Journal of Applied Analysis and Computation
Volume 14, Number 4, August 2024, 2082–2108

Website:http://www.jaac-online.com
DOI:10.11948/20230195

DUPIRE ITÔ'S FORMULA FOR THE EXPONENTIAL SYNCHRONIZATION OF STOCHASTIC SEMI-MARKOV JUMP SYSTEMS WITH MIXED DELAY UNDER IMPULSIVE CONTROL

Ning Zhang[1], Haodong Wang[1] and Wenxue Li[1,†]

Abstract This paper emphasizes the exponential synchronization for a class of stochastic semi-Markov jump systems with mixed delay via stochastic hybrid impulsive control. The impulsive sequence includes synchronous and asynchronous impulses with the impulsive gains being a sequence of stochastic variables. Inspired by the idea of average, a concept of "average stochastic impulsive gain" is used to qualify the impulse intensity. Our approach expands Dupire functional Itô's formula to the stochastic semi-Markov jump systems with mixed delay for the first time. Moreover, in view of the established Lyapunov functional, graph theory, and stochastic analysis theory, some exponential synchronization criteria for the systems are derived. The theoretical results are applied to a class of Chua's circuit systems with semi-Markov jump and mixed delay. Some synchronization criteria for the circuit systems are provided. The simulation results verify the effectiveness of the theoretical results.

Keywords Dupire functional Itô's formula, exponential synchronization, mixed delay, stochastic hybrid impulsive control, semi-Markov jump systems.

MSC(2010) 39B05, 93A10.

1. Introduction

Complex network systems (CNSs), being able to describe various actual networks such as neural networks [25], infectious disease spread networks [4], and circuit networks [18]. Numerous valuable approaches have been proposed to settle issues of CNSs [2, 15, 24, 27, 36, 38]. Moreover, dozens of uncertainties and stochastic disturbances resulting from unanticipated environmental noise always affect the evolution of CNSs, as a consequence, stochastic complex network systems (SCNSs) have been a fascinating study area worldwide [20,23,43,48,49]. Most notably, many SCNSs are unavoidably impacted by sudden stimulation like operational errors and fluctuation at random and perform abrupt changes in structure and parameter, which are generally characterized by Markovian jump systems. However, it is insufficient to explain

[†]The corresponding author.
[1]Department of Mathematics, Harbin Institute of Technology (Weihai), Weihai 264209, China
Email: ning13999245237@163.com(N. Zhang),
yezi521888@163.com(H. Wang), wenxue810823@163.com(W. Li).

the parameter jump phenomenon in actual systems since the states' transition rates in the corresponding systems are constant. As a result of this, the semi-Markov jump systems with time-varying state transition rates have evoked the enormous interest of scholars, which has been extensively explored in recent years [6, 29, 30]. Additionally, time delay is inescapable in SCNSs on account of wide-ranging existing communication disturbances, signal interference, network packet failure, and other issues. Hence, many scholars have in-depth investigated the stochastic semi-Markov jump systems (SSMJSs) with time delay [12, 17, 22, 33, 45]. For instance, in [12], the exponential synchronization criteria of SSMJSs with time-varying delay were introduced via adaptive aperiodically intermittent control. However, different from the single kind of time delay, the mixed delay can significantly enhance the systems' use of historical data which improves the adherence to the actual networks, and few pieces of literature discuss the SSMJSs with mixed delay. For theoretical and practical significance, it is meaningful and important to study SSMJSs with mixed delay.

Through the recent decades, synchronization acting as one of the most significant cooperative behaviors in both natural and synthetic networks has been popularly applied in diverse fields like power transmission [7], multi-vehicle collaboration [10] and communication security [32]. There have been many investigations on the synchronization of SCNSs. In [50], synchronization of hybrid switching diffusion delayed networks was investigated and in [40], bipartite synchronization of fractional-order multi-layer signed networks was investigated. In this paper, the theoretical significance and potential for practical applications of research on SSMJSs synchronization are really what drives our study.

For the sake of achieving synchronization of CNSs, some control strategies have been designed like intermittent control [3, 39], pinning control [13], event-triggered control [19], sampled-data control [31, 47] and impulsive control [41, 42]. Among them, the hybrid impulsive control containing synchronous and asynchronous impulses has been adequately and extensively utilized to investigate the synchronization of CNSs [14, 16, 34, 35, 46]. For example, in [35] Wang et al. proposed a new definition of "average impulsive gain" to estimate the intensity of hybrid impulses to discuss the synchronization of a kind of coupled neural networks. In the above-mentioned literature, the fixed impulsive intensity and density in the control scheme are taken into account. However, numerous actual systems are affected by random fluctuations, and the systems could be not clearly defined, thus stochastic hybrid impulsive controllers are designed to deal with the synchronization problem. Based on the above discussion, we naturally wonder whether stochastic hybrid impulsive control can be applied to resolve the exponential synchronization issue of SMJSs with mixed delay. In addition, how to deal with semi-Markov jump with time-varying state transition rate and stochastic hybrid impulsive control with the impulsive gain being a sequence of stochastic variables is a key issue that needs to be addressed.

On the other hand, as we all know, two famous methods have been established to deal with the stabilization or synchronization of delayed systems including the Razumikhin method and the Lyapunov functional method. However, it actually lacks the true sense of functional Itô's formula for the SCNSs with delay. In view of that, Dupire extended the Itô's formula to the case of stochastic functional differential equations [9]. Based on it, many results have been derived. Nguyen et al. studied almost sure stability, exponential stability of stochastic functional differential equations and gave some novel conditions for stability in terms of Lyapunov functionals by using Dupire's functional Itô's formula in [28]. And further, he established a new stability theory for stochastic functional differential systems with random switching in [8]. In our paper, we are aiming to investigate the exponential synchronization of SSMJSs with the mixed delay with the aid of Dupire functional Itô's formula, which has not been touched. We attempt to construct a proper Lyapunov-Krasovaskii functional to estimate the sign of operate $\mathbb{LV}$ by using the defined Dupire horizontal and vertical partial derivatives. Then through Dupire functional Itô's formula and Lyapunov theory, some synchronization criteria can be derived, avoiding using the Razumikhin method. Thus, how to construct a proper functional is another both appealing and challenging question to be addressed.

Motivated by the above analysis, in this paper, we concentrate on the exponential synchronization of SSMJSs with mixed delay via stochastic hybrid impulsive control. By applying Dupire functional Itô's formula, we shall derive the new exponential synchronization criteria of SSMJSs with mixed delay. Meanwhile, in order to confirm the applicability of the established outcomes, Chua's circuit systems are provided, and the numerical simulation results demonstrate the validity of derived theories. The chief contributions are presented below:

- Different from the single kind of time-varying delay, the mixed delay effectively utilizes the past information of SSMJSs which pre-eminently enhances the reliability of the results. Furthermore, the semi-Markov jump, which has a time-varying transfer rate as opposed to the Markov jump's constant transfer rate, is better capable of capturing the phenomenon of parameter jump in practical systems.
- Unlike the previous work [35], stochastic hybrid impulsive control includes impulsive gains being a sequence of random variables at different impulsive times and contains synchronous impulses and asynchronous impulses simultaneously, which is more adaptable in practical systems. Additionally, the notion of "average stochastic impulsive gain" is proposed to determine the magnitude of such stochastic hybrid impulsive intensity.
- Based on graph theory, a novel appropriate global Lyapunov functional is constructed via vertex Lyapunov functional. According to horizontal and vertical derivatives and with the help of Dupire functional Itô's formula, some sufficient conditions to achieve exponential synchronization of SSMJSs with mixed delay are given as the extension of [28].

The rest arrangements of this paper are organized as follows. In Section 2, some preliminaries and model description are displayed. Section 3 presents the main theoretical results containing some synchronization criteria. And a kind of Chua's circuit systems is demonstrated as the application of SSMJSs with mixed delay in Section 4. In Section 5, the numerical example is derived to illustrate our theoretical results.

Notations. Let $\mathbb{N} = \{1, 2, \cdots, N\}$, $\mathbb{S} = \{1, 2, \cdots, S\}$, $\mathbb{H} = \{1, 2, \cdots, H, \cdots\}$. And $\mathbb{R}_+$ denotes the set of non-negative real numbers, $\mathbb{R}^k$ is k-dimensional Euclidean space. For $a \in \mathbb{R}^k$, write $|\cdot|$ for the Euclidean norm of the vector. The superscript "T" stands for the transpose of a vector or a matrix. "tr" is the trace of a square matrix. For a fixed positive real number α, $C([-\alpha, 0]; \mathbb{R}^k)$ refers to the space consisting of continuous functions mapped from $[-\alpha, 0]$ to $\mathbb{R}^k$. For the continuous function $F(t)$, $\mathrm{D}^+F(t) = \lim_{\epsilon\to 0^+} \frac{F(t+\epsilon)-F(t)}{\epsilon}$ represents the right and upper Dini's derivative. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability space with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ satisfying usual conditions. $\mathbb{E}$ is the mathematical expectation about the probability $\mathbb{P}$.

2. Preliminaries and model description

2.1. Preliminaries

Some related knowledge about Dupire functional Itô's formula is presented as preliminaries in the following.

Consider the following stochastic functional differential equation with semi-Markov jump $\beta(t)$

$$\mathrm{d}x(t) = F(t, x_t, \beta(t))\mathrm{d}t + \Lambda(t, x(t), \beta(t))\mathrm{dB}(t),\ t \geq 0. \tag{2.1}$$

For $\chi \in C([-\alpha, 0]; \mathbb{R}^k)$, $x \geq 0$, $y \in \mathbb{R}^k$, its horizontal and vertical perturbations are defined as

$$\chi_x(\sigma) = \begin{cases} \chi(x+\sigma), & \sigma \in [-\alpha, -x], \\ \chi(0), & \sigma \in [-x, 0], \end{cases} \tag{2.2}$$

$$\chi^y(\sigma) = \begin{cases} \chi(\sigma), & \sigma \in [-\alpha, 0), \\ \chi(0) + y, & \sigma = 0. \end{cases} \tag{2.3}$$

Assume that $\mathbb{V} : C([-\alpha, 0]; \mathbb{R}^k) \times \mathbb{S} \mapsto \mathbb{R}_+$. The horizontal and vertical partial derivatives of $\mathbb{V}$ at (χ, s) are defined as

$$\mathbb{V}_t(\chi, s) = \lim_{x\to 0^+} \frac{\mathbb{V}(\chi_x, s) - \mathbb{V}(\chi, s)}{x},$$
$$\partial_i \mathbb{V}(\chi, s) = \lim_{x\to 0^+} \frac{\mathbb{V}(\chi^{xu_i}, s) - \mathbb{V}(\chi, s)}{x},$$

where u_i is the standard unit vector in $\mathbb{R}^k$, and its ith element is 1, but the other elements are 0. $\mathbb{V}$ is continuous with respect to the first argument. Derivatives $\mathbb{V}_t$, $\mathbb{V}_l = (\partial_i \mathbb{V})$, $\mathbb{V}_{ll} = (\partial_{ij} \mathbb{V})$ exist and are continuous. $\mathbb{V}$, $\mathbb{V}_t$, $\mathbb{V}_l$ is bounded on the bounded set $B_r = \{\chi \mid \|\chi\| \leq r, r > 0\}$. Define

$$\begin{aligned}\mathbb{L}\mathbb{V}(\chi, s) =& \mathbb{V}_t(\chi, s) + \mathbb{V}_l(\chi, s)F(t, \chi, s) + \frac{1}{2}\mathrm{tr}\left(\Lambda^{\mathrm{T}}(t, l, s)\mathbb{V}_{ll}(\chi, s)\Lambda(t, l, s)\right) \\ &+ \sum_{\hat{s}=1}^{S} \zeta_{s\hat{s}}(\epsilon(t)) \left(\mathbb{V}(\chi, \hat{s}) - \mathbb{V}(\chi, s)\right).\end{aligned}$$

Hence, one gets Dupire Itô's formula

$$\mathrm{d}\mathbb{V}(\chi, s) = \mathbb{L}\mathbb{V}(\chi, s)\mathrm{d}t + \mathbb{V}_l(\chi, s)\Lambda(t, l, s)\mathrm{dB}(t).$$

2.2. Model description

Consider the driving system

$$\begin{aligned}\mathrm{d}\phi_m(t) = &\left[\tilde{\Gamma}^1_m\left(t, \phi_m(t), \beta(t)\right) + \tilde{\Gamma}^2_m\left(t, \phi_m(t-\alpha_1(t))\right) + \int_{t-\alpha_2}^{t} \tilde{\Gamma}^3_m\left(\sigma, \phi_m(\sigma)\right)\mathrm{d}\sigma\right.\\ &\left. + \sum_{n=1}^{N}\Pi_{mn}(\beta(t))\tilde{\Theta}_{mn}\left(t, \phi_m(t), \phi_n(t), \beta(t)\right)\right]\mathrm{d}t\\ &+ \tilde{\Lambda}_m\left(t, \phi_m(t), \beta(t)\right)\mathrm{dB}(t),\ t \geq 0,\ m,\ n \in \mathbb{N},\end{aligned} \tag{2.4}$$

and the response system with impulsive control to synchronize with the driving system is given as below:

$$\begin{cases}\mathrm{d}\psi_m(t) = \left[\tilde{\Gamma}^1_m\left(t, \psi_m(t), \beta(t)\right) + \tilde{\Gamma}^2_m\left(t, \psi_m(t-\alpha_1(t))\right) + \int_{t-\alpha_2}^{t} \tilde{\Gamma}^3_m\left(\sigma, \psi_m(\sigma)\right)\mathrm{d}\sigma\right.\\ \qquad\left. + \sum_{n=1}^{N}\Pi_{mn}(\beta(t))\tilde{\Theta}_{mn}\left(t, \psi_m(t), \psi_n(t), \beta(t)\right)\right]\mathrm{d}t\\ \qquad + \tilde{\Lambda}_m\left(t, \psi_m(t), \beta(t)\right)\mathrm{dB}(t),\ t \geq 0,\ t \neq t_h,\\ \psi_m(t_h) - \phi_m(t_h) = I^m_{u(t_h)}(\psi(t_h^-) - \phi(t_h^-)),\ m,\ n \in \mathbb{N},\ h \in \mathbb{H},\end{cases} \tag{2.5}$$

where $\phi_m(t) \in \mathbb{R}^k$ is the state vector of the mth vertex for driving system (2.4) at time t, $\psi_m(t) \in \mathbb{R}^k$ is the state vector of the mth vertex for response system (2.5) at time t, $\beta(t)$ is the semi-Markov jump with the state space $\mathbb{S} = \{1, 2, \cdots, S\}$ and the state transfer probability $\mathbb{P}$ is described as

$$\mathbb{P}(\beta(t+\epsilon(t)) = \hat{s}|\beta(t) = s) = \begin{cases}\zeta_{s\hat{s}}(\epsilon(t))\epsilon(t) + o(\epsilon(t)), & s \neq \hat{s},\\ 1 + \zeta_{ss}(\epsilon(t))\epsilon(t) + o(\epsilon(t)), & s = \hat{s},\end{cases}$$

in which $\lim_{\epsilon(t)\to 0} o(\epsilon(t)) = 0$, $\zeta_{s\hat{s}}(\epsilon(t)) > 0$ $(s \neq \hat{s})$ is the transfer rate from state s to state $\hat{s}$, and $\zeta_{s\hat{s}}(\epsilon(t)) = -\sum_{\hat{s}=1,\hat{s}\neq s}^{S} \zeta_{s\hat{s}}(\epsilon(t))$, $\tilde{\Gamma}^1_m$: $\mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{S} \mapsto \mathbb{R}^k$, $\tilde{\Gamma}^2_m$, $\tilde{\Gamma}^3_m$: $\mathbb{R}_+ \times \mathbb{R}^k \mapsto \mathbb{R}^k$ are piecewise continuous functions, $\tilde{\Theta}_{mn}$: $\mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{S} \mapsto \mathbb{R}^k$ is the coupling function between the mth node and the nth node, representing the influence of the nth node on the mth node with the influence intensity being $\Pi_{mn}(\cdot) \geq 0$, $\tilde{\Lambda}_m$: $\mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{S} \mapsto \mathbb{R}^k$ is a stochastic perturbation function, $\mathrm{B}(t)$ is Brownian motion defined in the complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, $\alpha_1(t)$ and α_2 are time-varying discrete delay and distributed delay of the system, respectively such that $0 \leq \alpha_1(t) \leq \alpha_1$, $\dot{\alpha}_1(t) \leq \alpha_1^* < 1$, $\alpha_1, \alpha_2 \leq \alpha$. Besides, $\tilde{\Gamma}^1_m$, $\tilde{\Gamma}^2_m$, $\tilde{\Gamma}^3_m$, $\tilde{\Theta}_{mn}$, $\tilde{\Lambda}_m$ are all satisfied with the Lipschitz condition and the linear growth condition. $I^m_{u(t_h)}$: $\mathbb{R}^k \to \mathbb{R}^k$ is stochastic impulse intensity function, $\bar{\mathbb{H}} = \{t_1, t_2, \cdots, t_H, \cdots\}$ is stochastic impulse sequence, $\phi_m(t)$ and $\psi_m(t)$ satisfy $\phi_m(t_h^-) = \lim_{t\to t_h^-} \phi_m(t)$, $\phi_m(t_h^+) = \lim_{t\to t_h^+} \phi_m(t)$, $\psi_m(t_h^-) = \lim_{t\to t_h^-} \psi_m(t)$, $\psi_m(t_h^+) = \lim_{t\to t_h^+} \psi_m(t)$. They are right continuous, that is, $\phi_m(t_h) = \phi_m(t_h^+)$, $\psi_m(t_h) = \psi_m(t_h^+)$. Suppose that $e_m(t) = \psi_m(t) - \phi_m(t)$ is the error vector for the mth vertex at time t. Then the error system of driving system (2.4) and response system (2.5) can be depicted as

$$\begin{cases} \mathrm{d}e_m(t) = \Big[\Gamma_m^1\left(t, e_m(t), \beta(t)\right) + \Gamma_m^2\left(t, e_m(t-\alpha_1(t))\right) + \int_{t-\alpha_2}^{t} \Gamma_m^3\left(\sigma, e_m(\sigma)\right) \mathrm{d}\sigma \\ \qquad + \sum_{n=1}^{N} \Pi_{mn}(\beta(t))\Theta_{mn}\left(t, e_m(t), e_n(t), \beta(t)\right)\Big] \mathrm{d}t \\ \qquad + \Lambda_m\left(t, e_m(t), \beta(t)\right) \mathrm{dB}(t),\ t \geq 0,\ t \neq t_h, \\ e_m(t_h) = I_{u(t_h)}^m(e_m(t_h^-)),\ m,\ n \in \mathbb{N},\quad h \in \mathbb{H}. \end{cases} \tag{2.6}$$

Among them,

$$\begin{aligned}
&\Gamma_m^1\left(t, e_m(t), \beta(t)\right) = \tilde{\Gamma}_m^1\left(t, \psi_m(t), \beta(t)\right) - \tilde{\Gamma}_m^1\left(t, \phi_m(t), \beta(t)\right), \\
&\Gamma_m^2\left(t, e_m(t-\alpha_1(t))\right) = \tilde{\Gamma}_m^2\left(t, \psi_m(t-\alpha_1(t))\right) - \tilde{\Gamma}_m^2\left(t, \phi_m(t-\alpha_1(t))\right), \\
&\Gamma_m^3\left(t, e_m(t)\right) = \tilde{\Gamma}_m^3\left(t, \psi_m(t)\right) - \tilde{\Gamma}_m^3\left(t, \phi_m(t)\right), \\
&\Theta_{mn}\left(t, e_m(t), e_n(t), \beta(t)\right) = \tilde{\Theta}_{mn}\left(t, \psi_m(t), \psi_n(t), \beta(t)\right) - \tilde{\Theta}_{mn}\left(t, \phi_m(t), \phi_n(t), \beta(t)\right), \\
&\Lambda_m\left(t, e_m(t), \beta(t)\right) = \tilde{\Lambda}_m\left(t, \psi_m(t), \beta(t)\right) - \tilde{\Lambda}_m\left(t, \phi_m(t), \beta(t)\right).
\end{aligned}$$

In order to obtain the theoretical results, some assumptions, definitions, and a lemma about the error system are presented in the following.

Assumption 2.1. *There exist positive numbers $\gamma_m^1(s)$, γ_m^2, γ_m^3, $m \in \mathbb{N}$, $s \in \mathbb{S}$ such that*

$$\begin{aligned}
&e_m^{\mathrm{T}}\Gamma_m^1(t, e_m, s) \leq \gamma_m^1(s)|e_m|^2, \\
&|\Gamma_m^2(t, e_m)| \leq \gamma_m^2|e_m|, \\
&|\Gamma_m^3(t, e_m)|^2 \leq \gamma_m^3|e_m|^2.
\end{aligned}$$

Assumption 2.2. *There exist positive numbers $\theta_{mn}(s)$, $\lambda_m(s)$, $m \in \mathbb{N}$, $s \in \mathbb{S}$, and a sequence of stochastic variables $IMP_{u(t_h)}$, $h \in \mathbb{H}$, satisfying that*

$$\begin{aligned}
&|\Theta_{mn}(t, e_m, e_n, s)| \leq \theta_{mn}(s)(|e_m| + |e_n|), \\
&|\Lambda_m(t, e_m, s)|^2 \leq \lambda_m(s)|e_m|^2, \\
&\left|I_{u(t_h)}^m(e_m^h)\right|^2 \leq IMP_{u(t_h)}\left|e_m^h\right|^2.
\end{aligned}$$

Definition 2.1. If there exist positive numbers ε and L such that for any initial condition $\xi = \xi_\psi - \xi_\phi \in C([-\alpha, 0]; \mathbb{R}^{kN})$,

$$\mathbb{E}|e(t)|^2 \leq L\|\xi\|^2 \exp\{-\varepsilon t\},\ t \geq 0,$$

where $e(t) = (e_1^{\mathrm{T}}(t), e_2^{\mathrm{T}}(t), \cdots, e_N^{\mathrm{T}}(t))^{\mathrm{T}} \in \mathbb{R}^{kN}$. Then driving system (2.4) and response system (2.5) achieve mean-square exponential synchronization.

Definition 2.2. [26] Suppose that $NUM_H(t, 0)$ represents the number of impulse occurrences of the impulse sequence $\bar{\mathbb{H}}$ in the time period $(0, t)$. Then average impulsive interval AI of the impulse sequence $\bar{\mathbb{H}}$ at time interval $(0, t)$ is defined as

$$AI = \lim_{t \to \infty} \frac{t}{NUM_H(t, 0)}.$$

Definition 2.3. The average stochastic impulsive gain AG of the impulse sequence $\bar{\mathbb{H}}$ at time interval $(0,t)$ is

$$AG = \lim_{t\to\infty} \frac{\mathbb{E}IMP_{u(t_1)} + \mathbb{E}IMP_{u(t_2)} + \cdots \mathbb{E}IMP_{u(t_{NUM_H(t,0)})}}{NUM_H(t,0)}.$$

Remark 2.1. In most cases, the impulsive effects can be classified into two most common categories, i.e., synchronous impulses and asynchronous impulses. In [26], the definition of "average impulsive interval" has been introduced and some unified synchronization criteria both suitable for synchronous and asynchronous impulses were given. Besides, the definition of "average impulsive gain" was put forward in [35] and some synchronization criteria for an array of coupled neural networks were provided. Furthermore, it is worth noticing that the impulsive intensity is presumed to be predetermined in the above references, which is seldom to describe the stochastic factors universally appearing in the impulsive effect. Therefore, by introducing and adopting the novel definition "average stochastic impulsive gain", we shall calculate the intensity of the stochastic hybrid impulse that is being investigated in this paper. As a consequence, some synchronization criteria firmly linking to Definition 2.3 will be derived in the next section.

Lemma 2.1. *Assume $N \geq 2$, c_m is the cofactor of the mth diagonal element of the Laplacian matrix for matrix $THE = (THE_{mn})_{N\times N}$, it can be concluded that*

$$\sum_{m=1}^{N}\sum_{n=1}^{N} c_m THE_{mn}\Theta_{mn}(t,e_m,e_n) = \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) \sum_{(v,v')\in E(\mathcal{C}_{\mathcal{Q}})} \Theta_{mn}(t,e_{v'},e_v),$$

where Θ_{mn} is an arbitrary function, $\mathbb{Q}$ denotes the set consisting of spanning unicyclic graphs of (G,THE), $W(\mathcal{Q})$ is the weight of $\mathcal{Q}$, $\mathcal{C}_{\mathcal{Q}}$ represents the directed cycle of $\mathcal{Q}$. In particular, if (G,THE) is strongly connected, then $c_m > 0$, $m \in \mathbb{N}$.

Based on the above discussions, the following so-called Lyapunov-type theorem and Cofficient-type theorem will be presented for driving system (2.4) and response system (2.5) to achieve exponential synchronization.

3. Main results

In this section, we will give some synchronization criteria which are included in the last two theorems. And the first theorem illustrates a fact that ensures the validity of the theoretical results in this paper as basics.

To illustrate a fact, we will prove that functional derivative calculating by the defined horizontal movement and conventional method of calculating the derivative of the integral is equal.

Theorem 3.1. *Suppose $Y(t)$ is a continuous function on $[0,+\infty)$, for a fixed t, $Y_t \in C([-\alpha,0];\mathbb{R}^k)$, $Y_t(\sigma) = Y(t+\sigma)$, $\sigma\in[-\alpha,0]$. If $\mathbb{V}(Y_t) = \int_{-\alpha}^{0}\int_{t+\sigma}^{t} Y(\mu)\mathrm{d}\mu\mathrm{d}\sigma$, then one obtains*

$$\begin{aligned}\mathbb{V}_t(Y_t) &= \lim_{x\to 0^+} \frac{\mathbb{V}\left((Y_t)_x\right) - \mathbb{V}(Y_t)}{x}\\ &= \alpha Y(t) - \int_{t-\alpha}^{t} Y(\mu)\mathrm{d}\mu\end{aligned}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{-\alpha}^{0}\int_{t+\sigma}^{t} Y(\mu)\mathrm{d}\mu\mathrm{d}\sigma\right).$$

Proof. *Based on* (2.2)*, one gets*

$$(Y_t)_x(\sigma) = \begin{cases} Y_t(x+\sigma), & \sigma \in [-\alpha, -x], \\ Y_t(0), & \sigma \in [-x, 0]. \end{cases}$$

Then according to

$$\mathbb{V}(Y_t) = \int_{-\alpha}^{0}\int_{t+\sigma}^{t} Y(\mu)\mathrm{d}\mu\mathrm{d}\sigma,$$

one derives

$$\mathbb{V}((Y_t)_x) = \int_{-\alpha}^{-x}\int_{t+\sigma}^{t} Y(\mu+x)\mathrm{d}\mu\mathrm{d}\sigma + \int_{-x}^{0}\int_{t+\sigma}^{t} Y(0)\mathrm{d}\mu\mathrm{d}\sigma.$$

Moreover, one has

$$\begin{aligned}
\mathbb{V}_t(Y_t) = &\lim_{x\to 0^+}\frac{1}{x}\left(\int_{-\alpha}^{-x}\int_{t+\sigma}^{t} Y(\mu+x)\mathrm{d}\mu\mathrm{d}\sigma + \int_{-x}^{0}\int_{t+\sigma}^{t} Y(0)\mathrm{d}\mu\mathrm{d}\sigma\right.\\
&\left.-\int_{-\alpha}^{0}\int_{t+\sigma}^{t} Y(\mu)\mathrm{d}\mu\mathrm{d}\sigma\right)\\
=&\lim_{x\to 0^+} -\int_{t-x}^{t} Y(\mu+x)\mathrm{d}\mu + \int_{-\alpha}^{-x}\int_{t+\sigma}^{t}\dot{Y}(\mu+x)\mathrm{d}\mu\mathrm{d}\sigma + \int_{t-x}^{t} Y(0)\mathrm{d}\mu\\
=&\int_{-\alpha}^{0}\int_{t+\sigma}^{t}\dot{Y}(\mu)\mathrm{d}\mu\mathrm{d}\sigma\\
=&\int_{-\alpha}^{0}\left(Y(t) - Y(t+\sigma)\right)\mathrm{d}\sigma\\
=&\alpha Y(t) - \int_{t-\alpha}^{t} Y(\mu)\mathrm{d}\mu.
\end{aligned}$$

This completes the proof. □

Remark 3.1. Since Dupire functional Itô's formula was put forward in [9], numerous theoretical results have been inspired [8, 28], in which some stability criteria were derived for the stochastic functional differential equation and random switching system. Different from them, we consider the stochastic complex network with semi-Markov jump and distributed delay in this paper. Due to the existence of distributed delay, we construct a different functional $\mathbb{V}_m(t,\chi_m,s) = V_m^1(l_m,s) + V_m^2(t)$, $V_m^1(l_m,s) = \bar{w}_m(s)|l_m|^2$, $V_m^2(t) = \int_{t-\alpha_1(t)}^{t}\left|\Gamma_m^2(\sigma,e_m(\sigma))\right|^2\mathrm{d}\sigma + \int_{-\alpha_2}^{0}\int_{t+\sigma}^{t}\left|\Gamma_m^3(\mu,e_m(\mu))\right|^2\mathrm{d}\mu\mathrm{d}\sigma$. When using Dupire functional Itô's formula, $\mathbb{V}_t$ should be calculated and $\mathbb{V}_t$ depends on the defined Dupire horizontal partial derivative. In Theorem 3.1, we prove that the result is the same as the conventional method of calculating the derivative of integral, which is an additional and necessary result to ensure the validity of the theoretical results of this paper.

Theorem 3.2. *For* $s \in \mathbb{S}$*, suppose there exists a function* $\mathbb{V}_m(\chi,s)$ *defined on* $C([-\alpha,0];\mathbb{R}^k)\times\mathbb{S}$ *such that* $\mathbb{V}_m(\chi,s) = V_m^1(l_m,s) + V_m^2(t)$ *in which* $V_m^2(t_h) = 0$.

$\mathbb{V}_m(\chi, s)$ is a continuous function that is twice differentiable with respect to the first variable. If the following conditions hold:

WX1. *There exist positive constants $d_m^1(s)$, $d_m^2(s)$ satisfying that*

$$d_m^1(s)|l_m|^2 \leq V_m^1(l_m, s) \leq d_m^2(s)|l_m|^2. \tag{3.1}$$

WX2. *When $t \neq t_h$, there exist positive constants $\eta_m(s)$, $THE_{mn} \geq 0$ and function Θ_{mn} satisfying*

$$\mathbb{LV}_m((e_m)_t, s) \leq \eta_m(s)\mathbb{V}_m((e_m)_t, s) + \sum_{n=1}^{N} THE_{mn}\Theta_{mn}(t, e_m(t), e_n(t)). \tag{3.2}$$

When $t = t_h$, one derives

$$\mathbb{V}_m\left((e_m)_{t_h}, \beta(t_h)\right) \leq IMP_{u(t_h)}\mathbb{V}_m\left((e_m)_{t_h^-}, \beta(t_h^-)\right). \tag{3.3}$$

WX3. *Digraph (G, THE) is strongly connected, $THE = (THE_{mn})_{N\times N}$, $THE_{mn} = \max_{s\in\mathbb{S}}\{\Pi_{mn}(s)\theta_{mn}(s)\}$. For each digraph (G, THE), we have*

$$\sum_{(m,n)\in E(\mathcal{C}_\mathcal{Q})} \Theta_{mn}(t, e_m(t), e_n(t)) \leq 0.$$

WX4. *If the average stochastic impulsive gain AG and the average stochastic impulsive interval AI satisfy*

$$\frac{\ln AG}{AI} = k < k_1 < 0, k_1 + \eta < 0, \eta = \max_{m\in\mathbb{N}, s\in\mathbb{S}}\{\eta_m(s)\}.$$

Then driving system (2.4) and response system (2.5) achieve mean-square exponential synchronization.

Proof. *Suppose $\mathbb{V}(e_t, s) = \sum_{m=1}^{N} c_m\mathbb{V}_m((e_m)_t, s)$, here c_m represents the cofactor of the mth diagonal element of the* Laplacian *matrix for matrix THE. Assume that (G, THE) is strongly connected, we can get $c_m > 0$, $m \in \mathbb{N}$. Impulse instants and non-impulse instants are discussed separately below.*

When $t \neq t_h$, based on (3.2), it can be obtained that

$$\begin{aligned}
\mathbb{LV}(e_t, s) &= \sum_{m=1}^{N} c_m\mathbb{LV}_m((e_m)_t, s) \\
&\leq \sum_{m=1}^{N} c_m\left(\eta_m(s)\mathbb{V}_m((e_m)_t, s) + \sum_{n=1}^{N} THE_{mn}\Theta_{mn}(t, e_m(t), e_n(t))\right) \\
&\leq \eta\mathbb{V}(e_t, s) + \sum_{m=1}^{N}\sum_{n=1}^{N} c_m THE_{mn}\Theta_{mn}(t, e_m(t), e_n(t)).
\end{aligned} \tag{3.4}$$

By Lemma 2.1, $W(\mathcal{Q}) \geq 0$, it can be written as

$$\begin{aligned}
&\sum_{m=1}^{N}\sum_{n=1}^{N} c_m THE_{mn}\Theta_{mn}(t, e_m(t), e_n(t)) \\
&= \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) \sum_{(v,v')\in E(\mathcal{C}_\mathcal{Q})} \Theta_{mn}(t, e_{v'}(t), e_v(t)) \leq 0.
\end{aligned} \tag{3.5}$$

Substituting (3.5) *to* (3.4), *it can be concluded that*

$$\mathbb{L}\mathbb{V}(e_t, s) \le \eta \mathbb{V}(e_t, s).$$

It's given by Dupire Itô's formula that

$$\mathrm{D}^{+}\mathbb{E}\mathbb{V}(e_t, s) \le \eta \mathbb{E}\mathbb{V}(e_t, s). \tag{3.6}$$

When $t = t_h$, *according to* (3.4), *one gets*

$$\begin{aligned}\mathbb{V}\left(e_{t_h}, \beta(t_h)\right) &= \sum_{m=1}^{N} c_m \mathbb{V}_m\left((e_m)_{t_h}, \beta(t_h)\right) \\ &\le \sum_{m=1}^{N} c_m IMP_{u(t_h)} \mathbb{V}_m\left((e_m)_{t_h^-}, \beta(t_h^-)\right) \\ &= IMP_{u(t_h)} \mathbb{V}\left((e_m)_{t_h^-}, \beta(t_h^-)\right),\end{aligned} \tag{3.7}$$

then

$$\begin{aligned}\mathbb{E}\mathbb{V}\left((e_m)_{t_h}, \beta(t_h)\right) \le& \mathbb{E}\left[IMP_{u(t_h)} \mathbb{V}\left((e_m)_{t_h^-}, \beta(t_h^-)\right)\right] \\ =& \mathbb{E}\left[\mathbb{E}\left[IMP_{u(t_h)} \mathbb{V}\left((e_m)_{t_h^-}, \beta(t_h^-)\right) \mid \mathcal{F}_{h-1}\right]\right] \\ =& \mathbb{E}\left[\mathbb{V}\left((e_m)_{t_h^-}, \beta(t_h^-)\right)\right] \mathbb{E}\left[IMP_{u(t_h)} \mid \mathcal{F}_{h-1}\right] \\ =& \mathbb{E}\mathbb{V}\left((e_m)_{t_h^-}, \beta(t_h^-)\right) \mathbb{E}IMP_{u(t_h)}.\end{aligned}$$

Depending on the method of inductive, it holds that

$$\mathbb{E}\mathbb{V}\left(e_t, s\right) \le \mathbb{E}\mathbb{V}_o \prod_{h=1}^{H} \mathbb{E}IMP_{u(t_h)} \exp\{\eta t\},\ t \in [t_H, t_{H+1}), \tag{3.8}$$

where

$$\mathbb{E}\mathbb{V}_o = \sup_{t \in [-\alpha, 0]} \mathbb{E}\mathbb{V}\left(e_t, s\right).$$

When $t \in [-\alpha, 0]$, *one derives* $\mathbb{E}\mathbb{V}\left(e_t, s\right) \le \mathbb{E}V_o$, *and it obviously holds. When* $t \in (0, t_1)$, *from (3.6), we can derive the following inequalities:*

$$\mathbb{E}\mathbb{V}\left(e_t, s\right) \le \mathbb{E}\mathbb{V}\left(e_0, s\right) \exp\{\eta t\} \le \mathbb{E}\mathbb{V}_o \exp\{\eta t\}.$$

Thus, one concludes that

$$\mathbb{E}\mathbb{V}\left(e_{t_1}, \beta(t_1)\right) \le \mathbb{E}IMP_{u(t_1)} \mathbb{E}\mathbb{V}\left(e_{t_1^-}, \beta(t_1^-)\right) \le \mathbb{E}IMP_{u(t_1)} \mathbb{E}\mathbb{V}_o \exp\{\eta t_1\}.$$

When $t \in [t_1, t_2)$,

$$\mathbb{E}\mathbb{V}\left(e_t, s\right) \le \mathbb{E}\mathbb{V}\left(e_{t_1}, \beta(t_1)\right) \exp\{\eta(t - t_1)\} \le \mathbb{E}IMP_{u(t_1)} \mathbb{E}\mathbb{V}_o \exp\{\eta t\}.$$

Hence, when $H = 1$, (3.8) *holds. If* (3.8) *is true for* H, *then for* $H + 1$,

$$\begin{aligned}\mathbb{E}\mathbb{V}\left(e_{t_{H+1}}, \beta(t_{H+1})\right) &\le \mathbb{E}IMP_{u(t_{H+1})} \mathbb{E}\mathbb{V}\left(e_{t_{H+1}^-}, u(t_{H+1}^-)\right) \\ &\le \prod_{h=1}^{H+1} \mathbb{E}IMP_{u(t_h)} \mathbb{E}\mathbb{V}_o \exp\{\eta t_{H+1}\}.\end{aligned}$$

When $t \in [t_{H+1}, t_{H+2})$*,*

$$\mathbb{E}\mathbb{V}\left(e_t, s\right) \leq \mathbb{E}\mathbb{V}\left(e_{t_{H+1}}, \beta(t_{H+1})\right) \exp\{\eta(t - t_{H+1})\} = \prod_{h=1}^{H} \mathbb{E}IMP_{u(t_h)} \mathbb{E}\mathbb{V}_o \exp\{\eta t\}.$$

Therefore, (3.8) *holds for* $H + 1$*, and based on the method of inductive, we can conclude that* (3.8) *is valid for all* $H \geq 1$*, so*

$$\begin{aligned}\mathbb{E}\mathbb{V}\left(e_t, s\right) = & \prod_{h=1}^{NUM_H(t,0)} \mathbb{E}IMP_{u(t_h)} \mathbb{E}\mathbb{V}_o \exp\{\eta t\} \\ \leq & \mathbb{E}\mathbb{V}_o \nu^{NUM_H(t,0)} \exp\{\eta t\} \\ = & \mathbb{E}\mathbb{V}_o \exp\{NUM_H(t,0) \ln \nu\} \exp\{\eta t\} \\ \leq & \mathbb{E}\mathbb{V}_o \exp\left\{\frac{t \ln \nu}{t/NUM_H(t,0)}\right\} \exp\{\eta t\}, \end{aligned} \tag{3.9}$$

in which

$$\nu = \frac{\mathbb{E}IMP_{u(t_1)} + \mathbb{E}IMP_{u(t_2)} + \cdots \mathbb{E}IMP_{u\left(t_{NUM_H(t,0)}\right)}}{NUM_H(t,0)}.$$

Since $AG = \lim_{t\to\infty} \nu$ *and* $AI = \lim_{t\to\infty} \frac{t}{NUM_H(t,0)}$*, it can be described as follows:*

$$\left| \ln \nu / \frac{t}{NUM_H(t,0)} - \frac{\ln AG}{AI} \right| < k_1 - k. \tag{3.10}$$

Substituting (3.10) *into* (3.9)*, one derives*

$$\mathbb{E}\mathbb{V}\left(e_t, s\right) \leq \mathbb{E}\mathbb{V}_o \exp\{(k_1 + \eta)t\}, t \geq 0.$$

In view of $e_t(0) = e(t)$*, one gets*

$$\mathbb{E}V^1\left(e(t), s\right) \leq V_o^1 \exp\{(k_1 + \eta)t\}, t \geq 0,$$

where

$$V_o^1 = \sup_{t\in[-\alpha,0]} V^1\left(e(t), s\right).$$

According to condition **WX1***, it concludes*

$$\begin{aligned} V_o^1 &= \sum_{m=1}^{N} c_m (V_m^1)_o(e_m(t), s) \\ &\leq \sup_{t\in[-\alpha,0]} \sum_{m=1}^{N} c_m d_m^2(s) |e_m(t)|^2 \\ &\leq \max_{m\in\mathbb{N}, s\in\mathbb{S}} \{c_m d_m^2(s)\} \sup_{t\in[-\alpha,0]} |e(t)|^2 \triangleq d_2 \|\xi\|^2, \end{aligned}$$

$$\begin{aligned} \mathbb{E}V_o^1\left(e(t), s\right) &= \mathbb{E} \sum_{m=1}^{N} c_m (V_m^1)_o(e_m(t), s) \\ &\geq \mathbb{E} \sum_{m=1}^{N} c_m d_m^1(s) |e_m(t)|^2 \\ &\geq \min_{m\in\mathbb{N}, s\in\mathbb{S}} \{c_m d_m^1(s)\} \mathbb{E}|e(t)|^2 = d_1 \mathbb{E}|e(t)|^2. \end{aligned}$$

Consequently, we have

$$\mathbb{E}|e(t)|^2 \le \frac{d_2}{d_1}\|\xi\|^2 \exp\{(k_1+\eta)t\}, t \ge 0.$$

On the basis of condition **WX4**, $k_1 + \eta < 0$ *can be obtained, then driving system (2.4) and response system (2.5) can reach mean-square exponential synchronization.* □

Remark 3.2. There have been many investigations on the synchronization of coupled systems by graph theory and the Lyapunov method [21,37,44]. In [44], global Lyapunov function was constructed as $V(e,t) = \sum_{i=1}^{n} c_i V_i(e_i,t)$ through vertex Lyapunov function. Referring to this method, we construct the global Lyapunov functional as $\mathbb{V}(t, e_t, s) = \sum_{m=1}^{N} \mathbb{V}_m(t,(e_m)_t,s)$ through vertex Lyapunov functional, in which c_m is the cofactor of the mth diagonal element of the Laplacian matrix for matrix THE. From this condition, we can also see that the synchronization is related to the network's topological structure.

In the following, Coefficient-type theorem attaching to the coefficients in driving system (2.4) and response system (2.5) is derived to give some other synchronization criteria.

Theorem 3.3. *Suppose Assumption 2.1 and Assumption 2.2 hold and the following conditions are satisfied:*
HD1. *Directed graph* (G, THE) *is strongly connected,* $THE = (THE_{mn})_{N\times N}$, $THE_{mn} = \max_{s\in\mathbb{S}}\{\Pi_{mn}(s)\theta_{mn}(s)\}$.
HD2. *The average stochastic impulsive gain and the average stochastic impulsive interval satisfy that*

$$\frac{\ln AG}{AI} = k < k_1 < 0, k_1+\eta < 0, \eta = \max_{m\in\mathbb{N}, s\in\mathbb{S}}\{\eta_m(s)\}.$$

Then driving system (2.4) and response system (2.5) can reach mean-square exponential synchronization.

Proof. *Assume* $V_m^1(e_m,s) = \bar{w}_m(s)|e_m|^2$, $V_m^2(t) = \int_{t-\alpha_1(t)}^{t}\left|\Gamma_m^2(\sigma,e_m(\sigma))\right|^2 \mathrm{d}\sigma + \int_{-\alpha_2}^{0}\int_{t+\sigma}^{t}\left|\Gamma_m^3(\mu,e_m(\mu))\right|^2 \mathrm{d}\mu\mathrm{d}\sigma$, *condition* **WX1** *apparently holds. Based on Assumption 2.1, Assumption 2.2 and Lemma 2.1, computing* $\mathbb{L}\mathbb{V}_m$ *along system (2.6), we can get the following conclusions. When* $t \ne t_h$, *it has*

$$\begin{aligned}
&\mathbb{L}\mathbb{V}_m((e_m)_t,s)\\
=&2\bar{w}_m(s)e_m^{\mathrm{T}}(t)\Bigg[\Gamma_m^1(t,e_m(t),s)+\Gamma_m^2(t,e_m(t-\alpha_1(t))) \qquad (3.11)\\
&+\int_{t-\alpha_2}^{t}\Gamma_m^3(\sigma,e_m(\sigma))\mathrm{d}\sigma+\sum_{n=1}^{N}\Pi_{mn}(s)\Theta_{mn}(t,e_m(t),e_n(t),s)\Bigg]\\
&+\bar{w}_m(s)\mathrm{tr}\left[\Lambda_m^{\mathrm{T}}(t,e_m(t),s)\Lambda_m(t,e_m(t),s)\right]+\sum_{\hat{s}\in\mathbb{S}}\zeta_{s\hat{s}}(\epsilon(t))\bar{w}_m(\hat{s})|e_m(t)|^2\\
&+\left|\Gamma_m^2(t,e_m(t))\right|^2-(1-\dot{\alpha}_1(t))\left|\Gamma_m^2(t,e_m(t-\alpha_1(t)))\right|^2\\
&+\alpha_2\left|\Gamma_m^3(t,e_m(t))\right|^2-\int_{t-\alpha_2}^{t}\left|\Gamma_m^3(\sigma,e_m(\sigma))\right|^2\mathrm{d}\sigma
\end{aligned}$$

$$
\begin{aligned}
&\leq 2\bar{w}_m(s)\gamma_m^1(s)|e_m(t)|^2 + \bar{w}_m(s)\gamma_m^2|e_m(t)|^2 + \alpha_2\bar{w}_m(s)\gamma_m^3|e_m(t)|^2 \\
&\quad + 2\bar{w}_m(s)\Pi_{mn}(s)\theta_{mn}(s)|e_m(t)|^2 + 2\bar{w}_m(s)\Pi_{mn}(s)\theta_{mn}(s)|e_m(t)||e_n(t)| \\
&\quad + 2\bar{w}_m(s)e_m^{\mathrm{T}}(t)\Gamma_m^2(t,e_m(t-\alpha_1(t))) - (1-\alpha_1^*)\left|\Gamma_m^2(t,e_m(t-\alpha_1(t)))\right|^2 \\
&\quad + 2\bar{w}_m(s)e_m^{\mathrm{T}}(t)\int_{t-\alpha_2}^{t}\Gamma_m^3(\sigma,e_m(\sigma))\mathrm{d}\sigma - \int_{t-\alpha_2}^{t}\left|\Gamma_m^3(\sigma,e_m(\sigma))\right|^2\mathrm{d}\sigma \\
&\quad + \bar{w}_m(s)\lambda_m(s)|e_m(t)|^2 + \sum_{\hat{s}\in\mathbb{S}}\zeta_{s\hat{s}}(\epsilon(t))\frac{\bar{w}_m(\hat{s})}{\bar{w}_m(s)}\bar{w}_m(s)|e_m(t)|^2.
\end{aligned} \tag{3.12}
$$

In view of the following inequalities

$$
\begin{aligned}
&-\int_{t-\alpha_2}^{t}\left|\Gamma_m^3(\sigma,e_m(\sigma))\right|^2\mathrm{d}\sigma + 2\bar{w}_m(s)e_m^{\mathrm{T}}(t)\int_{t-\alpha_2}^{t}\Gamma_m^3(\sigma,e_m(\sigma))\mathrm{d}\sigma \\
\leq& -\left|\int_{t-\alpha_2}^{t}\Gamma_m^3(\sigma,e_m(\sigma))^2\mathrm{d}\sigma\right|^2 + 2\bar{w}_m(s)e_m^{\mathrm{T}}(t)\int_{t-\alpha_2}^{t}\Gamma_m^3(\sigma,e_m(\sigma))\mathrm{d}\sigma \\
=& -\left|\int_{t-\alpha_2}^{t}\Gamma_m^3(\sigma,e_m(\sigma))\mathrm{d}\sigma - \bar{w}_m(s)e_m^{\mathrm{T}}(t)\right|^2 + \bar{w}_m^2(s)|e_m(t)|^2 \\
\leq& \bar{w}_m^2(s)|e_m(t)|^2,
\end{aligned} \tag{3.13}
$$

$$
\begin{aligned}
&-(1-\alpha_1^*)\left|\Gamma_m^2(t,e_m(t-\alpha_1(t)))\right|^2 + 2\bar{w}_m(s)e_m^{\mathrm{T}}(t)\Gamma_m^2(t,e_m(t-\alpha_1(t))) \\
=& -\left|(1-\alpha_1^*)^{\frac{1}{2}}\Gamma_m^2(t,e_m(t-\alpha_1(t))) - \bar{w}_m(s)(1-\alpha_1^*)^{-\frac{1}{2}}e_m^{\mathrm{T}}(t)\right|^2 + \frac{\bar{w}_m^2(s)}{1-\alpha_1^*}|e_m(t)|^2 \\
\leq& \frac{\bar{w}_m^2(s)}{1-\alpha_1^*}|e_m(t)|^2,
\end{aligned} \tag{3.14}
$$

and

$$
2\Pi_{mn}(s)\theta_{mn}(s)|e_m(t)||e_n(t)| \leq \Pi_{mn}(s)\theta_{mn}(s)|e_m(t)|^2 + \Pi_{mn}(s)\theta_{mn}(s)|e_n(t)|^2, \tag{3.15}
$$

$$
\Pi_{mn}(s)\theta_{mn}(s)|e_n(t)|^2\Pi_{mn}(s)\theta_{mn}(s)|e_m(t)|^2 + \Pi_{mn}(s)\theta_{mn}(s)(|e_n(t)|^2 - |e_m(t)|^2), \tag{3.16}
$$

according to (3.13)-(3.16) *and* $V_m^2(t) > 0$*, we obtain*

$$
\begin{aligned}
&\mathbb{L}\mathbb{V}_m((e_m)_t,s) \\
&\leq 2\bar{w}_m(s)\gamma_m^1(s)|e_m(t)|^2 + \bar{w}_m(s)\gamma_m^2|e_m(t)|^2 + \alpha_2\bar{w}_m(s)\gamma_m^3|e_m(t)|^2 \\
&\quad + \bar{w}_m(s)\lambda_m(s)|e_m(t)|^2 + 4\bar{w}_m(s)\sum_{n=1}^{N}THE_{mn}|e_m(t)|^2 \\
&\quad + \frac{\bar{w}_m^2(s)}{1-\alpha_1^*}|e_m(t)|^2 + \bar{w}_m^2(s)|e_m(t)|^2 + \sum_{\hat{s}\in\mathbb{S}}\zeta_{s\hat{s}}(\epsilon(t))\frac{\bar{w}_m(\hat{s})}{\bar{w}_m(s)}\bar{w}_m(s)|e_m(t)|^2 \\
&\quad + \eta_m(s)V_m^2(t) + \sum_{n=1}^{N}THE_{mn}\Theta_{mn}(t,e_m(t),e_n(t))
\end{aligned}
$$

$$
\begin{aligned}
&\leq\left(2\gamma_m^1(s)+\gamma_m^2+\alpha_2\gamma_m^3+\lambda_m(s)+4\sum_{n=1}^{N}THE_{mn}+\frac{\bar{w}_m^2(s)}{1-\alpha_1^*}\right.\\
&\quad\left.+\bar{w}_m^2(s)+\sum_{\hat{s}\in\mathbb{S}}\zeta_{s\hat{s},s'}\frac{\bar{w}_m(\hat{s})}{\bar{w}_m(s)}\right)\bar{w}_m(s)|e_m(t)|^2\\
&\quad+\eta_m(s)V_m^2(t)+\sum_{n=1}^{N}THE_{mn}\Theta_{mn}(t,e_m(t),e_n(t))\\
&\leq\eta_m(s)\left(V_m^1(e_m(t),s)+V_m^2(t)\right)+\sum_{n=1}^{N}THE_{mn}\Theta_{mn}(t,e_m(t),e_n(t))\\
&=\eta_m(s)\mathbb{V}_m((e_m)_t,s)+\sum_{n=1}^{N}THE_{mn}\Theta_{mn}(t,e_m(t),e_n(t)),
\end{aligned}
$$

where

$$
\begin{aligned}
\eta_m(s)&=2\gamma_m^1(s)+\gamma_m^2+\alpha_2\gamma_m^3+\lambda_m(s)+4\sum_{n=1}^{N}THE_{mn}+\frac{\bar{w}_m^2(s)}{1-\alpha_1^*}+\bar{w}_m^2(s)\\
&\quad+\sum_{\hat{s}\in\mathbb{S}}\zeta_{s\hat{s},s'}\frac{\bar{w}_m(\hat{s})}{\bar{w}_m(s)},\\
\Theta_{mn}(t,e_m(t),e_n(t))&=\max_{s\in\mathbb{S}}\{\bar{w}_m(s)\}\left(|e_n(t)|^2-|e_m(t)|^2\right).
\end{aligned}
$$

When $t=t_h$, it follows that

$$
\begin{aligned}
\mathbb{V}_m\left((e_m)_{t_h},\beta(t_h)\right)&=V_m^1\left(e_m(t_h),\beta(t_h)\right)\\
&=\bar{w}_m(s)|e_m(t_h)|^2\\
&=\bar{w}_m(s)\left|I_{u(t_h)}^m(e_m(t_h^-))\right|^2\\
&\leq\bar{w}_m(s)IMP_{u(t_h)}\left|e_m(t_h^-)\right|^2\\
&=IMP_{u(t_h)}V_m^1\left(e_m(t_h^-),\beta(t_h^-)\right)\\
&=IMP_{u(t_h)}\left(V_m^1\left(e_m(t_h^-),\beta(t_h^-)\right)+V_2(t_h^-)\right)\\
&=IMP_{u(t_h)}\mathbb{V}_m\left(e_m(t_h^-),\beta(t_h^-)\right).
\end{aligned}
$$

Therefore, condition **WX2** *holds. In addition,* **HD1** *and* **HD2** *conclude that conditions* **WX3** *and* **WX4** *hold, respectively. Consequently, driving system (2.4) and response system (2.5) can achieve mean-square exponential synchronization.* □

Remark 3.3. Note that digraph (G, THE) in Theorem 3.3 is strongly connected implying there are directed paths between any two different nodes in the maximum graph (G, THE). According to $THE_{mn} = \max_{s\in\mathbb{S}}\{\Pi_{mn}(s)\theta_{mn}(s)\}$, it is not a requisite for each sub-network to be strongly connected. Besides, $\ln AG/AI < 0$ represents the average impulsive gain is less than 1, which means the impulse plays a synchronous effect on the whole. And $k_1 + \eta < 0$ indicates the impulse indeed synchronizes the response system to the driving system.

4. Application to stochastic semi-Markov jump Chua's circuit system with mixed delay

Chua's circuit systems are widely used in various fields and have received widespread attention in recent years. In this part, we will apply the theoretical results to a kind of semi-Markov jump Chua's circuit systems with mixed delay. To make the driving Chua's circuit system and the corresponding response Chua's circuit system achieve synchronization, we apply stochastic hybrid impulsive control to the response system. Additionally, some synchronization criteria for the circuit systems are given.

A single uncoupled Chua's circuit system is known to be described as follows.

$$\begin{cases} C_1\mathrm{d}U_1(t) = \left[\dfrac{1}{I}\left(-U_1(t)+U_2(t)\right)-\tilde{\Gamma}(U_1(t))\right]\mathrm{d}t, \\ C_2\mathrm{d}U_2(t) = \left[\dfrac{1}{I}\left(U_1(t)-U_2(t)\right)+U_3(t)\right]\mathrm{d}t, \\ M\mathrm{d}U_3(t) = -\left(U_2(t)+I_0U_3(t)\right)\mathrm{d}t, \end{cases} \tag{4.1}$$

where $U_1(t)$ and $U_2(t)$ are the voltages of capacitors C_1 and C_2, respectively, U_3 is the current through inductor M, I and I_0 represent linear resistors. $\tilde{\Gamma}(U_1(t)) = \nu_2U_1(t)+\frac{1}{2}(\nu_1-\nu_2)(|U_1(t)+1|-|U_2(t)-1|)$, where ν_1 and ν_2 represent the slopes of the inner region and the outer region, respectively. Next, we consider the following coupled stochastic semi-Markov jump Chua's circuit system with mixed delay as the driving system.

$$\begin{aligned} \begin{pmatrix} \mathrm{d}\Phi_{m1}(t) \\ \mathrm{d}\Phi_{m2}(t) \\ \mathrm{d}\Phi_{m3}(t) \end{pmatrix} &= \left[\begin{pmatrix} -\tau_{m1}(\beta(t)) & \tau_{m1}(\beta(t)) & 0 \\ \tau_{m2}(\beta(t)) & -\tau_{m2}(\beta(t)) & \tau_{m3}(\beta(t)) \\ 0 & \tau_{m4}(\beta(t)) & -\tau_{m5}(\beta(t)) \end{pmatrix}\begin{pmatrix} \Phi_{m1}(t) \\ \Phi_{m2}(t) \\ \Phi_{m3}(t) \end{pmatrix}\right. \\ &\quad + \begin{pmatrix} -\xi_{m1}\tilde{\Gamma}_m\left(t,\Phi_{m1}(t-\alpha_1(t))\right) \\ 0 \\ 0 \end{pmatrix} + \int_{t-\alpha_2}^{t}\tilde{\Gamma}_m^3(\sigma,\Phi_m(\sigma))\mathrm{d}\sigma \\ &\quad \left.+ \sum_{n=1}^{N}\Pi_{mn}(\beta(t))\tilde{\Theta}_{mn}(t,\Phi_m(t),\Phi_n(t),\beta(t))\right]\mathrm{d}t \\ &\quad + \tilde{\Lambda}_m(t,\Phi_m(t),\beta(t))\mathrm{dB}(t),\ t\geq 0,\ m,\ n\in\mathbb{N}, \end{aligned} \tag{4.2}$$

in which $\Phi_m(t) = (\Phi_{m1}(t),\Phi_{m2}(t),\Phi_{m3}(t))^{\mathrm{T}} \in \mathbb{R}^3$ is the state vector of the mth circuit system at time t. $\tau_{m1}(\beta(t)) = \frac{1}{I_m(\beta(t))C_{m1}}$, $\tau_{m2}(\beta(t)) = \frac{1}{I_m(\beta(t))C_{m2}(\beta(t))}$, $\tau_{m3}(\beta(t)) = \frac{1}{C_{m2}(\beta(t))}$, $\tau_{m4}(\beta(t)) = \frac{1}{M_m(\beta(t))}$, $\tau_{m5}(\beta(t)) = \frac{I_{m0}(\beta(t))}{M_m(\beta(t))}$, $\xi_{m1} = \frac{1}{C_{m1}}$. $\tilde{\Gamma}_m$ and $\tilde{\Gamma}_m^3$ are continuous function. $\tilde{\Theta}_{mn}$ is the coupling function between the mth circuit and the nth circuit, and represents the influence of the nth circuit on the mth circuit with the influence intensity being $\Pi_{mn}(\beta(t))$. $\tilde{\Lambda}_m$ is the stochastic perturbation function, and $\mathrm{B}(t)$ is a one-dimensional Brownian motion. In the following, the response system with stochastic impulsive control that makes the response system synchronize with driving system (4.2) is given as follows.

$$\begin{cases} \begin{pmatrix} \mathrm{d}\Psi_{m1}(t) \\ \mathrm{d}\Psi_{m2}(t) \\ \mathrm{d}\Psi_{m3}(t) \end{pmatrix} = \Bigg[\begin{pmatrix} -\tau_{m1}(\beta(t)) & \tau_{m1}(\beta(t)) & 0 \\ \tau_{m2}(\beta(t)) & -\tau_{m2}(\beta(t)) & \tau_{m3}(\beta(t)) \\ 0 & \tau_{m4}(\beta(t)) & -\tau_{m5}(\beta(t)) \end{pmatrix} \begin{pmatrix} \Psi_{m1}(t) \\ \Psi_{m2}(t) \\ \Psi_{m3}(t) \end{pmatrix} \\ \qquad + \begin{pmatrix} -\xi_{m1}\tilde{\Gamma}_m\left(t, \Psi_{m1}(t-\alpha_1(t))\right) \\ 0 \\ 0 \end{pmatrix} \\ \qquad + \int_{t-\alpha_2}^{t} \tilde{\Gamma}_m^3(\sigma, \Psi_m(\sigma))\mathrm{d}\sigma \\ \qquad + \sum_{n=1}^{N} \Pi_{mn}(\beta(t))\tilde{\Theta}_{mn}(t, \Psi_m(t), \Psi_n(t), \beta(t)) \Bigg] \mathrm{d}t \\ \qquad + \tilde{\Lambda}_m(t, \Psi_m(t), \beta(t))\mathrm{dB}(t),\ t \geq 0,\ t \neq t_h, \\ \Psi_m(t_h) - \Phi_m(t_h) = I_{u(t_h)}^m(\Psi_m(t_h^-) - \Phi_m(t_h^-)), m, n \in \mathbb{N}, h \in \mathbb{H}, \end{cases} \tag{4.3}$$

where $\Psi_m(t) = (\Psi_{m1}(t), \Psi_{m2}(t), \Psi_{m3}(t))^{\mathrm{T}} \in \mathbb{R}^3$ is the state vector of the mth response circuit system at time t. Suppose that $\bar{e}_{m1}(t) = \Psi_{m1}(t) - \Phi_{m1}(t)$, $\bar{e}_{m2}(t) = \Psi_{m2}(t) - \Phi_{m2}(t)$, $\bar{e}_{m3}(t) = \Psi_{m3}(t) - \Phi_{m3}(t)$, thus the error system of driving system (4.2) and response system (4.3) can be described as

$$\begin{cases} \begin{pmatrix} \mathrm{d}\bar{e}_{m1}(t) \\ \mathrm{d}\bar{e}_{m2}(t) \\ \mathrm{d}\bar{e}_{m3}(t) \end{pmatrix} = \Bigg[\begin{pmatrix} -\tau_{m1}(\beta(t)) & \tau_{m1}(\beta(t)) & 0 \\ \tau_{m2}(\beta(t)) & -\tau_{m2}(\beta(t)) & \tau_{m3}(\beta(t)) \\ 0 & \tau_{m4}(\beta(t)) & -\tau_{m5}(\beta(t)) \end{pmatrix} \begin{pmatrix} \bar{e}_{m1}(t) \\ \bar{e}_{m2}(t) \\ \bar{e}_{m3}(t) \end{pmatrix} \\ \qquad + \begin{pmatrix} -\xi_{m1}\Gamma_m\left(t, \bar{e}_{m1}(t-\alpha_1(t))\right) \\ 0 \\ 0 \end{pmatrix} + \int_{t-\alpha_2}^{t} \Gamma_m^3(\sigma, \bar{e}_m(\sigma))\mathrm{d}\sigma \\ \qquad + \sum_{n=1}^{N} \Pi_{mn}(\beta_1(t))\Theta_{mn}(t, \bar{e}_m(t), \bar{e}_n(t), \beta(t)) \Bigg] \mathrm{d}t \\ \qquad + \Lambda_m(t, \bar{e}_m(t), \beta(t))\mathrm{dB}(t),\ t \geq 0, \\ \bar{e}_m(t_h) = I_{u(t_h)}^m(e_m(t_h^-)), m, n \in \mathbb{N}, h \in \mathbb{H}, \end{cases} \tag{4.4}$$

where $\bar{e}_m(t) = (\bar{e}_{m1}(t), \bar{e}_{m2}(t), \bar{e}_{m3}(t))^{\mathrm{T}}$, $\Gamma_m\left(t, \bar{e}_{m1}(t-\alpha_1(t))\right)$, $\Gamma_m^3(t, \bar{e}_m(t))$, $\Theta_{mn}(t, \bar{e}_m(t), \bar{e}_n(t), \beta(t))$, $\Lambda_m(t, \bar{e}_m(t), \beta(t))$ can be denoted as

$$\Gamma_m\left(t, \bar{e}_{m1}(t-\alpha_1(t))\right) = \tilde{\Gamma}_m\left(t, \Psi_{m1}(t-\alpha_1(t))\right) - \tilde{\Gamma}_m\left(t, \Phi_{m1}(t-\alpha_1(t))\right),$$
$$\Gamma_m^3(t, \bar{e}_m(t)) = \tilde{\Gamma}_m^3(t, \Psi_m(t)) - \tilde{\Gamma}_m^3(t, \Phi_m(t)),$$
$$\Theta_{mn}(t, \bar{e}_m(t), \bar{e}_n(t), \beta(t)) = \tilde{\Theta}_{mn}(t, \Psi_m(t), \Psi_n(t), \beta(t)) - \tilde{\Theta}_{mn}(t, \Phi_m(t), \Phi_n(t), \beta(t)),$$
$$\Lambda_m(t, \bar{e}_m(t), \beta(t)) = \tilde{\Lambda}_m(t, \Psi_m(t), \beta(t)) - \tilde{\Lambda}_m(t, \Phi_m(t), \beta(t)).$$

Define

$$M_m(\beta(t)) = \begin{pmatrix} -\tau_{m1}(\beta(t)) & \tau_{m1}(\beta(t)) & 0 \\ \tau_{m2}(\beta(t)) & -\tau_{m2}(\beta(t)) & \tau_{m3}(\beta(t)) \\ 0 & \tau_{m4}(\beta(t)) & -\tau_{m5}(\beta(t)) \end{pmatrix},$$

$$\hat{\Gamma}_m\left(t, \bar{e}_{m1}(t-\alpha_1(t))\right) = \begin{pmatrix} -\xi_{m1}\Gamma_m\left(t, \bar{e}_{m1}(t-\alpha_1(t))\right) \\ 0 \\ 0 \end{pmatrix},$$

then system (4.4) can be written as

$$\begin{cases} \mathrm{d}\bar{e}_m(t) = \Big[M_m(\beta(t))\bar{e}_m(t) + \hat{\Gamma}_m\left(t, \bar{e}_m(t-\alpha_1(t))\right) + \int_{t-\alpha_2}^{t} \Gamma_m^3\left(\sigma, \bar{e}_m(\sigma)\right)\mathrm{d}\sigma \\ \qquad + \sum_{n=1}^{N} \Pi_{mn}(\beta(t))\Theta_{mn}\left(t, \bar{e}_m(t), \bar{e}_n(t), \beta(t)\right)\Big]\mathrm{d}t \\ \qquad + \Lambda_m\left(t, \bar{e}_m(t), \beta(t)\right)\mathrm{dB}(t), \ t \geq 0, \ t \neq t_h, \\ \bar{e}_m(t_h) = I^m_{u(t_h)}(\bar{e}_m(t_h^-)), \ m, \ n \in \mathbb{N}, \ h \in \mathbb{H}. \end{cases} \tag{4.5}$$

Some sufficient conditions for system (4.2) and system (4.3) reach synchronization are derived below.

Theorem 4.1. *If the following conditions are satisfied:*
ZN1. *There are positive numbers γ_m^3, $\theta_{mn}(s)$, $\lambda_m(s)$, $m, n \in \mathbb{N}$, $s \in \mathbb{S}$ and a sequence of stochastic variables $IMP_{u(t_h)}$, $h \in \mathbb{H}$ such that*

$$\begin{aligned} &|\Gamma_m^3\left(t, \bar{e}_m\right)|^2 \leq \gamma_m^3 |\bar{e}_m|^2, \\ &|\Theta_{mn}\left(t, \bar{e}_m, \bar{e}_n, s\right)| \leq \theta_{mn}(s)(|\bar{e}_m| + |\bar{e}_m|), \\ &|\Lambda_m\left(t, \bar{e}_m, s\right)|^2 \leq \lambda_m(s)|\bar{e}_m|^2, \\ &\left|I^m_{u(t_h)}\left(\bar{e}_m\left(t_m^h\right)\right)\right|^2 \leq IMP_{u(t_h)}\left|\bar{e}_m^h\right|^2. \end{aligned}$$

ZN2. *The directed graph (G, THE) is strongly connected, $THE = (THE_{mn})_{N\times N}$, $THE_{mn} = \max_{s\in\mathbb{S}}\{\Pi_{mn}(s)\theta_{mn}(s)\}$.*
ZN3. *The average stochastic impulsive gain and the average stochastic interval satisfy*

$$\frac{\ln AG}{AI} = k < k_1 < 0, k_1 + \eta < 0,$$

in which $\eta = \max_{m\in\mathbb{N}, s\in\mathbb{S}}\Big\{2\gamma_m^1(s) + \gamma_m^2 + \alpha_2\gamma_m^3 + \lambda_m(s) + 4\sum_{n=1}^{N} THE_{mn} + \frac{\bar{w}_m^2(s)}{1-\alpha_1^*} + \bar{w}_m^2(s) + \sum_{\hat{s}\in\mathbb{S}} \zeta_{s\hat{s},s'}\frac{\bar{w}_m(\hat{s})}{\bar{w}_m(s)}\Big\}$. Then driving system (4.2) and response system (4.3) can reach mean-square exponential synchronization.

Proof. *Consider*

$$\bar{e}_m^{\mathrm{T}}(t)M(s)\bar{e}_m(t)$$

$$
\begin{aligned}
&=(\bar{e}_{m1}(t),\bar{e}_{m2}(t),\bar{e}_{m3}(t))\begin{pmatrix}-\tau_{m1}(s) & \tau_{m1}(s) & 0\\ \tau_{m2}(s) & -\tau_{m2}(s) & \tau_{m3}(s)\\ 0 & \tau_{m4}(s) & -\tau_{m5}(s)\end{pmatrix}\begin{pmatrix}\bar{e}_{m1}(t)\\ \bar{e}_{m2}(t)\\ \bar{e}_{m3}(t)\end{pmatrix}\\
&\leq +\left(\frac{1}{2}\tau_{m1}(s)+\frac{1}{2}\tau_{m3}(s)+\frac{1}{2}\tau_{4}(s)-\tau_{m2}(s)\right)|\bar{e}_{m2}(t)|^2\\
&\quad+\left(\frac{1}{2}\tau_{m3}(s)+\frac{1}{2}\tau_{m4}(s)-\tau_{m5}(s)\right)|\bar{e}_{m3}(t)|^2-\frac{1}{2}\tau_{m1}(s)|\bar{e}_{m1}(t)|^2\\
&\leq \max\left\{-\frac{1}{2}\tau_{m1}(s),\frac{1}{2}\tau_{m1}(s)+\frac{1}{2}\tau_{m3}(s)+\frac{1}{2}\tau_{4}(s)-\tau_{m2}(s),\right.\\
&\quad\left.\frac{1}{2}\tau_{m3}(s)+\frac{1}{2}\tau_{m4}(s)-\tau_{m5}(s)\right\}|\bar{e}_{m}(t)|^2\\
&\leq \gamma_m^1(s)|\bar{e}_m(t)|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&|\hat{\Gamma}_m\left(t,\bar{e}_m(t-\alpha_1(t))\right)|=\xi_{m1}|\Gamma_m\left(t,\bar{e}_{m1}(t-\alpha_1(t))\right)|\\
=&\xi_{m1}\left|\nu_2\bar{e}_{m1}(t-\alpha_1(t))+\frac{1}{2}(\nu_2-\nu_1)\left(|\bar{e}_{m1}(t-\alpha_1(t))+1|-|\bar{e}_{m1}(t-\alpha_1(t))-1|\right)\right|\\
\leq&\xi_{m1}\left|\nu_2\bar{e}_{m1}(t-\alpha_1(t))\right|+\xi_{m1}\left|(\nu_2-\nu_1)\bar{e}_{m1}(t-\alpha_1(t))\right|\\
=&\xi_{m1}(2\nu_2-\nu_1)\left|\bar{e}_{m1}(t-\alpha_1(t))\right|\\
\leq&\gamma_m^2\left|\bar{e}_{m1}(t-\alpha_1(t))\right|,
\end{aligned}
$$

combining with **ZN1***, it can be seen that Assumption 2.1 and Assumption 2.2 are both valid. From* **ZN2** *and* **ZN3***, we can conclude that all the conditions in Theorem 3.3 are valid. Therefore, driving system (4.2) and response system (4.3) achieve mean-square exponential synchronization.* □

Remark 4.1. Circuit systems have become a relatively popular topic in recent years due to their wide range of practical applications [1, 5, 11]. Different from them, we consider semi-Markov jump and mixed delay in this paper. Besides, we give the synchronization criteria for the driving system and the response circuit system, which extends the theoretical results and practical applications of the circuit systems.

5. Numerical example

This section utilizes a numerical example to verify the theoretical results in Section 4.

Firstly, we consider driving system (4.2) and response system (4.3) on digraph G with $N = 18$, $S = 2$, and the topological structures are presented in Figure 1 considering $s = 1$ and $s = 2$. Moreover, the non-zero elements of the adjacency matrices $\Pi^1 = (\Pi_{mn}(1))_{18\times 18}$ and $\Pi^2 = (\Pi_{mn}(2))_{18\times 18}$ corresponding to the two states of the semi-Markov jump are chosen in Table 1, and the other elements are installed as zero, meaning there is no arc between the two nodes. In addition, the coupling functions are chosen as

$$\tilde{\Theta}_{mn}(t,\Phi_m(t),\Phi_n(t),1)=\sin(\Phi_m(t))-\sin(\Phi_n(t)),$$

$$\tilde{\Theta}_{mn}(t, \Phi_m(t), \Phi_n(t), 2) = 1.2(\sin(\Phi_m(t)) - \sin(\Phi_n(t))),$$
$$\tilde{\Theta}_{mn}(t, \Psi_m(t), \Psi_n(t), 1) = \sin(\Psi_m(t)) - \sin(\Psi_n(t)),$$
$$\tilde{\Theta}_{mn}(t, \Psi_m(t), \Psi_n(t), 2) = 1.2(\sin(\Psi_m(t)) - \sin(\Psi_n(t))),$$

and

$$\Theta_{mn}(t, \bar{e}_m(t), \bar{e}_n(t), 1) = \sin(\Psi_m(t)) - \sin(\Psi_n(t)) - \sin(\Phi_m(t)) + \sin(\Phi_n(t)),$$
$$\Theta_{mn}(t, \bar{e}_m(t), \bar{e}_n(t), 2) = 1.2(\sin(\Psi_m(t)) - \sin(\Psi_n(t)) - \sin(\Phi_m(t)) + \sin(\Phi_n(t))).$$

From **ZN1**, we get $\theta_{mn}(1) = 1$, $\theta_{mn}(2) = 1.2$. Let $THE_{mn} = \max_{s\in\mathbb{S}} \{\Pi_{mn}(s)\theta_{mn}(s)\}$, then (G, THE) is strongly connected, and the sketch map is depicted in Figure 2.

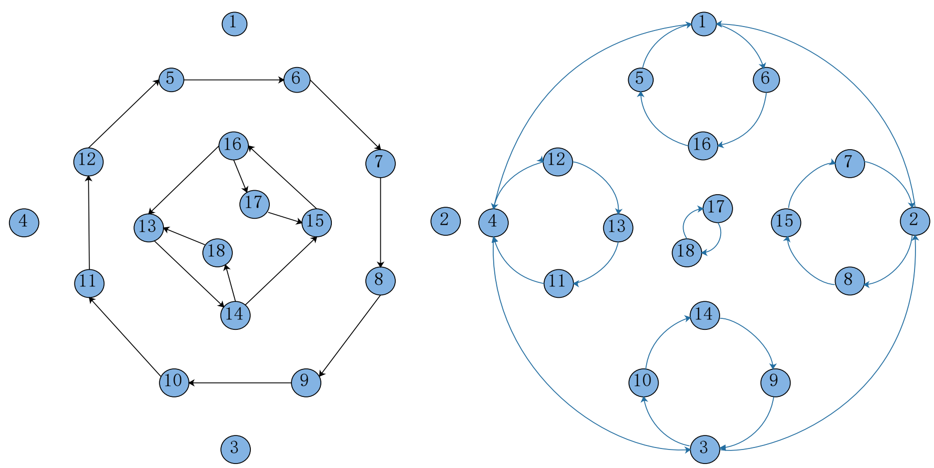

Figure 1. Digraph (G, Π^1) (left) and (G, Π^2) (right).

Table 1. The non-zero elements of adjacency matrices Π^1 and Π^2.

$\Pi_{5,6}(1)$	$\Pi_{6,7}(1)$	$\Pi_{7,8}(1)$	$\Pi_{8,9}(1)$	$\Pi_{9,10}(1)$	$\Pi_{10,11}(1)$	$\Pi_{11,12}(1)$	$\Pi_{12,5}(1)$	$\Pi_{13,14}(1)$
0.001	0.001	0.002	0.001	0.002	0.004	0.005	0.006	0.006
$\Pi_{14,15}(1)$	$\Pi_{15,16}(1)$	$\Pi_{16,13}(1)$	$\Pi_{14,18}(1)$	$\Pi_{18,13}(1)$	$\Pi_{16,17}(1)$	$\Pi_{17,15}(1)$	$\Pi_{1,2}(2)$	$\Pi_{2,3}(2)$
0.002	0.001	0.002	0.001	0.003	0.004	0.002	0.002	0.002
$\Pi_{3,4}(2)$	$\Pi_{4,1}(2)$	$\Pi_{1,6}(2)$	$\Pi_{6,16}(2)$	$\Pi_{16,5}(2)$	$\Pi_{5,1}(2)$	$\Pi_{2,8}(2)$	$\Pi_{8,15}(2)$	$\Pi_{15,7}(2)$
0.002	0.003	0.003	0.003	0.003	0.001	0.004	0.001	0.001
$\Pi_{3,10}(2)$	$\Pi_{10,14}(2)$	$\Pi_{14,9}(2)$	$\Pi_{4,12}(2)$	$\Pi_{12,13}(2)$	$\Pi_{13,11}(2)$	$\Pi_{11,4}(2)$	$\Pi_{17,18}(2)$	$\Pi_{18,17}(2)$
0.002	0.001	0.002	0.001	0.004	0.005	0.006	0.008	0.009

Next, we choose $\nu_1 = -0.45$, $\nu_2 = -0.21$, and the settings of $\tau_{m1}(s)$, $\tau_{m2}(s)$, $\tau_{m3}(s)$, $\tau_{m4}(s)$, $\tau_{m5}(s)$ at $s = 1$ and $s = 2$ are introduced in Table 2 and Table 3 respectively, as well as the values of ξ_{m1} are chosen in Table 4. Then we get $\gamma^1 = \max_{m\in\mathbb{N}, s\in\{1,2\}}\{\gamma_m^1(s)\} = \max_{m\in\mathbb{N}, s\in\{1,2\}}\{\tau_{m2}(s), \tau_{m1}(s) + \tau_{m3}(s) - \tau_{m4}(s), \tau_{m3}(s) + \tau_{m4}(s) - \tau_{m5}(s)\} = 0.0455$, $\gamma^2 = \max_{m\in\mathbb{N}}\{\gamma_m^2\} = \max_{m\in\mathbb{N}}\{\xi_{m1} (2\nu_2 - \nu_1)\} = 0.0830$. We suppose that $\alpha_1(t) = 0.01\cos^2 t$, $\alpha_2 = 0.01$, and we have $\alpha_1 = \alpha_1^* = 0.01$. The initial conditions are picked as $\Phi_{m1}(t) = -0.02$, $\Phi_{m2}(t) = 0.02$, $\Phi_{m3}(t) = -0.04$, $\Psi_{m1}(t) = -0.07$, $\Psi_{m2}(t) = 0.04$, $\Psi_{m3}(t) = -0.08$.

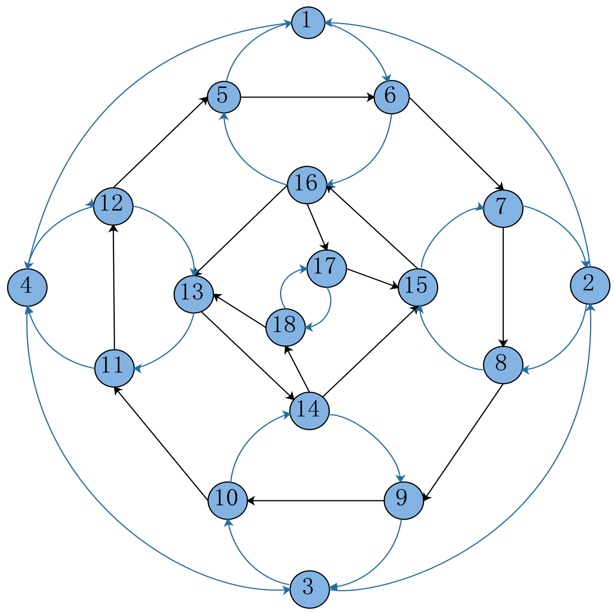

Figure 2. Digraph (G, THE).

Table 2. Settings of some parameters of system (4.5) at $s = 1$.

	m=1	m=2	m=3	m=4	m=5	m=6	m=7	m=8	m=9
$\tau_{m1}(1)$	0.6666	0.7943	0.9660	1.0233	0.6922	0.9108	0.7617	0.7544	0.7717
$\tau_{m2}(1)$	1.1694	1.4361	1.2232	1.3408	1.4569	1.3636	1.3558	1.5123	1.3847
$\tau_{m3}(1)$	0.6465	0.4910	0.7563	0.7340	0.6590	0.5006	0.4574	0.4469	0.5323
$\tau_{m4}(1)$	0.5403	0.6446	0.7160	0.5408	0.8139	0.6243	0.5961	0.6408	0.5822
$\tau_{m5}(1)$	0.9353	0.7483	0.7332	0.8288	0.7822	0.7714	0.7850	0.7671	0.7547
	m=10	m=11	m=12	m=13	m=14	m=15	m=16	m=17	m=18
$\tau_{m1}(1)$	0.9005	0.8279	1.1219	0.8034	0.5747	0.9848	0.5780	1.1202	1.0388
$\tau_{m2}(1)$	1.6119	1.4065	1.4166	1.4252	1.3521	1.5358	1.4707	1.3767	1.4685
$\tau_{m3}(1)$	0.4547	0.6118	0.4555	0.6788	0.6337	0.4934	0.6906	0.6920	0.5825
$\tau_{m4}(1)$	0.5229	0.7765	0.7333	0.5594	0.5376	0.6151	0.7057	0.5541	0.4557
$\tau_{m5}(1)$	0.7714	0.6692	0.8859	0.5736	0.6042	0.6790	0.8034	0.7819	0.8517

Besides, we set that

$$\tilde{\Gamma}_m\left(t, \Phi_{m1}(t-\alpha_1(t))\right) = \Phi_{m1}(t-\alpha_1(t)), \tilde{\Gamma}_m\left(t, \Psi_{m1}(t-\alpha_1(t))\right) = \Psi_{m1}(t-\alpha_1(t)),$$

and

$$\Gamma_m\left(t, \bar{e}_{m1}(t-\alpha_1(t))\right) = \Psi_{m1}(t-\alpha_1(t)) - \Phi_{m1}(t-\alpha_1(t)) = \bar{e}_{m1}(t-\alpha_1(t)).$$

In the following, we select

$$\tilde{\Gamma}^3_m(t, \Phi_m(t)) = \sin(\Phi_m(t)), \tilde{\Gamma}^3_m(t, \Psi_m(t)) = \sin(\Psi_m(t)),$$

and

$$\Gamma^3_m(t, \bar{e}_m(t)) = \sin(\Psi_m(t)) - \sin(\Phi_m(t)).$$

Table 3. Settings of some parameters of system (4.5) at $s = 2$.

	m=1	m=2	m=3	m=4	m=5	m=6	m=7	m=8	m=9
$\tau_{m1}(2)$	1.0061	1.0311	1.4162	1.5171	1.2910	1.2600	1.2155	0.9690	1.1897
$\tau_{m2}(2)$	1.7388	1.9025	1.8591	1.8568	1.6898	1.7398	1.8819	1.7484	1.8078
$\tau_{m3}(2)$	0.7400	0.8363	1.0356	0.7781	0.8733	0.9952	1.0159	0.8134	0.9331
$\tau_{m4}(2)$	0.8097	0.8524	1.0521	0.7899	0.9827	0.8602	0.9626	0.8611	0.8264
$\tau_{m5}(2)$	1.3353	1.1483	1.1332	1.2288	1.1822	1.1714	1.1850	1.1671	1.1547
	m=10	m=11	m=12	m=13	m=14	m=15	m=16	m=17	m=18
$\tau_{m1}(2)$	1.4147	1.1934	1.1356	1.2580	1.4495	1.2459	1.1230	1.0485	1.3224
$\tau_{m2}(2)$	1.9365	1.9054	1.9178	1.7826	1.8516	1.8068	1.7420	1.8056	1.8713
$\tau_{m3}(2)$	1.0500	0.9452	0.7234	0.9132	0.7308	0.7342	0.7951	0.8567	0.9544
$\tau_{m4}(2)$	0.8078	0.8108	0.8756	0.8923	0.8192	0.7281	1.1214	0.9101	0.9050
$\tau_{m5}(2)$	1.1714	1.0692	1.2859	0.9736	1.0042	1.0790	1.2034	1.1819	1.2517

Table 4. Settings of ξ_{m1} of system (4.5).

	m=1	m=2	m=3	m=4	m=5	m=6	m=7	m=8	m=9
ξ_{m1}	1.9714	1.3768	2.4213	1.0542	1.5858	1.3177	2.7668	2.2056	2.0735
	m=10	m=11	m=12	m=13	m=14	m=15	m=16	m=17	m=18
ξ_{m1}	1.5608	1.1651	1.4751	2.2815	2.6628	2.7065	1.4289	2.2735	2.5872

From **ZN1**, we have $\gamma_m^3 = 1$. The stochastic perturbation functions are picked as

$$\tilde{\Lambda}(t, \Phi_m(t), 1) = 0.2\Phi_m(t), \tilde{\Lambda}(t, \Phi_m(t), 2) = 0.4\Phi_m(t),$$

$$\tilde{\Lambda}(t, \Psi_m(t), 1) = 0.2\Psi_m(t), \tilde{\Lambda}(t, \Psi_m(t), 2) = 0.4\Psi_m(t),$$

and

$$\Lambda_m(t, \bar{e}_m(t), 1) = 0.2(\Psi_m(t) - \Phi_m(t)) = 0.2\bar{e}_m(t),$$
$$\Lambda_m(t, \bar{e}_m(t), 2) = 0.4(\Psi_m(t) - \Phi_m(t)) = 0.4\bar{e}_m(t).$$

According to **ZN1**, we derive $\lambda_m(1) = 0.1$, $\lambda_m(2) = 0.2$. Furthermore, we let the state transition rate of the semi-Markov jump is $0.1 \leq \zeta_{12}(\Delta(t)) \leq 0.25$ and $0.25 \leq \zeta_{21}(\Delta(t)) \leq 0.4$. We have $\zeta_{12,1} = 0.1$, $\zeta_{12,2} = 0.25$, $\zeta_{21,1} = 0.25$, $\zeta_{21,2} = 0.4$. And we choose $\bar{w}(1) = 0.1$, $\bar{w}(2) = 0.12$, then we can get

$$\begin{aligned}\eta &= \max_{m\in\mathbb{N}, s\in\mathbb{S}}\{\eta_m(s)\}\\ &= \max_{m\in\mathbb{N}, s\in\mathbb{S}}\left\{2\gamma_m^1(s) + \gamma_m^2 + \alpha_2\gamma_m^3 + \lambda_m(s) + 4\sum_{n=1}^{N} THE_{mn} + \frac{\bar{w}_m^2(s)}{1-\alpha_1^*} + \bar{w}_m^2(s)\right.\\ &\quad\left. + \sum_{\hat{s}\in\mathbb{S}} \zeta_{s\hat{s},s'}\frac{\bar{w}_m(\hat{s})}{\bar{w}_m(s)}\right\}\\ &=0.7496.\end{aligned}$$

We attempt to add the stochastic hybrid impulsive control on response system (4.3) to make it synchronize with driving system (4.2). The impulse gains are a sequence of stochastic variables valued from [0.2,1.2] and obey uniform distribution.

The impulsive intervals are valued from [0.3,0.5]. Then average impulsive gain can be calculated as $AG = 0.7 < 1$, $AI = 0.4$. It is obvious that $\frac{\ln AG}{AI} = k < k_1 = -0.8917 < 0$, $k_1 + \eta = -0.1421 < 0$. Thus, conditions in Theorem 4.1 are all satisfied and response system (4.3) can synchronize with driving system (4.2) in theory. Three-dimensional state trajectories of system (4.2) and system (4.3) are presented in Figure 3, Figure 5, Figure 7. Indeed, they reach synchronization and it can be seen from Figure 4, Figure 6, Figure 8, which performs the tending-to-0 state trajectories of error system (4.4). Besides, it also can be seen that the mean square state trajectories of error system (4.4) tend to 0. The above results illustrate the effectiveness and validity of the theoretical results.

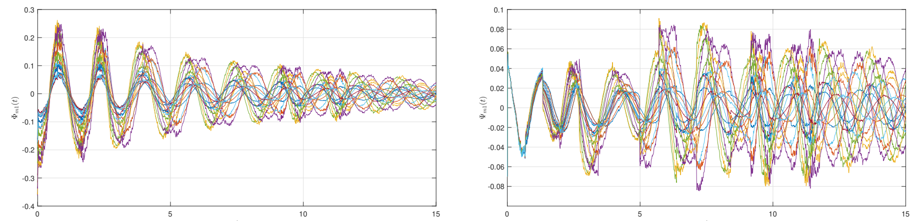

Figure 3. State trajectories $\Phi_{m1}(t)$ of drive system (4.2) (left) and $\Psi_{m1}(t)$ of response system (4.3) (right).

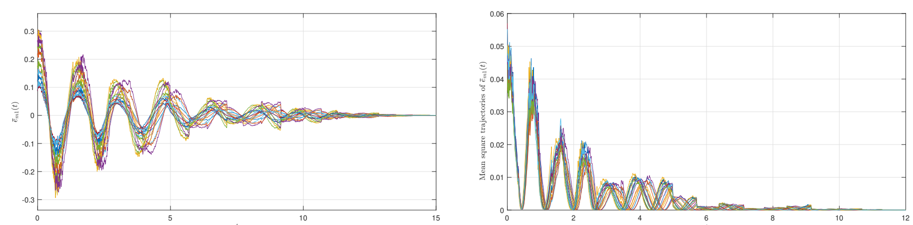

Figure 4. State trajectories $\bar{e}_{m1}(t)$ (left) and mean square trajectories $\mathbb{E}|\bar{e}_{m1}(t)|^2$ (right) of error system (4.4).

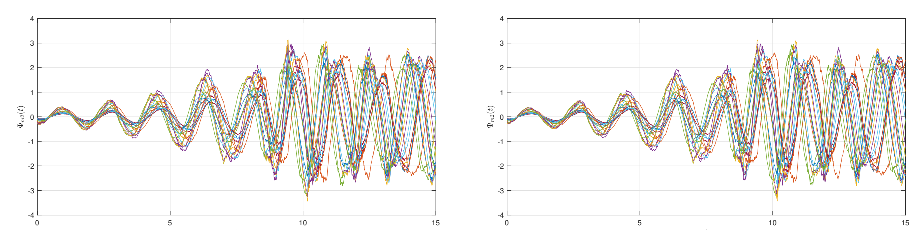

Figure 5. State trajectories $\Phi_{m2}(t)$ of drive system (4.2) (left) and $\Psi_{m2}(t)$ of response system (4.3) (right).

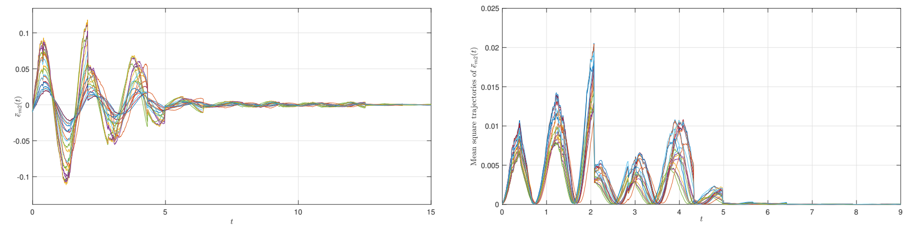

Figure 6. State trajectories $\bar{e}_{m2}(t)$ (left) and mean square trajectories $\mathbb{E}|\bar{e}_{m2}(t)|^2$ (right) of error system (4.4).

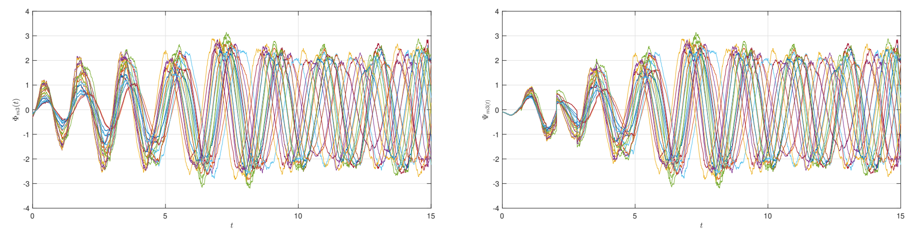

Figure 7. State trajectories $\Phi_{m3}(t)$ of drive system (4.2) (left) and $\Psi_{m3}(t)$ of response system (4.3) (right).

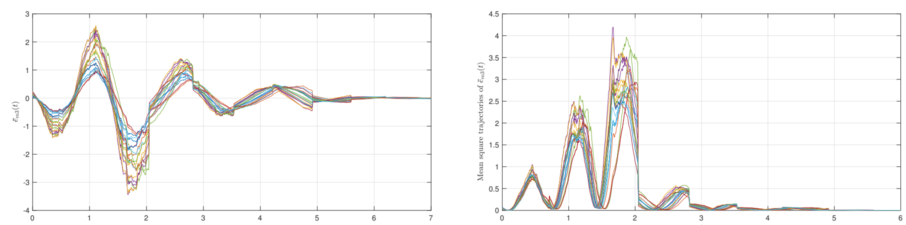

Figure 8. State trajectories $\bar{e}_{m3}(t)$ (left) and mean square trajectories $\mathbb{E}|\bar{e}_{m3}(t)|^2$ (right) of error system (4.4).

6. Conclusion

This paper investigated the exponential synchronization of stochastic semi-Markov jump systems with mixed delay via stochastic hybrid impulsive control. And Dupire functional Itô's formula has been firstly used in the synchronization of the mixed delayed systems under impulsive control. A definition of average stochastic impulsive gain has been put forward to estimate the strength of the stochastic mixed impulses. Based on that, some synchronization criteria for the systems have been provided, related to the topological structure, semi-Markov jump, stochastic disturbance intensity and impulsive control. The theoretical results have also been applied into a class of circuit systems and the related synchronization criteria have been derived. This study provides a new thought on the synchronization of mixed delayed systems and gives a further exploration on the impulsive control systems. And nodes of the complex systems may be connected in a variety of ways, which emerges the investigations on multi-links complex systems. In addition, the time-varying distributed delay is widely presented in communication networks and control systems.

It has some limitations when applying the theoretical results of this paper to the multi-links stochastic functional systems with time-varying distributed delay. In the future, we will explore the new method to investigate the synchronization problem of multi-links stochastic semi-Markov jump systems with time-varying distributed delay.

References

[1] D. A. Burbano-L., S. Yaghouti, C. Petrarca, M. de Magistris and M. di Bernardo, *Synchronization in multiplex networks of Chua's circuits: Theory and experiments*, IEEE Transactions on Circuits and Systems I-Regular Papers, 2020, 67(3), 927–938.

[2] J. Chen, Z. Zeng and P. Jiang, *Global Mittag-Leffler stability and synchronization of memristor-based fractional-order neural networks*, Neural Networks, 2014, 51, 1–8.

[3] T. Chen, X. Liu and Y. Wu, *Prescribed-time stabilization of complex networks with intermittent control*, IEEE Transactions on Circuits and Systems II: Express Briefs, 2023. DOI: 10.1109/TCSII.2023.3330851.

[4] V. K. R. Chimmula and L. Zhang, *Time series forecasting of COVID-19 transmission in Canada using LSTM networks*, Chaos Solitons & Fractals, 2020, 135, 109864.

[5] S. V. Dhople, B. B. Johnson, F. Doerfler and A. O. Hamadeh, *Synchronization of nonlinear circuits in dynamic electrical networks with general topologies*, IEEE Transactions on Circuits and Systems I-Regular Papers, 2014, 61(9), 2677–2690.

[6] S. Dong, G. Chen, M. Liu and Z. Wu, *Robust adaptive H-infinity control for networked uncertain semi-Markov jump nonlinear systems with input quantization*, Science China-Information Sciences, 2022, 65(8), 189201.

[7] A. Doria-Cerezo, J. M. Olm, M. Di Bernardo and E. Nuno, *Modelling and control for bounded synchronization in multi-terminal VSC-HVDC transmission networks*, IEEE Transactions on Circuits and Systems I-Regular Papers, 2016, 63(6), 916–925.

[8] N. Du, D. Nguyen, N. Nguyen and G. Yin, *Stability of stochastic functional differential equations with random switching and applications*, Automatica, 2021, 125, 109410.

[9] B. Dupire, *Functional Ito calculus*, Quantitative Finance, 2019, 19(5), 721–729.

[10] X. Ge, Q. Han, L. Ding, Y. Wang and X. Zhang, *Dynamic event-triggered distributed coordination control and its applications: A survey of trends and techniques*, 2020, 50(9), 3112–3125.

[11] N. Gunasekaran, G. Zhai and Q. Yu, *Sampled-data synchronization of delayed multi-agent networks and its application to coupled circuit*, Neurocomputing, 2020, 413, 499–511.

[12] B. Guo, Y. Xiao, C. Zhang and Y. Zhao, *Graph theory-based adaptive intermittent synchronization for stochastic delayed complex networks with semi-Markov jump*, Applied Mathematics and Computation, 2020, 366, 124739.

[13] W. Guo, Y. Zhang and W. Li, *Synchronization for the coupled stochastic strict-feedback nonlinear systems with delays under pinning control*, Nonlinear Analysis: Hybrid Systems, 2023, 48, 101326.

[14] Y. Guo and Y. Li, *Bipartite leader-following synchronization of fractional-order delayed multilayer signed networks by adaptive and impulsive controllers*, Applied Mathematics and Computation, 2022, 430, 127243.

[15] S. Hu and Q. Zhu, *Stochastic optimal control and analysis of stability of networked control systems with long delay*, Automatica, 2003, 39(11), 1877–1884.

[16] X. Ji, J. Lu, B. Jiang and K. Shi, *Distributed synchronization of delayed neural networks: Delay-dependent hybrid impulsive control*, IEEE Transactions on Network Science and Engineering, 2022, 9(2), 634–647.

[17] T. Jia, X. Chen, F. Zhao, J. Cao and J. Qiu, *Adaptive fixed-time synchronization of stochastic memristor-based neural networks with discontinuous activations and mixed delays*, Journal of the Franklin Institute, 2023, 360(4), 3364–3388.

[18] X. Jin, S. Wang, J. Qin, W. Zheng and Y. Kang, *Adaptive fault-tolerant consensus for a class of uncertain nonlinear second-order multi-agent systems with circuit implementation*, IEEE Transactions on Circuits and Systems I-Regular Papers, 2018, 65(7), 2243–2255.

[19] J. Li, H. Jiang, J. Wang, C. Hu and G. Zhang, *H infinity exponential synchronization of complex networks: aperiodic sampled-data-based event-triggered control*, IEEE Transactions on Cybernetics, 2022, 52(8), 7968–7980.

[20] S. Li, Y. Zheng and H. Su, *Almost sure synchronization of multilayer networks via intermittent pinning noises: A white-noise-based time-varying coupling*, IEEE Transactions on Circuits and Systems I-Regular Papers, 2021, 68(8), 3460–3473.

[21] X. Li and G. Yang, *Graph theory-based pinning synchronization of stochastic complex dynamical networks*, IEEE Transactions on Neural Networks and Learning Systems, 2017, 28(2), 427–437.

[22] H. Liu, F. Zhao, M. Wang, J. Qiu and X. Chen, *Finite-time pinning impulsive synchronization of stochastic complex networks with mixed delays*, Applied Mathematical Modelling, 2023, 117, 840–850.

[23] X. Liu, D. W. C. Ho, Q. Song and W. Xu, *Finite/Fixed-time pinning synchronization of complex networks with stochastic disturbances*, IEEE Transactions on Cybernetics, 2019, 49(6), 2398–2403.

[24] Y. Liu, J. Liu and W. Li, *Stabilization of highly nonlinear stochastic coupled systems via periodically intermittent control*, IEEE Transactions on Automatic Control, 2021, 66(10), 4799–4806.

[25] Y. Liu, Z. Wang and X. Liu, *Global exponential stability of generalized recurrent neural networks with discrete and distributed delays*, Neural Networks, 2006, 19(5), 667–675.

[26] J. Lu, D. W. C. Ho and J. Cao, *A unified synchronization criterion for impulsive dynamical networks*, Automatica, 2010, 46(7), 1215–1221.

[27] D. Nesic and A. Teel, *Input-output stability properties of networked control systems*, IEEE Transactions on Automatic Control, 2004, 49(10), 1650–1667.

[28] D. Nguyen and G. Yin, *Stability of stochastic functional differential equations with regime-switching: Analysis using Dupire's functional Ito formula*, Potential Analysis, 2020, 53(1), 247–265.

[29] H. Shen, M. Chen, Z. Wu, J. Cao and J. Park, *Reliable event-triggered asynchronous extended passive control for semi-Markov jump fuzzy systems and its application*, IEEE Transactions on Fuzzy Systems, 2020, 28(8), 1708–1722.

[30] H. Shen, F. Li, J. Cao, Z. Wu and G. Lu, *Fuzzy-model-based output feedback reliable control for network-based semi-Markov jump nonlinear systems subject to redundant channels*, IEEE Transactions on Cybernetics, 2020, 50(11), 4599–4609.

[31] X. Song, R. Zhang, C. Ahn and S. Song, *Synchronization for semi-Markovian jumping reaction-diffusion complex dynamical networks: A space-time sampled-data control scheme*, IEEE Transactions on Network Science and Engineering, 2022, 9(4), 2684–2696.

[32] J. Sun, M. Zang, P. Liu and Y. Wang, *A secure communication scheme of three-variable chaotic coupling synchronization based on DNA chemical reaction networks*, IEEE Transactions on Signal Processing, 2022, 70, 2362–2373.

[33] Y. Tian, H. Yan, H. Zhang, J. Cheng and H. Shen, *Asynchronous output feedback control of hidden semi-Markov jump systems with random mode-dependent delays*, IEEE Transactions on Automatic Control, 2022, 67(8), 4107–4114.

[34] H. Wang, S. Duan, T. Huang and J. Tan, *Synchronization of memristive delayed neural networks via hybrid impulsive control*, Neurocomputing, 2017, 267, 615–623.

[35] N. Wang, X. Li, J. Lu and F. E. Alsaadi, *Unified synchronization criteria in an array of coupled neural networks with hybrid impulses*, Neural Networks, 2018, 101, 25–32.

[36] P. Wang, B. Zhang and H. Su, *Stabilization of stochastic uncertain complex-valued delayed networks via aperiodically intermittent nonlinear control*, IEEE Transactions on Systems Man Cybernetics-Systems, 2019, 49(3), 649–662.

[37] Z. Wang, J. Cao, Z. Duan and X. Liu, *Synchronization of coupled Duffing-type oscillator dynamical networks*, Neurocomputing, 2014, 136, 162–169.

[38] Y. Wu, Z. Sun, G. Ran and L. Xue, *Intermittent control for fixed-time synchronization of coupled networks*, IEEE/CAA Journal of Automatica Sinica, 2023, 10(6), 1488–1490.

[39] D. Xu, J. Pang and H. Su, *Bipartite synchronization of signed networks via aperiodically intermittent control based on discrete-time state observations*, Neural Networks, 2021, 144, 307–319.

[40] Y. Xu, T. Lin, X. Liu and W. Li, *Exponential bipartite synchronization of fractional-order multilayer signed networks via hybrid impulsive control*, IEEE Transactions on Cybernetics, 2023, 53(6), 3926–3938.

[41] N. Yang, X. Gu and H. Su, *Event-triggered delayed impulsive control for functional differential systems on networks*, Communications in Nonlinear Science and Numerical Simulation, 2024. DOI: 10.1016/j.cnsns.2024.107850.

[42] X. Yang, X. Li, J. Lu and Z. Cheng, *Synchronization of time-delayed complex networks with switching topology via hybrid actuator fault and impulsive effects control*, IEEE Transactions on Cybernetics, 2020, 50(9), 4043–4052.

[43] W. Yu, G. Chen and J. Cao, *Adaptive synchronization of uncertain coupled stochastic complex networks*, Asian Journal of Control, 2011, 13(3), 418–429.

[44] C. Zhang, W. Li and K. Wang, *Exponential synchronization of stochastic coupled oscillators networks with delays*, Applicable Analysis, 2017, 96(6), 1058–1075.

[45] H. Zhang, Z. Qiu and L. Xiong, *Stochastic stability criterion of neutral-type neural networks with additive time-varying delay and uncertain semi-Markov jump*, Neurocomputing, 2019, 333, 395–406.

[46] N. Zhang, X. Wang and W. Li, *Stability for multi-linked stochastic delayed complex networks with stochastic hybrid impulses by Dupire Itô's formula*, Nonlinear Analysis-Hybrid Systems, 2022, 45, 101200.

[47] N. Zhang, X. Wang, C. Tong and W. Li, *Semi-global exponential stability of stochastic nonlinear functional sampling systems by emulation approach*, Communications in Nonlinear Science and Numerical Simulation, 2023, 125(6), 107336.

[48] W. Zhang, X. Yang and C. Li, *Fixed-time stochastic synchronization of complex networks via continuous control*, IEEE Transactions on Cybernetics, 2019, 49(8), 3099–3104.

[49] X. Zhao, L. Liu, H. Wang and M. Fan, *Ecological effects of predator harvesting and environmental noises on oceanic coral reefs*, Bulletin of Mathematical Biology, 2023, 85(7), 59.

[50] H. Zhou, S. Li and C. Zhang, *Synchronization of hybrid switching diffusions delayed networks via stochastic event-triggered control*, Neural Networks, 2023, 159, 1–13.