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NUMERICAL APPROACH FOR THE HUNTER SAXTON EQUATION ARISING IN LIQUID CRYSTAL MODEL THROUGH COCKTAIL PARTY GRAPHS CLIQUE POLYNOMIAL

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Abstract In this paper, a well-known nematic liquid crystal model, the Hunter Saxton equation, is solved by the new graph theoretic polynomial approach. At first, we extracted the clique polynomials from the cocktail party graph (CPG) and generated the generalized operational matrix of integration through the clique polynomials of CPG. Then, an effective computational technique called the cocktail party graphs clique polynomial collocation method (CCCM) is developed to obtain an approximate numerical solution for the nonlinear Hunter-Saxton equation (HSE). The operational matrix of CPG reduces the HSE into an algebraic system of nonlinear equations that makes the solution relatively superficial. The Newton-Raphson method solves these nonlinear algebraic equations to obtain the clique polynomial solution for HSE. The efficiency of the CCCM is illustrated by examining two numerical examples. The solution of the HSE is presented through figures and tables for different values of x,t, and N. The convergence analysis, tabulated results of numerical comparison of absolute errors of CCCM with the recent numerical methods, and error norms projected that, CCCM is considerably efficacious on the computational ground for higher accuracy and convergence of numerical solutions.

Keywords Hunter Saxton equation, graph-theoretic polynomial called Clique polynomial, cocktail party graph, operational matrix, convergence analysis.

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1. Introduction

In Mathematics, Graph theory is well known for its enormous applications in applied mathematics like the modeling of complicated chemical compounds, dynamic systems, geometric models, etc [20]. In 2010, Dehmer et al. [23] introduced graph polynomials (information polynomials). In general, graph polynomials encode graph-theoretic information of the underlying graph in various ways. Until now, plenty of graph polynomials have been introduced [22]. Only a few are used to study nonlinear mathematical models [12, 16, 18, 21, 30]. Clique polynomial is one among the

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polynomials that we're able to successfully operate in the study of various mathematical models such as time-fractional Klein-Gordon equations [7], fractional partial differential equations [1], convection flow problems [15], Schrödinger equations [10], distributed order integrodifferential equations [28], etc. In the literature survey, we observed that only clique polynomials of the complete graph were used to study the above-mentioned differential equations. Here, the author tried to extract the clique polynomial numerical method and implement it in studying nonlinear models. The present paper deals with the clique polynomial of the cocktail party graph.

In the modern context of sciences, most physical phenomena are mathematically modeled as differential equations. Mathematical models convert application zone problems into manageable mathematical formulations with a hypothetical and arithmetical analysis. Many mathematical models are governed by partial differential equations. PDEs are encountered in many branches of sciences, such as astrophysics, quantum mechanics, biology, fluid mechanics, environmental science, chemistry, particle physics, etc. Solving these PDEs is quite strenuous. Current mathematical methods fail to give closed-form solutions, and more advances are yet to be developed. Meanwhile, many numerical techniques have been developed for solving PDEs. Here, we developed an efficient numerical technique called a Generalized operational matrix based on the clique polynomial of the cocktail party graph to study the nonlinear liquid crystal model called the Hunter-Saxton equation (HSE).

In theoretical physics, HSE is an integrable PDE aroused in the study of nematic liquid crystals. The HSE describes the propagation of weakly nonlinear orientation waves in the nematic liquid crystal director field [11], and it is also called the high-frequencies ubiquitous Camassa–Holm equation [6]. The universal form of the Hunter Saxton equation [4, 29] is as follows:

$$\frac{\partial^2 \xi(x,t)}{\partial x\, \partial t} + \xi(x,t)\frac{\partial^2 \xi(x,t)}{\partial x^2} + \frac{1}{2}\left(\frac{\partial \xi(x,t)}{\partial x}\right)^2 = 0. \tag{1.1}$$

Where x is the space variable and t is the time variable.

The physical conditions are:

$$\xi(x,0) = \ v(x),\ \frac{\partial \xi(x,0)}{\partial x} = \ \mathcal{W}(x),\ \ \xi(0,t) = \ \mu(t). \tag{1.2}$$

Where $v(x), \mathcal{W}(x), \mu(t)$, are smooth functions. Many numerical methods have been proposed for the fairly accurate solution of HSE. For insane, HSE by Cubic trigonometric method [9], Bivariate Chebyshev method [19], Collocation method [13], Galerkin method [27], Finite difference and Haar wavelet quasilinearization method [3], Laguerre wavelet on time domains [17], Lipschitz metric [5], Time marching scheme [2], Adomain Decomposition method [14], Sinc collocation method [26], Mesh free collocation method [25], Fibonacci wavelet method [24]. Here, we ensure that the clique polynomial of the cocktail party graph scheme is new and not yet utilized in studying the Hunter-Saxton equation. Hence, we proposed the CCCM to find the approximate solution of HSE via a generalized operational integration matrix.

Arrangement of the article: The preliminaries section briefs the clique polynomials of the cocktail party graph and convergence analysis. The operational matrix section describes generating an operational matrix of clique polynomials. The method of solution section explains the clique polynomial-based numerical method. The numerical applications part shows the results of numerical examples and the comparison of current results with the other results. Finally, the Conclusion segment provides the summative result analysis.

2. Preliminaries of cocktail party graphs clique polynomial and convergence analysis

Clique Polynomial of Graph: In the given graph G, the clique polynomial $C(x)$ is defined as;

$$C(x) = a_0(x) + \sum_{\theta=1}^{\rho(G)} a_\theta x^\theta.$$

Where,

$$a_\theta = \theta, \text{ denotes the number of cliques in graph } G,$$
$$\rho(G) = \text{ the total number of cliques in graph } G,$$
$$a_0(x) = \text{ number of zero cliques in graph } G, \text{ and it is constant.}$$

Hajiabolhassan and Mehrabadi [8] showed that $C(x)$, always has a real root. The coefficient a_1 is the vertex count, a_2 is the edge count, and a_3 is the triangle count, etc in graph G.

Cocktail Party Graph: The n^{th} order cocktail party graph is the graph that contains n rows of paired vertices in which all vertices except the paired ones are not connected with the edge. The cocktail party graph of n^{th}order has also referred to as the Hyper octahedral graph, n-octahedron, Roberts graph, Complement of the ladder rung graph, dual hypercube graph, skeleton of the cross polytope complete n-partite graph and cocktail party graph is denoted with various notations such as $K_{n\times 2}$ and $K_{n(2)}$,$(2n, n)-$ Turán graph, since the cocktail party graphs are distance-transitive, they are distance-regular. The cocktail party graph arises in the handshake problem [24].

Some results on the Clique Polynomial of Cocktail Party graph and convergence analysis:

Theorem 2.1. *Let $G = (V, E)$ is the cocktail party graph (complete $n-$ partite graph), then its general form of the clique polynomial is $C\left(K_{n(2)}; x\right) = (1 + 2x)^n$. Where n denotes the number of partitions of V.*

Proof. Given that, $G = (V, E)$, be the complete $n-$ partite graph.

For $n = 1$, Graph $G_1 = \{A, B\}$ has one pair of vertices. In Graph G_1, the

Figure 1. Graph G_1

number of zero-cliques, i.e. $a_0 = 1$, and number of one − cliques $\{A, B\}$ i.e. $a_1 = 2$. Hence,the clique polynomial of the first-order cocktail party graph is

$$C\left(K_{1(2)}; x\right) = a_0 + \sum_{\theta=1}^{1} a_\theta x^\theta = a_0 + a_1 x^1 = 1 + 2x.$$

For $n = 2$, Graph G_2 has two pair of vertices. $G_2 = (V = \{\{A, B\}, \{C, D\}\}, E)$. In Graph G_2, the number of zero-cliques, i.e. $a_0 = 1$, number of one-cliques

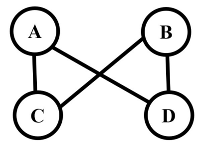

Figure 2. Graph G_2

$\{A, B, C, D\}$ i.e. $a_1 = 4$, number of two – cliques $\{AC, AD, BC, BD\}$ i.e $a_2 = 4$. Hence, the clique polynomial of the second-order cocktail party graph is,

$$C\left(K_{2(2)}; x\right) = a_0 + \sum_{\theta=1}^{2} a_\theta x^\theta = \; a_0 + a_1 x^1 + a_2 x^2 \; = 1 + 4x + 4x^2 = \; (1 + 2x)^2 .$$

For $n = 3$, Graph G_3 has three pair of vertices. That is,

$$G_3 = (V = \{\{A, B\}, \; \{C, D\}, \{E, F\}\}, E).$$

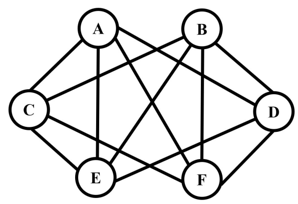

Figure 3. Graph G_3

In Graph G_3, the number of zero-cliques, i.e. $a_0 = 1$, number of one-cliques $\{A, B, C, D, E, F\}$ i.e. $a_1 = 6$, number of two-cliques $\{AB, AC, AE, AF, BD, BE, BF, CD, CE, CF, DE, DF\}$ i.e $a_2 = 12$, number of three-cliques $\{ABF, ACE, BDE, BDF, CDE, CDF, EAB, FAC\}$ i.e $a_3 = 8$. Hence, the clique polynomial of the third-order cocktail party graph is,

$$C\left(K_{3(2)}; x\right) = a_0 + \sum_{\theta=1}^{3} a_\theta x^\theta = \; a_0 + a_1 x^1 + a_2 x^2 + \; a_3 x^4 = 1 + 6x + 12x^2 + 8x^3$$
$$= \; (1 + 2x)^3 .$$

Similarly, we obtained the sequence of clique polynomials of $4^{\text{th}}, 5^{\text{th}}, 6^{\text{th}}, 7^{\text{th}}, \ldots,$ etc. ordered cocktail party graphs as,

For $n = 4$, $C\left(K_{4(2)}; x\right) = 1 + 8x + 24x^2 + 32x^3 + 16x^4 = \; (1 + 2x)^4 ;$

For $n = 5$, $C\left(K_{5(2)}; x\right) = 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5 = \; (1 + 2x)^5 ;$

For $n = 6$, $C\left(K_{6(2)}; x\right) = 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6$
$= (1 + 2x)^6 ;$

$$\text{For } n = 7,\ C\left(K_{7(2)}; x\right) = 1 + 14x + 84x^2 + 280x^3 + 560x^4 + 672x^5 + 448x^6 + 128x^7 = (1 + 2x)^7; \ldots \text{etc.}$$

With the above observations, in general, the clique polynomial of $n^{\text{th}}-$order cocktail party graph is of the form, $C\left(K_{n(2)}; x\right) = (1 + 2x)^n$. □

Theorem 2.2 ([16]). *Let $\xi(x,t)$ be the continuous function in $L^2(R \times R)$ on $[0,1] \times [0,1]$ and $\xi(x,t)$ is bounded by some positive real number λ, then clique polynomial expansion of $\xi(x,t)$,converges to it.*

Proof. Given that, $\xi(x,t)$, is the continuous function in $L^2(R \times R)$ on $[1,0] \times [0,1]$ and $\xi(x,t)$ is bounded by some positive real number λ. Consider,

$$\xi(x,t) = C(t)^T \text{ M C}(x),$$
$$\xi(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\, C\left(K_{i(2)}; t\right) C\left(K_{i(2)}; x\right).$$

Where, $a_{ij} = \left\langle \xi(x,t),\ C\left(K_{i(2)}; t\right) C\left(K_{i(2)}; x\right)\right\rangle$ and $\langle\ .\rangle$ indicate the inner product. The unknown coefficient of the clique polynomial function is defined as,

$$a_{\text{ij}} = \int_0^1\int_0^1 \xi(x,t)\, C\left(K_{i(2)}; t\right)\ C\left(K_{i(2)}; x\right)\ dx\, dt,$$
$$a_{\text{ij}} = \int_0^1\int_0^1 \xi(x,t)\, C\left(K_{i(2)}; t\right)\ dt\, C\left(K_{i(2)}; x\right)\ dx.$$

By the generalized mean value theorem, we have,

$$a_{ij} = \int_0^1 C\left(K_{i(2)}; x\right)\ dx \quad \int_0^1 \xi(n,t)\, C\left(K_{i(2)}; t\right)\ dt.$$

Here, $n \in [0,1]$. Since $C\left(K_{i(2)}; x\right)$ is a continuous and integrable function on $[0,1]$, we consider, $\int_0^1 C\left(K_{i(2)}; x\right)\ dx = Z$. Hence, we have,

$$a_{ij} = Z\int_0^1 \xi(n,t)\ \ C\left(K_{i(2)}; t\right) dt,$$

by generalized mean value theorem for integrals,

$$a_{\text{ij}} = Z\ \xi(n,\omega)\int_0^1 C\left(K_{i(2)}; t\right)\ dt.$$

Where, $\omega \in [0,1]$. Since $C\left(K_{i(2)}; t\right)$ is continuous and integrable on $[0,1]$.
Put, $\int_0^1 C\left(K_{i(2)}; t\right)\ dt = D$.

$$a_{\text{ij}} = Z\ D\ \xi(n,\omega).$$

Where, $n,\ \omega \in [0,1]$. Therefore, $|a_{ij}| = |Z|\ |D|\ |\xi(x,\omega)|$.

Since, $\xi(x,t)$ is bounded by λ. Therefore, $\left|a_{ij}\right| = \left|Z\right|\left|D\right|\lambda$. Therefore, $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}$ is absolutely convergent. Hence, clique polynomial expansion of $\xi(x,t)$ converges to it. □

Theorem 2.3 ([16]). *Let $C\left(K_{n(2)}; x\right)$ are the clique polynomial of the cocktail party graph of n vertices, then $C\left(K_{n(2)}; x\right)$ are uniformly continuous on $[0,1]$.*

3. Operational matrix of clique polynomial of complete graphs

Considering a few Clique polynomials of the Cocktail party graph,

$$\begin{aligned}
&C(K_{0(2)};x) = 1,\\
&C\left(K_{1(2)};x\right) = 1+2x,\\
&C\left(K_{2(2)};x\right) = 1+4x+4x^2,\\
&C\left(K_{3(2)};x\right) = 1+6x+12x^2+8x^3,\\
&C\left(K_{4(2)};x\right) = 1+8x+24x^2+32x^3+16x^4\ldots.
\end{aligned}$$

Where,

$$C_6(x) = \left[C\left(K_{0(2)};x\right), C\left(K_{1(2)};x\right), C\left(K_{2(2)};x\right), \ldots, C\left(K_{5(2)};x\right)\right]^T.$$

The definite integrals of the above polynomials and their matrix forms are as follows,

$$\begin{aligned}
&\int_0^x C\left(K_{0(2)};x\right)dx = x = \left[-\frac{1}{2},\frac{1}{2},0,0,0,0\right]C_6(x),\\
&\int_0^x C\left(K_{1(2)};x\right)dx = x+x^2 = \left[-\frac{1}{4},0,\frac{1}{4},0,0,0\right]C_6(x),\\
&\int_0^x C\left(K_{2(2)};x\right)dx = x+2x^2+\frac{4}{3}x^3 = \left[-\frac{1}{6},0,0,\frac{1}{6},0,0\right]C_6(x),\\
&\int_0^x C\left(K_{3(2)};x\right)dx = x+3x^2+4x^3+2x^4 = \left[-\frac{1}{8},0,0,0,\frac{1}{8},0\right]C_6(x),\\
&\int_0^x C\left(K_{4(2)};x\right)dx = x+4x^2+8x^3+8x^4+16x^5 = \left[-\frac{1}{10},0,0,0,0,\frac{1}{10}\right]C_6(x),\\
&\int_0^x C\left(K_{5(2)};x\right)dx = x+5x^2+\frac{40}{3}x^3+20x^4+16x^5+\frac{16}{3}x^6\\
&\qquad = \left[-\frac{1}{12},0,0,0,0,0\right]C_6(x)+\frac{1}{6}C\left(K_{6(2)};x\right).
\end{aligned}$$

Thus,

$$\int_0^x C_6(x)dx = Z_{6\times 6}C_6(x) + \overline{C_6(x)}.$$

Where,

$$Z_{6\times 6} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0\\ -\frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0\\ -\frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & 0\\ -\frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & 0\\ -\frac{1}{10} & 0 & 0 & 0 & 0 & \frac{1}{10}\\ -\frac{1}{12} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } \quad \overline{C_6(x)} = \begin{bmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ \frac{1}{6}C\left(K_{6(2)};x\right) \end{bmatrix}.$$

The double integral of the above polynomials and their matrix forms are as follows:

$$\int_0^x \int_0^x C\left(K_{0(2)}; x\right) dx\ dx = \frac{x^2}{2} = \left[\frac{1}{8}, -\frac{1}{4}, \frac{1}{8}, 0, 0, 0\right] C_6(x),$$

$$\int_0^x \int_0^x C\left(K_{1(2)}; x\right) dx\ dx = \frac{x^2}{2} + \frac{x^3}{3} = \left[\frac{1}{12}, -\frac{1}{8}, 0, \frac{1}{24}, 0, 0\right] C_6(x),$$

$$\int_0^x \int_0^x C\left(K_{2(2)}; x\right) dx\ dx = \frac{x^2}{2} + \frac{2x^3}{3} + \frac{x^4}{3} = \left[\frac{1}{16}, -\frac{1}{12}, \frac{1}{8}, 0, 0, \frac{1}{48}\right] C_6(x),$$

$$\int_0^x \int_0^x C\left(K_{3(2)}; x\right) dx\ dx = \frac{x^2}{2} + x^3 + x^4 + \frac{2}{5}x^5 = \left[\frac{1}{20}, -\frac{1}{16}, 0, 0, 0, \frac{1}{80}\right] C_6(x),$$

$$\int_0^x \int_0^x C\left(K_{4(2)}; x\right) dx\ dx = \frac{x^2}{2} + \frac{4}{3}x^3 + 2x^4 + \frac{8}{5}x^5 + \frac{8}{3}x^6$$

$$= \left[\frac{1}{24}, -\frac{1}{20}, 0, 0, 0, 0\right] C_6(x) + \frac{1}{120} C\left(K_{6(2)}; x\right),$$

$$\int_0^x \int_0^x C\left(K_{5(2)}; x\right) dx\ dx = \frac{x^2}{2} + \frac{5}{3}x^3 + \frac{10}{3}x^{\ 4} + 4x^5 + \frac{8}{3}x^6 + \frac{16}{21}x^7$$

$$= \left[\frac{1}{28}, -\frac{1}{24}, 0, 0, 0, 0\right] C_6(x) + \frac{1}{168} C\left(K_{7(2)}; x\right).$$

Thus,

$$\int_0^x \int_0^x C_6(x) dx\ dx = Z'_{6\times 6} C_6(x) + \overline{C'_6(x)}.$$

Where,

$$Z'_{6\times 6} = \begin{bmatrix} \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} & 0 & 0 & 0 \\ \frac{1}{12} & -\frac{1}{8} & 0 & \frac{1}{24} & 0 & 0 \\ \frac{1}{16} & -\frac{1}{12} & 0 & 0 & \frac{1}{48} & 0 \\ \frac{1}{20} & -\frac{1}{16} & 0 & 0 & 0 & \frac{1}{80} \\ \frac{1}{24} & -\frac{1}{20} & 0 & 0 & 0 & 0 \\ \frac{1}{28} & \frac{1}{24} & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } \overline{C'_6(x)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{120} C\left(K_{6(2)}; x\right) \\ \frac{1}{168} C\left(K_{7(2)}; x\right) \end{bmatrix}.$$

The generalized single integration of n-clique polynomials is denoted as:

$$\int_0^x C_n(x) dx = Z_{n\times n} C(x) + \overline{C(x)}.$$

Where,

$$Z_{n\times n}=\begin{bmatrix} -\frac{1}{2n-(2n-2)} & \frac{1}{2n-(2n-2)} & 0 & 0 & ... & 0\\ -\frac{1}{2n-(2n-4)} & 0 & \frac{1}{2n-(2n-4)} & 0 & ... & 0\\ -\frac{1}{2n-(2n-6)} & 0 & 0 & \frac{1}{2n-(2n-6)} & ... & 0\\ . & . & . & . & ... & .\\ . & . & . & . & ... & .\\ . & . & . & . & ... & .\\ -\frac{1}{2(n-1)} & 0 & 0 & 0 & 0\,0\,0 & \frac{1}{2(n-1)}\\ -\frac{1}{2n} & 0 & 0 & 0 & 0\,0\,0 & 0 \end{bmatrix},$$

$$C(x)=\begin{bmatrix} C\left(K_{0(2)};x\right)\\ C\left(K_{1(2)};x\right)\\ C\left(K_{2(2)};x\right)\\ \vdots\\ C\left(K_{n(2)};x\right)\end{bmatrix} \text{ and } \overline{C(x)}=\begin{bmatrix} 0\\ 0\\ 0\\ \vdots\\ \frac{1}{2n}C\left(K_{n(2)};x\right)\end{bmatrix}.$$

Similarly, the double integration of n-clique polynomials, in general, is denoted as:

$$\int_0^x\int_0^x C_n(x)\ dx\ dx = Z'_{n\times n}C_n(x)+\overline{C_n{}'(x)}.$$

Where,

$$Z'_{n\times n}=\begin{bmatrix} \frac{1}{4n-(4n-8)} & -\frac{1}{4n-(4n-4)} & 0\,0 ... & 0\\ \frac{1}{4n-(4n-12)} & -\frac{1}{4n-(4n-8)} & 0\,0 ... & 0\\ \frac{1}{4n-(4n-16)} & -\frac{1}{4n-(4n-12)} & 0\,0 ... & 0\\ . & . & ..... & .\\ . & . & ..... & .\\ . & . & ..... & .\\ \frac{1}{4(n-1)} & -\frac{1}{4(n-2)} & 0\,0\,0\,0\,0 & \frac{1}{4(n-1)4(n-2)}\\ \frac{1}{4n} & -\frac{1}{4(n-1)} & 0\,0\,0\,0\,0 & 0\\ \frac{1}{4(n+1)} & -\frac{1}{4n} & 0\,0\,0\,0\,0 & 0 \end{bmatrix},$$

$$C_n(x)=\begin{bmatrix} C\left(K_{0(2)};x\right) \\ C\left(K_{1(2)};x\right) \\ C\left(K_{2(2)};x\right) \\ \vdots \\ C\left(K_{n(2)};x\right) \end{bmatrix} \text{ and } \overline{C_n{}'(x)}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \frac{1}{4n(4n-1)}C\left(K_{n(2)};x\right) \end{bmatrix}.$$

4. Method of solution

The proposed method presents the clique polynomial-oriented numerical algorithm through the generalized operational matrix of integration to deal with the HSE under distinct ICs and BCs.

$$\frac{\partial^3\xi(x,t)}{\partial x^2\,\partial t}=C(t)^T M\,C(x), \tag{4.1}$$

here,

$$C(t)^T=\left[C\left(K_{0(2)};t\right),\ C\left(K_{1(2)};t\right),\ C\left(K_{2(2)};t\right),\ \ldots,\ C\left(K_{n(2)};t\right)\right],$$

$$C(x)=\left[C\left(K_{0(2)};x\right),\ C\left(K_{1(2)};x\right),C\left(K_{2(2)};x\right),\ldots,C\left(K_{n(2)};x\right)\right]^T.$$

$M=[a_{ij}]$ is the $n\times n$ matrix, where $i=0,1,2,\ldots,n$ and $j=0,1,2,\ldots,n$. Integrating equation (4.1) concerning t from 0 and t we get,

$$\begin{aligned}\frac{\partial^2\xi(x,t)}{\partial x^2}&=\frac{\partial^2\xi(x,0)}{\partial x^2}+\int_0^t C(t)^T M\,C(x)\,dt,\\ \frac{\partial^2\xi(x,t)}{\partial x^2}&=\frac{\partial^2\xi(x,0)}{\partial x^2}+\left[Z_{n\times n}C(t)+\overline{C(t)}\right]^T M\,C(x).\end{aligned} \tag{4.2}$$

Integrating equation (4.2) concerning x from 0 to x, we get,

$$\begin{aligned}\frac{\partial\xi(x,t)}{\partial x}&=\frac{\partial\xi(0,t)}{\partial x}+\frac{\partial\xi(x,0)}{\partial x}-\frac{\partial\xi(0,0)}{\partial x}+\left[Z_{n\times n}C(t)+\overline{C(t)}\right]^T M\int_0^x C(x)dx,\\ \frac{\partial\xi(x,t)}{\partial x}&=\frac{\partial\xi(x,0)}{\partial x}+\left[\frac{\partial\xi(0,t)}{\partial x}-\frac{\partial\xi(0,0)}{\partial x}\right]\\ &\quad+\left[Z_{n\times n}C(t)+\overline{C(t)}\right]^T M\left[Z_{n\times n}C(x)+\overline{C(x)}\right].\end{aligned} \tag{4.3}$$

Integrating equation (4.3) concerning x from 0 to x, we get,

$$\begin{aligned}\xi(x,t)&=\xi(0,t)+\xi(x,0)-\xi(0,0)+x\left\{\frac{\partial\xi(0,t)}{\partial x}-\frac{\partial\xi(0,0)}{\partial x}\right\}\\ &\quad+\int_0^x\left\{\left[Z_{n\times n}C(t)+\overline{C(t)}\right]^T M\left[Z_{n\times n}C(x)+\overline{C(x)}\right]\right\}dx,\end{aligned}$$

$$\xi(x,t)=\xi(0,t)+\ \xi(x,0)-\ \xi(0,0)+x\left\{\frac{\partial\xi(0,t)}{\partial x}-\ \frac{\partial\xi(0,0)}{\partial x}\right\}$$
$$+\left[Z_{n\times n}C(t)+\ \overline{C(t)}\right]^T M\left[Z'_{n\times n}C(x)+\overline{C'(x)}\right]. \tag{4.4}$$

Put $x=\alpha$ in equation (4.4) we get,

$$\xi(\alpha,t)=\ \xi(0,t)+\ \xi(\alpha,0)-\ \xi(0,0)+\alpha\left\{\frac{\partial\xi(0,t)}{\partial x}-\ \frac{\partial\xi(0,0)}{\partial x}\right\}$$
$$+\left[Z_{n\times n}\ C(t)+\ \overline{C(t)}\right]^T\ \mathrm{M}\ \left\{\left[Z'_{n\times n}\ C(x)+\ \overline{C'(x)}\right]\right\}_{x=\alpha},$$
$$\left\{\frac{\partial\xi(0,t)}{\partial x}-\ \frac{\partial\xi(0,0)}{\partial x}\right\}=\frac{1}{\alpha}\Bigg(\xi(\alpha,t)-\xi(0,t)-\xi(\alpha,0)+\xi(0,0) \tag{4.5}$$
$$-\left[Z_{n\times n}C(t)+\ \overline{C(t)}\right]^T$$
$$\times\ \mathrm{M}\ \left\{\left[Z'_{n\times n}C(x)+\ \overline{C'(x)}\right]\right\}_{x=\alpha}\Bigg).$$

On substituting equation (4.5) in equations (4.3) and (4.4) we obtained,

$$\frac{\partial\xi(x,t)}{\partial x}=\frac{\partial\xi(x,0)}{\partial x}+\frac{1}{\alpha}\Bigg(\xi(\alpha,t)-\xi(0,t)-\xi(\alpha,0)+\xi(0,0)$$
$$-\left[Z_{n\times n}C(t)+\ \overline{C(t)}\right]^T\ \mathrm{M}\ \left\{\left[Z'_{n\times n}C(x)+\ \overline{C'(x)}\right]\right\}_{x=\alpha}\Bigg) \tag{4.6}$$
$$+\left[Z_{n\times n}C(t)+\overline{C(t)}\right]^T M\left[Z_{n\times n}C(x)+\ \overline{C(x)}\right],$$

$$\xi(x,t)=\xi(0,t)+\ \xi(x,0)-\ \xi(0,0)+x\frac{1}{\alpha}\Bigg(\xi(\alpha,t)-\xi(0,t)-\xi(\alpha,0)+\xi(0,0)$$
$$-\left[Z_{n\times n}C(t)+\ \overline{C(t)}\right]^T\ \mathrm{M}\ \left\{\left[Z'_{n\times n}C(x)+\ \overline{C'(x)}\right]\right\}_{x=\alpha}\Bigg)$$
$$+\left[Z_{n\times n}C(t)+\ \overline{C(t)}\right]^T M\left[Z'_{n\times n}C(x)+\overline{C'(x)}\right]. \tag{4.7}$$

On differentiation of equation (4.7) concerning t we get,

$$\frac{\partial\xi(x,t)}{\partial t}=\frac{\partial\xi(0,t)}{\partial t}+\frac{\partial}{\partial t}\Bigg\{x\frac{1}{\alpha}\Bigg(\xi(\alpha,t)-\xi(0,t)-\xi(\alpha,0)+\xi(0,0)$$
$$-\left[Z_{n\times n}C(t)+\ \overline{C(t)}\right]^T\ \mathrm{M}\ \left\{\left[Z'_{n\times n}C(x)+\ \overline{C'(x)}\right]\right\}_{x=\alpha}\Bigg)\Bigg\} \tag{4.8}$$
$$+\frac{\partial}{\partial t}\left\{\left[Z_{n\times n}C(t)+\ \overline{C(t)}\right]^T M\left[Z'_{n\times n}C(x)+\overline{C'(x)}\right]\right\}.$$

Differentiating equation (4.8) concerning x we get,

$$\begin{aligned}\frac{\partial^2\xi(x,t)}{\partial x\partial t} =&\frac{\partial}{\partial x}\left[\frac{\partial}{\partial t}\left\{x\frac{1}{\alpha}\left(\xi(\alpha,t)-\xi(0,t)-\xi(\alpha,0)+\xi(0,0)\right.\right.\right.\\ &-\left[Z_{n\times n}C(t)+\ \overline{C(t)}\right]^T \ \mathrm{M}\ \left.\left.\left.\left\{\left[Z'_{n\times n}C(x)+\ \overline{C'(x)}\right]\right\}_{x=\alpha}\right)\right\}\right]\\ &+\frac{\partial}{\partial x}\left[\frac{\partial}{\partial t}\left\{\left[Z_{n\times n}C(t)+\ \overline{C(t)}\right]^T M\left[Z'_{n\times n}C(x)+\overline{C'(x)}\right]\right\}\right].\end{aligned} \tag{4.9}$$

Substituting equations (4.9), (4.7), (4.2), and (4.6) in HSE given in (4.1) and discretized by the grid points. $x_i = t_i = \frac{(2i-1)}{(2n^2)}$; $i = 1, 2, \ldots, n^2$, it reduced into a system of nonlinear algebraic equations. The coefficients of the clique polynomials were obtained by solving the resultant system of equations by Newton's method. On substituting these coefficients in equation (4.7), we get a graph of a theoretically oriented solution of the Hunter Saxton equation.

5. Numerical applications

Example 5.1. Consider the HS equation [9],

$$\frac{\partial^2\xi(x,t)}{\partial x\ \partial t}+\xi(x,t)\frac{\partial^2\xi(x,t)}{\partial x^2}+\frac{1}{2}\left(\frac{\partial\xi(x,t)}{\partial x}\right)^2=0, \tag{5.1.1}$$

with physical conditions,

$$\xi(x,0)=2x,\ \frac{\partial\xi(x,0)}{\partial t}=-2x,\ \xi(0,t)=0,\ \frac{\partial\xi(1,t)}{\partial t}=\frac{2}{(1+t)}.$$

The exact solution is

$$\xi(x,t)=\frac{2x}{(1+t)}.$$

Implementation of CCCM for N=2: Consider,

$$\frac{\partial^3\xi(x,t)}{\partial x^2\ \partial t}=C(t)^T\ \mathrm{M}\ \mathrm{C}(x). \tag{5.1.2}$$

Here,

$$C(t)^T=\ [1,1+2t];\ \ C(x)=\begin{bmatrix}1\\ 1+2x\end{bmatrix};\ \ M=\begin{bmatrix}a[1]\ a[2]\\ a[3]\ a[4]\end{bmatrix}. \tag{5.1.3}$$

Substituting equation (5.1.3) in equation (5.1.2) and simplifying, we get,

$$\frac{\partial^3\xi(x,t)}{\partial x^2\ \partial t}=a[1]+(1+2t)a[3]+(1+2x)\left(a[2]+(1+2t)a[4]\right). \tag{5.1.4}$$

Integrating equation (5.1.4) concerning t from 0 to t, we get

$$\frac{\partial^2\xi(x,t)}{\partial x^2}=ta[1]+t(1+t)a[3]+(1+2x)\left(\mathrm{t}a[2]+t(1+t)a[4]\right). \tag{5.1.5}$$

Integrating equation (5.1.5) concerning x from 0 to x, we get,

$$\frac{\partial\xi(x,t)}{\partial x} = \frac{2}{1+t} + \frac{1}{2}\Bigg(- ta[1] - t(1+t)a[3]\Bigg) + x\left(\text{ta}[1] + t(1+t)a[3]\right) \\ - \frac{5}{6}\Bigg(\text{ta}[2] + t(1+t)a[4]\Bigg) + x(1+x)\left(\text{ta}[2] + t(1+t)a[4]\right). \tag{5.1.6}$$

Integrating equation (5.1.6) concerning x from 0 to x, we get,

$$\xi(x,t) = 2x + \frac{1}{2}x^2\left(\text{ta}[1] + t(1+t)a[3]\right) + \frac{1}{6}x^2(3+2x)\left(\text{ta}[2] + t(1+t)a[4]\right) \\ \times x\left(-2 + \frac{2}{1+t} + \frac{1}{2}\left(-ta[1] - t(1+t)a[3]\right) - \frac{5}{6}\left(\text{ta}[2] + t(1+t)a[4]\right)\right). \tag{5.1.7}$$

Differentiating equation (5.1.7) partially, concerning t, we get,

$$\frac{\partial\xi(x,t)}{\partial t} \\ = \frac{1}{2}x^2\left(a[1] + ta[3] + (1+t)a[3]\right) + \frac{1}{6}x^2(3+2x)\left(a[2] + ta[4] + (1+t)a[4]\right) \\ + x\left(-\frac{2}{(1+t)^2} + \frac{1}{2}\left(-a[1] - ta[3] - (1+t)a[3]\right) \right. \\ \left. - \frac{5}{6}\left(a[2] + ta[4] + (1+t)a[4]\right)\right). \tag{5.1.8}$$

Differentiating equation (5.1.8) partially, concerning x, we get,

$$\frac{\partial^2\xi(x,t)}{\partial x\,\partial t} \\ = -\frac{2}{(1+t)^2} + \frac{1}{2}\left(-a[1] - ta[3] - (1+t)a[3]\right) + x\left(a[1] + ta[3] + (1+t)a[3]\right) \\ - \frac{5}{6}\left(a[2] + ta[4] + (1+t)a[4]\right) + \frac{1}{3}x^2\left(a[2] + ta[4] + (1+t)a[4]\right) \\ + \frac{1}{3}x(3+2x)\left(a[2] + ta[4] + (1+t)a[4]\right). \tag{5.1.9}$$

Substituting equations (5.1.9),(5.1.7),(5.1.6), and (5.1.5) in equation (5.1.1), we get,

$$\text{HSE} = -\frac{2}{(1+t)^2} - \frac{1}{2}\left(a[1] + a[3] + 2ta[3]\right) + x\left(a[1] + a[3] + 2ta[3]\right) \\ - \frac{5}{6}\left(a[2] + a[4] + 2ta[4]\right) + \frac{1}{3}x^2\left(a[2] + a[4] + 2ta[4]\right) \\ + \frac{1}{3}x(3+2x)\left(a[2] + a[4] + 2ta[4]\right) + \frac{1}{3}x(3+2x)\left(a[2] + a[4] + 2ta[4]\right) \\ + \frac{1}{6}tx\left(a[1] + (1+t)a[3] + (1+2x)\left(a[2] + (1+t)a[4]\right)\right)$$

$$\times\left(\frac{12}{1+t}+t(-1+x)\left(3a[1]+(5+2x)a[2]+(1+t)\left(3a[3]+(5+2x)a[4]\right)\right)\right)$$
$$+\frac{1}{2}\left(\frac{2}{1+t}+\frac{1}{6}t\left((-3+6x)a[1]+(-5+6x(1+x))\,a[2]\right.\right.$$
$$\left.\left.+(1+t)\left((-3+6x)a[3]+(-5+6x(1+x))\,a[4]\right)\right)\right)^2. \tag{5.1.10}$$

Discretizing the equation (5.1.10) with the defined collocation points, we get the following system of equations:

$$\frac{a[1]^2}{4096}+\frac{8869a[2]^2}{4718592}+\frac{81a[3]^2}{262144}+a[1]\left(-\frac{31}{72}+\frac{49a[2]}{32768}+\frac{9a[3]}{16384}+\frac{441a[4]}{262144}\right)$$
$$+a[3]\left(-\frac{17}{32}+\frac{3969a[4]}{2097152}\right)+a[2]\left(-\frac{1403}{1728}+\frac{441a[3]}{262144}+\frac{8869a[4]}{2097152}\right) \tag{5.1.11}$$
$$-a[4]+\frac{79821a[4]^2}{33554432}=0;$$

$$-\frac{63a[1]^2}{4096}-\frac{25259a[2]^2}{524288}-\frac{7623a[3]^2}{262144}+a[2]\left(-\frac{281}{2112}-\frac{19767a[3]}{262144}-\frac{277849a[4]}{2097152}\right)$$
$$+a[3]\left(-\frac{1}{32}-\frac{217437a[4]}{2097152}\right)+a[1]\left(\frac{1}{88}-\frac{1797a[2]}{32768}-\frac{693a[3]}{16384}-\frac{19767a[4]}{262144}\right)$$
$$-\frac{29a[4]}{96}-\frac{3056339a[4]^2}{33554432}=0; \tag{5.1.12}$$

$$-\frac{175a[1]^2}{4096}-\frac{981875a[2]^2}{4718592}-\frac{29575a[3]^2}{262144}$$
$$+a[2]\left(\frac{3505}{2496}-\frac{242125a[3]}{786432}-\frac{12764375a[4]}{18874368}\right)$$
$$+a[3]\left(\frac{39}{32}-\frac{3147625a[4]}{6291456}\right)+a[1]\left(\frac{73}{104}-\frac{18625a[2]}{98304}-\frac{2275a[3]}{16384}-\frac{242125a[4]}{786432}\right)$$
$$+\frac{115a[4]}{48}-\frac{165936875a[4]^2}{301989888}=0; \tag{5.1.13}$$

$$\frac{49a[1]^2}{4096}-\frac{45227a[2]^2}{4718592}+\frac{11025a[3]^2}{262144}+a[2]\left(\frac{10963}{2880}+\frac{11025a[3]}{262144}-\frac{226135a[4]}{6291456}\right)$$
$$+a[1]\left(\frac{37}{24}+\frac{735a[2]}{32768}+\frac{735a[3]}{16384}+\frac{11025a[4]}{262144}\right)+a[3]\left(\frac{103}{32}+\frac{165375a[4]}{2097152}\right)$$
$$+\frac{251a[4]}{32}-\frac{1130675a[4]^2}{33554432}=0. \tag{5.1.14}$$

Solving the system of above equations (5.1.11 − 5.1.14) by the Newton's method, we obtained the roots as, $a[1]=0; a[2]=0; a[3]=0; a[4]=0$. Substituting these unknown coefficients of clique polynomials of the Cocktail party graph in equation (5.1.7), we obtained the clique polynomial solution for Hunter Saxton equation with the given initial and boundary conditions in Example 5.1 as, $\xi(x,t) = \frac{2x}{1+t}$, which is the same as the exact solution. Here, the solution of the HSE is derived by the CCCM with different values of N (size of the operational matrix of integration) along with collocation points. The broad-spectrum steps to obtain the solution of the HSE for all the projected cases are illustrated in section 4. Tables 1-3 reveal that absolute error by the proposed method is better than the other methods in the literature. Table 4 assures the efficiency of the proposed method for discrete t with fixed x. Table 5 compares the error norms of the projected method with other methods mentioned in the literature. Table 6 indicates the error analysis of the CCCM method for fixed x. The graphical interpretation of the accurate solution and the CCCM solution is showcased in Figures 4-9. Figure 9 depicts the comparison of absolute errors of CCCM with the recent numerical methods in [17, 19, 24].

Table 1. Numerical comparison of absolute errors (AE) at $t = 0.1$ and $N = 10$.

$x/128$	Accurate Solution	Solution by the CCCM	AE by HWQA [3]	AE by BGFCF collocation method [19]	AE by B-spline collocation method [9]	AE by the CCCM
1	0.014204	0.014204	4.61×10^{-6}	7.75×10^{-16}	5.08×10^{-8}	0
3	0.042613	0.042613	3.43×10^{-5}	2.93×10^{-15}	5.84×10^{-8}	0
5	0.071022	0.071022	5.40×10^{-5}	1.45×10^{-15}	7.16×10^{-8}	0
7	0.099431	0.099431	7.52×10^{-5}	1.10×10^{-14}	6.60×10^{-8}	0
9	0.127840	0.127840	9.68×10^{-5}	1.82×10^{-14}	6.33×10^{-8}	0
59	0.838068	0.838068	6.35×10^{-4}	9.20×10^{-14}	5.93×10^{-8}	0
61	0.866477	0.866477	6.56×10^{-4}	8.73×10^{-14}	5.86×10^{-8}	0
63	0.894886	0.894886	6.78×10^{-4}	7.89×10^{-14}	5.79×10^{-8}	0
65	0.923295	0.923295	6.99×10^{-4}	6.86×10^{-14}	5.17×10^{-8}	0
67	0.951704	0.951704	7.21×10^{-4}	5.86×10^{-14}	5.53×10^{-8}	0
69	0.980113	0.980113	7.42×10^{-4}	5.09×10^{-14}	5.54×10^{-8}	0
119	1.690340	1.690340	1.28×10^{-3}	1.26×10^{-13}	1.29×10^{-8}	0
121	1.718750	1.718750	1.30×10^{-3}	1.20×10^{-13}	1.02×10^{-8}	0
123	1.747159	1.747159	1.32×10^{-3}	1.31×10^{-13}	7.40×10^{-9}	0
125	1.775568	1.775568	1.34×10^{-3}	1.52×10^{-13}	4.51×10^{-9}	0
127	1.803977	1.803977	1.37×10^{-3}	1.51×10^{-13}	1.52×10^{-9}	0

Example 5.2. Consider the HS equation [17],

$$\frac{\partial^2 \xi(x,t)}{\partial x\ \partial t} + \xi(x,t)\frac{\partial^2 \xi(x,t)}{\partial x^2} + \frac{1}{2}\left(\frac{\partial \xi(x,t)}{\partial x}\right)^2 = 0, \tag{5.2.1}$$

Table 2. Numerical comparison of absolute errors (AE) at $t = 0.01$ and $N = 10$.

$x/128$	Accurate Solution	Solution by the CCCM	AE by HWQA [3]	AE by BGFCF collocation method [19]	AE by B-spline collocation method [9]	AE by the CCCM
1	0.015470	0.015470	7.36×10^{-9}	6.76×10^{-17}	7.37×10^{-10}	0
3	0.046410	0.046410	5.21×10^{-8}	3.93×10^{-6}	5.34×10^{-9}	0
5	0.077351	0.077351	6.91×10^{-8}	3.19×10^{-16}	9.68×10^{-9}	0
7	0.108292	0.108292	1.10×10^{-7}	8.30×10^{-17}	1.03×10^{-8}	0
9	0.139232	0.139232	1.32×10^{-7}	3.83×10^{-17}	9.01×10^{-9}	0
59	0.912747	0.912747	8.88×10^{-7}	2.50×10^{-15}	8.26×10^{-9}	0
61	0.943688	0.943688	9.16×10^{-7}	3.17×10^{-15}	8.17×10^{-9}	0
63	0.974628	0.974628	9.46×10^{-7}	3.74×10^{-15}	8.07×10^{-9}	0
65	1.005569	1.005569	9.76×10^{-7}	4.11×10^{-15}	7.96×10^{-9}	0
67	1.036509	1.036509	1.01×10^{-6}	4.21×10^{-15}	7.85×10^{-9}	0
69	1.067450	1.067450	1.04×10^{-6}	4.03×10^{-15}	7.72×10^{-9}	0
119	1.840965	1.840965	1.79×10^{-6}	4.71×10^{-15}	1.80×10^{-9}	0
121	1.871905	1.871905	1.82×10^{-6}	3.29×10^{-15}	1.42×10^{-9}	0
123	1.902846	1.902846	1.85×10^{-6}	3.56×10^{-15}	1.03×10^{-9}	0
125	1.933787	1.933787	1.88×10^{-6}	6.09×10^{-15}	6.29×10^{-10}	0
127	1.964727	1.964727	1.91×10^{-6}	6.49×10^{-15}	2.13×10^{-10}	0

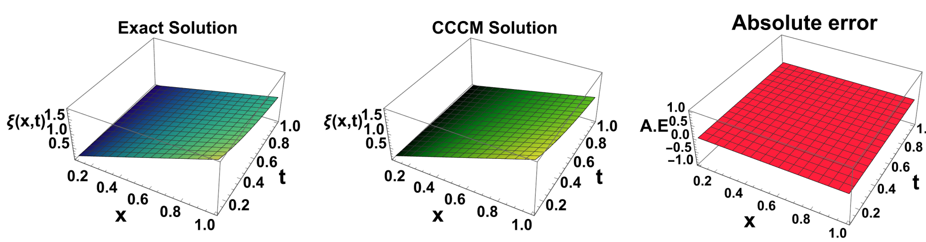

Figure 4. Graphical representation of the Exact, CCCM solution along with its absolute error at N = 4 (Example 5.1).

with physical conditions,

$$\xi(x,0) = (2+3x)^{\frac{2}{3}} + 2x + 2; \ \frac{\partial \xi(x,0)}{\partial x} = \frac{2(1+t) + 2^{\frac{2}{3}}(1+t)}{(1+t)^2} \ ; \ \xi(0,t) = \frac{2+2^{\frac{2}{3}}}{(1+t)^2} \tag{5.2.2}$$

$\xi(x,t) = \frac{2+2x(1+t)+(2+3x(1+t))^{\frac{2}{3}}}{(1+t)^2}$, is the exact solution of Example 5.2. The proposed method solution to this problem is illustrated in Tables 10 and 11 for discrete values of x and t concerning the fixed t and x respectively. Tables 7-9 display the absolute error analysis of recent numerical methods with the proposed CCCM

Table 3. Numerical comparison of absolute errors (AE) at $t = 0.001$ and $N = 10$.

$x/128$	Accurate Solution	Solution by the CCCM	AE by HWQA [3]	AE by BGFCF collocation method [19]	AE by B-spline collocation method [9]	AE by the CCCM
1	0.015609	0.015609	1.00×10^{-11}	7.19×10^{-19}	7.60×10^{-12}	0
3	0.046828	0.046828	5.00×10^{-11}	4.10×10^{-18}	6.70×10^{-11}	0
5	0.078046	0.078046	7.00×10^{-11}	3.38×10^{-18}	1.78×10^{-10}	0
7	0.109265	0.109265	1.00×10^{-10}	9.97×10^{-19}	3.27×10^{-10}	0
9	0.140484	0.140484	1.00×10^{-10}	6.05×10^{-19}	4.97×10^{-10}	0
59	0.920954	0.920954	1.10×10^{-9}	2.74×10^{-17}	8.46×10^{-10}	0
61	0.952172	0.952172	2.00×10^{-10}	3.42×10^{-17}	8.36×10^{-10}	0
63	0.983391	0.983391	4.00×10^{-10}	4.01×10^{-17}	8.32×10^{-10}	0
65	1.014610	1.014610	1.00×10^{-9}	4.38×10^{-17}	8.30×10^{-10}	0
67	1.045829	1.045829	1.00×10^{-9}	4.48×10^{-17}	8.29×10^{-10}	0
69	1.077047	1.077047	2.00×10^{-9}	4.28×10^{-17}	8.26×10^{-10}	0
119	1.857517	1.857517	4.00×10^{-9}	4.99×10^{-17}	1.83×10^{-10}	0
121	1.888736	1.888736	4.00×10^{-9}	3.55×10^{-17}	1.43×10^{-10}	0
123	1.919955	1.919955	4.00×10^{-9}	3.97×10^{-17}	1.02×10^{-10}	0
125	1.951173	1.951173	4.00×10^{-9}	6.72×10^{-17}	6.21×10^{-11}	0
127	1.982392	1.982392	4.00×10^{-9}	6.82×10^{-17}	2.08×10^{-11}	0

Table 4. Numerical comparison of absolute errors (AE) at $x = 0.1, 0.01$ and $0.001 (N = 10)$.

$t/128$	$x = 0.1$			$x = 0.01$			$x = 0.1$		
	Accurate Solution	Solution by the CCCM	AE by the CCCM	Accurate Solution	Solution by the CCCM	AE by the CCCM	Accurate Solution	Solution by the CCCM	AE by the CCCM
1	0.198449	0.198449	0	0.019844	0.019844	0	0.001984	0.001984	0
3	0.195419	0.195419	0	0.019541	0.019541	0	0.001954	0.001954	0
5	0.192481	0.192481	0	0.019248	0.019248	0	0.001924	0.001924	0
7	0.189629	0.189629	0	0.018962	0.018962	0	0.001896	0.001896	0
9	0.186861	0.186861	0	0.018686	0.018686	0	0.001868	0.001868	0
59	0.136898	0.136898	0	0.013689	0.013689	0	0.001368	0.001368	0
61	0.135449	0.135449	0	0.013544	0.013544	0	0.001354	0.001354	0
63	0.134031	0.134031	0	0.013403	0.013403	0	0.001340	0.001340	0
65	0.132642	0.132642	0	0.013264	0.013264	0	0.001326	0.001326	0
67	0.131282	0.131282	0	0.013128	0.013128	0	0.001312	0.001312	0
69	0.129949	0.129949	0	0.012994	0.012994	0	0.001299	0.001299	0
119	0.103643	0.103643	0	0.010364	0.010364	0	0.001036	0.001036	0
121	0.102811	0.102811	0	0.010281	0.010281	0	0.001028	0.001028	0
123	0.101992	0.101992	0	0.010199	0.010199	0	0.001019	0.001019	0
125	0.101185	0.101185	0	0.010118	0.010118	0	0.001011	0.001011	0
127	0.100392	0.100392	0	0.010039	0.010039	0	0.001003	0.001003	0

Table 5. Comparison of error norms at $t = 0.1, 0.01, 0.001$ for Example 5.1.

$t = 0.1$				
Error	**CCCM Method**	**HWQA method [6]**	**BGFCF collocation method [19]**	**Cubic trigonometric B-spline collocation method [9]**
$\mathbf{L_2}$	0	3.40×10^{-3}	3.56×10^{-13}	1.97×10^{-7}
RMS	0	8.52×10^{-4}	8.90×10^{-14}	4.93×10^{-8}
$\mathbf{L_\infty}$	0	1.37×10^{-3}	1.52×10^{-13}	7.16×10^{-8}
$t = 0.01$				
$\mathbf{L_2}$	0	4.76×10^{-6}	1.43×10^{-14}	2.65×10^{-8}
RMS	0	1.19×10^{-6}	3.59×10^{-15}	6.63×10^{-9}
$\mathbf{L_\infty}$	0	1.91×10^{-6}	6.49×10^{-15}	1.03×10^{-8}
$t = 0.001$				
$\mathbf{L_2}$	0	9.35×10^{-9}	1.54×10^{-16}	2.15×10^{-9}
RMS	0	2.33×10^{-9}	3.85×10^{-17}	5.38×10^{-10}
$\mathbf{L_\infty}$	0	4.00×10^{-9}	6.82×10^{-17}	8.46×10^{-10}

Table 6. Comparision of error norms at $x = 0.1, 0.01, 0.001$ for Example 5.1

Error	**CCCM Method**	**Error**	**CCCM Method**	**Error**	**CCCM Method**
$\mathbf{L_2}$	4.3884×10^{-17}	$\mathbf{L_2}$	5.2041×10^{-18}	$\mathbf{L_2}$	3.7557×10^{-19}
RMS	1.1331×10^{-17}	**RMS**	1.3437×10^{-18}	**RMS**	9.6974×10^{-20}
$\mathbf{L_\infty}$	2.7755×10^{-17}	$\mathbf{L_\infty}$	3.4694×10^{-18}	$\mathbf{L_\infty}$	2.1684×10^{-19}

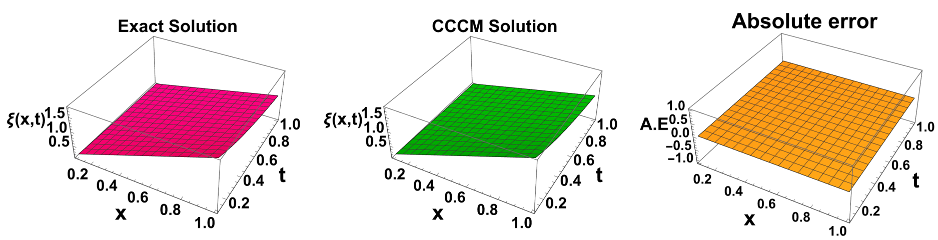

Figure 5. Graphical representation of the Exact, CCCM solution along with its absolute error at N = 10 (Example 5.1).

the obtained results were so close to the exact solution. Tables 10-11 provides the error analysis of the CCCM solution for different values of x and t. A graphical comparison of the accurate solution, CCCM solution, and the absolute error by the CCCM is shown in figures 10-12, and figures 13 and 14 indicate the error analysis.

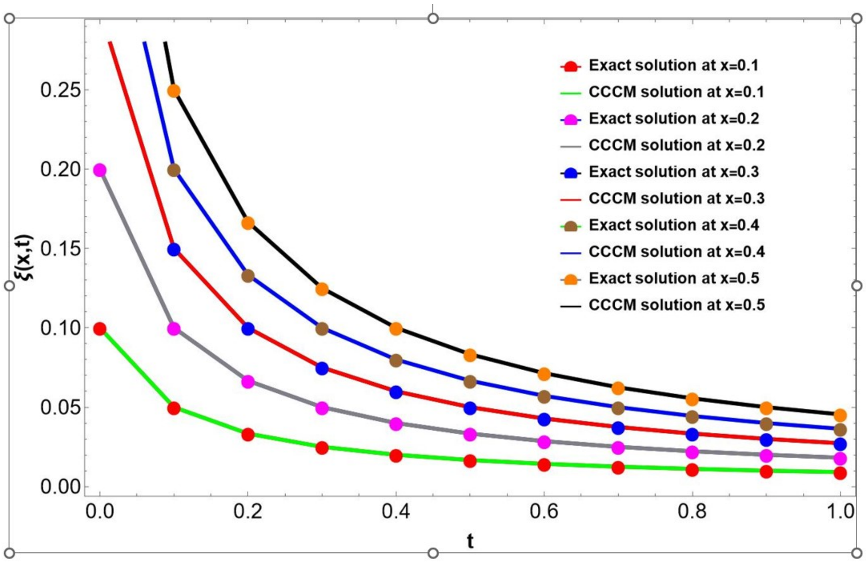

Figure 6. Graphical representation of the Exact, CCCM solution at fixed $x = 0.1, 0.2, 0.3, 0.4, 0.5$ and $t \in [0, 1], \Delta t = 0.1$ for Example 5.1($N = 4$).

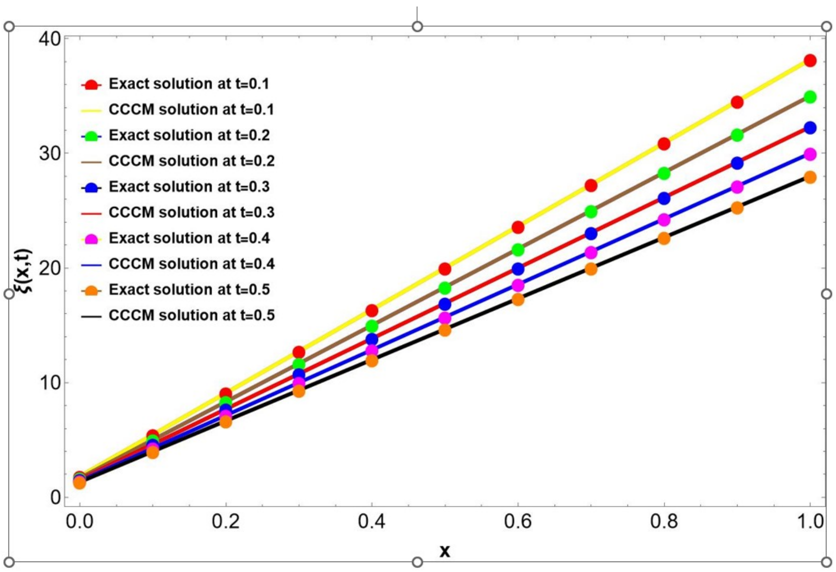

Figure 7. Graphical representation of the Exact CCCM solution at $N = 10, t = 0.1, 0.2, 0.3, 0.4, 0.5$ and $x \in [0, 1], \Delta x = 0.1$ for Example 5.1.

6. Conclusion

Here, we presented an efficient numerical method for the Hunter Saxton equation using the Clique polynomial of the cocktail party graph. As per the literature survey, the proposed method is a novel scheme for the Hunter-Saxton equation. Clique polynomials of Cocktail party graphs are effectively transformed into HSE through an operational matrix. The obtained numerical results are almost closer to

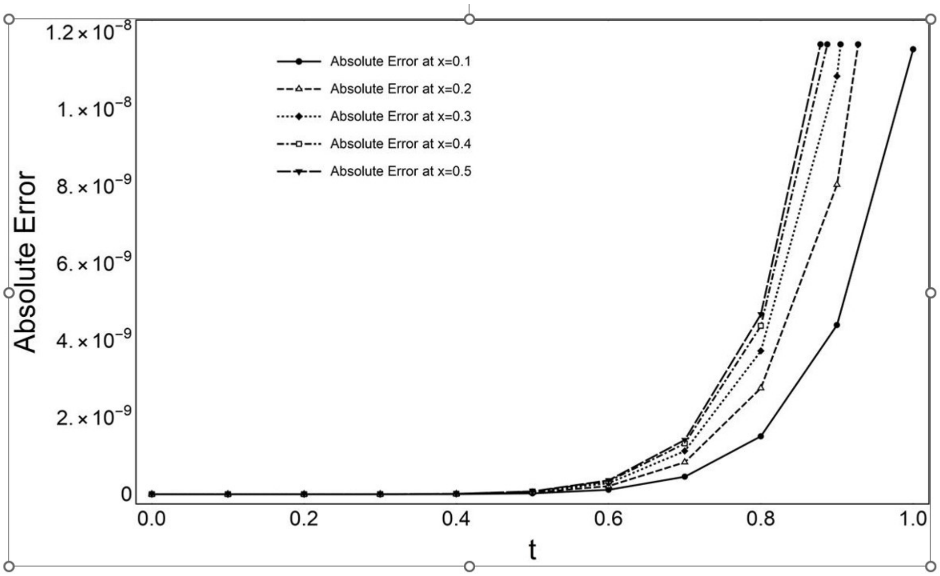

Figure 8. Graphical representation of Absolute errors of Example 5.1 at $N = 3$, $t \in [0, 1]$, $\Delta t = 0.1$,$x = 0.1, 0.2, 0.3, 0.4, 0.5$.

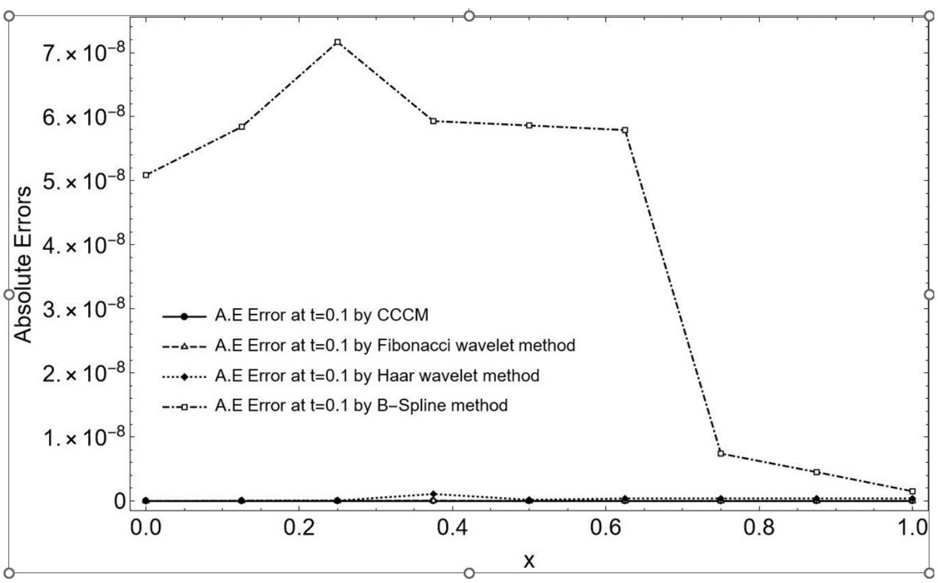

Figure 9. Graphical representation of comparison of absolute errors of various numerical methods for Example 5.1 at $x \in [0, 1], \Delta x = 0.1, t = 0.1, N = 3$.

the exact solution when compared with the HWQA Solution [3], BGFCF collocation method [9], B-Spline collocation method [19], Fibonacci wavelet method [24], Laguerre wavelet method [17]. Convergence analysis assured the efficiency of the CCCM method and produced in terms of theorems. Application of the method through numerical examples reflects good agreement with an exact solution in the

Table 7. Numerical comparison of absolute errors (AE) of the proposed CCCM with the FWM, LWM, and BSM of Example 5.2 for $\boldsymbol{t = 0.1, N = 4}$.

x	**Accurate Solution**	**Solution by the CCCM**	**AE by Fibbonocci wavelet method [24]**	**AE by Laguerre wavelet method [17]**	**AE by B-spline collocation method [19]**	**AE by the CCCM**
0.1	3.287218	3.287218	1.06×10^{-10}	1.80×10^{-10}	1.33×10^{-9}	1.07×10^{-14}
0.2	3.603134	3.603134	4.03×10^{-10}	1.51×10^{-10}	5.79×10^{-9}	1.37×10^{-14}
0.3	3.913602	3.913602	2.57×10^{-10}	3.92×10^{-9}	1.05×10^{-8}	6.20×10^{-15}
0.4	4.219412	4.219412	2.55×10^{-10}	1.47×10^{-8}	5.64×10^{-7}	6.10×10^{-14}
0.5	4.521173	4.521173	1.10×10^{-9}	1.67×10^{-8}	5.45×10^{-7}	1.54×10^{-13}
0.6	4.819368	4.819368	2.27×10^{-11}	1.90×10^{-8}	5.35×10^{-7}	2.77×10^{-13}
0.7	5.114386	5.114386	1.99×10^{-10}	3.53×10^{-8}	5.82×10^{-8}	3.97×10^{-13}
0.8	5.406549	5.406549	2.14×10^{-10}	8.83×10^{-8}	3.52×10^{-8}	4.58×10^{-13}
0.9	5.696123	5.696123	1.09×10^{-9}	2.43×10^{-8}	1.18×10^{-8}	3.69×10^{-13}

Table 8. Numerical comparison of absolute errors (AE) of the proposed CCCM with the FWM, LWM, and BSM of Example 5.2 for $\boldsymbol{t = 0.01, N = 4}$.

x	**Accurate Solution**	**Solution by the CCCM**	**AE by Fibbonocci wavelet method [24]**	**AE by Laguerre wavelet method [17]**	**AE by B-spline collocation method [19]**	**AE by the CCCM**
0.1	3.868180	3.868180	8.87×10^{-11}	2.58×10^{-10}	3.81×10^{-9}	4.54×10^{-16}
0.2	4.213040	4.213040	3.37×10^{-10}	3.98×10^{-10}	8.65×10^{-8}	3.15×10^{-15}
0.3	4.552303	4.552303	2.20×10^{-10}	2.69×10^{-9}	8.03×10^{-8}	9.26×10^{-15}
0.4	4.886736	4.886736	2.40×10^{-10}	7.10×10^{-9}	3.27×10^{-8}	1.95×10^{-14}
0.5	5.216939	5.216939	6.34×10^{-10}	8.98×10^{-9}	8.18×10^{-8}	3.36×10^{-14}
0.6	5.543393	5.543393	1.90×10^{-10}	2.05×10^{-9}	8.74×10^{-9}	4.97×10^{-14}
0.7	5.866489	5.866489	1.41×10^{-10}	2.46×10^{-9}	9.45×10^{-9}	6.35×10^{-14}
0.8	6.186554	6.186554	1.64×10^{-10}	1.56×10^{-9}	5.73×10^{-9}	6.77×10^{-14}
0.9	6.503863	6.503863	7.01×10^{-10}	8.65×10^{-9}	1.93×10^{-9}	5.16×10^{-14}

given domain. The absolute error analysis of example 1 of the proposed method solution with the exact solution is zero, confirming the CCCM's ability. Increasing the values of N (size of the matrix) leads to a better result, as seen in Tables 10-11.

Conflict of interest

The authors declare that they have no competing interests.

Table 9. Numerical comparison of absolute errors (AE) of the proposed CCCM with the FWM, LWM, and BSM of Example 5.2 for $t = 0.001, N = 4$.

x	Accurate Solution	Solution by the CCCM	AE by Fibbonocci wavelet method [24]	AE by Laguerre wavelet method [17]	AE by B-spline collocation method [19]	AE by the CCCM
0.1	3.934893	3.934893	8.77×10^{-11}	1.35×10^{-10}	7.29×10^{-9}	7.52×10^{-17}
0.2	4.282935	4.282935	3.33×10^{-10}	5.31×10^{-10}	6.15×10^{-9}	3.87×10^{-16}
0.3	4.625364	4.625364	2.18×10^{-10}	8.25×10^{-10}	7.05×10^{-9}	1.05×10^{-15}
0.4	4.962946	4.962946	2.39×10^{-10}	1.05×10^{-10}	8.99×10^{-9}	2.13×10^{-15}
0.5	5.296278	5.296278	6.00×10^{-10}	1.36×10^{-10}	8.87×10^{-9}	3.59×10^{-15}
0.6	5.625842	5.625842	1.90×10^{-10}	2.05×10^{-10}	8.74×10^{-9}	5.23×10^{-15}
0.7	5.952031	5.952031	1.37×10^{-10}	6.20×10^{-11}	9.97×10^{-10}	6.66×10^{-15}
0.8	6.275169	6.275169	1.61×10^{-10}	1.25×10^{-11}	6.06×10^{-10}	7.02×10^{-15}
0.9	6.595531	6.595531	6.73×10^{-10}	1.80×10^{-10}	2.04×10^{-10}	5.33×10^{-15}

Table 10. Numerical comparison of absolute errors (AE) at $t = 0.1, 0.01, 0.001$ and $N = 10$, for Example 5.2.

$x/128$	Solution by CCCM	AE by the CCCM	Solution by CCCM	AE by the CCCM	Solution by CCCM	AE by the CCCM
1	2.990248	5.39×10^{-15}	3.544440	1.01×10^{-17}	3.608211	4.17×10^{-18}
3	3.041016	1.34×10^{-14}	3.599747	2.50×10^{-16}	3.664018	4.06×10^{-17}
5	3.091598	1.79×10^{-14}	3.654870	8.77×10^{-16}	3.719639	1.13×10^{-16}
7	3.142003	1.89×10^{-14}	3.709813	1.86×10^{-15}	3.775081	2.22×10^{-16}
9	3.192235	1.65×10^{-14}	3.764582	3.19×10^{-15}	3.830348	3.67×10^{-16}
59	4.403746	9.18×10^{-13}	5.088424	1.20×10^{-13}	5.166543	1.23×10^{-14}
61	4.450784	9.76×10^{-13}	5.139900	1.27×10^{-13}	5.218508	1.30×10^{-14}
63	4.497732	1.03×10^{-12}	5.191282	1.33×10^{-13}	5.270378	1.36×10^{-14}
65	4.544592	1.09×10^{-12}	5.242572	1.39×10^{-13}	5.322156	1.42×10^{-14}
67	4.591367	1.15×10^{-12}	5.293772	1.46×10^{-13}	5.373842	1.49×10^{-14}
69	4.638056	1.20×10^{-12}	5.344883	1.52×10^{-13}	5.425440	1.55×10^{-14}
119	5.781627	9.03×10^{-13}	6.597568	1.02×10^{-13}	6.690140	1.03×10^{-14}
121	5.826547	7.40×10^{-13}	6.646799	8.33×10^{-14}	6.739846	8.42×10^{-15}
123	5.871412	5.56×10^{-13}	6.695971	6.25×10^{-14}	6.789493	6.32×10^{-15}
125	5.916223	3.51×10^{-13}	6.745085	3.94×10^{-14}	6.839081	3.98×10^{-15}
127	5.960979	1.23×10^{-13}	6.794140	1.37×10^{-14}	6.888610	1.39×10^{-15}

Author's contribution

Both authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Table 11. Numerical comparison of absolute errors (AE) at $\boldsymbol{t = 0.1, 0.01, 0.001}$ and $\boldsymbol{N = 6}$, for Example 5.2.

$x/128$	Solution by CCCM	AE by the CCCM	Solution by CCCM	AE by the CCCM	Solution by CCCM	AE by the CCCM
1	2.990248	7.69×10^{-15}	3.544440	2.60×10^{-17}	3.608211	4.29×10^{-18}
3	3.041016	1.97×10^{-15}	3.599747	2.21×10^{-16}	3.664018	4.25×10^{-17}
5	3.091598	2.75×10^{-14}	3.654870	8.53×10^{-16}	3.719639	6.14×10^{-15}
7	3.142003	3.12×10^{-14}	3.709813	1.85×10^{-15}	3.775081	2.31×10^{-17}
9	3.192235	3.10×10^{-14}	3.764582	3.20×10^{-15}	3.830348	3.78×10^{-16}
59	4.403746	7.18×10^{-13}	5.088424	9.70×10^{-14}	5.166543	9.94×10^{-15}
61	4.450784	7.54×10^{-13}	5.139900	1.00×10^{-13}	5.218508	1.03×10^{-14}
63	4.497732	7.89×10^{-13}	5.191282	1.04×10^{-13}	5.270378	1.06×10^{-14}
65	4.544592	8.23×10^{-13}	5.242572	1.07×10^{-13}	5.322156	1.10×10^{-14}
67	4.591367	8.54×10^{-13}	5.293772	1.11×10^{-13}	5.373842	1.13×10^{-14}
69	4.638056	8.84×10^{-13}	5.344883	1.14×10^{-13}	5.425440	1.16×10^{-14}
119	5.781627	3.76×10^{-13}	6.597568	4.55×10^{-14}	6.690140	4.62×10^{-15}
121	5.826547	2.98×10^{-13}	6.646799	3.60×10^{-14}	6.739846	3.66×10^{-15}
123	5.871412	2.15×10^{-13}	6.695971	2.61×10^{-14}	6.789493	2.66×10^{-15}
125	5.916223	1.31×10^{-13}	6.745085	1.59×10^{-14}	6.839081	1.62×10^{-15}
127	5.960979	4.41×10^{-14}	6.794140	5.38×10^{-15}	6.888610	5.47×10^{-16}

Table 12. Error Analysis of example 2 at $\boldsymbol{t = 0.1, 0.01, 0.001 (N = 4)}$.

Error	CCCM Method	Error	CCCM Method	Error	CCCM Method
$\mathbf{L_2}$	2.9421×10^{-12}	$\mathbf{L_2}$	3.6835×10^{-13}	$\mathbf{L_2}$	3.7549×10^{-14}
RMS	7.3552×10^{-13}	**RMS**	9.2087×10^{-14}	**RMS**	9.3873×10^{-15}
$\mathbf{L_\infty}$	1.2065×10^{-14}	$\mathbf{L_\infty}$	1.5203×10^{-13}	$\mathbf{L_\infty}$	1.5503×10^{-14}

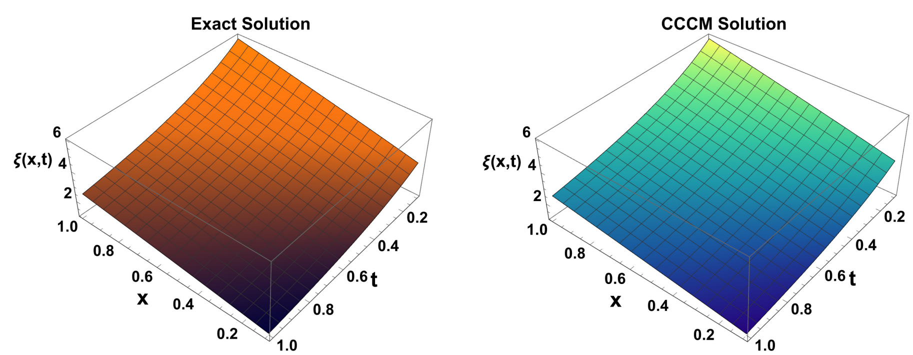

Figure 10. Graphical representation of the Exact, CCCM solution obtained at $N = 4, t = 0.1, x \in [0, 1], \Delta x = 0.1$ for Example 5.2.

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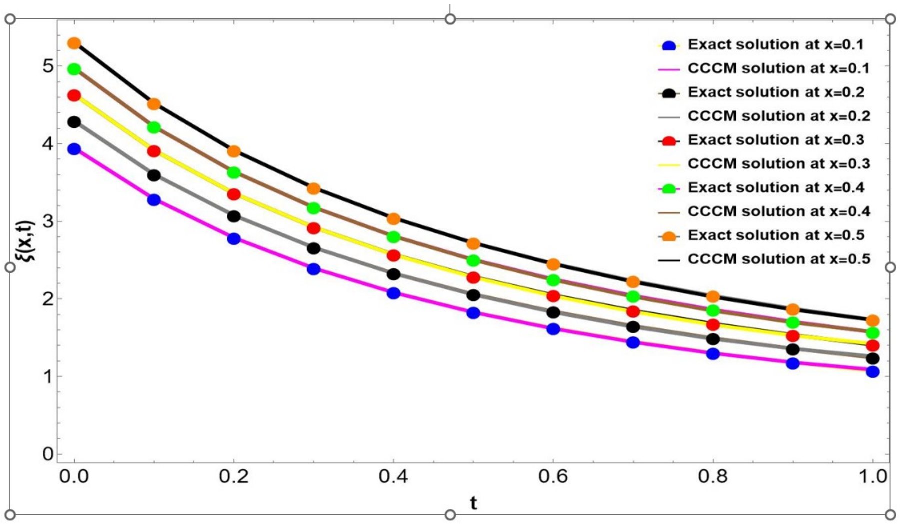

Figure 11. Graphical representation of the Exact, and CCCM solution at $x = 0.1, 0.2, 0.3, 0.4, 0.5.(N = 4)$ for Example 5.2.

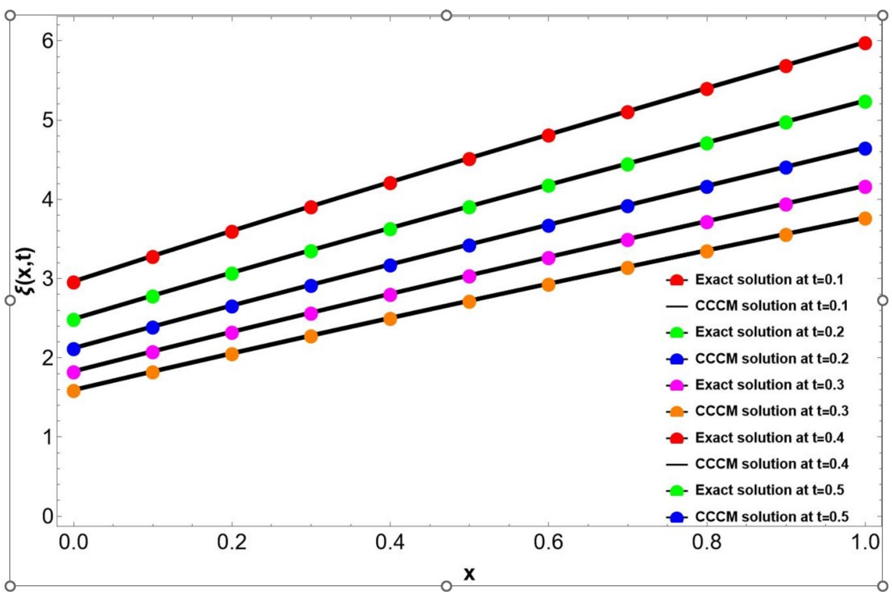

Figure 12. Graphical representation of the Exact, CCCM solution at $t = 0.1, 0.2, 0.3, 0.4, 0.5.$ for Example 5.2 ($N = 4$).

Data Availibility

The data supporting this study's findings are available in the article.

Disclosure statement

No potential conflict of interest was supported by the author(s).

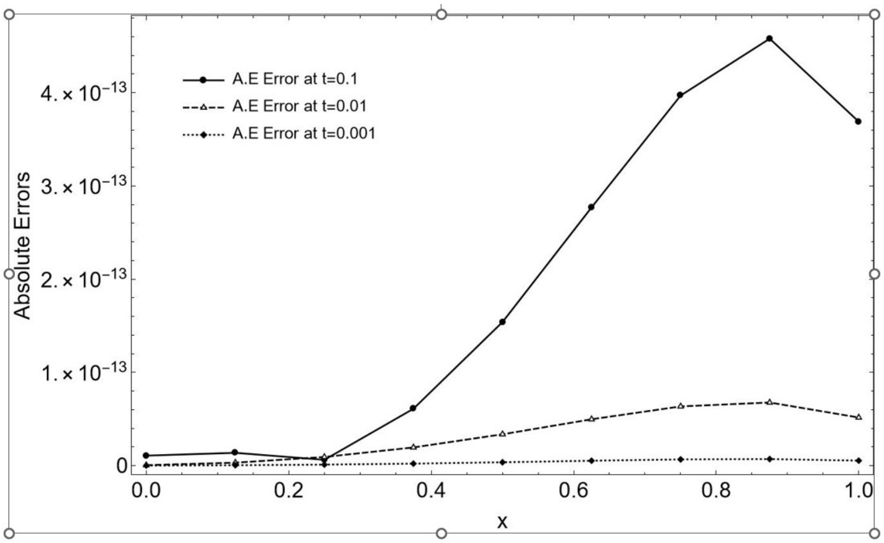

Figure 13. Graphical representation of absolute errors for Example 5.2 at $x \in [0,1], t = 0.1, 0.01, 0.001, N = 4$.

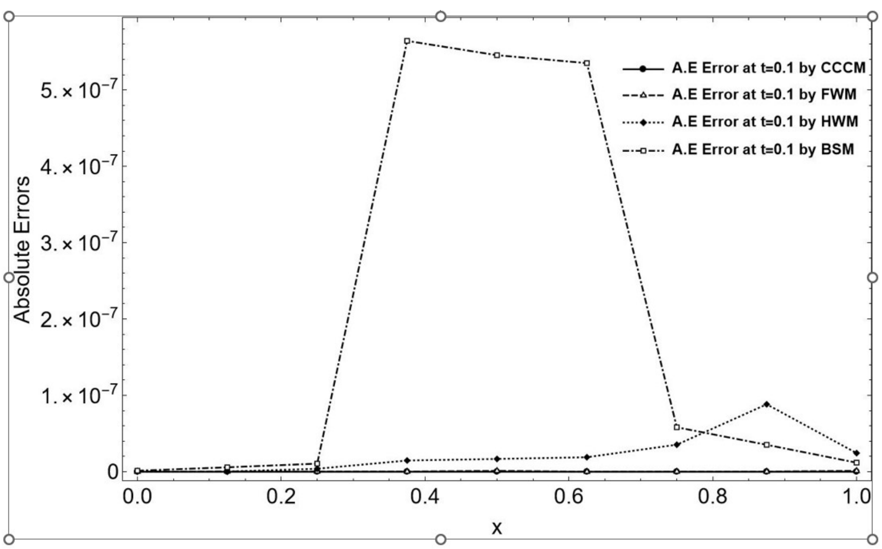

Figure 14. Graphical representation of comparison of absolute errors of CCCM, Fibonacci wavelet method(FWM), Haar wavelet method(HWM)), B-spline method(BSM) at $t = 0.1, N = 2, x \in [0,1]$ for Example 5.2.

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