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DYNAMICAL BEHAVIOR OF THE GENERALIZED COMPLEX LORENZ CHAOTIC SYSTEM*

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Abstract The purpose of this paper is to investigate the boundedness and global attractivity of the complex Lorenz system:

$$\dot{x} = \alpha (y - x), \dot{y} = \gamma x - cy - dxz, \dot{z} = -\beta z + \frac{1}{2} (\bar{x} y + x \bar{y}),$$

where $\alpha, \beta, \gamma, c, d$ are real parameters, x and y are complex variables, z is a real variable, an overbar denotes complex conjugate variable and dots represent derivatives with respect to time. This system arises in many important applications in laser physics and rotating fluids dynamics. It is very interesting that we find that this system exhibits chaos phenomenon for the given parameters. Using generalized Lyapunov-like functions, we prove the existence of the ultimate bound set and the globally exponentially attractive set in this generalized complex Lorenz system. The rate of the trajectories is also obtained. Numerical simulations show the effectiveness and correctness of the conclusions. Finally, we present an application of our results that obtained in this paper.

Keywords Complex Lorenz chaotic system, chaotic attractor, Lyapunov exponent, Lyapunov dimension, global attractivity.

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1. Introduction

In 1963, Edward Lorenz [23] introduced the real Lorenz chaotic system:

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$$\begin{cases} \dot{x} = \alpha \left(y - x \right), \\ \dot{y} = \gamma x - y - xz, \\ \dot{z} = -\beta z + xy, \end{cases} \tag{1.1}$$

where α, β, γ are real parameters of the Lorenz system as stated in several papers [3,23,31]. The Lorenz system can describe the thermal convection in fluids [3,23,31]. The Lorenz chaotic system has inspired many researchers to study new chaotic systems and chaotic phenomena [2,4,11–14,24,25,28,32]. Since then, many methods have been proposed to study chaotic behaviors of chaotic systems [1,7,9,10,14,15, 15–20,33–40,42].

A chaotic system is a nonlinear deterministic system that displays complex and unpredictable behaviors. Since the pioneering work by A.C. Fowler et al. [5], complex chaotic systems have become an interesting field of research over the last few decades [6,8,22,26,27,30,41]. The complex Lorenz system is as follows:

$$\begin{cases} \dot{x} = \alpha \left(y - x \right), \\ \dot{y} = \gamma x - y - xz, \\ \dot{z} = -\beta z + \dfrac{1}{2} \left(\bar{x} y + x \bar{y} \right), \end{cases} \tag{1.2}$$

where x and y are complex variables, z is a real variable and α, β, γ are real parameters, an overbar denotes complex conjugate variable and dots represent derivatives with respect to time. Variables x, y, z in system (1.2) are related respectively to electric field, the atomic polarization amplitudes and the population inversion in a ring laser system of two-level atoms, for more details, see [5, 25]. It is reported in the literature [30] that the complex Lorenz system (1.2) is often used to describe and simulate the physics of detuned lasers. The complex Lorenz model applies to the description of detuned single mode, homogeneously broadened lasers when a certain constraint on the parameters is observed [30]. The complex Lorenz system also has many important applications in physics, for example, in laser physics and rotating fluids dynamics [6,8,26,27]. Nonlinear dynamical behaviors of the complex Lorenz system, such as bifurcation, limit cycle, analytic solution, the stability of equilibrium point, synchronous behavior, geometric structure, have been studied in [5, 6, 8, 22, 26, 27, 30, 41].

Boundedness is an important concept in the study of chaotic dynamical systems which can be applied to analyze the Lyapunov dimension of chaotic attractors [12, 16], chaos control and chaos synchronization [21, 35]. The bounds of the Lorenz system were studied by Leonov et al. in [15, 17]. Inspired by Leonov' idea, Liao et al. have proposed the concept of the global exponential attractive set of a chaotic system and have obtained the global exponential attractive sets of the Lorenz system [21]. It is reported in the literatures [29, 42] that how to get the bounds of the Chen system and the Lu system is considered as an open problem. Bounds of the Chen system and the Lu system have been addressed in [36, 37].

The rest of this paper is organized as follows. The new generalized complex Lorenz chaotic system is proposed in Section 2. In Section 3, we will study chaotic behaviour of the five-dimensional Lorenz system (2.2). In Section 4, we will study the ultimate boundedness of the five-dimensional Lorenz system (2.2). In Section

5, we will study global attractivity of the five-dimensional Lorenz system (2.2). In Section 6, we will give conclusion remarks.

2. Mathematical model

According to the complex Lorenz system (1.2), we propose a generalized complex Lorenz system as follows

$$\begin{cases} \dot{x} = \alpha\left(y - x\right), \\ \dot{y} = \gamma x - cy - dxz, \\ \dot{z} = -\beta z + \dfrac{1}{2}\left(\bar{x}y + x\bar{y}\right), \end{cases} \tag{2.1}$$

where $x = u_1 + iu_2, y = u_3 + iu_4$ are complex variables, $z = u_5$ is a real state variable, $\alpha, \beta, \gamma, c, d$ are real parameters, an overbar denotes complex conjugate variable, $i^2 = -1$ and dots represent derivatives with respect to time. Variables x, y, z of system (2.1) are related respectively to electric field, the atomic polarization amplitudes and the population inversion in a ring laser system of two-level atoms, for more details, see [5, 25]. System (2.1) has many important applications in laser physics and rotating fluids dynamics [5, 6, 8, 22, 26, 27, 30, 41]. The real version of (2.1) is described by

$$\begin{cases} \dot{u}_1 = \alpha\left(u_3 - u_1\right), \\ \dot{u}_2 = \alpha\left(u_4 - u_2\right), \\ \dot{u}_3 = \gamma u_1 - cu_3 - du_1u_5, \\ \dot{u}_4 = \gamma u_2 - cu_4 - du_2u_5, \\ \dot{u}_5 = u_1u_3 + u_2u_4 - \beta u_5, \end{cases} \tag{2.2}$$

where $\alpha > 0, \beta > 0, c > 0, d > 0, \gamma \in R$ are real parameters of system (2.2).

3. Chaos phenomenon

When the parameters α= 0.0046,β= 0.0008,γ= 0.03,$c = 0.001, d = 0.009$, we have calculated the Lyapunov exponents of system (2.2) as $\lambda_1 = 0.0314, \lambda_2 = 0.0075, \lambda_3 = -0.0058, \lambda_4 = -0.0136, \lambda_5 = -0.0136$ by using the algorithm [7]. The Lyapunov exponents of system (2.2) is shown in Fig. 1.

The Lyapunov dimension of system (2.2) is given by [7, 28]

$$D_L = j + \frac{\sum\limits_{i=1}^{j} \lambda_i}{|\lambda_{j+1}|}, \tag{3.1}$$

such that j is the largest integer that guarantees the inequality $\sum\limits_{i=1}^{j} \lambda_i > 0$. According to the above formula (3.1), the Lyapunov dimension of system (2.2) is calculated as

$$D_L = 4 + \frac{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4}{|\lambda_5|} = 4.6171.$$

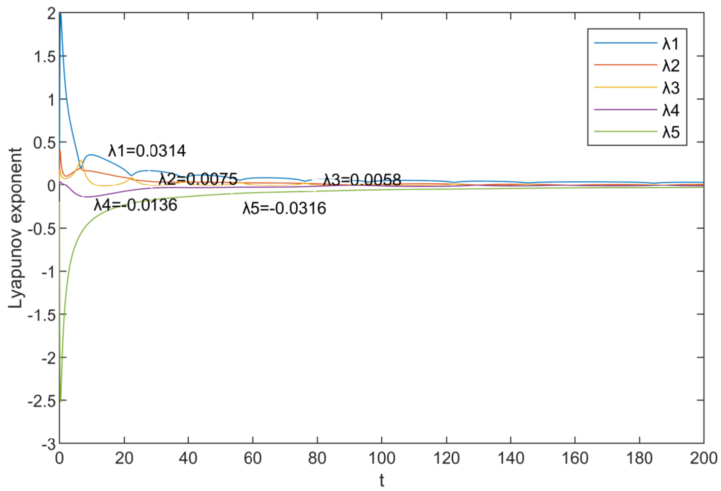

Figure 1. Lyapunov exponents of system (2.2).

The Lyapunov dimension of system (2.2) is a fractional number which ensures the presence of a strange attractor.

Since the largest the Lyapunov exponents of system (2.2) is $\lambda_1 = 0.0314 > 0$ and the Lyapunov dimension of system (2.2) is a fractional number, so the system (2.2) shows chaotic behaviour for parameters α= 0.0046,β= 0.0008,γ= 0.03,$c = 0.001, d = 0.009$.

Remark 3.1. The Lyapunov exponents of the system (2.2) are $\lambda_1 = -6.1757, \lambda_2 = -6.2097, \lambda_3 = -6.3292, \lambda_4 = -7.5025, \lambda_5 = -7.4897$ when α= 14,β= 3.7,γ= 35,$c = 1, d = 1$. Since all Lyapunov exponents of system (2.2) are negative, the system (2.2) is not chaotic when the parameters α= 14,β= 3.7,γ= 35,$c = 1, d = 1$.

In the following part, we will study the boundedness and global attractivity of the five-dimensional Lorenz system (2.2).

4. Boundedness

In this section, we will study the boundedness of the five-dimensional Lorenz system (2.2). Firstly, let us introduce the following Lemma 4.1 and Lemma 4.2 that will be used in the following section.

Lemma 4.1. *Define*

$$\Gamma_1 = \left\{ (x_1, x_2, y_1, y_2, z) | \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{(z-c)^2}{c^2} + \frac{y_1^2}{d^2} + \frac{y_2^2}{e^2} = 1 \right\}$$

and

$$H_1(x_1, x_2, y_1, y_2, z) = x_1^2 + x_2^2 + y_1^2 + y_2^2 + (z - 2c)^2, (x_1, x_2, y_1, y_2, z) \in \Gamma_1.$$

Then, we get

$$\max_{(x_1,x_2,y_1,y_2,z)\in\Gamma_1} H_1(x_1,x_2,y_1,y_2,z) = \begin{cases} \dfrac{a^4}{a^2-c^2}, & a \ge b, a \ge d, a \ge e, a \ge \sqrt{2}c, \\ \dfrac{b^4}{b^2-c^2}, & b > a, b \ge d, b > e, b \ge \sqrt{2}c, \\ \dfrac{d^4}{d^2-c^2}, & d > a, d > b, d \ge e, d \ge \sqrt{2}c, \\ \frac{e^4}{e^2-c^2}, & e > a, e \ge b, e > d, e \ge \sqrt{2}c, \\ 4c^2, a < \sqrt{2}c, b < \sqrt{2}c, d < \sqrt{2}c, e < \sqrt{2}c. \end{cases}$$

Proof. It can be easily proved by the Lagrange multiplier method. □

Another lemma is given as follows.

Lemma 4.2. *Define*

$$\Gamma_2 = \left\{ (x,y,z) | \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(z-c)^2}{c^2} = 1 \right\}, \tag{4.1}$$

and

$$H_2(x,y,z) = x^2 + y^2 + (z-2c)^2, \ (x,y,z) \in \Gamma_2.$$

Then, we get

$$\max_{(x,y,z)\in\Gamma_2} H_2(x,y,z) = \begin{cases} \dfrac{a^4}{a^2-c^2}, & a \ge b, \ a \ge \sqrt{2}c, \\ \dfrac{b^4}{b^2-c^2}, & b > a, \ b \ge \sqrt{2}c, \\ 4c^2, \ a < \sqrt{2}c, \ b < \sqrt{2}c. \end{cases}$$

Proof. It can be easily proved by the Lagrange multiplier method. □

By Lemma 4.1 and Lemma 4.2, we can get the ultimate bound and positively invariant set of the five-dimensional Lorenz system (2.2).

Theorem 4.1. *For any parameters the following set with two parameters $\alpha > 0, \beta > 0, c > 0, d > 0, \gamma \in R$, the following set with two parameters λ and m*

$$\Omega_{\lambda,m} = \left\{ U | \, mu_1^2 + mu_2^2 + \lambda u_3^2 + \lambda u_4^2 + \lambda d \left(u_5 - \frac{\lambda\gamma + \alpha m}{\lambda d} \right)^2 \le R_{\lambda,m}^2 \right\} \tag{4.2}$$

is the ultimate bound set and positively invariant set of the five-dimensional Lorenz system (2.2), where

$$U = (u_1, u_2, u_3, u_4, u_5),$$

$$R_{\lambda,m}^2 = \begin{cases} \dfrac{\beta^2(\lambda\gamma + \alpha m)^2}{4\alpha d(\beta - \alpha)\lambda}, & c \ge \alpha, \ \beta \ge 2\alpha, \\ \dfrac{\beta^2(\lambda\gamma + \alpha m)^2}{4cd(\beta - c)\lambda}, & \beta \ge 2c, \ \alpha > c, \\ \dfrac{(\lambda\gamma + \alpha m)^2}{\lambda d}, & \beta < 2\alpha, \ \beta < 2c. \end{cases}$$

Proof. Construct the Lyapunov-like function

$$V_{\lambda,m}(U) = mu_1^2 + mu_2^2 + \lambda u_3^2 + \lambda u_4^2 + \lambda d\left(u_5 - \frac{\lambda\gamma + \alpha m}{\lambda d}\right)^2, \tag{4.3}$$

where $\forall\lambda > 0, \forall m > 0, U = (u_1, u_2, u_3, u_4, u_5)$. The derivative $V_{\lambda,m}(U)$ of along the trajectory of (2.2) is

$$\begin{aligned}
&\left.\frac{dV_{\lambda,m}(U)}{dt}\right|_{(2.2)}\\
&= 2mu_1\frac{du_1}{dt} + 2mu_2\frac{du_2}{dt} + 2\lambda u_3\frac{du_3}{dt} + 2\lambda u_4\frac{du_4}{dt} + 2\lambda d\left(u_5 - \frac{\lambda\gamma+\alpha m}{\lambda d}\right)\frac{du_5}{dt}\\
&= 2\alpha m u_1(u_3 - u_1) + 2\alpha m u_2(u_4 - u_2) + 2\lambda u_3(\gamma u_1 - cu_3 - du_1u_5)\\
&\quad + 2\lambda u_4(\gamma u_2 - cu_4 - du_2u_5) + 2\lambda d\left(u_5 - \frac{\lambda\gamma+\alpha m}{\lambda d}\right)(u_1u_3 + u_2u_4 - \beta u_5)\\
&= -2\alpha m u_1^2 - 2\alpha m u_2^2 - 2\lambda c u_3^2 - 2\lambda c u_4^2 - 2\lambda d\beta u_5^2 + 2\beta(\lambda\gamma + \alpha m)u_5\\
&= -2\alpha m u_1^2 - 2\alpha m u_2^2 - 2\lambda c u_3^2 - 2\lambda c u_4^2 - 2\lambda d\beta\left(u_5 - \frac{\lambda\gamma+\alpha m}{2\lambda d}\right)^2\\
&\quad + \frac{\beta(\lambda\gamma+\alpha m)^2}{2\lambda d}.
\end{aligned}$$

Let

$$\Gamma = \left\{U \,|\, \alpha m u_1^2 + \alpha m u_2^2 + \lambda c u_3^2 + \lambda c u_4^2 + \lambda d\beta\left(u_5 - \frac{\lambda\gamma+\alpha m}{2\lambda d}\right)^2 = \frac{\beta(\lambda\gamma+\alpha m)^2}{4\lambda d}\right\}, \tag{4.4}$$

then Γ is an ellipsoid in R^5 for $\alpha > 0, \beta > 0, c > 0, d > 0, \gamma \in R$. Outside Γ, $\dot{V}_{\lambda,m}(U) < 0$, while inside Γ, Γ, $\dot{V}_{\lambda,m}(U) > 0$. Since $V_{\lambda,m}(U)$ is a generalized positively definite and radially unbounded continuous function and Γ is a bounded close set, then the maximum value $\max\limits_{U\in\Gamma} V_{\lambda,m}(U) = R_{\lambda,m}^2$ of the function $V_{\lambda,m}(U)$ exists. Obviously, $\left\{U|\, V_{\lambda,m}(U) \le \max\limits_{U\in\Gamma} V_{\lambda,m}(U) = R_{\lambda,m}^2\right\}$ contains the solutions of system (2.2). In order to get the maximum value $\max\limits_{U\in\Gamma} V_{\lambda,m}(U) = R_{\lambda,m}^2$, we have to solve the following optimization problem:

$$\begin{cases}
\max\ V_{\lambda,m}(U) = \max\left\{mu_1^2 + mu_2^2 + \lambda u_3^2 + \lambda u_4^2 + \lambda d\left(u_5 - \dfrac{\lambda\gamma+\alpha m}{\lambda d}\right)^2\right\},\\
s.t. \alpha m u_1^2 + \alpha m u_2^2 + \lambda c u_3^2 + \lambda c u_4^2 + \lambda d\beta\left(u_5 - \dfrac{\lambda\gamma+\alpha m}{2\lambda d}\right)^2 = \dfrac{\beta(\lambda\gamma+\alpha m)^2}{4\lambda d}.
\end{cases} \tag{4.5}$$

In order to use Lemma 4.1 to solve problem (4.5), let us take $\sqrt{m}u_1 = x_1, \sqrt{m}u_2 = x_2, \sqrt{\lambda}u_3 = y_1, \sqrt{\lambda}u_4 = y_2, \sqrt{\lambda d}u_5 = z$ as new variables, then optimization problem

(4.5) transforms into:

$$\begin{cases} \max V_{\lambda,m}(U) = \max\left\{x_1^2 + x_2^2 + y_1^2 + y_2^2 + \left(z - \frac{\lambda\gamma + \alpha m}{\sqrt{d\lambda}}\right)^2\right\}, \\ s.t. \frac{x_1^2}{\frac{\beta(\lambda\gamma+\alpha m)^2}{4\lambda\alpha d}} + \frac{x_2^2}{\frac{\beta(\lambda\gamma+\alpha m)^2}{4\lambda\alpha d}} + \frac{y_1^2}{\frac{\beta(\lambda\gamma+\alpha m)^2}{4\lambda c d}} + \frac{y_2^2}{\frac{\beta(\lambda\gamma+\alpha m)^2}{4\lambda c d}} + \frac{\left(z - \frac{\lambda\gamma+\alpha m}{2\sqrt{d\lambda}}\right)^2}{\frac{(\lambda\gamma+\alpha m)^2}{4\lambda d}} = 1. \end{cases}$$

We can easily get the optimal solution of the above optimization problem by Lemma 4.1,

$$\max_{U\in\Gamma} V_{\lambda,m}(U) = R_{\lambda,m}^2 = \begin{cases} \frac{\beta^2(\lambda\gamma+\alpha m)^2}{4\alpha d(\beta-\alpha)\lambda}, & c \geq \alpha,\ \beta \geq 2\alpha, \\ \frac{\beta^2(\lambda\gamma+\alpha m)^2}{4cd(\beta-c)\lambda}, & \beta \geq 2c,\ \alpha > c, \\ \frac{(\lambda\gamma+\alpha m)^2}{\lambda d}, & \beta < 2\alpha,\ \beta < 2c. \end{cases}$$

This completes the proof. □

Remark 4.1. i) Let us take $m = 1$ in Theorem 4.1, then we can get that

$$\Omega_{\lambda,1} = \left\{(u_1, u_2, u_3, u_4, u_5) \middle|\, u_1^2 + u_2^2 + \lambda u_3^2 + \lambda u_4^2 + \lambda d\left(u_5 - \frac{\lambda\gamma+\alpha}{\lambda d}\right)^2 \leq l_\lambda^2\right\},$$

is the ultimate bound set and positively invariant set of the complex Lorenz system (2.2), where

$$l_\lambda^2 = \begin{cases} \frac{\beta^2(\lambda\gamma+\alpha)^2}{4\alpha d(\beta-\alpha)\lambda}, & c \geq \alpha,\ \beta \geq 2\alpha, \\ \frac{\beta^2(\lambda\gamma+\alpha)^2}{4cd(\beta-c)\lambda}, & \beta \geq 2c,\ \alpha > c, \\ \frac{(\lambda\gamma+\alpha)^2}{\lambda d}, & \beta < 2\alpha,\ \beta < 2c. \end{cases}$$

ii) Let us take $\lambda = 1$ in Theorem 4.1, then we can get that

$$\Omega_{1,m} = \left\{(u_1, u_2, u_3, u_4, u_5) \middle|\, mu_1^2 + mu_2^2 + u_3^2 + u_4^2 + d\left(u_5 - \frac{\gamma+\alpha m}{d}\right)^2 \leq L_m^2\right\}$$

is the ultimate bound set and positively invariant set of the complex Lorenz system (2.2), where

$$L_m^2 = \begin{cases} \frac{\beta^2(\gamma+\alpha m)^2}{4\alpha d(\beta-\alpha)}, & c \geq \alpha,\ \beta \geq 2\alpha, \\ \frac{\beta^2(\gamma+\alpha m)^2}{4cd(\beta-c)}, & \beta \geq 2c,\ \alpha > c, \\ \frac{(\gamma+\alpha m)^2}{d}, & \beta < 2\alpha,\ \beta < 2c. \end{cases}$$

iii) Let us take $\lambda = 1, m = 1$ in Theorem 4.1, then we can get that

$$\Omega_{1,1} = \left\{ (u_1, u_2, u_3, u_4, u_5) | \, u_1^2 + u_2^2 + u_3^2 + u_4^2 + d\left(u_5 - \frac{\gamma + \alpha}{d} \right)^2 \leq r^2 \right\},$$

is the ultimate bound set and positively invariant set of the complex Lorenz system (2.2), where

$$r^2 = \begin{cases} \dfrac{\beta^2 (\gamma + \alpha)^2}{4\alpha d \, (\beta - \alpha)}, & c \geq \alpha, \ \beta \geq 2\alpha, \\ \dfrac{\beta^2 (\gamma + \alpha)^2}{4cd \, (\beta - c)}, & \beta \geq 2c, \ \alpha > c, \\ \dfrac{(\gamma + \alpha)^2}{d}, & \beta < 2\alpha, \ \beta < 2c. \end{cases}$$

Let us take $\alpha = 14$, $\beta = 3.7$, $\gamma = 35$, $c = 1$, $d = 1$, then we can obtain that

$$\Omega_{1,1} = \left\{ (u_1, u_2, u_3, u_4, u_5) | \, u_1^2 + u_2^2 + u_3^2 + u_4^2 + (u_5 - 49)^2 \leq 55.2^2 \right\}$$

is the ultimate bound set and positively invariant set of the complex Lorenz system (2.2).

Fig. 2. shows the projection of $\Omega_{1,1}$ into the (u_2, u_3, u_4) space. Fig. 3. shows the projection of $\Omega_{1,1}$ into the (u_3, u_4, u_5) space. Projection of $\Omega_{1,1}$ onto the (u_3, u_5) plane is shown in Figure 4. Projection of $\Omega_{1,1}$ onto the (u_4, u_5) plane is shown in Figure 5.

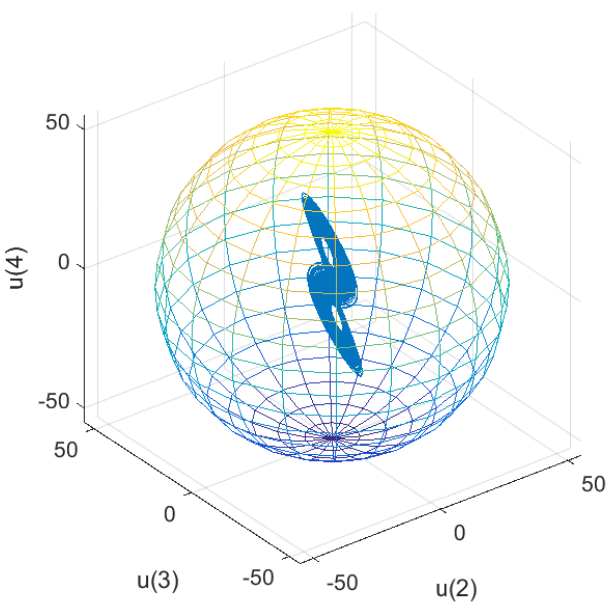

Figure 2. Projection of $\Omega_{1,1}$ into the (u_2, u_3, u_4) space.

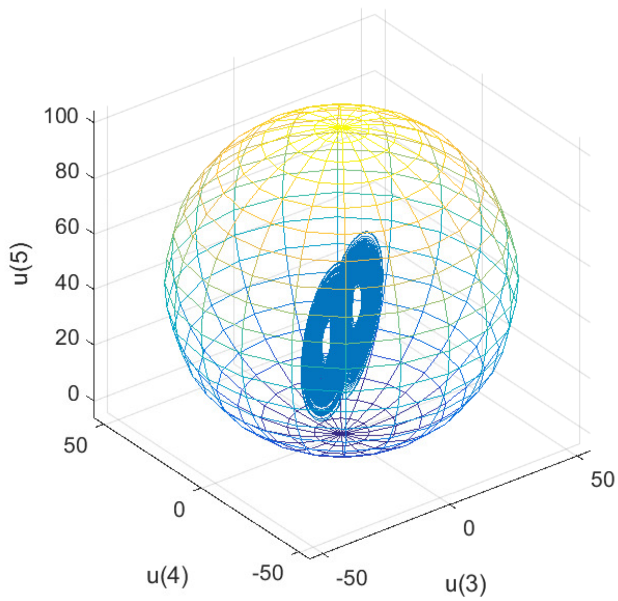

Figure 3. Projection of $\Omega_{1,1}$ into the (u_3, u_4, u_5) space.

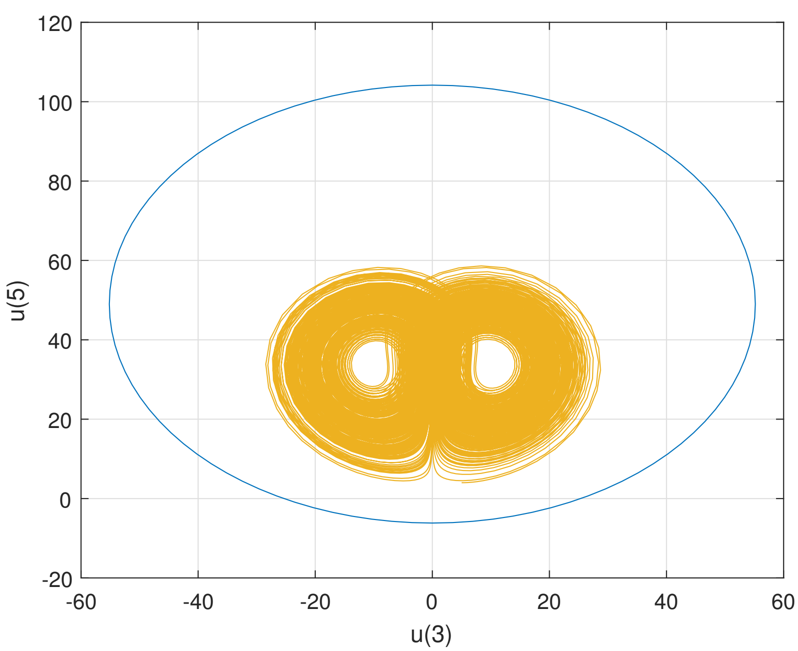

Figure 4. Projection of $\Omega_{1,1}$ onto the (u_3, u_5) plane.

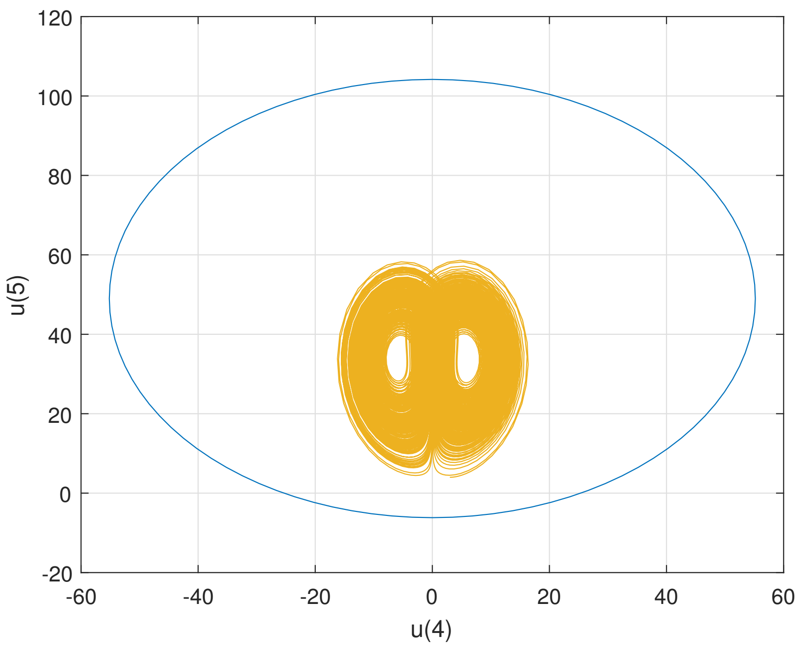

Figure 5. Projection of $\Omega_{1,1}$ onto the (u_4, u_5) plane.

Theorem 4.2. *For any parameters* $\alpha > 0$, $\beta > 0$, $c > 0$, $d > 0$, $\gamma \in R$, *the following set*

$$\Lambda = \left\{ (u_1, u_2, u_3, u_4, u_5) |\, u_3^2 + u_4^2 + d\Big(u_5 - \frac{\gamma}{d}\Big)^2 \leq r_0^2, u_1^2 + u_2^2 \leq r_0^2 \right\} \tag{4.6}$$

is the bounds for the five-dimensional Lorenz system (2.2), where

$$r_0^2 = \begin{cases} \dfrac{\beta^2\gamma^2}{4cd\,(\beta - c)}, & \beta \geq 2c, \\ \dfrac{\gamma^2}{d}, & \beta < 2c. \end{cases}$$

Proof. Construct the Lyapunov-like function

$$V_1\,(U) = u_3^2 + u_4^2 + d\Big(u_5 - \frac{\gamma}{d}\Big)^2.$$

The derivative of $V_1\,(U)$ along the trajectory of (2.2) is

$$\begin{aligned} \frac{dV_1}{dt} =& 2u_3\frac{du_3}{dt} + 2u_4\frac{du_4}{dt} + 2d\left(u_5 - \frac{\gamma}{d}\right)\frac{du_5}{dt} \\ =& 2u_3\,(\gamma u_1 - cu_3 - du_1u_5) + 2u_4\,(\gamma u_2 - cu_4 - du_2u_5) \\ &+ 2d\left(u_5 - \frac{\gamma}{d}\right)(-\beta u_5 + u_1u_3 + u_2u_4) \end{aligned}$$

$$= -\, 2cu_3^2 - 2cu_4^2 - 2d\beta u_5^2 + 2\gamma\beta u_5.$$

Let

$$\Gamma_3 = \ \{(u_3, u_4, u_5) \,|\, cu_3^2 + cu_4^2 + d\beta\Big(u_5 - \frac{\gamma}{2d}\Big)^2 = \frac{\beta\gamma^2}{4d}\,\},$$

then Γ_3 is an ellipsoid in R^3 for $\alpha > 0, \beta > 0, c > 0, d > 0, \gamma \in R$. Outside Γ_3, $\dot{V}_1(U) < 0$, while inside Γ_3, $\dot{V}_1(U) > 0$. Since $V_1(U)$ is a generalized positively definite and radially unbounded continuous function and Γ_3 is a bounded close set, then the maximum value $\max\limits_{U\in\Gamma_3} V_1(U) = r_0^2$ of the function $V_1(U)$ exists. In order to get the maximum value $\max\limits_{U\in\Gamma_3} V_1(U) = r_0^2$, we have to solve the following optimization problem:

$$\begin{cases} \max\ \ V_1(U) = \max\left\{u_3^2 + u_4^2 + d\Big(u_5 - \dfrac{\gamma}{d}\Big)^2\right\}, \\ s.t. cu_3^2 + cu_4^2 + d\beta\Big(u_5 - \dfrac{\gamma}{2d}\Big)^2 = \dfrac{\beta\gamma^2}{4d}. \end{cases} \tag{4.7}$$

The following optimization problem is equivalent to

$$\begin{cases} \max\ \ V_1(U) = \max\left\{u_3^2 + u_4^2 + d\Big(u_5 - \dfrac{\gamma}{d}\Big)^2\right\}, \\ s.t. \dfrac{u_3^2}{\frac{\beta\gamma^2}{4dc}} + \dfrac{u_4^2}{\frac{\beta\gamma^2}{4dc}} + \dfrac{d\big(u_5 - \frac{\gamma}{2d}\big)^2}{\frac{\gamma^2}{4d}} = 1. \end{cases} \tag{4.8}$$

In order to use Lemma 4.2 to solve problem (4.8), let us take $u_3 = z_1, u_4 = z_2, \sqrt{d}u_5 = z_3$ as new variables, then optimization problem (4.8) transforms into:

$$\begin{cases} \max\ \ V_1(U) = \max\left\{z_1^2 + z_2^2 + \left(z_3 - \dfrac{\gamma}{\sqrt{d}}\right)^2\right\}, \\ s.t. \dfrac{z_1^2}{\frac{\beta\gamma^2}{4dc}} + \dfrac{z_2^2}{\frac{\beta\gamma^2}{4dc}} + \dfrac{\left(z_3 - \frac{\gamma}{2\sqrt{d}}\right)^2}{\frac{\gamma^2}{4d}} = 1. \end{cases}$$

We can easily get the optimal solution of the above optimization problem by Lemma 4.2,

$$\max_{U\in\Gamma_3} V_1(U) = r_0^2 = \begin{cases} \dfrac{\beta^2\gamma^2}{4cd\,(\beta - c)}, & \beta \geq 2c, \\ \dfrac{\gamma^2}{d}, & \beta < 2c. \end{cases}$$

Construct the Lyapunov-like function

$$V_2(U) = u_1^2 + u_2^2.$$

The derivative of $V_2(U)$ along the trajectory of (2.2) is

$$\frac{dV_2}{dt} = 2u_1\frac{du_1}{dt} + 2u_2\frac{du_2}{dt}$$

$$\begin{aligned}
&= 2\alpha u_1 (u_3 - u_1) + 2\alpha u_2 (u_4 - u_2) \\
&= -2\alpha u_1^2 - 2\alpha u_2^2 + 2\alpha u_1 u_3 + 2\alpha u_2 u_4 \\
&= -2\alpha u_1^2 - 2\alpha u_2^2 + \alpha (2u_1 u_3 + 2u_2 u_4) \\
&\le -2\alpha u_1^2 - 2\alpha u_2^2 + \alpha \left(u_1^2 + u_3^2 + u_2^2 + u_4^2\right) \\
&= -\alpha u_1^2 - \alpha u_2^2 + \alpha \left(u_3^2 + u_4^2\right) \\
&\le -\alpha u_1^2 - \alpha u_2^2 + \alpha r_0^2 \\
&= -\alpha \left[V_2 (U) - r_0^2\right].
\end{aligned}$$

Thus, we have

$$V_2 (U (t)) - r_0^2 \le \left[V_2 (U (t_0)) - r_0^2\right] e^{-\alpha(t-t_0)}.$$

So,

$$\overline{\lim_{t\to+\infty}} V_2 (U (t)) \le r_0^2.$$

This completes the proof. □

5. Global exponential attractive domain

Though Theorem 4.1 and Theorem 4.2 point out that the solution of the system (2.2) is ultimately bounded, they do not give the rate of the trajectories going from the exterior of the trapping region into the interior trapping region. The rate of the trajectories going from the exterior of the trapping region into the interior trapping region of system (2.2) is given in the following Theorem 5.1.

Theorem 5.1. *For any* $\alpha > 0, \beta > 0, c > 0, d > 0, \gamma \in R$, *with*

$$V_{\lambda,m} (U) = mu_1^2 + mu_2^2 + \lambda u_3^2 + \lambda u_4^2 + \lambda d\left(u_5 - \frac{\lambda\gamma + \alpha m}{\lambda d}\right)^2,$$

$$\eta = \min (\alpha, c, \beta) > 0,\ L_{\lambda,m} = \frac{\beta(\lambda\gamma + \alpha m)^2}{\lambda d\eta}.$$

When $V_{\lambda,m} (U (t)) > L_{\lambda,m}, V_{\lambda,m} (U (t_0)) > L_{\lambda,m}$, *we can get an exponential inequality of system (2.2), given by*

$$V_{\lambda,m} (U (t)) - L_{\lambda,m} \le [V_{\lambda,m} (U (t_0)) - L_{\lambda,m}] e^{-\eta(t-t_0)}.$$

Hence, the set

$$\begin{aligned}
&\Delta_{\lambda,m} \\
&= \{U | V_{\lambda,m} (U) \le L_{\lambda,m}\} \\
&= \left\{(u_1, u_2, u_3, u_4, u_5) \,|\, mu_1^2 + mu_2^2 + \lambda u_3^2 + \lambda u_4^2 + \lambda d\left(u_5 - \frac{\lambda\gamma + \alpha m}{\lambda d}\right)^2 \le \frac{\beta(\lambda\gamma + \alpha m)^2}{\lambda d\eta}\right\},
\end{aligned} \tag{5.1}$$

is the global exponential attractive set of the five-dimensional Lorenz system (2.2).

Proof. Define the Lyapunov-like function

$$V_{\lambda,m}(U) = mu_1^2 + mu_2^2 + \lambda u_3^2 + \lambda u_4^2 + \lambda d\left(u_5 - \frac{\lambda\gamma + \alpha m}{\lambda d}\right)^2,$$

where $\lambda > 0, m > 0, U = (u_1, u_2, u_3, u_4, u_5)$.

When $V_{\lambda,m}(U(t)) > L_{\lambda,m}, V_{\lambda,m}(U(t_0)) > L_{\lambda,m}$, the derivative of $V_{\lambda,m}(U)$ along the trajectory of (2.2) is

$$\begin{aligned}
&\frac{dV_{\lambda,m}(U)}{dt}\Big|_{(2.2)} \\
&= 2mu_1\frac{du_1}{dt} + 2mu_2\frac{du_2}{dt} + 2\lambda u_3\frac{du_3}{dt} + 2\lambda u_4\frac{du_4}{dt} + 2\lambda d\left(u_5 - \frac{\lambda\gamma + \alpha m}{\lambda d}\right)\frac{du_5}{dt} \\
&= 2\alpha m u_1(u_3 - u_1) + 2\alpha m u_2(u_4 - u_2) + 2\lambda u_3(\gamma u_1 - cu_3 - du_1u_5) \\
&\quad +2\lambda u_4(\gamma u_2 - cu_4 - du_2u_5) + 2\lambda d\left(u_5 - \frac{\lambda\gamma + \alpha m}{\lambda d}\right)(u_1u_3 + u_2u_4 - \beta u_5) \\
&= -2\alpha m u_1^2 - 2\alpha m u_2^2 - 2\lambda c u_3^2 - 2\lambda c u_4^2 - 2\lambda d\beta u_5^2 + 2\beta(\lambda\gamma + \alpha m)u_5 \\
&\le -\alpha m u_1^2 - \alpha m u_2^2 - \lambda c u_3^2 - \lambda c u_4^2 - \lambda d\beta u_5^2 + 2\beta(\lambda\gamma + \alpha m)u_5 \\
&= -\alpha m u_1^2 - \alpha m u_2^2 - \lambda c u_3^2 - \lambda c u_4^2 - \lambda d\beta\left(u_5 - \frac{\lambda\gamma + \alpha m}{\lambda d}\right)^2 + \frac{\beta(\lambda\gamma + \alpha m)^2}{\lambda d} \\
&\le -\eta V_{\lambda,m}(U) + \frac{\beta(\lambda\gamma + \alpha m)^2}{\lambda d} \\
&< 0.
\end{aligned}$$

That is equivalent to say,

$$\frac{dV_{\lambda,m}(U)}{dt}\Big|_{(2.2)} \le -\eta\left(V_{\lambda,m}(U) - \frac{\beta(\lambda\gamma + \alpha m)^2}{\lambda d\eta}\right). \tag{5.2}$$

From the above inequality (5.2), we can get

$$\begin{aligned}
V_{\lambda,m}(U(t)) &\le V_{\lambda,m}(U_0)e^{-\eta(t-t_0)} + \int_{t_0}^{t} e^{-\eta(t-\tau)}\frac{\beta(\lambda\gamma + \alpha m)^2}{\lambda d\eta}d\tau \\
&= V_{\lambda,m}(U_0)e^{-\eta(t-t_0)} + L_{\lambda,m}\left(1 - e^{-\eta(t-t_0)}\right).
\end{aligned}$$

We have the following exponential inequality

$$V_{\lambda,m}(U(t)) - L_{\lambda,m} \le [V_{\lambda,m}(U_0) - L_{\lambda,m}]e^{-\eta(t-t_0)}.$$

Taking limit on both sides of the above inequality as $t \to +\infty$ results in

$$\overline{\lim_{t\to+\infty}} V_{\lambda,m}(U(t)) \le L_{\lambda,m}.$$

Namely, the set $\Delta_{\lambda,m}$ is the global exponential attractive set of system (2.2).

This completes the proof. □

Remark 5.1. Let us take $\lambda = 1, m = 0$, then we can get that the following set

$$\Psi_{1,0} = \left\{ (u_3, u_4, u_5) \,|u_3^2 + u_4^2 + \left(u_5 - \frac{\gamma}{d}\right)^2 \le \delta^2 \right\}, \tag{5.3}$$

is the global exponential attractive set of system (2.2), where $\eta = \min(c, \beta) > 0, \delta^2 = \frac{\beta\gamma^2}{d\eta}$.

The proved method is similar to the above Theorem 5.1.

In the following part, we will present an application of the results that obtained in this paper. We will apply above results to show that the equilibrium point $O(0,0,0,0,0)$ of the system (2.2) is the globally exponentially stable when $\alpha > 0, \beta > 0, c > 0, d > 0, \gamma < 0$.

Theorem 5.2. *If real parameters $\alpha > 0, \beta > 0, c > 0, d > 0, \gamma < 0$, then the equilibrium point $O(0,0,0,0,0)$ of system (2.2) is the globally exponentially stable.*

Proof. When $\alpha > 0, \beta > 0, c > 0, d > 0, \gamma < 0$, let us choose $m = -\gamma, \lambda = \alpha$ in the above Theorem 5.1. Then, we can get $L_{\lambda,m} = L_{\alpha,-\gamma} = 0$ according to Theorem Theorem 5.1. And the exponential inequality in Theorem 5.1 becomes

$$\begin{aligned} &\left[-\gamma u_1^2(t) - \gamma u_2^2(t) + \alpha u_3^2(t) + \alpha u_4^2(t) + \alpha d u_5^2(t)\right] \\ &\le \left[-\gamma u_1^2(t_0) - \gamma u_2^2(t_0) + \alpha u_3^2(t_0) + \alpha u_4^2(t_0) + \alpha d u_5^2(t_0)\right] e^{-\eta(t-t_0)} \end{aligned} \tag{5.4}$$

where $\eta = \min(\alpha, c, \beta) > 0$. The above inequality (5.4) shows that the equilibrium point $O(0,0,0,0)$ of system (2.2) is globally exponentially stable.

This completes the proof. □

Remark 5.2. The results of this paper can also be used for chaos synchronization, chaos control and the estimation of the Hausdorff dimension of attractors. The applications of the boundedness of chaotic systems in chaos control and chaos synchronization can be referred to the papers [28, 35]. The applications of the boundedness of chaotic systems in the estimation of the Hausdorff dimension of attractors can be referred to the papers [9, 11, 16].

6. Conclusions

In this paper, a new generalized complex Lorenz system was proposed and studied by using the theory of chaotic systems. Boundedness and the global exponential attractive set of the complex Lorenz system are obtained. The corresponding boundedness is numerically verified by the computer. Numerical simulations are presented to show the effectiveness of the theoretical research results. Finally, the theoretical results obtained in this paper are used to study the globally exponential stability of the equilibrium point of system (2.2).

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