@qq.com(Jiaoping Chen), chenjq_iapcm@126.com(Jianqing Chen)

with $f(s) = s^p$ and $\beta = -1$, where $1 \le p < 4$ and $p = \frac{m}{n}$ with m and n being relatively prime and n is odd. A series of works on soliton solutions, rogue wave solutions, soliton and rogue wave mixed solutions as well as their numerical simulations, have been obtained by Ma et al. [13, 14]. Other results on Cauchy problem have been investigated in [8, 19].

For GKP without a nonhomogeneous term $\rho(y,t)$ in higher spatial dimensions, Xuan [17] has investigated the existence of solitary waves of

$$w_t + w_{xxx} + (f(w))_x = D_x^{-1}\Delta_y w, \tag{1.3}$$

where $(t,x,y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^{N-1}$, $y = (y_1, \cdots, y_{N-1})$, $N \ge 3$. The operator D_x^{-1} denotes $D_x^{-1}g(x,y) = \int_{-\infty}^{x} g(s,y)ds$ and $\Delta_y = \sum_{k=1}^{N-1} \frac{\partial^2}{\partial y_k^2}$.

Finding a solitary wave of (1.3) is equivalent to study the existence of solutions in a suitable function space to the following equation

$$-u_{xx} + D_x^{-2}\Delta_y u + u = f(u). \tag{1.4}$$

In [12], the authors have investigated the existence of solutions to the equation

$$-u_{xx} + D_x^{-2}\Delta_y u + cu = Q(x,y)|u|^{p-2}u.$$

It has been proved that the existence of solutions depends strongly on the properties of the coefficient $Q(x,y)$. Several other results on the existence of solutions to various kinds of GKP equation can be found in [1, 3, 9, 15]. The purpose of the present paper is study the existence and multiplicity of solutions to (GKP) with a potential and concave-convex nonlinearity of the form

$$\begin{cases} -u_{xx} + D_x^{-2}\Delta_y u + (1 + V(x,y))u = \lambda h(x,y)|u|^{q-2}u + |u|^{p-2}u, \\ u \to 0 \text{ as } |(x,y)| \to \infty, \end{cases} \tag{1.5}$$

where $\lambda > 0$, $1<q<2<p<\bar{N} = \frac{2(2N-1)}{2N-3}$ and $V(x,y)$ is a nonnegative function. Before stating the main results, we give several conditions and definitions.

(V). $V(x,y)$ satisfies $\inf_{(x,y)\in\mathbb{R}\times\mathbb{R}^{N-1}} V(x,y) \ge a > 0$, and for any $M > 0$, $\mu(\{(x,y) \in \mathbb{R}\times\mathbb{R}^{N-1} : V(x,y) \le M\}) < +\infty$. Here and after μ denotes the Lebesgue measure.

(H). For any $(x,y) \in \mathbb{R}\times\mathbb{R}^{N-1}$, $h(x,y) > 0$, and $h \in L^{\frac{p}{p-q}}(\mathbb{R}^N)$.

Definition 1.1. On $Y := \{g_x : g \in C_0^\infty(\mathbb{R}^N)\}$, we define the inner product

$$(u,v)_Y := \int_{\mathbb{R}^N} \left(u_x v_x + D_x^{-1}\nabla_y u \cdot D_x^{-1}\nabla_y v + uv\right) dxdy,$$

where $\nabla_y = \left(\frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_{N-1}}\right)$, and the corresponding norm

$$\|u\| := \left(\int_{\mathbb{R}^N} \left(u_x^2 + |D_x^{-1}\nabla_y u|^2 + u^2\right) dxdy\right)^{\frac{1}{2}}.$$

A function $u : \mathbb{R}^N \to \mathbb{R}$ belongs to X if there exists $(u_n) \subset Y$ such that:

(1) $u_n \to u$ a.e. on $\mathbb{R}^N$;

(2) $\|u_j - u_k\| \to 0, \quad j, k \to \infty.$

The space X with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$ is a Hilbert space.

Definition 1.2. Denote

$$X_V := \left\{ u \in X \ : \ \int_{\mathbb{R}^N} V u^2 dxdy < \infty \right\},$$

then the X_V with inner product $(\cdot, \cdot)_V$ and norm $\|\cdot\|_V$ is a Hilbert space, where $(u, v)_V := (\cdot, \cdot) + \int_{\mathbb{R}^N} V(x, y) uv dxdy$ and $\|u\|_V^2 := \|u\|^2 + \int_{\mathbb{R}^N} V(x, y) u^2 dxdy$.

The main results are the following two theorems.

Theorem 1.1. *There exists $\lambda_1 > 0$ such that for any $\lambda \in (0, \lambda_1)$, the (1.5) admits infinitely many solutions with positive energy in X_V.*

Theorem 1.2. *There exists $\lambda_2 > 0$ such that for any $\lambda \in (0, \lambda_2)$, the (1.5) possesses infinitely many solutions with negative energy in X_V.*

The proofs of Theorem1.1 and Theorem 1.2 are based on the variational method. We will define a suitable Euler-Lagrange functional on the space X_V and show that this functional is well defined and C^1 on X_V. Then we use abstract critical point theorems to prove our main results.

This paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we prove the existence of infinitely many solutions with positive energy. In Section 4, with the help of *genus*, we prove the existence of infinitely many solutions with negative energy.

2. Preliminaries

In this section, we give some preliminaries. A starting point is the following embedding relations.

Lemma 2.1. [12, 17]:

(*i*) *If $2 \le p \le \bar{N}$, then $X \hookrightarrow L^p\left(\mathbb{R}^N\right)$ is continuous;*

(*ii*) *If $2 \le p < \bar{N}$, then $X \hookrightarrow\hookrightarrow L^p_{loc}\left(\mathbb{R}^N\right)$ is compact.*

With the help of Lemma 2.1 and the definition of X_V, we easily get the following proposition.

Proposition 2.1. *If $2 \le p \le \bar{N}$, then $X_V \hookrightarrow L^p\left(\mathbb{R}^N\right)$ is continuous; if $2 \le p < \bar{N}$, then $X_V \hookrightarrow\hookrightarrow L^p_{loc}\left(\mathbb{R}^N\right)$ is compact.*

Moreover, under the assumption of (V), we are able to prove the following compact embedding.

Lemma 2.2. *Under the condition (V), the following embedding is compact:*

$$X_V \hookrightarrow\hookrightarrow L^p\left(\mathbb{R}^N\right), \ 2 \le p < \bar{N}. \tag{2.1}$$

Proof. To prove this lemma, we use an idea from [2]. Suppose that $(u_n) \subset X_V$ is bounded. Thus after passing to a subsequence, we may assume there is a $u_0 \in X_V$, such that

$$\begin{aligned} u_n &\rightharpoonup u_0 \qquad \text{weakly in } X_V, \\ u_n &\to u_0 \qquad \text{strongly in } L^p_{loc}(\mathbb{R}^N). \end{aligned}$$

We first show that $u_n \to u_0$ strongly in $L^2(\mathbb{R}^N)$. For this it suffices to prove that $\alpha_n := |u_n|_{L^2} \to |u_0|_{L^2}$. Suppose $\alpha_n \to \alpha$ along a subsequence, so $\alpha \geq |u_0|_{L^2}$. We claim that for every $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{\mathbb{R}^N \setminus D_R} u^2 dxdy < \varepsilon, \tag{2.2}$$

where $D_R = \{(x,y) \in \mathbb{R} \times \mathbb{R}^{N-1} : |x| \leq R, |y| \leq R\}$. If (2.2) holds, then $u_n \to u_0$ strongly in $L^2(\mathbb{R}^N)$, because $u_n|_{D_R} \to u_0|_{D_R}$ strongly in $L^2(D_R)$. Hence

$$\begin{aligned} |u_0|_{L^2(\mathbb{R}^N)} &= |u_0|_{L^2(D_R)} + |u_0|_{L^2(\mathbb{R}^N \setminus D_R)} \\ &\geq \lim_{n\to\infty} |u_n|_{L^2(D_R)} \\ &\geq \lim_{n\to\infty} |u_n|_{L^2(D_R)} - \lim_{n\to\infty} |u_n|_{L^2(\mathbb{R}^N \setminus D_R)} \\ &\geq \alpha - \varepsilon. \end{aligned}$$

It remains to prove (2.2). We fixed $\varepsilon > 0$ and choose positive constant $M > \frac{2}{\varepsilon} \sup \|u_n\|_V^2$, $s \in (1, \frac{\bar{N}}{2})$ and

$$c_0 \geq \sup_{u \in X_V \setminus \{0\}} \frac{|u_n|^2_{L^{2s}(\mathbb{R}^N)}}{\|u_n\|_V^2}. \tag{2.3}$$

Let s' satisfy $\frac{1}{s} + \frac{1}{s'} = 1$. The property of the potential V implies that for $R > 0$ large enough

$$\mu\left(\{(x,y) \in \mathbb{R} \times \mathbb{R}^{N-1} \setminus D_R : V(x,y) < M\}\right) \leq \left(\frac{\varepsilon}{2c_0 \sup\limits_n \|u_n\|_V^2}\right)^{s'}. \tag{2.4}$$

We set

$$A := \{(x,y) \in \mathbb{R} \times \mathbb{R}^{N-1} \setminus D_R : V(x,y) \geq M\},$$

and

$$B := \{(x,y) \in \mathbb{R} \times \mathbb{R}^{N-1} \setminus D_R : V(x,y) < M\}.$$

Then by our choice of M, we can get

$$\int_A {u_n}^2 dxdy \leq \int_{\mathbb{R}^N} \frac{V(x,y)}{M} {u_n}^2 dxdy \leq \frac{\|u_n\|_V^2}{M} \leq \frac{\varepsilon}{2}.$$

Moreover, the Hölder inequality, (2.3) and (2.4) imply

$$\begin{aligned} \int_B {u_n}^2 dxdy &\leq \left(\int_B |u_n|^{2s} dxdy\right)^{\frac{1}{s}} \left(\int_B 1 dxdy\right)^{\frac{1}{s'}} \\ &= |u_n|^2_{L^{2s}} \cdot \mu(B)^{\frac{1}{s'}} \\ &\leq c_0 \|u_n\|^2_{X_V} \cdot \mu(B)^{\frac{1}{s'}} \\ &< \frac{\varepsilon}{2}. \end{aligned}$$

Therefore we obtain

$$\int_{\mathbb{R}^N \setminus D_R} {u_n}^2 dxdy = \int_A {u_n}^2 dxdy + \int_B {u_n}^2 dxdy \le \varepsilon.$$

Thus we have proved that $u_n \to u_0$ strongly in $L^2(\mathbb{R}^N)$. We can use the Anisotropic Sobolev inequality [7] in order to see that $u_n \to u_0$ strongly in $L^p(\mathbb{R}^N)$ for $2 \le p < \bar{N}$. The proof is complete.

We end this section with the following minimization problem.

$$S := \inf \left\{ \|u\|_V^2 : u \in X_V, \int_{\mathbb{R}^N} |u|^p dxdy = 1,\ \ 2 < p < \bar{N} = \frac{2(2N-1)}{2N-3} \right\}.$$

From Lemma 2.2, it is easy to see that S is positive and can be achieved by a function in X_V. □

3. Infinitely many solutions with positive energy

The aim of this section is to show the existence of infinitely many solutions of (1.5) with positive energy. That is to say, we will prove Theorem 1.1 holds. On X_V, we define the following energy functional

$$I(u) = \frac{1}{2}\|u\|_V^2 - \frac{\lambda}{q}\int_{\mathbb{R}^N} h(x,y)|u|^q dxdy - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dxdy. \tag{3.1}$$

Then $I \in C^1(X_V, \mathbb{R})$. And we only need to prove the existence of infinitely many positive critical values of I on X_V.

Lemma 3.1. *The functional I satisfies (PS) condition on X_V. That is, for any $c \in \mathbb{R}$, any sequence $(u_n) \subset X_V$ satisfying $I(u_n) \to c$, $I'(u_n) \to 0$, the (u_n) contains a convergent subsequence in X_V.*

Proof. For n large enough, there is a positive constant C such that

$$\begin{aligned} c + 1 + \|u_n\|_V &\ge I(u_n) - \frac{1}{p}(I'(u_n), u_n) \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|_V^2 - \left(\frac{1}{q} - \frac{1}{p}\right)\lambda \int_{\mathbb{R}^N} h(x,y)|u_n|^q dxdy \\ &\ge \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|_V^2 - \left(\frac{1}{q} - \frac{1}{p}\right) C S^{-\frac{q}{2}} |h|_{L^{\frac{p}{p-q}}} \|u_n\|_V^q. \end{aligned}$$

Therefore, (u_n) is bounded on X_V. Going if necessary to a subsequence, we may assume that

$$\begin{aligned} u_n &\rightharpoonup u_0 &&\text{in } X_V, \\ u_n &\to u_0 &&a.e. \text{ in } \mathbb{R}^N, \\ u_n &\to u_0 &&\text{in } L^p_{loc}(\mathbb{R}^N). \end{aligned}$$

Then

$$\begin{aligned} P_n :=& I'(u_n)(u_n - u_0) \\ =& (u_n, u_n - u_0)_V - \int_{\mathbb{R}^{\mathbb{N}}} \left(\lambda h(x,y)|u_n|^{q-2}u_n + |u_n|^{p-2}u_n\right)(u_n - u_0)\, dxdy. \end{aligned}$$

Since $I'(u_n) \to 0$ as $n \to \infty$, the $P_n \to 0$ as $n \to \infty$. Moreover, the fact that $u_n \rightharpoonup u_0$ in X_V implies $Q_n := (u_0, u_n - u_0)_V \to 0$ as $n \to \infty$.

Note that from the Hölder inequality and Lemma 2.2 that

$$\int_{\mathbb{R}^N} h(x,y)|u_n|^{q-1}|u_n - u_0|dxdy \to 0, \tag{3.2}$$

$$\int_{\mathbb{R}^N} |u_n|^{p-1}|u_n - u_0|dxdy \to 0. \tag{3.3}$$

It is deduced from (3.2) and (3.3) that $P_n - Q_n = (u_n - u_0, u_n - u_0)_V \to 0$ as $n \to \infty$. Therefore $\|u_n - u_0\|_V \to 0$ as $n \to \infty$. Since $c \in \mathbb{R}$ is arbitrarily, I satisfies the (PS) condition on X_V. The proof is completed. □

Lemma 3.2. [16] *Let E be an infinite dimensional real Banach space, $I \in C^1(E, \mathbb{R})$ be even and satisfies (PS) condition and $I(0) = 0$. Assume $E = W \bigoplus Z$, W is finite dimensional, I satisfies:*

(1) There exist constants $\rho,\ \alpha > 0$ such that $I(u) \geq \alpha$ on $\partial B_\rho \cap Z$.

(2) For each finite dimensional subspace $X_0 \subset E$, there is an $R_0 = R_0(X_0)$ such that $I(u) \leq 0$ on $X_0 \setminus B_{R_0}$, where $B_r = \{u \in E : \|u\|_E < r\}$.

Then $I(u)$ possesses an unbounded sequence of critical values.

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We will use Lemma 3.2 to prove Theorem 1.1. Choosing $E = X_V$, then it is easy to see that the functional $I(u)$ defined in (3.1) is even in X_V. By Lemma 3.1, the functional I satisfies (PS) condition. Next, we prove that I satisfies (1) and (2). In the first place, by Proposition 2.1 and the condition (H), there is a positive constant C such that for any $u \in X_V$,

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|_V^2 - \frac{\lambda}{q} C S^{-\frac{q}{2}} |h|_{L^{\frac{p}{p-q}}} \|u\|_V^q - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dxdy \\ &\geq \frac{1}{2}\|u\|_V^2 - \frac{\lambda}{q} C S^{-\frac{q}{2}} |h|_{L^{\frac{p}{p-q}}} \|u\|_V^q - C\|u\|_V^p. \end{aligned}$$

Denote $\phi(z) = z^2 \left(\frac{1}{2} - \frac{\lambda}{q} C S^{-\frac{q}{2}} |h|_{L^{\frac{p}{p-q}}} z^{q-2} - C z^{p-2}\right)$, $z > 0$. Then, there exist $\lambda_1, z_1, \alpha > 0$ such that $\phi(z_1) \geq \alpha$ for any $\lambda \in (0, \lambda_1)$. Let $\rho = z_1$, we have $I(u) \geq \alpha$ with $\|u\|_V = \rho$ and $\lambda \in (0, \lambda_1)$. So the condition (1) is satisfied.

In the second place, for any finite dimensional subspace $X_0 \subset X_V$, we assert that there is a constant $R_0 > \rho$ such that $I < 0$ on $X_0 \setminus B_{R_0}$. Otherwise, there exists a sequence $(u_n) \subset X_0$ such that $\|u_n\|_V \to \infty$ and $I(u_n) \geq 0$. Hence

$$\frac{1}{2}\|u_n\|_V^2 \geq \frac{\lambda}{q}\int_{\mathbb{R}^N} h(x,y)|u_n|^q dxdy - \frac{1}{p}\int_{\mathbb{R}^N} |u_n|^p dxdy. \tag{3.4}$$

Set $\omega_n = \frac{u_n}{\|u_n\|_V}$. Then up to a sequence, we can assume $\omega_n \rightharpoonup \omega$ in X_V, $\omega_n \to \omega$ a.e. in $\mathbb{R}^N$. Denote $\Omega = \{(x,y) \in \mathbb{R} \times \mathbb{R}^{N-1} : \omega(x,y) \neq 0\}$. Assume $|\Omega| > 0$, Clearly, $u_n(x,y) \to \infty$ in Ω. There is

$$\frac{\lambda}{q}\int_{\mathbb{R}^N} h(x,y)|u_n|^q dxdy \leq \frac{\lambda}{q} C S^{-\frac{q}{2}} |h|_{L^{\frac{p}{p-q}}} \|u_n\|^q.$$

Therefore

$$\|u_n\|_V^{-2}\int_{\mathbb{R}^N} h(x,y)|u_n|^q dxdy \le CS^{-\frac{q}{2}}|h|_{L^{\frac{p}{p-q}}}\|u_n\|^{q-2} \to 0 \quad \text{as} \quad n\to\infty.$$

And it clearly that $\|u_n\|_V^{-2}\int_{\mathbb{R}^N}|u_n|^p dxdy \le C\|u_n\|^{p-2}\to\infty \quad as \quad n\to\infty$. Therefore, multiplying (3.4) by $\|u_n\|_V^{-2}$ and passing to the limit as $n\to\infty$ and by the equivalence of all the norms in X_0, show that $\frac{1}{2}\ge\infty$. This is impossible. So $|\Omega|=0$ and $\omega(x,y)=0$ a.e. on $\mathbb{R}^N$. Using the same property as above, we know there exists a constant $\beta>0$ such that

$$\int_{\mathbb{R}^N}|u|^p dxdy \ge \beta^p\|u\|_V^p, \quad \forall u\in X_0,$$

and

$$\int_{\mathbb{R}^N}|u_n|^p dxdy \ge \beta^p\|u_n\|_V^p, \quad \forall n\in\mathbb{N}.$$

Hence

$$0<\beta^p\le\limsup_{n\to\infty}\int_{\mathbb{R}^N}|\omega_n|^p dxdy \le \int_{\mathbb{R}^N}\limsup_{n\to\infty}|\omega_n|^p dxdy = \int_{\mathbb{R}^N}|\omega|^p dxdy = 0.$$

This is a contradiction. So there exists a constant R_0 such that $I<0$ on $X_0\setminus B_{R_0}$.

By Lemma 3.2, we know that the functional I possesses an unbounded sequence of critical values. The proof of Theorem 1.1 is complete. □

4. Infinitely many solutions to (1.5) with negative energy

In this section, we will construct a min-max class of critical points by using the classical concept and properties of the *genus*. Let F be a Banach space, and Σ be the class of closed and symmetric with respect to the origin subsets of $F\setminus\{0\}$. For $A\in\Sigma$, we define the genus $\gamma(A)$ by

$$\gamma(A)=\min\left\{k\in N:\exists\phi\in C(A;\mathbb{R}^k\setminus\{0\}),\phi(x)=-\phi(-x)\right\}.$$

If such a minimum does not exist then we define $\gamma(A)=+\infty$. The main properties of the genus are following:

Lemma 4.1. [10] *Let $A,B\in\Sigma$. Then*

(1) *If $A\subset B$, then $\gamma(A)\le\gamma(B)$;*

(2) $\gamma(A\cup B)\le\gamma(A)+\gamma(B)$;

(3) *For any consecutive odd map $\varphi:F\to F$, and any $A\in\Sigma$, there is $\gamma(A)\le\gamma(\overline{\varphi(A)})$;*

(4) *If $A\in\Sigma$ is compact, then there exists a closed neighborhood N about A, such that $\gamma(A)=\gamma(N)$;*

(5) *If $A\subset\mathbb{R}^n$, A is closed, symmetric and $0\notin A$, then $\gamma(A)<n$;*

(6) *Let F_1 be an m-dimensional subspace of F, S be the unit sphere of F, then $\gamma(F_1\cap S)=m$;*

(7) *If there exists* $f \in C(A, B)$*, odd, then* $\gamma(A) < \gamma(B)$*.*

Given the functional I, defined by (3.1), using Proposition 2.1, we obtain

$$I(u) \geq \frac{1}{2}\|u\|_V^2 - \frac{\lambda}{q} C_1 S^{-\frac{q}{2}} |h|_{L^{\frac{p}{p-q}}} \|u\|_V^q - C_2 \|u\|_V^p, \quad u \in X_V$$

for some positive constant C_1 and C_2. If we define

$$g(s) = \frac{1}{2}s^2 - \frac{\lambda}{q} C S^{-\frac{q}{2}} |h|_{L^{\frac{p}{p-q}}} s^q - C s^p,$$

then

$$I(u) \geq g(\|u\|_V).$$

There exists $\lambda_2 > 0$ such that, if $0 < \lambda < \lambda_2$, g attains its positive maximum.

Let us assume $0 < \lambda < \lambda_2$, choosing R_0 and R_1 as $\|u\|_V < R_0$, $g < 0$ and $R_0 < \|u\|_V < R_1$, $g > 0$. we make the following truncation of the functional I:

Take $\tau : R^+ \to [0, 1]$; nonincreasing and C^∞, such that

$$\begin{cases} \tau(x) = 1 & \text{if} \quad x \leq R_0, \\ \tau(x) = 0 & \text{if} \quad x \geq R_1. \end{cases}$$

Let $\varphi(u) = \tau(\|u\|_V)$. We consider the truncated functional :

$$J(u) = \frac{1}{2}\|u\|_V^2 - \frac{\lambda}{q} \int_{\mathbb{R}^N} h(x, y)|u|^q dxdy - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p \varphi(u) dxdy.$$

So $J(u) \geq \overline{h}(\|u\|_V)$, with $\overline{h}(s) = \frac{1}{2}s^2 - \frac{\lambda}{q} C S^{-\frac{q}{2}} |h|_{L^{\frac{p}{p-q}}} s^q - C s^p \tau(x)$. The main properties of $J(u)$ are given by the following Lemma 4.2:

Lemma 4.2. (1) $J \in C^1(X_V, R)$;

(2) *If* $J(u) \leq 0$*, then* $\|u\|_V < R_0$*, and* $I(u) = J(v)$ *for all* v *in a small enough neighbourhood of* u;

(3) *There exists* $\lambda_2 > 0$*, such that, if* $0 < \lambda < \lambda_2$*, then* J *verifies (PS) condition for* $c \leq 0$*.*

Proof. (1) and (2) are immediate. (3) is a corollary of Lemma 3.1. □

Observe that, by (2), if we find a negative critical value for the functional J, then we have got a negative critical value of I.

Now, we will construct an appropriate mini-max sequence of negative critical value for the functional J.

Lemma 4.3. *Given* $n \in N$*, there is* $\varepsilon = \varepsilon(n) > 0$*, such that*

$$\gamma(\{u \in X_V : J(u) \leq -\varepsilon\}) \geq n.$$

Proof. Fix n, let E_n be an n-dimensional subspace of X_V. We take $u_n \in E_n$, with norm $\|u_n\|_V = 1$. For $0 < \rho < R_0$, we have

$$J(\rho u_n) = I(\rho u_n) = \frac{1}{2}\rho^2 - \frac{\lambda}{q}\rho^q \int_{\mathbb{R}^N} h(x)|u_n|^q dxdy - \frac{1}{p}\rho^p \int_{\mathbb{R}^N} |u_n|^p dxdy.$$

E_n is a space of finite dimension; so, all the norms are equivalent. Then, if we define

$$\alpha_n = \inf\left\{\int_{\mathbb{R}^N} |u|^q dxdy : u \in E_n, \|u_n\|_V = 1\right\} > 0,$$
$$\beta_n = \inf\left\{\int_{\mathbb{R}^N} |u|^p dxdy : u \in E_n, \|u_n\|_V = 1\right\} > 0,$$

we have

$$J(\rho u_n) \leq \frac{1}{2}\rho^2 - \frac{\lambda}{q}|h|_{L^{\frac{p}{p-q}}}\alpha_n\rho^q - \frac{1}{p}\beta_n\rho^p.$$

and we can choose ε (which depends on n), and $\eta < R_0$, such that $J(\eta u) \leq -\varepsilon$ if $u \in E_n$, and $\|u\|_V = 1$. Let $S_\eta = \{u \in X_V : \|u\|_V = \eta\}$. $S_\eta \cap E_n \subset \{u \in X_V : J(u) \leq -\varepsilon\}$; therefore, by (6) of Lemma 4.1, we know:

$$\gamma(\{u \in X_V : J(u) \leq -\varepsilon\}) \geq \gamma(S_\eta \cap E_n) = n.$$

□

This lemma allows us to prove the existence of critical points.

Lemma 4.4. *Let $\Sigma_k = \{C \subset X_V \setminus \{0\}, C \text{ is closed}, C = -C, \gamma(C) \geq k\}$. Let $c_k = \inf\limits_{C\in\Sigma_k}\sup\limits_{u\in C} J(u)$, $K_c = \{u \in X_V : J'(u) = 0, J(u) = c\}$, and suppose $0 < \lambda < \lambda_2$, where λ_2 is the parameter of Lemma 3.2. Then, if $c = c_k = c_{k+1} = \cdots = c_{k+r}$, $\gamma(K_c) \geq r+1$. (In particular, the c_k is critical values of J.)*

Proof. The proof can be finished by Lemma 4.3 and a classical deformation lemma. □

Proof of Theorem 1.2. For simplicity, we define $J^{-\varepsilon} = \{u \in X_V : J(u) \leq -\varepsilon\}$. By Lemma 4.3, we know $\forall k \in N$, $\exists\varepsilon(k) > 0$ such that $\gamma(J^{-\varepsilon}) \geq k$. Due to J is continuous and even, $J^{-\varepsilon} \in \Sigma_k$; then, $c_k \leq -\varepsilon(k) < 0$, $\forall k$. While J is bounded from below; hence, $c_k > -\infty$, $\forall k$.

Let us assume that $c = c_k = c_{k+1} = \cdots = c_{k+r}$, then $c < 0$; therefore, J verifies the Palais-Smale condition in K_c, and it is easy to see that K_c is a compact set.

Because $0 \notin K_c$, depending on the characteristic (4) and (5) of Lemma 4.1. We know $\exists\delta > 0$, than $\gamma\left(\overline{(K_c)_{3\delta}}\right) = \gamma(K_c) < \infty$.

Let $S = X_V \setminus (K_c)_{3\delta}$, and $N = \overline{(K_c)_{3\delta}}$. By the deformation lemma, there exists a map σ such that:

$$\begin{aligned}&\sigma \in C([0,1]\times X_V, X_V);\\ &\sigma(t,\cdot) \text{ is an odd homoembryonic;}\\ &\sigma(1, J^{c+\varepsilon}\cap S) \subset J^{c-\varepsilon}.\end{aligned}$$

It is clear from the definition that, $\exists A \in \Sigma_{k+r}$, $\sup\limits_A J \leq c+\varepsilon$, and $k+r \leq \gamma(A) \leq \gamma(A\setminus N) + \gamma(N)$.

Because $A \subset J^{c+\varepsilon}$, we can know $A \setminus N \subset J^{c+\varepsilon}\cap S$. So

$$\begin{aligned}k + r \leq \gamma(A) &\leq \gamma(K_c) + \gamma(\sigma(1, A\setminus N))\\ &\leq \gamma(K_c) + \gamma(J^{c-\varepsilon}).\end{aligned}$$

For the Lemma 4.4 to hold, it is need to satisfy $\gamma(J^{c-\varepsilon}) \leq k-1$ is sufficient. Using the inverse, if $\gamma(J^{c-\varepsilon}) \geq k$, then $J^{c-\varepsilon} \in \Sigma_k$, so there is $c = c_k \leq \sup\limits_{J^{c-\varepsilon}} J \leq c-\varepsilon$. This is a contradiction.

The proof of Theorem 1.2 is complete. $\square$

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