STABILITY OF TRAVELING WAVE FRONTS FOR NONLOCAL DIFFUSIVE SYSTEMS

Shengqiang Zhang¹, Zhixian Yu² and Yanling Meng^{1,†}

Abstract The paper is concerned with stability of traveling wave fronts for nonlocal diffusive systems. We adopt L^1 -weighted, L^1 - and L^2 -energy estimates for the perturbation systems, and show that all solutions of the Cauchy problem for the considered systems converge exponentially to traveling wave fronts provided that the initial perturbations around the traveling wave fronts belong to a suitable weighted Sobolev space.

Keywords Exponential stability, nonlocal dispersals, upper and lower solutions, traveling wave fronts, comparison principle, weighted energy.

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1. Introduction

Incorporating spatial variation into models is clearly central to understanding many biological and physical systems. Diffusion has been frequently used to model movement in spatially deterministic models [31,38]. Diffusion is a local process in which particles move infinitesimal distances in infinitesimal units of time. Thus, Lee et al. [15] argued that, for processes where the spatial scale for movement is large in comparison with its temporal scale, nonlocal models using integro-differential may allow for better estimation of parameters from data and provide more insight into the biological system. Usually, the classic nonlocal model can be described by the following single equation

$$\frac{\partial u(t,x)}{\partial t} = D\left[\int_{\mathbb{R}} J(x-y)u(t,y)dy - u(t,x)\right] + f(u(t,x)),\tag{1.1}$$

where the kernel J(x) is a probability density. The nonlocal model (1.1) with monostable nonlinearity has been widely investigated by authors (see [2,6,8–10,37]).

The purpose of this work is to investigate the exponential stability of the epidemic model with nonlocal dispersals

$$\begin{cases} u_t(x,t) = d_1[(J_1 * u)(x,t) - u(x,t)] + h(u(x,t), v(x,t-\tau_1)), \\ v_t(x,t) = d_2[(J_2 * v)(x,t) - v(x,t)] + g(u(x,t-\tau_2), v(x,t)), \end{cases} \quad x \in \mathbb{R}, \ t > 0,$$

$$(1.2)$$

[†]The corresponding author.

¹College of Sciences, Shanghai Institute of Technology, 100 Haiquan Road, Shanghai 201418, China

 $^{^2 \}mathrm{Department}$ of Mathematics, Shanghai Normal University, Shanghai 200234, China

Email: sqzhang@sit.edu.cn(S. Zhang), zxyu0902@163.com(Z. Yu),

ylmeng0321@163.com(Y. Meng)

where $\tau_1 > 0$ and $\tau_2 > 0$ represent the time delays, u(x,t) and v(x,t) represent the spatial concentration of the bacteria and the infective population at a point $x \in \mathbb{R}$ and time $t \ge 0$, respectively. $d_1 \ge 0$ and $d_2 \ge 0$ are diffusion coefficients. $(J_1*u)(x,t)$ $(J_2*v)(x,t)$ represent the total number of the bacteria and the infective population arriving at x from all possible locations y at time t, respectively.

If the diffusion kernel

$$J_i(x) = \delta(x) + \delta''(x)$$

with δ being the Dirac delta function (see [25]), $h(u, v) = -\alpha_1 u + h(v)$ and $g(u, v) = -\alpha_2 v + g(u)$, then (1.2) reduces to the traditional reaction diffusion systems

$$\begin{cases} u_t(x,t) = d_1 u_{xx}(x,t) - \alpha_1 u(x,t) + h(v(x,t)), \\ v_t(x,t) = d_2 v_{xx}(x,t) - \alpha_2 v(x,t) + g(u(x,t)), \end{cases} \quad x \in \mathbb{R}, \ t > 0.$$
(1.3)

Hsu and Yang [12] investigated the existence, uniqueness and asymptotic behavior of traveling waves for (1.3). See also [3, 5, 43, 49, 50] for some special cases. More recently, Hsu et al. [13] extended (1.3) to more general systems and obtained the existence and stability of traveling waves.

If $d_1 = d_2 = 0$, $h(u, v) = -\alpha_1 u + av$ and $g(u, v) = -\alpha_2 v + g(u)$ for some constant a > 0, (1.2) reduces to the classic ODE epidemic model

$$\begin{cases} u_t(x,t) = -\alpha_1 u(x,t) + av(x,t), \\ v_t(x,t) = -\alpha_2 v(x,t) + g(u(x,t)), \end{cases} \quad x \in \mathbb{R}, \ t > 0, \tag{1.4}$$

which was proposed in [4] to model the cholera epidemic spread.

From the view of mathematics, letting u = v, $d_1 = d_2$, $J_1 = J_2$ and $h(u, v) = g(u, v) = -\alpha u + h(u)$, (1.2) is equivalent to the following single equation with the nonlocal dispersal

$$u_t(x,t) = d[(J * u)(x,t) - u(x,t)] - \alpha u(x,t) + h(u(x,t)),$$
(1.5)

which can usually be used to describe the growth and spatial spread of single species population. Yu and Yuan [56] investigated the existence of traveling waves for (1.5). Especially, when $h(u) = pue^{-qu}$, Pan [32] showed the existence of traveling waves of (1.5). We refer to some references about the more general nonlocal monostable equation with delays or without delays, see [32–34, 40, 47, 48, 55, 57] and some references cited therein.

In this article, we are mainly concerned with the existence and exponential stability of traveling wave solutions for (1.2). More precisely, following the ideas from [12, 13], we can also construct a pair of suitable upper and lower solutions relying on careful local analysis near the stationary solutions. By using the theory in [33], the existence of traveling wave solutions connecting two equilibria is admitted. On the other hand, the stability on traveling waves is an important and interesting project. The stability problems of traveling waves for some specific reaction-diffusion have been widely studied, by using the spectral analysis method [36, 42], a squeezing technique via the upper and lower solutions comparison [7, 23, 39, 44] and the weighted–energy method [13, 14, 19–21, 24, 27, 35, 45, 49, 50, 52, 54] and many references cited therein. By using weighted–energy method, authors [21, 34] also investigated the stability of traveling wave fronts for single equation with nonlocal diffusion. Recently, there have been many studies on the stability of other types

of equations, see [1, 11, 16, 22, 46, 51, 53, 58, 59] and many references cited therein. However, the stability of traveling wave solutions for multi-component systems with nonlocal dispersals is less reported, see [54]. Motivated by the work of [13, 26, 29], we will use the weighted energy method to establish the L^1 -weighted, L^1 - and L^2 energy estimates for the perturbations between solutions of (1.2) and the traveling wave solutions, and show that all solutions of the Cauchy problem for the considered systems converge exponentially to traveling wave fronts provided that the initial perturbations around the traveling wave fronts belong to a suitable weighted Sobolev spaces.

The rest of our paper is organized as follows. In Section 2, we introduce some notations and main results. In Section 3, by using the the weighted energy method and the comparison principle, we study the asymptotic stability of traveling wave fronts of (1.2). In Section 4, we give an application.

2. Main results

A traveling wave solution of (1.2) is a pair of solutions with the form $u_1(x,t) = \phi_1(x+ct)$ and $u_2(x,t) = \phi_2(x+ct)$ for some functions $\phi_i(\cdot) \in C^2(\mathbb{R},\mathbb{R})$, i = 1, 2, where c > 0 is a constant corresponding to the wave speed and $\xi := x + ct$ is the moving coordinate. Substituting $(u_1(x,t), u_2(x,t)) = (\phi_1(\xi), \phi_2(\xi))$ into the system (1.2), we can derive the following wave profile equations

$$\begin{cases} c\phi_1'(\xi) = d_1 \left(\int_{\mathbb{R}} J_1(\xi - y)\phi_1(y)dy - \phi_1(\xi) \right) + h(\phi_1(\xi), \phi_2(\xi - c\tau_1)), \\ c\phi_2'(\xi) = d_2 \left(\int_{\mathbb{R}} J_2(\xi - y)\phi_2(y)dy - \phi_2(\xi) \right) + g(\phi_1(\xi - c\tau_2), \phi_2(\xi)). \end{cases}$$
(2.1)

Our goal is to prove the stability of monotone solutions of (2.1) satisfying the following conditions:

$$\lim_{\xi \to -\infty} (\phi_1(\xi), \phi_2(\xi)) = \mathbf{0} \text{ and } \lim_{\xi \to +\infty} (\phi_1(\xi), \phi_2(\xi)) = \mathbf{K}.$$
 (2.2)

For convenience, let us denote the coefficients of the linear parts of h(u, v) and g(u, v) at the equilibrium $\mathbf{0} = (0, 0)$ and $\mathbf{K} = (k_1, k_2)$, respectively, by

$$\begin{aligned} \alpha_1 &= \partial_u h(0,0), \ \alpha_2 &= \partial_v g(0,0), \ \beta_1 &= \partial_v h(0,0), \ \beta_2 &= \partial_u g(0,0), \\ \bar{\alpha}_1 &= \partial_u h(k_1,k_2), \ \bar{\alpha}_2 &= \partial_v g(k_1,k_2), \ \bar{\beta}_1 &= \partial_v h(k_1,k_2), \ \bar{\beta}_2 &= \partial_u g(k_1,k_2). \end{aligned}$$

Two vectors $(u_1, \dots, u_n) \leq (v_1, \dots, v_n)$ in \mathbb{R}^n means $u_i \leq v_i$ for $i = 1, 2, \dots, n$. An interval of \mathbb{R}^n is defined according to this order. For convenience, denote by ∂_i the first differential operator with respect to the *i*-th variables, and ∂_{ij} the second differential operator with respect to the *i*-th and *j*-th variables.

In order to state our main results, throughout this article, we assume the nonlinearities $h(\cdot)$ and $g(\cdot)$ satisfy the following assumptions.

- (J) $J_i \in C(\mathbb{R}), J_i(x) = J_i(-x) \ge 0, \int_{\mathbb{R}} J_i(y) dy = 1, i = 1, 2, \text{ and } \int_{\mathbb{R}} |x|^j J_i(x) e^{-\lambda x} dx < \infty \text{ for every } \lambda > 0, j = 0, 1, 2, i = 1, 2.$
- (H1) Assume $h_2 := \partial_2 h \ge 0$, $g_1 := \partial_1 g \ge 0$ on the interval $[(0,0), (k_1, k_2)]$.
- (H2) Assume $\alpha_i < 0, \ \bar{\alpha}_i < 0$,

$$\alpha_1 \alpha_2 < \beta_1 \beta_2$$
 and $\bar{\alpha}_1 \bar{\alpha}_2 > \bar{\beta}_1 \bar{\beta}_2$.

Since $h(\cdot)$, $g(\cdot)$ are C^2 , the assumption (H1) is equivalent to the following quasimonotonicity assumption:

there exist constants $m_i > 0, i = 1, 2$ such that the functions $\hat{h}(u, v) := m_1 u + h(u, v)$ and $\hat{g}(u, v) := m_2 v + g(u, v)$ satisfy

$$\hat{h}(\phi_1(\xi), \phi_2(\xi - c\tau_1)) \ge \hat{h}(\psi_1(\xi), \psi_2(\xi - c\tau_1)),$$
(2.3)

$$\hat{g}(\phi_1(\xi - c\tau_2), \phi_2(\xi)) \ge \hat{g}(\psi_1(\xi - c\tau_2), \psi_2(\xi)),$$
(2.4)

for any $\phi(\xi) := (\phi_1(\xi), \phi_2(\xi)), \ \psi(\xi) := (\psi_1(\xi), \psi_2(\xi)) \in C(\mathbb{R}, \mathbb{R}^2)$ satisfying $0 \le \psi_i(\xi) \le \phi_i(\xi) \le K_i$ for all $\xi \in \mathbb{R}$ and i = 1, 2.

The assumption (H2) can help us to investigate the characteristic roots of the linearized equations for the profile equations (2.1) at the equilibria $\mathbf{0}$ and \mathbf{K} , respectively.

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The c_* is actually the threshold speed such that the linearized equation of (2.1) at **0** has positive eigenvalues. Given a fixed c > 0, let $\lambda_1(c)$ be the smallest positive eigenvalue of the linearized equation of (2.1) at **0**, and $\lambda_2(c)$ be the largest negative eigenvalue of the linearized equation of (2.1) at **K**.

Now we recall the known result on the existence of traveling wave fronts (see [17, 30]).

Theorem 2.1 (Existence). Assume (J) and (H1)–(H2) hold. There exists a positive constant $c_* > 0$ such that (1.2) admits a positive traveling wave front ($\phi_1(x + ct), \phi_2(x + ct)$) with the wave speed $c \ge c_*$ and satisfying (2.2). For $0 < c < c_*$, the system (1.2) has no positive monotone traveling wave solution satisfying (1.2).

Next, we state the stability result of traveling wave fronts derived in Theorem 4.1. Before that, let us introduce the following notations.

- Let I be an interval, especially $I = \mathbb{R}$, then we denote $L^2(I)$ by the space of the square integrable functions on I.
- The space $H^k(I)$ $(k \ge 0)$ means the Sobolev space of the L^2 -functions f(x) defined on I whose derivatives $\frac{d^i}{dx^i}f(i=1,\cdots,k)$ also belong to $L^2(I)$.
- Let us write $L^2_{\omega}(I)$ and $W^{k,p}_{\omega}(I)$ by the weight L^2 -space and weight Sobolev space with positive weighted function $\omega(x) : \mathbb{R} \to \mathbb{R}$, respectively. For any $f \in L^2_{\omega}(I)$ or $W^{k,p}_{\omega}(I)$, its norm is given (resp.) by

$$\begin{split} \|f\|_{L^2_w(I)} &= \Big(\int_I w(x)|f(x)|^2 dx\Big)^{1/2} \text{ or } \|f\|_{W^{k,p}_\omega(I)} \\ &= \Big(\sum_{i=0}^k \int_I \omega(x)|\frac{d^i}{dx^i}f(x)|^p dx\Big)^{1/p}. \end{split}$$

Furthermore, we set $H^k_w(I) := W^{k,2}_\omega(I)$.

• Letting T > 0 and \mathcal{B} be a Banach space, we denote by $C^0([0,T];\mathcal{B})$ the space of the \mathcal{B} -valued continuous functions on [0,T] and $L^2([0,T];\mathcal{B})$ as the space of the \mathcal{B} -valued L^2 -function on [0,T]. The corresponding spaces of the \mathcal{B} -valued functions on $[0,\infty)$ are defined similarly. Define the weight function $\omega(\cdot)$ by

$$\omega(\xi) = \begin{cases} \omega_1(\xi), & \text{for } \xi \le \xi_0, \\ 1, & \text{for } \xi > \xi_0, \end{cases} \text{ with } \omega_1(\xi) := e^{-\gamma(\xi - \xi_0)}, \tag{2.5}$$

where γ and ξ_0 is large enough, which will be determined later.

In order to obtain the stability, we assume the nonlinearities $h(\cdot)$ and $g(\cdot)$ satisfy the following assumptions.

- (H3) $\partial_{ij}h \leq 0$ and $\partial_{ij}g \leq 0$ for i, j = 1, 2 on the interval $[(0,0), (k_1, k_2)]$.
- (H4) Assume $\bar{\alpha}_1 + \bar{\beta}_2 < 0, \ \bar{\alpha}_2 + \bar{\beta}_1 < 0,$

$$2\bar{\alpha}_1 + \bar{\beta}_1 + \bar{\beta}_2 < 0$$
 and $2\bar{\alpha}_2 + \bar{\beta}_1 + \bar{\beta}_2 < 0$.

Let $(\phi_1(\xi), \phi_2(\xi))$ be a traveling wave solution of (1.2) satisfying (2.2) with the wave speed $c > c_*$. Motivated by the work of [13,26,29], we will adopt the weighted energy method to establish the L^1 -weighted, L^1 - and L^2 -energy estimates (see Section 4) for the perturbations between solutions of (1.2) and $(\phi_1(\xi), \phi_2(\xi))$.

Moreover, we recall the following lemmas from [12, 17], which will play an important role in establishing the L^1 -weighted, L^1 - and L^2 -energy estimates.

Lemma 2.1. (1) If $c > c_*$, $\Delta(c, \lambda) = 0$ has two positive roots $\lambda_1(c) < \lambda_2(c)$ in $(0, \lambda_m^c)$. Moreover, $f_i(c, \lambda_1(c) + \varepsilon) < 0$ for i = 1, 2 and $\Delta(c, \lambda_1(c) + \varepsilon) > 0$ when $\varepsilon > 0$ is small enough, where

$$f_i(c,\lambda) = d_i \int_{\mathbb{R}} J_i(y) e^{-\lambda y} dy - c\lambda - d_i + \alpha_i, \ i = 1, 2.$$

(2) Let $A = (a_{ij})$ be a two by two matrix such that $a_{ii} < 0$, i = 1, 2 and $a_{ij} > 0$ for $i \neq j$. Then the system of the following equalities

$$\begin{cases} a_{11}x_1 + a_{12}x_2 < 0 \ (>0, resp.), \\ a_{21}x_1 + a_{22}x_2 < 0 \ (>0, resp.), \end{cases}$$
(2.6)

has a solution (x_1, x_2) with $x_i > 0$, i = 1, 2, if and only if det A > 0 (< 0, resp.).

Then, by the comparison principle and Hölder inequality, we can obtain the following stability result.

Theorem 2.2 (Asymptotic stability). Assume that (J) and (H1)–(H4) hold. Let $\tau := \max\{\tau_1, \tau_2\}$. For a given traveling wave front $(\phi_1(x + ct), \phi_2(x + ct))$ of (1.2) satisfying (2.2) with the wave speed $c > c_*$. If the Cauchy problem (1.2) with the initial data $(u_0(x, s), v_0(x, s))$ satisfying the following conditions

$$u_{0}(x,s) - \phi_{1}(x+cs), \ v_{0}(x,s) - \phi_{2}(x+cs) \in C(L^{1}_{\omega}(\mathbb{R}) \cap H^{1}(\mathbb{R})),$$

$$\mathbf{0} \le (u_{0}(x,s), v_{0}(x,s)) \le \mathbf{K} \ for \ (x,s) \in \mathbb{R} \times [-\tau, 0],$$
(2.7)

then the solution of (1.2) with initial data $(u_0(x,s), v_0(x,s))$ uniquely exists and satisfies

$$\sup_{x \in \mathbb{R}} |u(x,t) - \phi_1(x+ct)| \le Ce^{-\mu t}, \quad \sup_{x \in \mathbb{R}} |v(x,t) - \phi_2(x+ct)| \le Ce^{-\mu t}, \quad t \ge 0$$

for some positive constants μ and C.

3. Exponential stability of traveling wave fronts

This section is devoted to proving the exponential stability of noncritical traveling wave fronts of (1.2) and (2.2) with an exponential convergence rate. Throughout this section, it is assumed that (J) and (H1)–(H4) hold. We first give some auxiliary statements about the global solutions of the Cauchy problem (1.2) and the comparison principle. Via the standard energy method and continuity extension method (see, [27, 28]), we have the following result.

Proposition 3.1. Assume that (J) and (H1)–(H4) hold. If the initial data $(u_0(x,s), v_0(x,s))$ satisfies (2.7), then (1.2) admits a unique solution (u(x,t), v(x,t)) such that

$$u(\cdot,t) - \phi_1(\cdot + ct), \quad v(\cdot,t) - \phi_2(\cdot + ct) \in C(L^1_{\omega}(\mathbb{R}) \cap H^1(\mathbb{R})) \text{ for } t \in [0,\infty) \text{ and}$$

 $\mathbf{0} \leq (u(x,t), v(x,t)) \leq \mathbf{K} \text{ for } (x,t) \in \mathbb{R} \times [0,\infty).$

Similar to the proofs of Proposition 3 in [24], Lemma 3.2 in [41] and Lemma 3 in [18], we easily obtain the following comparison principle.

Proposition 3.2. Assume that (J) and (H1)–(H2) hold. Let $(u^-(x,t), v^-(x,t))$ and $(u^+(x,t), v^+(x,t))$ be the solutions of (1.2) with the initial data $(u^-_0(x,s), v^-_0(x,s))$ and $(u^+_0(x,s), v^+_0(x,s))$, respectively. If

$$(u_0^-(x,s), v_0^-(x,s)) \le (u_0^+(x,s), v_0^+(x,s)) \text{ for } (x,s) \in \mathbb{R} \times [-\tau, 0].$$

Then

$$(u^{-}(x,t),v^{-}(x,t)) \le (u^{+}(x,t),v^{+}(x,t))$$
 for $(x,t) \in \mathbb{R} \times \mathbb{R}_{+}$

Assume the initial data $(u_0(x,s), v_0(x,s))$ satisfies the assumptions of Theorem 2.2. Let $u_0^-(x,s) \triangleq \min\{u_0(x,s), \phi_1(x+cs)\}, v_0^-(x,s) \triangleq \min\{v_0(x,s), \phi_2(x+cs)\}, u_0^+(x,s) \triangleq \max\{u_0(x,s), \phi_1(x+cs)\}, v_0^+(x,s) \triangleq \max\{v_0(x,s), \phi_2(x+cs)\}$ and $(u^{\pm}(x,t), v^{\pm}(x,t))$ be the nonnegative solutions of system (1.2) with the initial data $(u_0^{\pm}(x,s), v_0^{\pm}(x,s))$. Then it follows from Proposition 3.2 (the comparison principle) that

$$0 \le u^{-}(x,t) \le u(x,t), \ \phi_{1}(x+ct) \le u^{+}(x,t) \le k_{1}, \\ 0 \le v^{-}(x,t) \le v(x,t), \ \phi_{2}(x+ct) \le v^{+}(x,t) \le k_{2}$$

for $(x,t) \in \mathbb{R} \times \mathbb{R}_+$.

Denote

$$U^{+}(\xi,t) \triangleq u^{+}(\xi - ct, t) - \phi_{1}(\xi), \ V^{+}(\xi, t) \triangleq v^{+}(\xi - ct, t) - \phi_{2}(\xi)$$

and

$$U^{-}(\xi,t) \triangleq \phi_{1}(\xi) - u^{-}(\xi - ct, t), \ V^{-}(\xi,t) \triangleq \phi_{2}(\xi) - v^{-}(\xi - ct, t),$$

where $\xi = x + ct$. Furthermore, since

$$(U_0^-(\xi,s), V_0^-(\xi,s)) \le (U_0(\xi,s), V_0(\xi,s)) \le (U_0^+(\xi,s), V_0^+(\xi,s)), \ (\xi,s) \in \mathbb{R} \times [-\tau, 0],$$

by the Comparison Theorem, we have

$$(U^{-}(x,t),V^{-}(x,t)) \le (U(x,t),V(x,t)) \le (U^{+}(x,t),V^{+}(x,t)), \ (x,t) \in \mathbb{R} \times [0,\infty).$$

Therefore, our goal is to show that there exist positive constants C and μ such that

$$\sup_{\xi \in \mathbb{R}} |U^{\pm}(\xi, t)|, \quad \sup_{\xi \in \mathbb{R}} |V^{\pm}(\xi, t)| \le Ce^{-\mu t}, \quad t \ge 0.$$

$$(3.1)$$

For convenience, we denote the column vectors

$$X(\xi,t) := (U(\xi,t), V(\xi - c\tau_1, t - \tau_1))^T, \quad Y(\xi,t) := (U(\xi - c\tau_2, t - \tau_2), V(\xi, t))^T,$$

$$\Phi(\xi) := (\phi_1(\xi), \phi_2(\xi - c\tau_1))^T, \qquad \Psi(\xi) := (\phi_1(\xi - c\tau_2), \phi_2(\xi))^T.$$

For the sake of convenience, let us simply denote $U^+(\xi, t)$, $V^+(\xi, t)$ by $U(\xi, t)$, $V(\xi, t)$. Hence, $U(\xi, t)$ and $V(\xi, t)$ satisfy

$$\begin{cases} \partial_t U(\xi,t) + c\partial_{\xi} U(\xi,t) - d_1[(J_1 * U)(\xi,t) - U(\xi,t)] \\ = h(U(\xi,t) + \phi_1(\xi), V(\xi - c\tau_1, t - \tau_1) + \phi_2(\xi - c\tau_1)) - h(\phi_1(\xi), \phi_2(\xi - c\tau_1)), \\ \partial_t V(\xi,t) + c\partial_{\xi} V(\xi,t) - d_2[(J_2 * V)(\xi,t) - V(\xi,t)] \\ = g(U(\xi - c\tau_2, t - \tau_2) + \phi_1(\xi - c\tau_2), V(\xi,t) + \phi_2(\xi)) - g(\phi_1(\xi - c\tau_2), \phi_2(\xi)), \end{cases}$$

$$(3.2)$$

with the initial data

$$\begin{cases} U_0(\xi, s) \triangleq u_0^+(\xi, s) - \phi_1(\xi + cs), \\ V_0(\xi, s) \triangleq u_0^+(\xi, s) - \phi_2(\xi + cs), \end{cases}$$

where $(\xi, s) \in \mathbb{R} \times [-\tau, 0]$. Obviously, $U_0(x, s), V_0(x, s) \in \mathcal{X}_{\omega}$ and Proposition 3.1 implies that the solution $U(\xi, t), V(\xi, t) \in \mathcal{N}_{\omega_1}$ for each $t \in [0, +\infty)$.

According to (H3), it is easy to see that (3.2) is equivalent to the following system:

$$\begin{cases} \partial_t U(\xi,t) + c \partial_{\xi} U(\xi,t) - d_1 [(J_1 * U)(\xi,t) - U(\xi,t)] - \nabla h(\Phi(\xi)) X(\xi,t) \\ = \frac{1}{2} X(\xi,t)^T A(\bar{\Phi}(\xi)) X(\xi,t) \le 0, \\ \partial_t V(\xi,t) + c \partial_{\xi} V(\xi,t) - d_2 [(J_2 * V)(\xi,t) - V(\xi,t)] - \nabla g(\Psi(\xi)) Y(\xi,t) \\ = \frac{1}{2} Y(\xi,t)^T B(\bar{\Psi}(\xi)) Y(\xi,t) \le 0, \end{cases}$$
(3.3)

where $\Phi(\xi) \leq \overline{\Phi}(\xi) \leq \Phi(\xi) + X(\xi, t)$ and $\Psi(\xi) \leq \overline{\Psi}(\xi) \leq \Psi(\xi) + Y(\xi, t)$. To obtain the estimations of (3.1), we first establish the $L^1_{w_1}$ -energy, L^1 -energy and L^2 -energy estimates for $(U(\xi, t), V(\xi, t))$ in the following subsections.

3.1. $L^1_{\omega_1}$ -energy and L^1 -energy estimates

Then we have the following results.

Lemma 3.1. Assume that (J) and (H1)–(H4) hold. For any $c > c_*$ and $\gamma = \lambda_1(c) + \varepsilon$ ($\varepsilon > 0$ small enough), there exist positive constants μ and C such that

$$e^{\mu t} (\|U(\cdot,t)\|_{L^{1}_{\omega_{1}}(\mathbb{R})} + \|V(\cdot,t)\|_{L^{1}_{\omega_{1}}(\mathbb{R})}) + \int_{0}^{t} e^{\mu s} (\|U(s)\|_{L^{1}_{\omega_{1}}(\mathbb{R})} + \|V(s)\|_{L^{1}_{\omega_{1}}(\mathbb{R})}) ds \le C$$

for each $t \ge 0$, where $\omega_1(\xi) = e^{-\gamma(\xi - \xi_0)}$.

Proof. Multiplying the equation (3.3) by $e^{\mu t}\omega_1(\xi)$ for some $\mu > 0$, respectively, and integrating it over $\mathbb{R} \times [0, t]$, we can obtain

$$\begin{split} 0 &\geq \int_{0}^{t} \int_{-\infty}^{\infty} e^{\mu s} \omega_{1} \partial_{s} U(\xi, s) + e^{\mu s} \omega_{1} \Big(c \partial_{\xi} U(\xi, s) - d_{1} (J_{1} * U)(\xi, s) + d_{1} U(\xi, s) \\ &- a_{1} U(\xi, s) - \beta_{1} V(\xi - c\tau_{1}, s - \tau_{1}) \Big) d\xi ds \\ &= \int_{0}^{t} \int_{-\infty}^{\infty} (e^{\mu s} \omega_{1} U(\xi, s))_{s} - \mu e^{\mu s} \omega_{1} U(\xi, s) + (c e^{\mu s} \omega_{1} U(\xi, s))_{\xi} + c \gamma e^{\mu s} \omega_{1} U(\xi, s) \\ &- \omega_{1} \Big(d_{1} (J_{1} * U)(\xi, s) - (d_{1} - \alpha_{1}) U(\xi, s) + \beta_{1} V(\xi - c\tau_{1}, s - \tau_{1}) \Big) d\xi ds \\ &= e^{\mu t} \| U(\cdot, t) \|_{L^{1}_{\omega_{1}}(\mathbb{R})} - \| U_{0}(0) \|_{L^{1}_{\omega_{1}}(\mathbb{R})} - \int_{0}^{t} \int_{-\infty}^{\infty} e^{\mu s} \omega_{1} \beta_{1} V(\xi - c\tau_{1}, s - \tau_{1}) d\xi ds \\ &+ \int_{0}^{t} \int_{-\infty}^{\infty} e^{\mu s} \omega_{1} U(\xi, s) \Big(-\mu + c\gamma - d_{1} \int_{\mathbb{R}} J_{1}(y) e^{-\gamma y} dy + (d_{1} - \alpha_{1}) \Big) d\xi ds \\ &\geq e^{\mu t} \| U(\cdot, t) \|_{L^{1}_{\omega_{1}}(\mathbb{R})} - \| U_{0}(0) \|_{L^{1}_{\omega_{1}}(\mathbb{R})} \\ &+ \int_{0}^{t} \int_{-\infty}^{\infty} e^{\mu s} \omega_{1} U(\xi, s) \Big(-\mu + c\gamma - d_{1} \int_{\mathbb{R}} J_{1}(y) e^{-\gamma y} dy + (d_{1} - \alpha_{1}) \Big) d\xi ds \\ &- \int_{0}^{t} \int_{-\infty}^{\infty} \beta_{1} e^{\mu (s + \tau_{1})} e^{-\gamma c \tau_{1}} \omega_{1} V(\xi, s) d\xi ds \\ &- \int_{-\tau_{1}}^{0} \int_{-\infty}^{\infty} \beta_{1} e^{\mu (s + \tau_{1})} e^{-\gamma c \tau_{1}} \omega_{1} V(\xi, s) d\xi ds. \end{split}$$

Hence, we have

$$e^{\mu t} \|U(\cdot,t)\|_{L^{1}_{\omega_{1}}(\mathbb{R})} + \int_{0}^{t} \int_{-\infty}^{\infty} e^{\mu s} \omega_{1} U(\xi,s) \Big(-\mu + c\gamma - d_{1} \int_{\mathbb{R}} J_{1}(y) e^{-\gamma y} dy + (d_{1} - \alpha_{1}) \Big) d\xi ds \\ - \int_{0}^{t} \int_{-\infty}^{\infty} \beta_{1} e^{\mu(s+\tau_{1})} e^{-\gamma c\tau_{1}} \omega_{1} V(\xi,s) d\xi ds \leq C_{1}$$
(3.4)

for some constant $C_1 > 0$. Similarly, it follows from the second equation of (3.3) that

$$\begin{split} 0 &\geq \int_{0}^{t} \int_{-\infty}^{\infty} e^{\mu s} \omega_{1} \partial_{s} V(\xi, s) + e^{\mu s} \omega_{1} \Big(c \partial_{\xi} V(\xi, s) - d_{2} (J_{2} * V)(\xi, s) + d_{2} V(\xi, s) \\ &- \alpha_{2} V(\xi, s) - \beta_{2} U(\xi - c \tau_{2}, s - \tau_{2}) \Big) d\xi ds \\ &= \int_{0}^{t} \int_{-\infty}^{\infty} (e^{\mu s} \omega_{1} V(\xi, s))_{s} - \mu e^{\mu s} \omega_{1} V(\xi, s) + (c e^{\mu s} \omega_{1} V(\xi, s))_{\xi} + c \gamma e^{\mu s} \omega_{1} V(\xi, s) \\ &- \omega_{1} \Big(d_{2} (J_{2} * V)(\xi, s) - (d_{2} - \alpha_{2}) V(\xi, s) + \beta_{2} U(\xi - c \tau_{2}, s - \tau_{2}) \Big) d\xi ds \\ &= e^{\mu t} \| V(\cdot, t) \|_{L^{1}_{\omega_{1}}(\mathbb{R})} - \| V_{0}(0) \|_{L^{1}_{\omega_{1}}(\mathbb{R})} - \int_{0}^{t} \int_{-\infty}^{\infty} e^{\mu s} \omega_{1} \beta_{1} U(\xi - c \tau_{2}, s - \tau_{2}) d\xi ds \\ &+ \int_{0}^{t} \int_{-\infty}^{\infty} e^{\mu s} \omega_{1} V(\xi, s) \Big(-\mu + c \gamma - d_{2} \int_{\mathbb{R}} J_{2}(y) e^{-\gamma y} dy + (d_{2} - \alpha_{2}) \Big) d\xi ds \end{split}$$

$$\geq e^{\mu t} \|V(\cdot,t)\|_{L^{1}_{\omega_{1}}(\mathbb{R})} - \|V_{0}(0)\|_{L^{1}_{\omega_{1}}(\mathbb{R})} + \int_{0}^{t} \int_{-\infty}^{\infty} e^{\mu s} \omega_{1} V(\xi,s) \Big(-\mu + c\gamma - d_{2} \int_{\mathbb{R}} J_{2}(y) e^{-\gamma y} dy + (d_{2} - \alpha_{2}) \Big) d\xi ds - \int_{0}^{t} \int_{-\infty}^{\infty} \beta_{2} e^{\mu(s+\tau_{2})} e^{-\gamma c\tau_{2}} \omega_{1} U(\xi,s) d\xi ds - \int_{-\tau_{2}}^{0} \int_{-\infty}^{\infty} \beta_{2} e^{\mu(s+\tau_{2})} e^{-\gamma c\tau_{2}} \omega_{1} U(\xi,s) d\xi ds.$$

Thus, it holds

$$e^{\mu t} \|V(\cdot,t)\|_{L^{1}_{\omega_{1}}(\mathbb{R})} + \int_{0}^{t} \int_{-\infty}^{\infty} e^{\mu s} \omega_{1} V(\xi,s) \Big(-\mu + c\gamma - d_{2} \int_{\mathbb{R}} J_{2}(y) e^{-\gamma y} dy + (d_{2} - \alpha_{2})\Big) d\xi ds \\ - \int_{0}^{t} \int_{-\infty}^{\infty} \beta_{2} e^{\mu(s+\tau_{2})} e^{-\gamma c\tau_{2}} \omega_{1} U(\xi,s) d\xi ds \leq C_{2}$$
(3.5)

for some constant $C_2 > 0$. Let $\gamma = \lambda_1 + \varepsilon$, where $\varepsilon > 0$ is small enough such that $f_i(\lambda_1 + \varepsilon) < 0$ for i = 1, 2. By Lemma 2.1, there are two positive constants p and q such that

$$pf_1(\gamma) + q\beta_2 e^{-\gamma c\tau_2} = p\Big(-c\gamma + d_1 \int_{\mathbb{R}} J_1(y) e^{-\gamma y} dy - (d_1 - \alpha_1)\Big) + q\beta_2 e^{-\gamma c\tau_2} < 0$$

and

$$p\beta_1 e^{-\gamma c\tau_1} + qf_2(\gamma) + = p\beta_1 e^{-\gamma c\tau_1} + q\Big(-c\gamma + d_2 \int_{\mathbb{R}} J_2(y) e^{-\gamma y} dy - (d_2 - \alpha_2)\Big) < 0.$$

Multiplying (3.4)–(3.5) by p and q, respectively, and adding them, we can obtain

$$e^{\mu t} \Big(p \| U(\cdot, t) \|_{L^{1}_{\omega_{1}}(\mathbb{R})} + q \| V(\cdot, t) \|_{L^{1}_{\omega_{1}}(\mathbb{R})} \Big) \\ - \Big(p \mu + p f_{1}(\gamma) + q \beta_{2} e^{-\gamma c \tau_{2}} \Big) \int_{0}^{t} \| U(\cdot, s) \|_{L^{1}_{\omega_{1}}(\mathbb{R})} ds \\ - \Big(q \mu + q f_{2}(\gamma) + p \beta_{1} e^{-\gamma c \tau_{1}} \Big) \int_{0}^{t} \| V(\cdot, s) \|_{L^{1}_{\omega_{1}}(\mathbb{R})} ds \\ \leq p C_{1} + q C_{2}, \tag{3.6}$$

where $f_i(\gamma) = -c\gamma + d_i \int_{\mathbb{R}} J_i(y) e^{-\gamma y} dy - (d_i - \alpha_i)$, i = 1, 2. By taking $\mu > 0$ small enough, it follows that

$$-\left(p\mu + pf_1(\gamma) + q\beta_2 e^{-\gamma c\tau_2}\right) > 0 \text{ and } -\left(q\mu + qf_2(\gamma) + p\beta_1 e^{-\gamma c\tau_1}\right) > 0.$$

Then we establish the key energy estimate

$$\begin{split} \|U(\cdot,t)\|_{L^{1}_{\omega_{1}}(\mathbb{R})} + \|V(\cdot,t)\|_{L^{1}_{\omega_{1}}(\mathbb{R})} \\ + \int_{0}^{t} e^{\mu(s-t)} \Big(\|U(s)\|_{L^{1}_{\omega_{1}}(\mathbb{R})} + \|V(s)\|_{L^{1}_{\omega_{1}}(\mathbb{R})} \Big) ds \leq C e^{-\mu t}. \\ \text{tes the proof.} \end{split}$$

This completes the proof.

Using the $L^1_{\omega_1}$ -estimate of Lemma 3.1, we further have the following L^1 -estimate.

Lemma 3.2. Assume that (J) and (A1)–(A2) hold, in addition, (A3) holds. For any $c > c_*$, there exist positive constants μ , ξ_0 and C such that

$$e^{\mu t}(\|U(\cdot,t)\|_{L^1(\mathbb{R})} + \|V(\cdot,t)\|_{L^1(\mathbb{R})}) \le C \text{ for all } t \ge 0.$$

Proof. Multiplying the inequalities (3.3) by $e^{\mu t}$ and integrating it over $\mathbb{R} \times [0, t]$, we can obtain

$$\begin{split} 0 &\geq \int_{0}^{t} \int_{-\infty}^{\infty} e^{\mu s} \partial_{s} U(\xi, s) + c e^{\mu s} \partial_{\xi} U(\xi, s) + e^{\mu s} \Big(-d_{1}(J_{1} * U)(\xi, s) + d_{1}U(\xi, s) \Big) \\ &- e^{\mu s} \Big(h_{1}(\Phi(\xi))U(\xi, s) + h_{2}(\Phi(\xi))V(\xi - c\tau_{1}, s - \tau_{1}) \Big) d\xi ds \\ &= \int_{0}^{t} \int_{-\infty}^{\infty} (e^{\mu s} U(\xi, s))_{s} - \mu e^{\mu s} U(\xi, s) + (c e^{\mu s} U(\xi, s))_{\xi} + e^{\mu s} \Big(-d_{1}(J_{1} * U)(\xi, s) \\ &+ d_{1}U(\xi, s) - h_{1}(\Phi(\xi))U(\xi, s) - h_{2}(\Phi(\xi))V(\xi - c\tau_{1}, s - \tau_{1}) \Big) d\xi ds \\ &= e^{\mu t} \|U(\cdot, t)\|_{L^{1}(\mathbb{R})} - \|U(0)\|_{L^{1}(\mathbb{R})} + \int_{0}^{t} \int_{-\infty}^{\infty} e^{\mu s} \Big(-\mu - h_{1}(\Phi(\xi)) \Big) U(\xi, s) d\xi ds \\ &- \int_{0}^{t} \int_{-\infty}^{\infty} e^{\mu s} h_{2}(\Phi(\xi))V(\xi - c\tau_{1}, s - \tau_{1}) d\xi ds \\ &\geq e^{\mu t} \|U(\cdot, t)\|_{L^{1}(\mathbb{R})} - \|U(0)\|_{L^{1}(\mathbb{R})} + \int_{0}^{t} \int_{-\infty}^{\infty} e^{\mu s} \Big(-\mu - h_{1}(\Phi(\xi)) \Big) U(\xi, s) d\xi ds \\ &- \int_{0}^{t} \int_{-\infty}^{\infty} e^{\mu (s + \tau_{1})} h_{2}(\Phi(\xi + c\tau_{1}))V(\xi, s) d\xi ds \\ &- \int_{-\tau_{1}}^{0} \int_{-\infty}^{\infty} e^{\mu (s + \tau_{1})} h_{2}(\Phi(\xi + c\tau_{1}))V(\xi, s) d\xi ds \\ &= e^{\mu t} \|U(\cdot, t)\|_{L^{1}(\mathbb{R})} - \|U(0)\|_{L^{1}(\mathbb{R})} - \int_{-\tau_{1}}^{0} \int_{-\infty}^{\infty} e^{\mu (s + \tau_{1})} h_{2}(\Phi(\xi + c\tau_{1}))V(\xi, s) d\xi ds \\ &= e^{\mu t} \|U(\cdot, t)\|_{L^{1}(\mathbb{R})} - \|U(0)\|_{L^{1}(\mathbb{R})} - \int_{-\tau_{1}}^{0} \int_{-\infty}^{\infty} e^{\mu (s + \tau_{1})} h_{2}(\Phi(\xi + c\tau_{1}))V(\xi, s) d\xi ds \\ &= e^{\mu t} \|U(\cdot, t)\|_{L^{1}(\mathbb{R})} - \|U(0)\|_{L^{1}(\mathbb{R})} - \int_{-\tau_{1}}^{0} \int_{-\infty}^{\infty} e^{\mu (s + \tau_{1})} h_{2}(\Phi(\xi + c\tau_{1}))V(\xi, s) d\xi ds \\ &= e^{\mu t} \|U(\cdot, t)\|_{L^{1}(\mathbb{R})} - \|U(0)\|_{L^{1}(\mathbb{R})} - \int_{-\tau_{1}}^{0} \int_{-\infty}^{\infty} e^{\mu (s + \tau_{1})} h_{2}(\Phi(\xi + c\tau_{1}))V(\xi, s) d\xi ds \\ &= e^{\mu t} \|U(\cdot, t)\|_{L^{1}(\mathbb{R})} - \|U(0)\|_{L^{1}(\mathbb{R})} - \int_{-\tau_{1}}^{0} \int_{-\infty}^{\infty} e^{\mu (s + \tau_{1})} h_{2}(\Phi(\xi + c\tau_{1}))V(\xi, s) d\xi ds \\ &= e^{\mu t} \|U(\cdot, t)\|_{L^{1}(\mathbb{R})} - \|U(0)\|_{L^{1}(\mathbb{R})} + \int_{0}^{0} \int_{-\tau_{1}}^{\infty} e^{\mu (s + \tau_{1})} h_{2}(\Phi(\xi + c\tau_{1}))V(\xi, s) d\xi ds \\ &+ \int_{0}^{t} \Big(\int_{-\infty}^{\xi_{0}} + \int_{\xi_{0}}^{\infty} \Big) e^{\mu s} \Big(\mathcal{Q}_{1}(\xi)U(\xi, s) + \mathcal{Q}_{2}(\xi)V(\xi, s) \Big) d\xi ds, \end{aligned}$$

where $Q_1(\xi) := -\mu - h_1(\Phi(\xi))$ and $Q_2(\xi) := -e^{\mu\tau_1}h_2(\Phi(\xi + c\tau_1))$. Since $\omega_1(\xi) \ge 1$ for $\xi \le \xi_0$, by Lemma 3.1, we have

$$\left| \int_{0}^{t} \int_{-\infty}^{\xi_{0}} e^{\mu s} \Big(\mathcal{Q}_{1}(\xi) U(\xi, s) + \mathcal{Q}_{2}(\xi) V(\xi, s) \Big) d\xi ds \right| \\
\leq C_{4} \int_{0}^{t} e^{\mu s} \Big(\| U(\cdot, s) \|_{L^{1}_{\omega_{1}}(-\infty, \xi_{0}]} + \| V(\cdot, s) \|_{L^{1}_{\omega_{1}}(-\infty, \xi_{0}]} \Big) ds \\
\leq C_{5} \tag{3.8}$$

for some positive constants C_4 and C_5 . Then it follows from (3.7) and (3.8), we have

$$e^{\mu t} \| U(\cdot, t) \|_{L^1(\mathbb{R})} + \int_0^t \int_{\xi_0}^\infty e^{\mu s} \Big(\mathcal{Q}_1(\xi) U(\xi, s) + \mathcal{Q}_2(\xi) V(\xi, s) \Big) d\xi ds \le C_6$$
(3.9)

for some positive constant C_6 . Similarly, there exists a constant $C_7 > 0$ such that

$$e^{\mu t} \|V(\cdot,t)\|_{L^1(\mathbb{R})} + \int_0^t \int_{\xi_0}^\infty e^{\mu s} \Big(\mathcal{U}_1(\xi)U(\xi,s) + \mathcal{U}_2(\xi)V(\xi,s)\Big) d\xi ds \le C_7, \quad (3.10)$$

where $\mathcal{U}_1(\xi) := -e^{\mu\tau_2}g_1(\Psi(\xi + c\tau_2))$ and $\mathcal{U}_2(\xi) := -\mu - g_2(\Psi(\xi))$. Summing (3.9) and (3.10), there exists a constant C > 0 such that

$$e^{\mu t} (\|U(\cdot,t)\|_{L^{1}(\mathbb{R})} + \|V(\cdot,t)\|_{L^{1}(\mathbb{R})}) + \int_{0}^{t} \int_{\xi_{0}}^{+\infty} e^{\mu s} \Big(\Big[\mathcal{Q}_{1}(\xi) + \mathcal{U}_{1}(\xi) \Big] U(\xi,s) + \Big[\mathcal{Q}_{2}(\xi) + \mathcal{U}_{2}(\xi) \Big] V(\xi,s) \Big) d\xi ds \leq C.$$
(3.11)

Taking $\mu = 0$, according to the assumption (H4), we see that

$$\lim_{\xi \to +\infty} (\mathcal{Q}_1(\xi) + \mathcal{U}_1(\xi)) = -h_1(k_1, k_2) - g_1(k_1, k_2) = -\bar{\alpha}_1 - \bar{\beta}_2 > 0$$

and

$$\lim_{\xi \to +\infty} (\mathcal{Q}_2(\xi) + \mathcal{U}_2(\xi)) = -h_2(k_1, k_2) - g_2(k_1, k_2) = -\bar{\alpha}_2 - \bar{\beta}_1 > 0.$$

Then choosing $\xi_0 > 0$ large enough and $\mu > 0$ small enough, for $\xi > \xi_0$, we have

$$e^{\mu t} \Big(\| U(\cdot,t) \|_{L^1(\mathbb{R})} + \| V(\cdot,t) \|_{L^1(\mathbb{R})} \Big) \leq C \text{ for all } t \geq 0.$$

This completes the proof.

3.2. L^2 –energy estimate

Now we begin to establish the following L^2 -energy estimate.

Lemma 3.3. Assume that (J) and (H1)–(H4) hold. For any $c > c_*$, there exist positive constants ξ_0 and C such that for $t \ge 0$, we have

$$||U(\cdot,t)||_{L^{2}(\mathbb{R})}^{2} + ||V(\cdot,t)||_{L^{2}(\mathbb{R})}^{2} \le C.$$

Proof. Multiplying the inequalities (3.3) by $U(\xi, t)$ and $V(\xi, s)$, respectively, and integrating them over $\mathbb{R} \times [0, t]$, we can obtain

$$\begin{split} 0 &\geq \int_{0}^{t} \int_{-\infty}^{+\infty} \Big\{ (U^{2}(\xi,s))_{s} + c(U^{2}(\xi,s))_{\xi} - 2d_{1} \int_{\mathbb{R}} J_{1}(y)U(\xi - y,s)U(\xi,s)dy \\ &+ 2d_{1}U^{2}(\xi,s) - 2U(\xi,s) \Big(h_{1}(\Phi(\xi))U(\xi,s) + h_{2}(\Phi(\xi))V(\xi - c\tau_{1}, s - \tau_{1}) \Big) \Big\} d\xi ds \\ &\geq \int_{0}^{t} \int_{-\infty}^{+\infty} \Big\{ (U^{2}(\xi,s))_{s} + c(U^{2}(\xi,s))_{\xi} + \int_{\mathbb{R}} -d_{1}J_{1}(y)U^{2}(\xi - y,s)dy \\ &- d_{1} \int_{\mathbb{R}} J_{1}(y)U^{2}(\xi,s)dy + 2d_{1}U^{2}(\xi,s) - 2h_{1}(\Phi(\xi))U^{2}(\xi,s) \\ &- h_{2}(\Phi(\xi))U^{2}(\xi,s) - h_{2}(\Phi(\xi))V^{2}(\xi - c\tau_{1}, s - \tau_{1}) \Big\} d\xi ds \\ &\geq \|U(\cdot,t)\|_{L^{2}(\mathbb{R})}^{2} - \|U(0)\|_{L^{2}(\mathbb{R})}^{2} - \int_{-\tau_{1}}^{0} \int_{-\infty}^{+\infty} h_{2}(\Phi(\xi + c\tau_{1}))V^{2}(\xi,s)d\xi ds \\ &+ \int_{0}^{t} \int_{-\infty}^{+\infty} \Big((-2h_{1}(\Phi(\xi)) - h_{2}(\Phi(\xi)))U^{2}(\xi,s) \\ &+ (-h_{2}(\Phi(\xi + c\tau_{1})))V^{2}(\xi,s) \Big) d\xi ds \end{split}$$

$$= \|U(\cdot,t)\|_{L^{2}(\mathbb{R})}^{2} - \|U(0)\|_{L^{2}(\mathbb{R})}^{2} - \int_{-\tau_{1}}^{0} \int_{-\infty}^{+\infty} h_{2}(\Phi(\xi+c\tau_{1}))V^{2}(\xi,s)d\xi ds + \int_{0}^{t} \Big(\int_{-\infty}^{\xi_{0}} + \int_{\xi_{0}}^{+\infty}\Big) \Big(\mathcal{F}_{1}(\xi)U^{2}(\xi,s) + \mathcal{F}_{2}(\xi)V^{2}(\xi,s)\Big)d\xi ds,$$
(3.12)

where $\mathcal{F}_1(\xi) := -2h_1(\Phi(\xi)) - h_2(\Phi(\xi))$ and $\mathcal{F}_2(\xi) := -h_2(\Phi(\xi + c\tau_1))$. Since $\omega_1(\xi) \ge 1$ for $\xi \le \xi_0$ and $0 \le U(\xi, t) \le k_1$, Lemma 3.1 can guarantee that

$$\int_{-\infty}^{\xi_0} U^2(\xi, t) d\xi \le k_1 \int_{-\infty}^{\xi_0} \omega_1 U(\xi, t) d\xi \le k_1 \| U(\cdot, t) \|_{L^1_{\omega_1}(\mathbb{R})} \le C e^{-\mu t} \quad \text{for } t > 0.$$

Similarly, it yields

$$\int_{-\infty}^{\xi_0} V^2(\xi, t) d\xi \le C e^{-\mu t} \quad \text{for } t > 0.$$

Then,

$$| \int_{0}^{t} \int_{-\infty}^{\xi_{0}} \left(\mathcal{F}_{1}(\xi) U^{2}(\xi, s) + \mathcal{F}_{2}(\xi) V^{2}(\xi, s) \right) d\xi ds |$$

$$\leq C_{8} \int_{0}^{t} \int_{-\infty}^{\xi_{0}} \left(U^{2}(\xi, s) + V^{2}(\xi, s) \right) d\xi ds$$

$$\leq C_{9},$$

where C_8 and C_9 are positive constants. Thus, it holds

$$\|U(\cdot,t)\|_{L^{2}(\mathbb{R})}^{2} + \int_{0}^{t} \int_{\xi_{0}}^{+\infty} \left(\mathcal{F}_{1}(\xi)U^{2}(\xi,s) + \mathcal{F}_{2}(\xi)V^{2}(\xi,s)\right) d\xi ds \le C_{10}$$
(3.13)

for some positive constant C_{10} .

Similarly, there exists a constant $C_{11} > 0$ such that

$$\|V(\cdot,t)\|_{L^{2}(\mathbb{R})}^{2} + \int_{0}^{t} \int_{\xi_{0}}^{+\infty} \left(\mathcal{R}_{1}(\xi)U^{2}(\xi,s) + \mathcal{R}_{2}(\xi)V^{2}(\xi,s)\right) d\xi ds \leq C_{11}, \quad (3.14)$$

where $\mathcal{R}_1(\xi) := -g_1(\Psi(\xi + c\tau_2))$ and $\mathcal{R}_2(\xi) := -g_1(\Psi(\xi)) - 2g_2(\Psi(\xi))$. Summing (3.13) and (3.14), it follows

$$\begin{aligned} \|U(\cdot,t)\|_{L^{2}(\mathbb{R})}^{2} + \|V(\cdot,t)\|_{L^{2}(\mathbb{R})}^{2} \\ + \int_{0}^{t} \int_{\xi_{0}}^{+\infty} \Big\{ \Big(\mathcal{F}_{1}(\xi) + \mathcal{R}_{1}(\xi)\Big) U^{2}(\xi,s) + \Big(\mathcal{F}_{2}(\xi) + \mathcal{R}_{2}(\xi)\Big) V^{2}(\xi,s) \Big\} d\xi ds &\leq C. \end{aligned}$$

$$(3.15)$$

According to (H4), it yields

$$\lim_{\xi \to +\infty} \left(\mathcal{F}_1(\xi) + \mathcal{R}_1(\xi) \right) = -2h_1(k_1, k_2) - h_2(k_1, k_2) - g_1(k_1, k_2) = -2\bar{\alpha}_1 - \bar{\beta}_1 - \bar{\beta}_2 > 0$$

and

$$\lim_{\xi \to +\infty} \left(\mathcal{F}_2(\xi) + \mathcal{R}_2(\xi) \right) = -h_2(k_1, k_2) - g_1(k_1, k_2) - 2g_2(k_1, k_2) = -2\bar{\alpha}_2 - \bar{\beta}_1 - \bar{\beta}_2 > 0.$$

Thus for $\xi_0 > 0$ large enough and $\xi \ge \xi_0$, it holds

$$\mathcal{F}_1(\xi) + \mathcal{R}_1(\xi) > 0 \text{ and } \mathcal{F}_2(\xi) + \mathcal{R}_2(\xi) > 0.$$

By (3.15), it holds

$$||U(\cdot,t)||_{L^{2}(\mathbb{R})}^{2} + ||V(\cdot,t)||_{L^{2}(\mathbb{R})}^{2} \le C.$$

Then the assertion of this lemma follows. This completes the proof.

In order to derive a L^2 -energy estimate for $(U_{\xi}(\xi, t), V_{\xi}(\xi, t))$, we differentiate the system (3.2) with respect to ξ , we can obtain

$$\begin{cases} \partial_{t\xi} U(\xi,t) + c\partial_{\xi\xi} U(\xi,t) - d_1 \Big((J_1 * U_{\xi})(\xi,t) - U_{\xi}(\xi,t) \Big) \\ -\nabla h(X(\xi,t) + \Phi(\xi)) X_{\xi}(\xi,t) \\ = \Big(\nabla h(X(\xi,t) + \Phi(\xi)) - \nabla h(\Phi(\xi)) \Big) \Phi'(\xi) := \mathcal{H}_h(\xi,t) \le 0, \\ \partial_{t\xi} V(\xi,t) + c\partial_{\xi\xi} V(\xi,t) - d_2 \Big((J_2 * V_{\xi})(\xi,t) - V_{\xi}(\xi,t) \Big) \\ -\nabla g(Y(\xi,t) + \Psi(\xi)) Y_{\xi}(\xi,t) \\ = \Big(\nabla g(Y(\xi,t) + \Psi(\xi)) - \nabla g(\Psi(\xi)) \Big) \Psi'(\xi) := \mathcal{H}_g(\xi,t) \le 0. \end{cases}$$
(3.16)

Similar to the process of Lemmas 3.1–3.3, we can obtain the following result.

Lemma 3.4. Assume that (J) and (H1)–(H4) hold. For any $c > c_*$, there exist positive constants ξ_0 and C such that for $t \ge 0$, it holds

$$||U_{\xi}(\cdot,t)||_{L^{2}(\mathbb{R})}^{2} + ||V_{\xi}(\cdot,t)||_{L^{2}(\mathbb{R})}^{2} \le C.$$

3.3. Proof of Theorem 2.2

Lemma 3.5. Assume that (J) and (H1)–(H4) hold. For any $c > c_*$, it holds

$$\|U(\cdot,t)\|_{L^{\infty}(\mathbb{R})} \le M_1 e^{-\frac{1}{3}\mu_1 t},$$
$$\|V(\cdot,t)\|_{L^{\infty}(\mathbb{R})} \le M_2 e^{-\frac{1}{3}\mu_2 t}$$

for some positive constants μ_1 , μ_2 , M_1 , M_2 and t > 0.

Proof. It is easily checked that

$$\begin{aligned} \|U(\cdot,t)\|_{L^{2}(\mathbb{R})}^{2} &= \int_{\mathbb{R}} U^{2}(\cdot,t)d\xi \\ &\leq \sup_{\xi \in \mathbb{R}} |U(\xi,t)| \int_{\mathbb{R}} |U(\cdot,t)| d\xi \\ &= \|U(\cdot,t)\|_{L^{\infty}(\mathbb{R})} \cdot \|U(\cdot,t)\|_{L^{1}(\mathbb{R})} \end{aligned}$$
(3.17)

for any $t \geq 0$. Since $U(\cdot, t) \in H^2(\mathbb{R}) \hookrightarrow C^1(\mathbb{R})$, by Hölder inequality, we have

$$U^{2}(\cdot,t) = 2 \int_{-\infty}^{\xi} U_{\xi}(\cdot,t) U(\cdot,t) d\xi$$

$$\leq 2 \Big(\int_{-\infty}^{\xi} |U_{\xi}(\cdot,t)|^{2} d\xi \Big)^{\frac{1}{2}} \Big(\int_{-\infty}^{\xi} |U(\cdot,t)|^{2} d\xi \Big)^{\frac{1}{2}}$$

$$\leq 2 \| U(\cdot, t) \|_{L^2(-\infty,\xi)} \cdot \| U_{\xi}(\cdot, t) \|_{L^2(-\infty,\xi)}$$

for any $\xi \in \mathbb{R}$ and t > 0. Then it follows that

$$\|U(\cdot,t)\|_{L^{\infty}(\mathbb{R})}^{2} \leq 2\|U(\cdot,t)\|_{L^{2}(\mathbb{R})} \cdot \|U_{\xi}(\cdot,t)\|_{L^{2}(\mathbb{R})}, \ \forall t > 0.$$
(3.18)

Combining (3.17) and (3.18), we have

$$\|U(\cdot,t)\|_{L^{\infty}(\mathbb{R})} \le 2^{\frac{2}{3}} \|U(\cdot,t)\|_{L^{1}(\mathbb{R})}^{\frac{1}{3}} \cdot \|U_{\xi}(\cdot,t)\|_{L^{2}(\mathbb{R})}^{\frac{2}{3}}, \ \forall t > 0.$$

According to Lemmas 3.2 and 3.4, there exist positive constants μ_1 and M_1 such that

$$|U(\cdot,t)||_{L^{\infty}(\mathbb{R})} \le M_1 e^{-\frac{1}{3}\mu_1 t}, \ \forall t > 0.$$

Similarly, there exist $\mu_2 > 0$ and $M_2 > 0$ such that

$$||V(\cdot,t)||_{L^{\infty}(\mathbb{R})} \le M_2 e^{-\frac{1}{3}\mu_2 t}, \ \forall t > 0.$$

This completes the proof.

Proof of Theorem 2.2. By Lemma 3.5, it is easily see that

$$\sup_{x \in \mathbb{R}} |u_i^+(x,t) - \phi_i(x+ct)| \le C e^{-\mu t} \ (i=1,2), \ \forall \ t \ge 0$$

Similarly, we can verify that for any $c > c_*$, it holds

$$\sup_{x \in \mathbb{R}} |u_i^-(x,t) - \phi_i(x+ct)| \le Ce^{-\mu t} \ (i=1,2), \ \forall \ t \ge 0.$$

Since

$$0 \le u_i^-(x,t) \le u_i(x,t), \phi_i(x+ct) \le u_i^+(x,t) \le k_i \ (i=1,2),$$

the squeezing argument implies that

$$\sup_{x \in \mathbb{R}} |u_i(x,t) - \phi_i(x+ct)| \le Ce^{-\mu t} \ (i=1,2), \ \forall \ t \ge 0.$$

This completes the proof of Theorem 2.2.

4. An application

In this section, we give an application as follow. If $h(u, v) = -\alpha_1 u + h(v)$ and $g(u, v) = -\alpha_2 v + g(u)$, then (1.2) reduces to the traditional reaction diffusion systems for $x \in \mathbb{R}, t > 0$:

$$\begin{cases} u_t(x,t) = d_1[(J_1 * u)(x,t) - u(x,t)] - \alpha_1 u(x,t) + h(v(x,t-\tau_1)), \\ v_t(x,t) = d_2[(J_2 * v)(x,t) - v(x,t)] - \alpha_2 v(x,t) + g(u(x,t-\tau_2)), \end{cases}$$
(4.1)

where u(x,t) and v(x,t) represent the spatial concentration of the bacteria and the infective population at a point $x \in \mathbb{R}$ and time $t \ge 0$, respectively. $d_1 \ge 0$ and $d_2 \ge 0$ are diffusion coefficients. $(J_1 * u)(x,t)$ and $(J_2 * v)(x,t)$ represent the total number of the bacteria and the infective population arriving at x from all possible

locations y at time t, respectively. $-\alpha_1 u$ is the natural death rate of the bacterial population and the nonlinearity h(v) is the contribution of the infective humans to the growth rate of the bacterial. $-\alpha_2 v$ is the natural diminishing rate of the infective population due to the finite mean duration of the infectious population and g(u)is the infection rate of the human population under the assumption that the total susceptible human population is constant during the evolution of the epidemic.

Before applying the results of stability, we give the following assumptions.

- (J) $J_i \in C(\mathbb{R}), J_i(x) = J_i(-x) \geq 0, \int_{\mathbb{R}} J_i(y) dy = 1, i = 1, 2, \text{ and}$ $\int_{\mathbb{R}} |x|^j J_i(x) e^{-\lambda x} dx < \infty \text{ for every } \lambda > 0, j = 0, 1, 2, i = 1, 2.$ Furthermore, J_1 and J_2 are compactly supported.
- (A1) h(u) and g(u) are nondecreasing on $(0, +\infty)$ and h''(u) < 0 and g''(u) < 0 for all $u \in (0, +\infty)$.
- (A2) $h, g \in C^2(\mathbb{R}^+, \mathbb{R}^+), h(0) = g(0) = 0, k_2 = g(k_1)/\alpha_2, h(g(k_1)/\alpha_2) = \alpha_1 k_1$ and $h(g(u)/\alpha_2) > \alpha_1 u$ for $u \in (0, k_1)$, where k_1 is a positive constant.
- (A3) $\min\{\alpha_1, \alpha_2\} > \max\{\overline{\beta}_1, \overline{\beta}_2\}.$

The existence result can be found in [17] without delays, the existence of system 4.1 can be obtained similarly, here we just review the result of existence.

Theorem 4.1 (Existence). Assume (J) and (A1)–(A2) hold. There exists a positive constant $c_* > 0$ such that (4.1) admits a positive traveling wave front $(\phi_1(x + ct), \phi_2(x + ct))$ with the wave speed $c \ge c_*$ and satisfying

$$\lim_{\xi \to -\infty} (\phi_1(\xi), \phi_2(\xi)) = \mathbf{0} \ and \ \lim_{\xi \to +\infty} (\phi_1(\xi), \phi_2(\xi)) = \mathbf{K}.$$
 (4.2)

For $0 < c < c_*$, the system (4.1) has no positive monotone traveling wave solution satisfying (4.2).

In order to apply the stability result, we first give a description of the hypothetical conditions.

It is obvious that the assumption (A1) is equivalent to the assumptions (H1) and (H3). From (A2), it is easy to know that $\beta_1\beta_2 > \alpha_1\alpha_2$ holds. And assumption (A3) implies that $-\alpha_1 + \bar{\beta}_2 < 0$, $-\alpha_2 + \bar{\beta}_1 < 0$, $-2\alpha_1 + \bar{\beta}_1 + \bar{\beta}_2 < 0$ and $-2\alpha_2 + \bar{\beta}_1 + \bar{\beta}_2 < 0$ hold, in other words, the assumptions (A2) and (A3) are equivalent to the assumptions (H2) and (H4).

Similar to the process of Lemmas 3.1–3.3, we can also obtain the following results.

Lemma 4.1. Assume that (J) and (A1)–(A3) hold. For any $c > c_*$, there exist positive constants ξ_0 and C such that for $t \ge 0$, it holds

$$||U(\cdot,t)||^{2}_{L^{2}(\mathbb{R})} + ||V(\cdot,t)||^{2}_{L^{2}(\mathbb{R})} \le C$$

and

$$||U_{\xi}(\cdot,t)||^{2}_{L^{2}(\mathbb{R})} + ||V_{\xi}(\cdot,t)||^{2}_{L^{2}(\mathbb{R})} \le C$$

Then, by applying the techniques of weighted energy method, comparison principle and the squeezing argument, we can get

$$\sup_{x \in \mathbb{R}} |u(x,t) - \phi_1(x+ct)| \le Ce^{-\mu t} \quad \text{and}$$

$$\sup_{x \in \mathbb{R}} |v(x,t) - \phi_2(x+ct)| \le Ce^{-\mu t}, \quad \forall \ t \ge 0.$$

This means that all solutions of the Cauchy problem for the considered systems 4.1 converge exponentially to traveling wave solutions provided that the initial perturbations around the traveling wave fronts belong to a suitable weighted Sobolev space. This completes the explain of application.

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