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THE SOLVABILITY OF SOME KINDS OF SINGULAR INTEGRAL EQUATIONS OF CONVOLUTION TYPE WITH VARIABLE INTEGRAL LIMITS

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Abstract In this paper, we discuss several classes of convolution type singular integral equations with variable integral limits in class H_1^*. By means of the theory of complex analysis, Fourier analysis and integral transforms, we can transform singular integral equations with variable integral limits into the Riemann boundary value problems with discontinuous coefficients. Under the solvability conditions, the existence and uniqueness of the general solutions can be obtained. Further, we analyze the asymptotic properties of the solutions at the nodes. Our work improves the Noether theory of singular integral equations and boundary value problems, and develops the knowledge architecture of complex analysis.

Keywords Singular integral equations, variable integral limits, Riemann boundary value problems, Wiener-Hopf type, dual type, Fourier integral transform.

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1. Introduction

Singular integral equations (SIEs) are closely related to the classical theory of boundary value problems for analytic functions, which has a wide range of applications in many fields, such as quantum mechanics, asymptotic analysis, statistical physics and orthogonal polynomial theory. Many scholars have systematically researched convolution type SIEs and formed a rigorous theoretical system.

Gakhov [10–12] studied the general solutions of Riemann boundary value problems. Litvinchuk [29] obtained the explicit solutions of SIEs in more general cases, and further developed the Noether theory of the general SIEs. Muskhelishvilli [36] investigated the conditions of solvability for SIEs with convolution kernels and discontinuous coefficients. Lu [30, 31] considered the explicit solutions and the solvability theory of convolution SIEs with constant coefficients, and obtained some worthwhile results. Du and Shen [5] further dealt with the integral equations of convolution type with variable coefficients. Subsequently, Li and Ren [21, 22, 27, 28] developed the theory of solvability and asymptotic theory for singular integral equa-

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tions with the mixture of convolution kernel and singular integral kernel in the case of non-normal type.

Based on the above work, in this paper, we will deal with SIEs of convolution type with variable integral limits, and we investigate the asymptotic properties and Noether solvability theory of solutions for such SIEs with variable integral limits under the solvability conditions. The main aim of this paper is to solve the following three classes SIEs with convolution kernel and variable integral limits.

(1) SIEs with two convolution kernels

$$\begin{aligned} &A\psi(\tau)+\frac{B}{\sqrt{2\pi}}\int_0^\tau \psi(t)h(\tau-t)dt+\frac{C}{\sqrt{2\pi}}\int_\tau^0 \psi(t)k(\tau-t)dt \\ &+\frac{D}{\pi i}\int_{\mathbb{R}}\frac{\psi(t)}{t-\tau}dt=n(\tau), \quad \tau\in\mathbb{R}. \end{aligned} \tag{1.1}$$

(2) SIEs of Wiener-Hopf with convolution kernels

$$A\psi(\tau)+\frac{B}{\sqrt{2\pi}}\int_0^\tau \psi(t)h(\tau-t)dt+\frac{C}{\pi i}\int_{\mathbb{R}}\frac{\psi(t)}{t-\tau}dt=n(\tau), \quad \tau\in\mathbb{R}^+. \tag{1.2}$$

(3) Dual SIEs with convolution kernels

$$\begin{cases} A\psi(\tau)+\dfrac{B}{\sqrt{2\pi}}\displaystyle\int_0^\tau \psi(t)h_1(\tau-t)dt+\dfrac{C}{\pi i}\displaystyle\int_{\mathbb{R}}\frac{\psi(t)}{t-\tau}dt=n(\tau), \quad \tau\in\mathbb{R}^+; \\ A\psi(\tau)+\dfrac{B}{\sqrt{2\pi}}\displaystyle\int_\tau^0 \psi(t)h_2(\tau-t)dt+\dfrac{C}{\pi i}\displaystyle\int_{\mathbb{R}}\frac{\psi(t)}{t-\tau}dt=n(\tau), \quad \tau\in\mathbb{R}^-. \end{cases} \tag{1.3}$$

In Eqs. (1.1)-(1.3), A, B, C, D are constants. The known functions $h, k, n, h_j \in H_1, j = 1, 2$. We require the unknown function $\psi \in H_1$. The notations mentioned above can be referred to Section 2.

It is well known that SIEs with variable integral limits, an important class of equations in physics, is closely related to the Riemann boundary value problems. By the Sokhotski-Plemelj formula and the principle of analytic continuation, we transform Eqs. (1.1)-(1.3) into linear Riemann-Hilbert problems with discontinuous coefficients, and prove the existence and uniqueness of analytic solutions given by integral-form. We propose a novel approach different from the one used in the classical Riemann-Hilbert problems. In view of the theory of classical boundary value problems for analytic functions, we study the properties of solutions at nodes, and obtain the solvability conditions and asymptotic properties of the general solutions. Therefore, this paper develops the theory of Noether solvability of SIEs and boundary value problems, and extends the ones in [8, 9, 32–35].

This paper is arranged as follows. In Section 2, we give properties of the function classes $H_1^*(H_2^*, H_3^*), H_1(H_2, H_3)$, and the relation between Cauchy type integrals and Fourier transform. In Sections 3-5, by using the boundary value theory, complex analysis and the system of linear algebra, the explicit solutions and asymptotic properties are obtained under the conditions of solvability, and the properties of solutions at nodes are further analyzed.

2. Preliminaries

$\mathbb{R}, \mathbb{C}$ denote the sets of real and complex numbers respectively, $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. As usual, $\mathrm{C}(\mathbb{R}), \mathrm{C}(\bar{\mathbb{R}})$ denote the sets of continuous functions on $\mathbb{R}$ and $\bar{\mathbb{R}}$ respectively. For $1 \leq q < \infty$, the space of Lebesgue integrable functions $L^q(\mathbb{R}) = \left\{\Phi \mid \|\Phi\|_q < \infty\right\}$ with the standard norm

$$\|\Phi\|_q = \left(\int_{\mathbb{R}} |\Phi(s)|^q ds\right)^{\frac{1}{q}},$$

where $\mathbb{R}^- = (-\infty, 0), \mathbb{R}^+ = (0, \infty)$, $\mathbb{R} = (-\infty, \infty)$.

Moreover, we respectively denote by $H(\mathbb{R})$, $\hat{H}(\overline{\mathbb{R}})$ as the spaces of Hölder continuous functions on $\mathbb{R}$ and $\bar{\mathbb{R}}$.

In the following, we give some necessary preliminary knowledge and notations.

Definition 2.1. Let $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, denote the Fourier integral transform $\mathcal{W}$ and inverse transform $\mathcal{W}^{-1}$ as

$$\mathcal{W}[\psi(\tau)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{is\tau} \psi(\tau) d\tau, \quad \mathcal{W}^{-1}[\Phi(s)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-is\tau} \Phi(s) ds. \tag{2.1}$$

For simplicity, (2.1) can be denoted as

$$\mathcal{W}[\psi(\tau)] = \Phi(s), \quad \mathcal{W}^{-1}[\Phi(s)] = \psi(\tau). \tag{2.2}$$

From (2.1) and (2.2), we know that

$$\mathcal{W} : L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \to L^2(\mathbb{R}). \tag{2.3}$$

Since $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, from the Planchere theorem [10, 15], the operator $\mathcal{W}$ can be uniquely extended to a self-mapping

$$\mathcal{W} : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \tag{2.4}$$

and

$$\|\Phi\|_2 = \|\psi\|_2 . \tag{2.5}$$

We introduce the concepts of several classes H_1, H_2, H_3 and H_1^*, H_2^*, H_3^*.

Definition 2.2. If $\Phi \in L^2(\mathbb{R}) \cap \hat{H}(\overline{\mathbb{R}})$, we say $\Phi \in H_1^*$. If $\Phi \in H_1^*$, then $\mathcal{W}^{-1}\Phi = \psi \in H_1$.

Definition 2.3. If (1) $\Phi \in \hat{H}(\overline{\mathbb{R}})$; (2) For $\ell > \frac{1}{2}$, $s \in N_\infty$, $\Phi(s) = O(|s|^{-\ell})$, we say $\Phi \in {H_2^*}^{\ell}$ or H_2^*, where $N_\infty = \left\{s \in \mathbb{R} \mid |s| > \delta^{-1}, \forall \delta > 0\right\}$.

For $\Phi \in {H_2^*}^{\ell}$ or H_2^*, we say that $\mathcal{W}^{-1}\Phi = \psi \in H_2^\ell$ or H_2.

Definition 2.4. If (1) $\Phi \in \hat{H}(\overline{\mathbb{R}}) \cap H^\ell(N_\infty), \ell > \frac{1}{2}$; (2) $\Phi(\infty) = 0$, then $\Phi \in {H_3^*}^{\ell}$ or H_3^*, and $\psi \in H_3^\ell$ or H_3.

For $s \in N_\infty$, we put

$$h_1 = \left\{\Phi \mid \Phi(s) = O(|s|^{-\ell})\right\}, \quad h_2 = H^\ell(N_\infty), \quad h_3 = \{\Phi \mid \Phi(\infty) = 0\}.$$

If $\Phi \in h_2 \cap h_3$, we have

$$|\Phi(s_1) - \Phi(s_2)| \leq a \left|\frac{1}{s_1} - \frac{1}{s_2}\right|^{\ell}, \quad s_1, s_2 \in N_\infty, \tag{2.6}$$

where $a \in \mathbb{R}^+, \ell \in (0,1]$. In (2.6), we let $s_2 \to \infty$, since $\lim_{s_2 \to \infty} \Phi(s_2) = 0$, thus we have

$$|\Phi(s_1)| \leq a|\frac{1}{s_1}|^\ell,$$

which implies $\Phi \in h_1$. Further, we get

$$\int_{\mathbb{R}} |\Phi(s)|^2 ds \leq a^2 \int_{\mathbb{R}} |s|^{-2\ell} ds. \tag{2.7}$$

When $\ell > \frac{1}{2}$, it is easily seen that

$$\|\Phi\|_2 < \infty, \quad i.e., \quad \Phi \in L^2(\mathbb{R}).$$

Notice that, $H_2^* = \hat{H} \cap h_1, H_3^* = \hat{H} \cap h_2 \cap h_3$, hence $H_3^* \subset H_2^* \subset H_1^* \subset H \cap L^2$, further, $H_3 \subset H_2 \subset H_1$.

Definition 2.5. Let $\psi, g \in L^2(\mathbb{R})$, then their convolution is defined by

$$\psi * g(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(t) g(\tau - t) dt. \tag{2.8}$$

Obviously, we have

$$\psi * g(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(t) g(\tau - t) dt = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(t) \psi(\tau - t) dt = g * \psi(\tau), \tag{2.9}$$

this implies, the convolution is commutative. From the Hölder inequality, we know that $\psi * g \in L^2(\mathbb{R})$. By the convolution theorem [4, 36], we have

$$\mathcal{W}[\psi * g(\tau)] = \mathcal{W}[\psi(\tau)] \cdot \mathcal{W}[g(\tau)] = \Phi(s) G(s). \tag{2.10}$$

Definition 2.6. Denote the Cauchy principal integral operator $\mathcal{V}$ as follow

$$\mathcal{V}\psi(\tau) = P.V. \frac{1}{\pi i} \int_{\mathbb{R}} \frac{\psi(\tau)}{\tau - t} d\tau = \lim_{\substack{\varepsilon \to +0 \\ X \to +\infty}} \frac{1}{\pi i} \int_{[-X,X] \setminus (t-\varepsilon, t+\varepsilon)} \frac{\psi(\tau)}{\tau - t} d\tau. \tag{2.11}$$

The operator $\mathcal{V}$ is a self-mapping under the modified notion of Cauchy principal value integrals. Moreover, in view of the Poincaré-Bertrand formula and the Riesz theorem [23, 28], we know that $\mathcal{V}$ is an involution in $L^2(\mathbb{R})$, namely, $\mathcal{V}^2 = \mathcal{I}$, where $\mathcal{I}$ is a unit operator.

Definition 2.7. We denote the operators $\mathcal{A}, \mathcal{K}$ as follows

$$\mathcal{A}\psi(\tau) = \psi(-\tau), \quad \mathcal{K}\psi(\tau) = \psi(\tau) \mathrm{sgn}(\tau).$$

Obviously, we have $\mathcal{A}^2 = \mathcal{K}^2 = \mathcal{I}$.

Lemma 2.1. *For the function $\psi(\tau) = \mathcal{W}^{-1}[\Phi(s)]$, we can get*

$$\mathcal{W}[\frac{1}{\pi i} \int_{\mathbb{R}} \frac{\psi(\tau)}{\tau - t} d\tau] = -\Phi(s) sgn(s), \quad s \in \mathbb{R}. \tag{2.12}$$

Proof. Denote Cauchy type integral $\psi(z)$ as

$$\psi(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\psi(t)}{t-z}dt, \quad z \in \mathbb{C}\setminus\mathbb{R}. \tag{2.13}$$

Substituting (2.2) into (2.13), it follows that

$$\psi(z) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \Phi(s)ds\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{e^{-ist}}{t-z}dt.$$

From the generalized residue theorem [10–12, 22], for $z \in \mathbb{C}^+$, we have

$$\psi^+(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\psi(t)}{t-z}dt = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^-} \Phi(s)e^{-isz}ds, \tag{2.14}$$

and for $z \in \mathbb{C}^-$, we have

$$\psi^-(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\psi(t)}{t-z}dt = \frac{-1}{\sqrt{2\pi}}\int_{\mathbb{R}^+} \Phi(s)e^{-isz}ds. \tag{2.15}$$

By the Sokhotski-Plemelj formula, we obtain

$$\mathcal{V}\psi(\tau) = \psi^+(t) + \psi^-(t) = \frac{-1}{\sqrt{2\pi}}\int_{\mathbb{R}} \Phi(s)\mathrm{sgn}(s)e^{-ist}ds = \mathcal{W}^{-1}[\mathcal{K}\Phi(s)],$$

where

$$\psi^+(t) = \lim_{\substack{z\to t\\ z\in\mathbb{C}^+}} \psi^+(z), \qquad \psi^-(t) = \lim_{\substack{z\to t\\ z\in\mathbb{C}^-}} \psi^-(z).$$

The proof is complete. □

Lemma 2.2. *For the operators $\mathcal{W}, \mathcal{A}, \mathcal{V}, \mathcal{K}$, we have*

$$(1)\ \mathcal{W}\mathcal{K}\mathcal{V} + \mathcal{V}\mathcal{K}\mathcal{W} = 0; \quad (2)\ \mathcal{V}\mathcal{W}\mathcal{A} - \mathcal{A}\mathcal{V}\mathcal{W} = 0.$$

Proof. (1) Note that, we may write (2.12) in the following form

$$\mathcal{W}\mathcal{V} = -\mathcal{K}\mathcal{W}.$$

Similarly, we can verify that

$$\mathcal{W}\mathcal{K} = \mathcal{V}\mathcal{W}, \tag{2.16}$$

hence

$$\mathcal{W}\mathcal{K}\mathcal{V} = \mathcal{V}\mathcal{W}\mathcal{V} = -\mathcal{V}\mathcal{K}\mathcal{W}.$$

(2) From (2.1) and (2.2), we know that

$$\mathcal{W}[\Phi(s)] = \mathcal{W}^{-1}[\Phi(-s)] = \psi(-\tau), \quad \mathcal{W}^{-1}[\psi(\tau)] = \mathcal{W}[\psi(-\tau)] = \Phi(-s),$$

which implies

$$\mathcal{W}^{-1} = \mathcal{A}\mathcal{W}, \qquad \mathcal{A}^{-1} = \mathcal{W}^2, \qquad \mathcal{K}\mathcal{A} - \mathcal{A}\mathcal{K} = 0, \tag{2.17}$$

it gives rise to

$$\mathcal{A}\mathcal{V}\mathcal{W} = \mathcal{A}\mathcal{W}\mathcal{K} = \mathcal{A}\mathcal{W}^3\mathcal{A}\mathcal{K} = \mathcal{W}\mathcal{K}\mathcal{A} = \mathcal{V}\mathcal{W}\mathcal{A}.$$

This completes the proof. □

Lemma 2.3. *If $\psi \in H_1$, and $\mathcal{W}\psi(0) = 0$, we have $\mathcal{W}\mathcal{V}\psi \in H_1^*$, further $\mathcal{V}\psi \in H_1$.*

Proof. By assumption, we have $\mathcal{W}\psi = \Phi \in H_1^*$. Notice that

$$\lim_{s\to\infty} \Phi(s) = 0,$$

and $\Phi(0) = 0$, one has $\mathcal{K}\Phi \in \hat{H}$ and

$$\int_{\mathbb{R}} |\Phi(s)\mathrm{sgn}s|^2 ds < \infty, \quad i.e., \quad \mathcal{K}\Phi \in L^2(\mathbb{R}),$$

thus we get $\mathcal{W}\mathcal{V}\psi \in H_1^*$, further, $\mathcal{V}\psi \in H_1$. □

Lemma 2.4 (Lemma 2.1, [24])**.** *If $\psi, g \in H_1(H_2, H_3)$, then $\psi * g \in H_1(H_2, H_3)$, thus we have $\Phi G \in H_1^*(H_2^*, H_3^*)$; if $\psi \in H_1, g \in H_2(H_3)$, then $\psi * g \in H_2(H_3)$, further, $\Phi G \in H_2^*(H_3^*)$.*

Lemma 2.5. *Let $\Phi \in \hat{H}$, we denote the Cauchy type integral as follows*

$$\Phi(z) = \int_{\mathbb{R}} \frac{\Phi(s)}{s - z} ds, \qquad z \notin \mathbb{R}, \tag{2.18}$$

then we can get

$$\mathcal{W}[\psi^+(\tau)] = \Phi^+(s), \quad \mathcal{W}[\psi^-(\tau)] = \Phi^-(s), \quad \Phi^+(s) - \Phi^-(s) = \Phi(s), \tag{2.19}$$

where

$$\psi^+(\tau) = \begin{cases} \psi(\tau), & \tau \geq 0; \\ 0, & \tau < 0, \end{cases} \qquad \psi^-(\tau) = \begin{cases} 0, & \tau \geq 0; \\ -\psi(\tau), & \tau < 0. \end{cases}$$

Proof. From (2.16) and Sokhotski-Plemelj formula, we have

$$\Phi^+(s) = \frac{1}{2}\mathcal{W}[\psi(\tau)] + \frac{1}{2}\mathcal{W}[\mathcal{K}\psi(\tau)] = \frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}} \psi(\tau)(1 + \mathrm{sgn}(\tau))e^{is\tau} d\tau = \mathcal{W}[\psi^+(\tau)],$$

for the negative boundary value, we have

$$\Phi^-(s) = \frac{-1}{2}\mathcal{W}[\psi(\tau)] + \frac{1}{2}\mathcal{W}[\mathcal{K}\psi(\tau)] = \frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}} \psi(\tau)(\mathrm{sgn}(\tau) - 1)e^{is\tau} d\tau = \mathcal{W}[\psi^-(\tau)].$$

Further,

$$\Phi^+(s) - \Phi^-(s) = \mathcal{W}[\psi^+(\tau)] - \mathcal{W}[\psi^-(\tau)] = \mathcal{W}[\psi(\tau)] = \Phi(s).$$

This proof is complete. □

From (2.19), we know that the positive and negative boundary values $\Phi^\pm(s)$ of $\Phi(z)$ are the single sided Fourier integral transforms of $\psi^\pm(\tau)$, respectively.

3. SIEs with two convolution kernels

We solve the following SIEs with two convolution kernels and variable integral limits

$$\begin{aligned} &A\psi(\tau) + \frac{B}{\sqrt{2\pi}} \int_0^\tau \psi(t)h(\tau - t)dt + \frac{C}{\sqrt{2\pi}} \int_\tau^0 \psi(t)k(\tau - t)dt \\ &+ \frac{D}{\pi i} \int_{\mathbb{R}} \frac{\psi(t)}{t - \tau} dt = n(\tau), \quad \tau \in \mathbb{R}, \end{aligned} \tag{3.1}$$

where A, B, C, D are constants, and $D \neq 0$. The functions $h, k, n \in H_1$, and the unknown function $\psi \in H_1$.

Expanding t to $t \in \mathbb{R}$, then Eq. (3.1) can be transformed into

$$\begin{aligned} &A\psi(\tau) + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(t) h_1(\tau - t) \mathrm{sgn}(t) dt + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(t) k_1(\tau - t) dt \\ &+ \frac{D}{\pi i} \int_{\mathbb{R}} \frac{\psi(t)}{t - \tau} dt = n(\tau), \quad \tau \in \mathbb{R}, \end{aligned} \tag{3.2}$$

where

$$h_1(\tau) = \frac{B}{2} h^+(\tau) + \frac{C}{2} k^-(\tau), \qquad k_1(\tau) = \frac{B}{2} h^+(\tau) - \frac{C}{2} k^-(\tau),$$

in which

$$h^{\pm}(\tau) = \frac{\mathrm{sgn}(\tau) \pm 1}{2} h(\tau), \qquad k^{\pm}(\tau) = \frac{\mathrm{sgn}(\tau) \pm 1}{2} k(\tau),$$

that is to say,

$$h_1(\tau) = \begin{cases} \frac{B}{2} h(\tau), & \tau \geq 0; \\ -\frac{C}{2} k(\tau), & \tau < 0, \end{cases} \qquad k_1(\tau) = \begin{cases} \frac{B}{2} h(\tau), & \tau \geq 0; \\ \frac{C}{2} k(\tau), & \tau < 0. \end{cases}$$

Applying the Fourier transforms to both sides of (3.2), then we have

$$A\Phi(s) + \frac{H(s)}{\pi i} \int_{\mathbb{R}} \frac{\Phi(t)}{t - s} dt + K(s)\Phi(s) - D\mathrm{sgn}(s)\Phi(s) = N(s), \tag{3.3}$$

where

$$\mathcal{W}[h_1(\tau)] = H(s),\ \mathcal{W}[k_1(\tau)] = K(s),\ \mathcal{W}[n(\tau)] = N(s),\ \mathcal{W}[\psi(\tau)] = \Phi(s).$$

Denote the Cauchy type integral

$$\Phi(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\Phi(t)}{t - z} dt, \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{3.4}$$

by the Sokhotski-Plemelj formula, we can get

$$\Phi^+(s) - \Phi^-(s) = \Phi(s), \quad \Phi^+(s) + \Phi^-(s) = \frac{1}{\pi i} \int_{\mathbb{R}} \frac{\Phi(t)}{t - s} dt. \tag{3.5}$$

Substituting (3.5) into (3.3), then we shall solve the following Riemann problem in place of (3.3)

$$\Phi^+(s) = J(s)\Phi^-(s) + N_0(s), \quad s \in \mathbb{R}, \tag{3.6}$$

where

$$J(s) = \frac{A - H(s) + K(s) - D\mathrm{sgn}(s)}{A + H(s) + K(s) - D\mathrm{sgn}(s)}, \quad N_0(s) = \frac{N(s)}{A + H(s) + K(s) - D\mathrm{sgn}(s)}.$$

Moreover, we can write $J(s), N_0(s)$ in the forms

$$J(s) = \begin{cases} \dfrac{A - H(0) + K(0)}{A + H(0) + K(0)}, & s = 0, \\ 1 - \dfrac{2H(s)}{A + H(s) + K(s) - D}, & s \in \mathbb{R}^+, \\ 1 - \dfrac{2H(s)}{A + H(s) + K(s) + D}, & s \in \mathbb{R}^-, \end{cases}$$

$$N_0(s)=\begin{cases}\dfrac{N(0)}{A+H(0)+K(0)}, & s=0,\\ \dfrac{N(s)}{A+H(s)+K(s)-D}, & s\in\mathbb{R}^+,\\ \dfrac{N(s)}{A+H(s)+K(s)+D}, & s\in\mathbb{R}^-.\end{cases}$$

Since $H,K,N\in H_1^*$, then we know that

$$\lim_{s\to\infty}H(s)=\lim_{s\to\infty}K(s)=\lim_{s\to\infty}N(s)=0,$$

which implies $\lim\limits_{s\to\infty}J(s)=1$. In this case, $s=\infty$ is not a node of (3.6).

Let $J_1(s)=A+H(s)+K(s)-D\mathrm{sgn}(s)$ have some zero-points $d_1,d_2,\cdots,d_m$ with the orders $\alpha_1,\alpha_2,\cdots,\alpha_m$ respectively; let $J_2(s)=A-H(s)+K(s)-D\mathrm{sgn}(s)$ have some zero-points $e_1,e_2,\cdots,e_n$ with the orders $\beta_1,\beta_2,\cdots,\beta_n$ respectively, where $d_i\neq e_j$, $\alpha_i,\beta_j(i=1,2,\cdots,m;j=1,2,\cdots,n)$ are non-negative integers. Let

$$E_1(s)=\prod_{i=1}^{m}(s-d_i)^{\alpha_i},\qquad E_2(s)=\prod_{j=1}^{n}(s-e_j)^{\beta_j},$$

and

$$\alpha_1+\alpha_2+\cdots+\alpha_m=M_1,\quad \beta_1+\beta_2+\cdots+\beta_n=M_2.$$

Next, we only consider the case of non-normal type, that is, $M_1>0,M_2>0$.

Without loss of generality, we take any fixed points $z_0=a+ib\in\mathbb{C}^+$, and $z_0^*=a-ib\in\mathbb{C}^-$, and rewrite (3.6) as

$$\Phi^+(s)=\frac{E_2(s)(s-z_0^*)^{M_1}}{E_1(s)(s-z_0)^{M_2}}P(s)\Phi^-(s)+N_0(s),\quad s\in\mathbb{R},\tag{3.7}$$

where

$$J(s)=\frac{E_2(s)(s-z_0^*)^{M_1}}{E_1(s)(s-z_0)^{M_2}}P(s),\qquad P(s)\neq 0,$$

and $P\in H$. Note that, the solution $\Phi(s)$ of (3.6) is continuous along $\mathbb{R}$, and $\lim\limits_{s\to\infty}\Phi(s)=0$.

Denote

$$\gamma_0=\lambda_0+i\eta_0,\quad \lambda_0=\frac{1}{2\pi}\arg\frac{P(-0)}{P(+0)},\quad \eta_0=-\frac{1}{2\pi}\ln|\frac{P(-0)}{P(+0)}|,\tag{3.8}$$

and we say κ the index of (3.7) which satisfies $\kappa=[\lambda_0]$. Let

$$\lambda=\lambda_0-\kappa,\qquad \gamma=\gamma_0-\kappa,$$

hence $\gamma=\lambda+i\eta_0,\lambda\in[0,1)$.

In order to prove the solvability of (3.6), we first introduce the sectionally holomorphic function

$$X(z)=\begin{cases}(z-z_0^*)^{-\kappa}e^{\Gamma_0(z)}, & \mathrm{Im}z>0;\\ (z-z_0)^{-\kappa}e^{\Gamma_0(z)}, & \mathrm{Im}z<0,\end{cases}\tag{3.9}$$

in which we have put

$$\Gamma_0(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \ln P_0(s)\frac{ds}{s-z}, \quad z \in \mathbb{C}\setminus\mathbb{R}, \tag{3.10}$$

and

$$P_0(s) = \left(\frac{s-z_0^*}{s-z_0}\right)^{\kappa} P(s), \quad P(s) = \frac{X^+(s)}{X^-(s)}.$$

Obviously,

$$\kappa \ln\frac{s-z_0^*}{s-z_0} = \ln P_0(s) - \ln P(s), \tag{3.11}$$

where $\ln P_0 \in \hat{H}$, and we take a continuous single-valued branch of $\ln\frac{s-z_0^*}{s-z_0}$ which satisfies

$$\ln\frac{\pm\infty - z_0^*}{\pm\infty - z_0} = 0, \quad \text{or} \quad \ln\frac{\pm 0 - z_0^*}{\pm 0 - z_0} = \pm\pi i.$$

Therefore, we rewrite (3.7) in the form

$$\Phi^+(s) = \frac{E_2(s)(s-z_0^*)^{M_1}X^+(s)}{E_1(s)(s-z_0)^{M_2}X^-(s)}\Phi^-(s) + N_0(s), \tag{3.12}$$

for convenience, we deal with (3.12) in the problem R_{-1}, that is, $\Phi(\infty) = 0$.

For homogeneous problem of (3.12), we consider the function

$$\Omega(z) = \begin{cases} \dfrac{\Phi(z)}{X(z)E_2(z)}(z-z_0^*)^{-M_1}, & \mathrm{Im}z > 0, \\ \dfrac{\Phi(z)}{X(z)E_1(z)}(z-z_0)^{-M_2}, & \mathrm{Im}z < 0. \end{cases} \tag{3.13}$$

From the principle of analytic continuation, we know that $\Omega(z)$ is analytic in $\mathbb{C}$. Moreover, by the generalized Liouville theorem, we have

$$\Omega(z) = U_{\kappa-M_1-M_2-1}(z) = \sum_{t=0}^{\kappa-M_1-M_2-1} C_{\kappa-M_1-M_2-1-t}z^t,$$

where $C_0, \cdots, C_{\kappa-M_1-M_2-1}$ are arbitrary complex constants. Therefore, the analytic solution of (3.12) is given by

$$\Phi_1(z) = \begin{cases} E_2(z)(z-z_0^*)^{M_1}X(z)U_{\kappa-M_1-M_2-1}(z), & \mathrm{Im}z > 0, \\ E_1(z)(z-z_0)^{M_2}X(z)U_{\kappa-M_1-M_2-1}(z), & \mathrm{Im}z < 0. \end{cases}$$

When $\kappa \le M_1 + M_2$, we have $\Omega(z) = 0$, thus $\Phi_1(z) \equiv 0$.

To solve the non-homogeneous problem (3.12), we introduce the following function

$$Y(z) = \begin{cases} X_0(z), & \mathrm{Im}z > 0, \\ (z-z_0^*)^{\kappa}(z-z_0)^{-\kappa}X_0(z), & \mathrm{Im}z < 0, \end{cases}$$

where $X_0(z) = e^{\Gamma_0(z)}$, then (3.12) is written as

$$\frac{\Phi^+(s)E_1(s)}{Y^+(s)(s-z_0^*)^{M_1}} = \frac{\Phi^-(s)E_2(s)}{Y^-(s)(s-z_0)^{M_2}} + \frac{N_0(s)E_1(s)}{Y^+(s)(s-z_0^*)^{M_1}}. \tag{3.14}$$

Define

$$\Psi(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \Psi^*(s)\frac{ds}{s-z}, \quad z \in \mathbb{C}\setminus\mathbb{R}, \tag{3.15}$$

where

$$\Psi^*(s) = \frac{N_0(s)E_1(s)}{Y^+(s)(s-z_0^*)^{M_1}}, \quad \Psi^* \in \hat{H}.$$

Via the generalized Liouville theorem, we know that

$$\Psi^+(s) - \Psi^-(s) = \Psi^*(s).$$

Again define

$$W(z) = \begin{cases} \dfrac{\Phi(z)E_1(z)}{Y(z)(z-z_0^*)^{M_1}} - \Psi(z), & \mathrm{Im}z > 0, \\ \dfrac{\Phi(z)E_2(z)}{Y(z)(z-z_0)^{M_2}} - \Psi(z), & \mathrm{Im}z < 0, \end{cases} \tag{3.16}$$

obviously, $W(z)$ is analytic in $\mathbb{C}\setminus\{z_0^*\}$, thus we can obtain the solution of (3.12) with singularities at d_i and e_j. Hence, we shall construct a Hermite interpolation polynomial

$$\Omega_\varrho(z) = b_0 z^\varrho + \cdots + b_{\varrho-1}z + b_\varrho, \tag{3.17}$$

where $\varrho = M_1+M_2-1$, and $b_1, \cdots, b_\varrho \in \mathbb{C}$. Note that, $\Omega_\varrho(z)$ is uniquely determined, and it has zero-points of the orders $\alpha_i (i = 1, 2, \cdots, m)$, $\beta_j (j = 1, 2, \cdots, n)$ at d_i, e_j, respectively, which implies

$$\begin{aligned} &[\Psi(z)(z-z_0^*)^\kappa]^{(L_1)} \mid_{z=d_i} = \Omega_\varrho^{(L_1)}(z) \mid_{z=d_i}, \\ &[\Psi(z)(z-z_0^*)^\kappa]^{(L_2)} \mid_{z=e_j} = \Omega_\varrho^{(L_2)}(z) \mid_{z=e_j}, \end{aligned}$$

where

$$L_1 = 1, 2, \cdots, \alpha_i - 1,\ i = 1, 2, \cdots, m; \quad L_2 = 1, 2, \cdots, \beta_j - 1,\ j = 1, 2, \cdots, n.$$

By the Riemann boundary value theory, (3.12) has the particular solution

$$\Phi_2(z) = \begin{cases} \dfrac{Y(z)(z-z_0^*)^{M_1}}{E_1(z)(z-z_0^*)^\kappa}\left[\Psi(z)(z-z_0^*)^\kappa - \Omega_\varrho(z)\right], & \mathrm{Im}z > 0, \\ \dfrac{Y(z)(z-z_0)^{M_2}}{E_2(z)(z-z_0^*)^\kappa}\left[\Psi(z)(z-z_0^*)^\kappa - \Omega_\varrho(z)\right], & \mathrm{Im}z < 0, \end{cases} \tag{3.18}$$

where

$$(z-z_0^*)^\kappa X(z) = Y(z) \quad z \in \mathbb{C}\setminus\mathbb{R}.$$

In view of linearity, we obtain the general solution of (3.12) as follows

$$\Phi(z) = \begin{cases} X(z)(z-z_0^*)^{M_1}E_1^{-1}(z)F(z), & \mathrm{Im}z > 0, \\ X(z)(z-z_0^*)^{M_2}E_2^{-1}(z)F(z), & \mathrm{Im}z < 0, \end{cases} \tag{3.19}$$

where

$$F(z) = \Psi(z)(z-z_0^*)^\kappa - \Omega_\varrho(z) + E_1(z)E_2(z)U_{\kappa-M_1-M_2-1}(z).$$

Obviously, $\Phi_2(z)$ has the singularity at $z = z_0^*$ when $\kappa < 0$. Due to $\Phi \in H_1^*$, we must have

$$\int_{\mathbb{R}} \Psi^*(s) \frac{ds}{(s - z_0^*)^\nu} = 0, \quad \nu = 1, 2, \cdots, |\kappa|. \tag{3.20}$$

Taking the boundary values of $Y(z)$, then we obtain

$$Y^+(s) = \sqrt{P_0(s)} X_0(s), \quad Y^-(s) = \frac{X_0(s)}{\sqrt{P_0(s)}}, \tag{3.21}$$

hence we have

$$\Phi^+(s) = \frac{N_0(s)}{2} + \frac{X_0(s)\sqrt{P_0(s)}}{E_1(s)(s - z_0^*)^{\kappa - M_1}} F_*(s),$$

and

$$\Phi^-(s) = -\frac{N_0(s)E_1(s)(s - z_0^*)^{M_2 - M_1}}{2E_2(s)P_0(s)} + \frac{X_0(s)(s - z_0^*)^{M_2 - \kappa}}{\sqrt{P_0(s)}E_2(s)} F_*(s),$$

where

$$F_*(s) = F(s) - \Omega_\varrho(s) + E_1(s)E_2(s)U_{\kappa - M_1 - M_2 - 1}(s).$$

By (2.19) we get the following closed-form solution

$$\begin{aligned}\Phi(s) =& \frac{N_0(s)}{2} + \frac{N_0(s)(s - z_0^*)^{M_2 - M_1} E_1(s)}{2P_0(s)E_2(s)} \\ &+ \frac{X_0(s)F_*(s)}{(s - z_0^*)^\kappa \sqrt{P_0(s)}} \left(\frac{P_0(s)}{E_1(s)} (s - z_0^*)^{M_1} - \frac{(s - z_0^*)^{M_2}}{E_2(s)} \right).\end{aligned} \tag{3.22}$$

Next, we consider the properties of the solutions at $s = 0$.

Suppose that $s = 0$ is an ordinary node, that is, $\lambda \in (0, 1)$, then $\gamma \neq 0$. Since

$$\lim_{s \to 0^+} \Phi(s) = \lim_{s \to 0^-} \Phi(s) = \Phi(0),$$

then by (3.21) we can prove that

$$Y^+(s) = s^\gamma \sqrt{P_0(s)} e^{\Gamma_*(s)}, \qquad Y^-(s) = \frac{s^\gamma e^{\Gamma_*(s)}}{\sqrt{P_0(s)}}, \tag{3.23}$$

where $\Gamma_*(s) = \Gamma_0(s) - \gamma \ln s$, and $\Gamma_* \in H$. From (3.9) and (3.10), we have

$$\ln P_0(\pm 0) = \ln P(\pm 0) + \kappa \ln \frac{\pm 0 - z_0^*}{\pm 0 - z_0} = \ln P(\pm 0) \pm \kappa \pi i, \tag{3.24}$$

again from (3.9) and (3.24), we obtain

$$\sqrt{P_0(+0)} = e^{\frac{\kappa}{2}\pi i} \sqrt{P(+0)} = e^{\kappa \pi i} e^{\frac{1}{2} \ln \frac{P(+0)}{P(-0)}} e^{-\frac{\kappa \pi i}{2}} e^{\frac{1}{2} \ln P(-0)} = e^{-\gamma \pi i} \sqrt{P_0(-0)}. \tag{3.25}$$

According to [8, 30, 31, 36, 40], when $s > 0$, it is clear that

$$\Psi(s) = \frac{e^{-\gamma \pi i} s^{-\gamma}}{2ie^{\Gamma_*(s)}} \Lambda_0 + \Psi_0(s), \tag{3.26}$$

where

$$\Lambda_0 = e^{\gamma \pi i} \Lambda(+0) \cot \gamma \pi - \Lambda(-0) \csc \gamma \pi, \quad \Lambda(s) = \frac{N_0(s)E_1(s)}{\sqrt{P_0(s)}(s - z_0^*)^{M_1}},$$

and $\Psi_0(s)|s|^{\lambda^*} \in H$, $\lambda^* \in (0,\lambda)$. Therefore, substituting (3.23)-(3.26) into $\Phi^+(s)$, one has

$$\Phi^+(+0) = \frac{N_0(+0)\csc\gamma\pi}{2ie^{2\gamma\pi i}}\left[e^{3\gamma\pi i} - \frac{N_0(-0)}{N_0(+0)}\right]. \tag{3.27}$$

As in [6,8,25,29,42], when $s<0$, we have

$$\Psi(s) = \frac{e^{\gamma\pi i}s^{-\gamma}}{2ie^{\Gamma_*(s)}}\Lambda_1 + \Psi_0(s), \tag{3.28}$$

where

$$\Lambda_1 = -\cot\gamma\pi e^{-\gamma\pi i}\Lambda(-0) + \Lambda(+0)\csc\gamma\pi,$$

it gives rise to

$$\Phi^+(-0) = \frac{N_0(+0)\csc\gamma\pi}{2ie^{\gamma\pi i}}\left[e^{3\gamma\pi i} - \frac{N_0(-0)}{N_0(+0)}\right]. \tag{3.29}$$

By comparing (3.27) with (3.29), and $e^{\gamma\pi i} \neq 1$, we know that $\Phi^+(+0) = \Phi^+(-0)$ if and only if

$$e^{3\gamma\pi i}N_0(+0) = N_0(-0). \tag{3.30}$$

Since $\Phi(s)$ is continuous at $s=0$, then we have

$$\Phi^-(0) = \Phi^+(0) = 0,$$

and $\Phi(0)=0$. Further, $N_0(0)=0$, and so

$$N(0) = 0. \tag{3.31}$$

If (3.31) is valid, for $s \in N_0 = \{|s|<\epsilon, \forall\epsilon>0\}$, we have $\Phi^\pm \in H_1^*$.

Suppose that $s=0$ is a special node, then $\lambda=0$, $\gamma=i\eta_0$. We have $\Psi_0 \in H$ near $s=0$. If $\eta_0 \neq 0$, from [24,30,31,36], (3.27) and (3.29) are still fulfilled, and $\Psi_0(\pm 0)$ exist and may not be equal. Return to (3.27) and (3.29), we should modify $\Phi^+(\pm 0)$ to

$$\Phi^+(+0) = \frac{N_0(+0)\csc\gamma\pi}{2ie^{2\gamma\pi i}}\left[e^{3\gamma\pi i} - \frac{N_0(-0)}{N_0(+0)}\right] + \sigma(0)\sqrt{P_0(+0)}\lim_{s\to+0}s^{i\eta_0}\left[\Psi_0(s)-\Lambda\right], \tag{3.32}$$

and

$$\Phi^+(-0) = \frac{N_0(+0)\csc\gamma\pi}{2ie^{\gamma\pi i}}\left[e^{3\gamma\pi i} - \frac{N_0(-0)}{N_0(+0)}\right] + \sigma(0)\sqrt{P_0(-0)}\lim_{s\to-0}s^{i\eta_0}\left[\Psi_0(s)-\Lambda\right], \tag{3.33}$$

where

$$\sigma(s) = \frac{e^{\Gamma_*(s)}}{E_1(s)}(s-z_0^*)^{M_1},$$

and when $\kappa > \varrho + 1$,

$$\Lambda = (-1)^\kappa \frac{b_\varrho - E_1(0)E_2(0)C_{\kappa-M_1-M_2-1}}{(z_0^*)^\kappa}.$$

Since $\Phi^+(+0) = \Phi^+(-0)$, and $\Psi_0(s)$ is continuous at $s=0$, then we have

$$\Psi_0(\pm 0) = \Lambda,$$

in this case, (3.30) holds, and so (3.31) holds. On the other hand, if (3.31) is valid, thus $\Phi(s)$ is continuous at $s=0$. When $\kappa > \varrho+1$, we have the condition of solvability $F(0)=0$, that is to say,

$$b_\varrho = E_1(0)E_2(0)C_{\kappa-M_1-M_2-1} + (-z_0^*)^\kappa \Psi(0); \tag{3.34}$$

when $\kappa \leq \varrho+1$, the constant b_ϱ takes the value

$$b_\varrho = \frac{(-z_0^*)^\kappa}{2\pi i}\int_{\mathbb{R}} \frac{\Psi^*(s)}{s} ds. \tag{3.35}$$

If $\eta_0 = 0$, then $\gamma = 0$, therefore $J(+0) = J(-0)$. We can also prove that $\Phi(0) = 0$ if and only if (3.31) is valid. Under the condition (3.31), we have $\Phi \in H$, and $\Phi(0) = 0$. Moreover, when $\kappa < \varrho + 1$, we have the following conditions of solvability

$$b_0 = \cdots = b_{\varrho-\kappa} = 0, \tag{3.36}$$

then $\Omega_\varrho(z)$ is a polynomial with the degree $\kappa - 1$. When $\kappa = 1$, $\Omega_\varrho(z) = b_\varrho$. In this case, we require that

$$\Psi(s)(s - z_0^*)\mid_{s=d_i} = b_\varrho; \qquad \Psi(s)(s - z_0^*)\mid_{s=e_j} = b_\varrho, \tag{3.37}$$

and

$$\int_{\mathbb{R}} \Psi^*(s)(s-d_i)^{-v_1-1} ds = 0; \qquad \int_{\mathbb{R}} \Psi^*(s)(s-e_j)^{-v_2-1} ds = 0, \tag{3.38}$$

are satisfied, where

$$v_1 = 1, 2, \cdots, \alpha_i - 1,\ i = 1, 2, \cdots, m; \quad v_2 = 1, 2, \cdots, \beta_j - 1,\ j = 1, 2, \cdots, n.$$

When $\kappa \leq 0$, $\Omega_\varrho(z) = 0$. Moreover, when $\kappa < 0$, in order that $\Phi_2 \in H_1^*$, we need (3.22) and the solvability conditions as follows

$$\int_{\mathbb{R}} \frac{\Psi^*(s)}{s-d_i} ds = 0; \qquad \int_{\mathbb{R}} \frac{\Psi^*(s)}{s-e_j} ds = 0. \tag{3.39}$$

Combining (3.38) and (3.39), when $\kappa < 0$, we require that the following (3.40) holds

$$\int_{\mathbb{R}} \Psi^*(s)(s-d_i)^{-v_1} ds = 0; \qquad \int_{\mathbb{R}} \Psi^*(s)(s-e_j)^{-v_2} ds = 0, \tag{3.40}$$

where

$$v_1 = 1, 2, \cdots, \alpha_i,\ i = 1, 2, \cdots, m; \quad v_2 = 1, 2, \cdots, \beta_j,\ j = 1, 2, \cdots, n.$$

Under the assumptions and solvability conditions, (3.1) has the solution

$$\psi(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Phi(s) e^{-is\tau} ds, \tag{3.41}$$

where $\Phi(s)$ is given by (3.22).

The results about the solutions of (3.1) are formulated in the following theorem.

Theorem 3.1. *Under the case of non-normal type, the necessary condition of solvability for problem (3.6) is (3.31).*

(1) If $s = 0$ is an ordinary node, (3.31) holds. When $\kappa > \varrho + 1$, Eq. (3.1) has $\kappa - \varrho - 1$ linearly independent solutions. When $\kappa = \varrho + 1$, (3.1) has a unique solution. When $\kappa < \varrho + 1$, (3.1) is solvable if (3.36) holds. When $\kappa = 1$, (3.37) and (3.38) hold, and when $\kappa < 0$, (3.20) and (3.40) hold.

(2) If $s = 0$ is a special node, (3.1) is solvable if (3.31) and (3.34) are valid. The solutions of (3.1) belong to the class H_1.

Remark 3.1. Suppose that $J_1(s), J_2(s)$ have common zero-points $j_1, \cdots, j_{k'}$ with the orders $\delta_1, \cdots, \delta_{k'}$ respectively, where $j_p (1 \leq p \leq k')$ are different from $d_i (1 \leq i \leq m)$ and $e_j (1 \leq j \leq n)$, the additional solvability condition should be fulfilled

$$N^{(l)}(j_p) = 0, \quad l = 1, \cdots, \delta_p - 1.$$

Remark 3.2. SIE with a convolution kernel and variable integral limits is a special case of Eq. (3.1), i.e., $h(\tau) = 0$ or $k(\tau) = 0$. In this case, the process of analysis is not fundamentally different from (3.1), and will be omitted.

4. SIEs of Wiener-Hopf with convolution kernels

Next, we consider SIEs of Wiener-Hopf with convolution kernels and variable integral limits

$$A\psi(\tau) + \frac{B}{\sqrt{2\pi}} \int_0^\tau \psi(t) h(\tau - t) dt + \frac{C}{\pi i} \int_{\mathbb{R}} \frac{\psi(t)}{t - \tau} dt = n(\tau), \quad \tau \in \mathbb{R}^+, \tag{4.1}$$

where A, B, C are constants, and $C \neq 0$. The known functions $h, n \in H_3^\ell$, $\ell \in (\frac{1}{2}, 1)$, and the unknown function $\psi \in H_1$. Extending τ to $\tau \in \mathbb{R}$ by adding $\psi^-(\tau)$ to the right side of (4.1), we have

$$A\psi(\tau) + \frac{B}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi^+(t) h^+(\tau - t) dt + \frac{C}{\pi i} \int_{\mathbb{R}} \frac{\psi(t)}{t - \tau} dt = n(\tau) + \psi^-(\tau), \quad \tau \in \mathbb{R}. \tag{4.2}$$

By applying the operator $\mathcal{W}$ to (4.2), we get

$$\Phi^+(s) = J(s)\Phi^-(s) + N_0(s), \qquad s \in \mathbb{R}, \tag{4.3}$$

in which

$$J(s) = \frac{1}{A + BH^+(s) - C\mathrm{sgn}(s)}, \qquad N_0(s) = \frac{N(s)}{A + BH^+(s) - C\mathrm{sgn}(s)},$$

and

$$\mathcal{W}[h^+(\tau)] = H^+(s),\ \mathcal{W}[n(\tau)] = N(s),\ \mathcal{W}[\psi^\pm(\tau)] = \Phi^\pm(s),\ N_0(s) = J(s)N(s).$$

It follows from $H, N \in h_3$ that

$$\lim_{s \to +\infty} J(s) = \frac{1}{A - C}, \quad \lim_{s \to -\infty} J(s) = \frac{1}{A + C}, \quad \lim_{s \to \infty} N_0(s) = 0, \tag{4.4}$$

hence $s = 0, \infty$ are nodes of problem (4.3).

Now, we give the definitions of γ_∞ and γ_0. Define

$$\gamma_\infty = \lambda_\infty + i\eta_\infty = \frac{1}{2\pi i}\ln\frac{J(-\infty)}{J(+\infty)} = \frac{1}{2\pi i}\ln\frac{A-C}{A+C}, \tag{4.5}$$

where $\ln J(s)$ takes a continuous branch such that $\lambda_\infty \in [0,1)$. From [30,31,36], we know that $\gamma_\infty \neq 0$ since $C \neq 0$. Again denote

$$\gamma_0 = \lambda_0 + i\eta_0 = \frac{1}{2\pi i}\ln\frac{J(-0)}{J(+0)} = \frac{1}{2\pi i}\ln\frac{A+BH^+(0)-C}{A+BH^+(0)+C}. \tag{4.6}$$

We call κ the index of (4.3) which satisfies $\lambda_0 - 1 < \kappa \leq \lambda_0$. Suppose that $J^{-1}(s)$ has some zero-points $d_i (1 \leq i \leq m)$ with the orders $\alpha_i (1 \leq i \leq m)$ respectively, then we put

$$E(s) = \sum_{i=1}^{m}(s-d_i)^{\alpha_i}, \quad M = \alpha_1 + \cdots + \alpha_m. \tag{4.7}$$

Here, we still only discuss the case $M > 0$. Rewrite (4.3) as

$$\Phi^+(s) = \frac{X^+(s)(s-z_0^*)^M}{X^-(s)E(s)}\Phi^-(s) + N_0(s). \tag{4.8}$$

By the principle of analytic continuation and the generalized Liouville theorem, (4.3) has the analytic solution

$$\Phi_1(z) = \begin{cases} X(z)(z-z_0^*)^M P_{\kappa-M-1}(z), & \mathrm{Im}z > 0, \\ X(z)E(z)P_{\kappa-M-1}(z), & \mathrm{Im}z < 0. \end{cases} \tag{4.9}$$

Next, we consider the non-homogeneous problem (4.8). Put

$$\Phi_2(z) = \begin{cases} \dfrac{Y(z)}{E(z)(z-z_0^*)^{\kappa-M}}[\Psi(z)(z-z_0^*)^\kappa - \Omega_\rho(z)], & \mathrm{Im}z > 0, \\ \dfrac{Y(z)}{(z-z_0^*)^\kappa}[\Psi(z)(z-z_0^*)^\kappa - \Omega_\rho(z)], & \mathrm{Im}z < 0, \end{cases} \tag{4.10}$$

where

$$\Omega_\rho(z) = b_\rho + b_{\rho-1}z + \cdots + b_0 z^\rho \tag{4.11}$$

has zero-points of the orders α_i at d_i, and $\rho = M-1$, this implies

$$\Omega_\rho^{(L)}(z) = [\Psi(z)(z-z_0^*)^\kappa]^{(L)}, \quad 1 \leq L \leq \alpha_i - 1, \ 1 \leq i \leq m. \tag{4.12}$$

Using the theory of classical boundary value, we can verify that (4.10) is the particular solution of (4.3). Therefore, (4.3) has the general solution

$$\Phi(z) = \begin{cases} \dfrac{X(z)(z-z_0^*)^M}{E(z)}[\Psi(z)(z-z_0^*)^\kappa - \Omega_\rho(z) + E(z)P_{\kappa-M-1}(z)], & \mathrm{Im}z > 0, \\ X(z)[\Psi(z)(z-z_0^*)^\kappa - \Omega_\rho(z) + E(z)P_{\kappa-M-1}(z)], & \mathrm{Im}z < 0. \end{cases} \tag{4.13}$$

Further process is the same as section 3 and will be omitted.

Now, we discuss the properties of solutions at $s=\infty$. By [8, 26, 36], we have, near $s \in N_\infty$,

$$Y(s)s^{\lambda_\infty} \in h_2.$$

Notice that, in (4.13), we shall write $P_{\kappa-M}(z)$ instead of $P_{\kappa-M-1}(z)$.

Suppose that $s=\infty$ is an ordinary node, then $\lambda_\infty \in (0,1)$, further $\gamma_\infty \neq 0$. Consider the positive boundary value $\Phi^+(s)$ as follow

$$\Phi^+(s) = Y^+(s)[\Psi(s)Q_1(s) + (s-z_0^*)^{M-\kappa}Q_2(s)] + \frac{N_0(s)}{2}, \tag{4.14}$$

where

$$Q_1(s) = (s-z_0^*)^M E^{-1}(s), \qquad Q_2(s) = P_{\kappa-M}(s) - \Omega_\rho(s)E^{-1}(s). \tag{4.15}$$

We can see that $Q_1(s)$ is bounded near $s \in N_\infty$. Moreover, when $\kappa \geq M$, one has

$$Q_2(s) = O(|s-z_0^*|^{\kappa-M}), \quad Y^+(s)Q_2(s)(s-z_0^*)^{M-\kappa} = O(|s|^{-\lambda_\infty}). \tag{4.16}$$

When $\frac{1}{2} < \lambda_\infty < \ell < 1$, we have $|\Psi^*(s)| \leq r_0$ near $s \in N_\infty$, then

$$|\Psi(s)| \leq \frac{r_0}{2\pi}\int_{\mathbb{R}} |\frac{1}{s}||1-\frac{t}{s}|^{-1}|dt| = \frac{r_0}{2\pi}\int_{\mathbb{R}} |\sum_{n=0}^{\infty}\frac{t^n}{s^{n+1}}|\,|dt| \leq r_0^*,$$

where r_0, r_0^* take constants. Further, we obtain

$$Y^+(s)\Psi(s) = O(|s|^{-\lambda_\infty}), \tag{4.17}$$

and by (4.15)-(4.17), when $\kappa \geq M$, it gives rise to

$$\Phi^+(s) = O(|s|^{-\lambda_\infty}). \tag{4.18}$$

When $\kappa < M$, $P_{\kappa-M}(z) = 0$, then we require the solvability conditions as follows

$$b_j = 0, \qquad j = 0, \cdots, \rho-\kappa-1. \tag{4.19}$$

In this case, $\Omega_\rho(z) = \sum_{j=0}^{\kappa} b_{\rho-j}z^j$. Moreover, when $\kappa > 0$, (4.18) is still valid; when $\kappa = 0$, $\Omega_\rho(z) = b_\rho$, one must have

$$\Psi(d_i)(d_i - z_0^*) = b_\rho, \quad 1 \leq i \leq m, \tag{4.20}$$

when $\kappa < 0$, we need that (3.20) and the following (4.21) hold

$$\int_{\mathbb{R}} \Psi^*(s)(s-d_i)^{-v}ds = 0, \tag{4.21}$$

where $1 \leq v \leq \alpha_i, 1 \leq i \leq m$.

When $\frac{1}{2} < \ell \leq \lambda_\infty < 1$, there exists $\epsilon > 0$ such that $\lambda_\infty - \epsilon > \frac{1}{2}$. Therefore, we have

$$Y^+(s)\Psi(s) = O(|s|^{-\lambda_\infty+\epsilon}), \quad s \in N_\infty. \tag{4.22}$$

Similarly, we can get

$$\Phi^+(s) = O(|s|^{-\lambda_\infty+\epsilon}), \quad s \in N_\infty. \tag{4.23}$$

Combining (4.18) and (4.23), when $\lambda_\infty > \frac{1}{2}$, it turns out that

$$\Phi^+(s) = O(|s|^{-r}), \qquad s \in N_\infty, \tag{4.24}$$

where $r > \min\{\ell, \lambda_\infty - \epsilon\}$, thus $r > \frac{1}{2}$. Moreover, $\Phi^+ \in H_2^*$.

When $0 < \lambda_\infty \leq \frac{1}{2}$, by [8, 25] we know that

$$Y^+(s)\Psi(s) = O(|s|^{-\ell}).$$

When $\kappa \geq M$, in order that $\Phi^+ \in L^2(\mathbb{R})$, (4.18) needs to be satisfied. When $\kappa < M$, (4.19) holds. Further, when $\kappa \geq 0$, we must have

$$b_{\rho-\kappa} = 0, \tag{4.25}$$

when $\kappa < 0$, we require that (4.19) and (4.21) are satisfied.

Under the assumptions and conditions of solvability, from [8, 23, 31], it is easily known that $\Phi^+ \in L^2(\mathbb{R})$.

Suppose that $s = \infty$ is a special node, then $\lambda_\infty = 0$. It follows from (4.4) that $\gamma_\infty \neq 0$, thus we can transform it to the case $\lambda_\infty < \frac{1}{2}$. Further discussions are the same as above.

In conclusion, we state the following result.

Theorem 4.1. *Under the case of non-normal type, (3.34) is a necessary condition of solvability for Eq. (4.3).*

(1) If $\lambda_\infty > \frac{1}{2}$, in (4.13), we write $P_{\kappa-M}(z)$ in place of $P_{\kappa-M-1}(z)$. When $\kappa > M$, (4.1) has $\kappa - M$ linearly independent solutions. When $\kappa = M$, (4.3) has the unique solution. When $\kappa < M$, (4.19) and (4.25) hold, in this case, (4.3) has the general solution. When $\kappa > 0$, (4.18) holds; when $\kappa = 0$, (4.20) holds; when $\kappa < 0$, (3.20) and (4.21) hold.

(2) If $\lambda_\infty \leq \frac{1}{2}$, when $\kappa \geq M$, (4.18) holds. When $\kappa < M$, (4.19) holds. When $\kappa \geq 0$, (4.25) holds, and when $\kappa < 0$, (4.19) and (4.21) hold.

Under the conditions of solvability, (4.1) has the general solutions

$$\psi^+(\tau) = \mathcal{W}^{-1}[\Phi^+(s)], \tag{4.26}$$

where $\Phi^+(s)$ is given by (4.14).

5. Dual SIEs with convolution kernels

Now we deal with dual SIEs with convolution kernels and variable integral limits

$$\begin{cases} A\psi(\tau) + \dfrac{B}{\sqrt{2\pi}} \displaystyle\int_0^\tau \psi(t)h_1(\tau - t)dt + \dfrac{C}{\pi i} \displaystyle\int_{\mathbb{R}} \dfrac{\psi(t)}{t-\tau}dt = n(\tau), \quad \tau \in \mathbb{R}^+, \\ A\psi(\tau) + \dfrac{B}{\sqrt{2\pi}} \displaystyle\int_\tau^0 \psi(t)h_2(\tau - t)dt + \dfrac{C}{\pi i} \displaystyle\int_{\mathbb{R}} \dfrac{\psi(t)}{t-\tau}dt = n(\tau), \quad \tau \in \mathbb{R}^-, \end{cases} \tag{5.1}$$

where A, B, C are constants, and $C \neq 0$. The functions $h_1, h_2, n \in H_1$, and the unknown function $\psi \in H_1$. Combining the two equations in (5.1) to the following (5.2):

$$\begin{aligned} &A\psi(\tau) + \frac{B}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(t)h(\tau - t)dt + \frac{B}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(t)k(\tau - t)\mathrm{sgn}(t)dt \\ &+ \frac{C}{\pi i} \int_{\mathbb{R}} \frac{\psi(t)}{t-\tau}dt = n(\tau), \qquad \tau \in \mathbb{R}, \end{aligned} \tag{5.2}$$

where

$$h(\tau) = \frac{1}{2}[h_1^+(\tau) - h_2^-(\tau)], \qquad k(\tau) = \frac{1}{2}[h_1^+(\tau) + h_2^-(\tau)],$$

then $h, k \in H_1$. Obviously,

$$h(\tau) = \begin{cases} \frac{1}{2}h_1(\tau), & \tau \geq 0; \\ \frac{1}{2}h_2(\tau), & \tau < 0, \end{cases} \qquad k(\tau) = \begin{cases} \frac{1}{2}h_1(\tau), & \tau \geq 0, \\ -\frac{1}{2}h_2(\tau), & \tau < 0. \end{cases}$$

We apply the operator $\mathcal{W}$ to (5.2) and obtain

$$A\Phi(s) + BH(s)\Phi(s) + \frac{BK(s)}{\pi i}\int_{\mathbb{R}} \frac{\Phi(t)}{t-s}dt - C\Phi(s)\mathrm{sgn}(s) = N(s), \quad s \in \mathbb{R}, \quad (5.3)$$

where

$$\mathcal{W}[h(\tau)] = H(s), \quad \mathcal{W}[k(\tau)] = K(s), \quad \mathcal{W}[n(\tau)] = N(s), \quad \mathcal{W}[\psi(\tau)] = \Phi(s),$$

and $H, K, N, \Phi \in H_1^*$. We shall solve the following (5.4) instead of (5.3)

$$\Phi^+(s) = J(s)\Phi^-(s) + N_0(s), \qquad (5.4)$$

where

$$J(s) = \frac{A + B[H(s) - K(s)] - C\mathrm{sgn}(s)}{A + B[H(s) + K(s)] - C\mathrm{sgn}(s)}, \; N_0(s) = \frac{N(s)}{A + B[H(s) + K(s)] - C\mathrm{sgn}(s)}.$$

From the analysis above, it is easy to prove that (5.4) has a unique node $s = 0$. Further process is similar to section 3 and will be omitted.

6. Conclusions

In this paper, we mainly focus on three classes SIEs with variable integral limits in the case of non-normal type. By means of the theory of complex variable functions and classical boundary value problems, Eqs. (1.1)-(1.3) are transformed into the linear Riemann problems with discontinuous property in class H_1^*, further, we obtain the general solution given by integral-form. Moreover, the novel method in this paper can effectively solve such equations mentioned in [1, 3, 13, 14, 16, 18, 37, 39], and we may also prove the existence and stability of solutions for Eqs. (1.1)-(1.3) in Clifford analysis (see [2, 7, 17, 19, 20, 38, 41]).

Recently, many scholars have studied SIEs with convolution kernels and nonlinear SIEs in multidimensional and hyper-singular fields, and gained a lot of excellent results. Based on our results in this paper, we may consider the solvability theory of Eqs. (1.1)-(1.3) in these areas. More precise details will be omitted now.

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Availability of data and materials

Our manuscript has no associated data.

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