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EXISTENCE OF THE GENERALIZED EXPONENTIAL ATTRACTOR FOR COUPLED SUSPENSION BRIDGE EQUATIONS WITH DOUBLE NONLOCAL TERMS*

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Abstract We investigate the long-time dynamical behavior of coupled suspension bridge equations with double nonlocal terms by using the quasi-stable methods. We first establish the well-posedness of the solutions by means of the monotone operator theory. Secondly, the dissipation of solution semigroup $\{S(t)\}_{t\geq 0}$ is obtained, and then, the asymptotic smoothness of solution semigroup $\{S(t)\}_{t\geq 0}$ is verified by the energy reconstruction method; ultimately, we prove the existence of global attractor. Finally, we show the existence of the generalized exponential attractor.

Keywords Coupled suspension bridge equations, double nonlocal terms, global attractor, generalized exponential attractor.

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1. Introduction

The early suspension bridge equation is derived from the mathematical model of one-dimensional simple support beam suspended by hangers, which describes the deflection of the roadbed in the vertical plane, see [12,17]. As a new problem in the field of nonlinear analysis in 1990, Lazer and McKenna [13] introduced the following one-dimensional suspension bridge equation

$$\begin{cases} u_{tt} + EIu_{xxxx} + \delta u_t + ku^+ = W(x) + \varepsilon f(x,t), \quad (x,t) \in (0,L) \times \mathbb{R}^+, \\ u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0, \quad t \geq 0. \end{cases} \tag{1.1}$$

In 1998, Ahmed and Harbi [1] made a rigorous mathematical analysis for the coupled suspension bridge equations, which studied the dynamical behavior of system under the different conditions, and gave the relevant simulation and physical interpretation.

In recent years, a series of important researches have been made on long-time dynamics of suspension bridge equations, see [2–5, 8–11, 14–16, 18, 19, 21–23, 25] and

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the references therein. Ma and Zhong [15] first obtained the global attractor of the weak solution for coupled suspension bridge equations in 2005, and they further studied the existence of strong solution and strong global attractor for the following beam-string coupling system in [16]

$$\begin{cases} u_{tt}+\alpha u_{xxxx}+\delta_1 u_t+k(u-v)^+ + f_B(u)=h_B(x), & x\in[0,L],\\ v_{tt}-\beta v_{xx}+\delta_2 v_t-k(u-v)^+ + f_S(v)=h_S(x), & x\in[0,L]. \end{cases} \tag{1.2}$$

Bochicchio, Giorgi and Vuk [5] proved the existence and regularity of the global attractor with finite fractal dimension for the extensible suspension bridge equation

$$\partial_{tt}u+\partial_{xxxx}u+(p-\|\partial_x u\|^2_{L^2(0,1)})\partial_{xx}u+\partial_t u+k^2u^+=f, \tag{1.3}$$

where $p\in\mathbb{R}$. Park and Kang [19] proved existence of global attractor for suspension bridge equation with nonlinear damping. Wang and Ma obtained the existence of pullback attractors for non-autonomous suspension bridge with time delay in [22], Hajjej et al. investigate the stability of the energy for suspension bridge with a localized structural damping in [9]. Recently, Zhao et al. [24] considered the following extensible beam equations with nonlocal weak damping

$$u_{tt}-\triangle^2u-m(\|\nabla u\|^2)\triangle u+\|u_t\|^pu_t+f(u)=h, \qquad (x,t)\in\Omega\times\mathbb{R}^+, \tag{1.4}$$

with two kinds of boundary conditions, namely, clamped or hinged boundary conditions

$$u|_{\partial\Omega}=\frac{\partial u}{\partial\nu}\Big|_{\partial\Omega}=0, \quad \text{or} \quad u|_{\partial\Omega}=\triangle u|_{\partial\Omega}=0.$$

And they showed the existence of solution and global attractor for (1.4) by the monotone operator theory and the energy reconstruction method.

It is worth noting that most of the researches on the long-time behavior of the solutions for suspension bridge equation only obtain the existence of the attractors, while the fractal dimension of the attractors and the existence of exponential attractor are relatively less. Based on the above-mentioned works, we are concerned with the existence of global attractor with finite fractal dimension and generalized exponential attractor for the following coupled suspension bridge equations with double nonlocal terms

$$\begin{cases} u_{tt}+u_{xxxx}+\|u_t\|^pu_t+k^2(u-v)^+ +\|u\|^qu=h_B(x), & (x,t)\in[0,L]\times\mathbb{R}^+,\\ v_{tt}-v_{xx}+\|v_t\|^pv_t-k^2(u-v)^+ +\|v\|^qv=h_S(x), & (x,t)\in[0,L]\times\mathbb{R}^+, \end{cases} \tag{1.5}$$

with initial-boundary value conditions

$$\begin{cases} u(0,t)=u(L,t)=u_{xx}(0,t)=u_{xx}(L,t)=0, & t\in\mathbb{R}^+,\\ v(0,t)=v(L,t)=0, & t\in\mathbb{R}^+,\\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), & x\in[0,L],\\ v(x,0)=v_0(x), v_t(x,0)=v_1(x), & x\in[0,L], \end{cases} \tag{1.6}$$

where $u = u(x,t)$, $v = v(x,t)$ are the unknown function and denote the downward deflections of the roadbed and the cable, respectively. $\|u_t\|^p u_t, \|v_t\|^p v_t$ are the nonlocal weak damping terms, $\|u\|^q u, \|v\|^q v$ are the nonlocal functions, $p, q \geq 0$, and the simplest function to model the restoring force of the stays in the suspension bridge can be denoted by multiplying the constant k^2 by $u-v$, where $k^2 > 0$ denotes the spring coefficient, $(u-v)^+ = \max\{(u-v), 0\}$, namely, the expansion if $u-v$ is positive, but zero, if $u-v$ is negative, corresponding to compression. Moreover, the external forcing term $h_B, h_S \in L^2([0,L])$ (here we can give two examples of the external forcing term for this work: $h(x) = \cos(\frac{2\pi x}{L})$, or $h(x) = \sin(\frac{2\pi x}{L}) \in L^2([0,L])$). For brevity, we denote $\Omega = [0,L]$, $\triangle^2 u = u_{xxxx}$, $-\triangle v = -v_{xx}$.

Our main object in this paper is to investigate the existence of global attractor with finite fractal dimension and generalized exponential attractor for beam-string coupled equations with double nonlocal functions. Since the coupling of the equations is reflected in the semilinear term $(u-v)^+$, the double nonlocal terms don't effect the energy reconstruction method proposed in [24], so we don't meet the new difficulties in dealing with existence of global attractor. Different from [24], we further obtain the finite fractal dimension of global attractor and the existence of generalized exponential attractor.

This paper is organized as follows. In section 2, we recall several definition and abstract results in theory of nonlinear dynamical systems that will be useful to discuss our problem, and obtain the well-posedness results by means of the monotone operator theory and show that the problem (1.5)-(1.6) generates a dynamical system $(\mathcal{H}, S(t))$ in the space $\mathcal{H} = (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$. In the next section, we give the dissipativity, and then prove the existence of global attractor for (1.5)-(1.6). Finally, we obtain the existence of generalized exponential attractor with finite fractal dimension in section 4.

Explaining in here, all C throughout the paper represent a normal numbers, and each C is not exactly the same.

2. Preliminaries

Let $V_0 = L^2(\Omega)$, $V_1 = H_0^1(\Omega)$, $V_2 = H^2(\Omega) \cap H_0^1(\Omega)$. Then we define the phase space

$$\mathcal{H} = V_2 \times V_0 \times V_1 \times V_0,$$

and endowed with the norms

$$\|(u, u_t, v, v_t)\|_{\mathcal{H}} = \left(\frac{1}{2}(\|\triangle u\|^2 + \|u_t\|^2 + \|\nabla v\|^2 + \|v_t\|^2)\right)^{\frac{1}{2}},$$

where $\|\nabla \cdot\|$ and $\|\triangle \cdot\|$ stand for the norm of V_1 and V_2, respectively. Denote $A = \triangle^2$ with domain $D(A) = \{u \in H^4(\Omega) \cap H_0^1(\Omega) | u_{xx}(0,t) = u_{xx}(L,t) = 0\}$.

Suppose that $\lambda_1 > 0$ is the first eigenvalue of $\triangle^2$ with $u(0) = u(L) = u_{xx}(0) = u_{xx}(L) = 0$, then $\lambda_1^{\frac{1}{2}}$ is the first eigenvalue of $-\triangle$ with $u(0) = u(L) = 0$, and there holds

$$\|\triangle u\|^2 \geq \lambda_1 \|u\|^2, \ \forall u \in V_2, \qquad \|\nabla u\|^2 \geq \lambda_1^{\frac{1}{2}} \|u\|^2, \ \forall u \in V_1. \tag{2.1}$$

Lemma 2.1. [20] *Let X be a separable Banach space. We denote by $L_p(a,b;X)$ $(1 \leq p \leq \infty)$ the space of (equivalence classes of) Bochner measurable functions*

$f : [a, b] \to X$ such that $\|f(\cdot)\|_X \in L_p(a, b)$. Each $L_p(a, b; X)$ is a Banach space with the norms

$$\|f\|_{L_p(a,b;X)} = \Big(\int_a^b \|f(t)\|_X^p dt\Big)^{\frac{1}{p}}, 1 \le p < \infty,$$
$$\|f\|_{L_\infty(a,b;X)} = esssup\{\|f(t)\|_X : t \in [a, b]\}.$$

We also denote by $C(a, b; X)$ the space of strongly continuous functions with values in X and use the space

$$W^{1,p}(a, b; X) = \{f \in C(a, b; X) : f' \in L_p(a, b; X)\},$$

where $f'(t)$ is a distributional derivative of $f(t)$ with respect to t. We note that the space $W^{1,1}(a, b; X)$ coincides with the set absolutely continuous functions from $[a, b]$ into X.

Definition 2.1. [6, 24] A function $(u(t), v(t)) \in C([0, T]; V_2 \times V_1)$ possessing the initial data $u(0) = u_0, u_t(0) = u_1, v(0) = v_0, v_t(0) = v_1$ is said to be

- (S) a strong solution of (1.5)-(1.6) on the interval $[0, T]$ if and only if
- (i) $u \in W^{1,1}(a, b; V_2)$, $v \in W^{1,1}(a, b; V_1)$, $u_t \in W^{1,1}(a, b; V_2)$ and $v_t \in W^{1,1}(a, b; V_1)$ for any $0 < a < b < T$;
- (ii) $Au(t) + A^{\frac{1}{2}}v(t) + Du_t(t) + Dv_t(t) \in V_0'$ for almost all $t \in [0, T]$, where the operator D satisfies [6, Assumption 1.1], and $Du_t(t) = \|u_t\|^p u_t$, $Dv_t(t) = \|v_t\|^p v_t$;
- (iii) Eq. (1.5) is satisfied in V_0' for almost all $t \in [0, T]$.
- (G) a generalized solution of (1.5)-(1.6) on the interval $[0, T]$ if and only if there exists sequence of the strong solution $\{u_n(t)\}$, $\{v_n(t)\}$ of (1.5)-(1.6) with initial data $(u_{0n}, u_{1n}, v_{0n}, v_{1n})$ instead of (u_0, u_1, v_0, v_1) such that

$$\lim_{n\to\infty} \max_{t\in[0,T]} \Big\{|\partial_t u(t) - \partial_t u_n(t)| + |A^{\frac{1}{2}}u(t) - A^{\frac{1}{2}}u_n(t)| + |\partial_t v(t) - \partial_t v_n(t)| + |A^{\frac{1}{4}}v(t) - A^{\frac{1}{4}}v_n(t)|\Big\} = 0.$$

Remark 2.1. For the convenience of readers, [6, Assumption 1.1] is given as follow: the operator $D : D(A^{\frac{1}{2}}) \to [D(A^{\frac{1}{2}})]'$ is assumed the monotone semicontinuous with $D(0) = 0$, i.e., $(Du - Dv, u - v) \ge 0$ for all $u, v \in D(A^{\frac{1}{2}})$, and $\lambda \mapsto (D(u + \lambda v), v)$ is a continuous function from $\mathbb{R}$ into itself. Moreover, we assume that there exists a set $W \subset D(A^{\frac{1}{2}})$ such that $D(w) \subset V'$ for every $w \in W$ and W is dense in V.

Lemma 2.2. [24] *Let H be a Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|_H$, $u, v \in H$. Then there exists some constant C_γ which depends on γ such that*

$$(\|u\|_H^{\gamma-2}u - \|v\|_H^{\gamma-2}v, u - v) \ge \begin{cases} C_\gamma\|u - v\|_H^\gamma, & \gamma \ge 2, \\ C_\gamma \dfrac{\|u - v\|_H^2}{(\|u\|_H + \|v\|_H)^{2-\gamma}}, & 1 \le \gamma \le 2. \end{cases} \tag{2.2}$$

Corollary 2.1. *Denote $D(\mu_t) = \|\mu_t\|^p \mu_t$ and by Lemma 2.2, we obtain*

$$\big(D(\mu_t) - D(\vartheta_t), \mu_t - \vartheta_t\big) \ge C_p\|\mu_t - \vartheta_t\|^{p+2}, \qquad p \ge 0,\ \mu_t, \vartheta_t \in V_0, \tag{2.3}$$

i.e., the damping operator D is strong monotone. Moreover, the damping operator D satisfies [6, Assumption 1.1].

Theorem 2.1. *Let $T > 0$ be arbitrary, the following statements hold:*

(i) For every $(u_0, u_1, v_0, v_1) \in V_2 \times V_2 \times V_1 \times V_1$, such that $Au_0 + A^{\frac{1}{2}} v_0 + Du_1 + Dv_1 \in L^2(\Omega)$, there exists a unique strong solution of problem (1.5)-(1.6) *on the interval $[0, T]$ such that*

$$
\begin{aligned}
&(u_t, u_{tt}, v_t, v_{tt}) \in L^\infty([0,T]; V_2 \times V_0 \times V_1 \times V_0),\\
&(u_t, v_t) \in C_r([0,T]; V_2 \times V_1), \quad (u_{tt}, v_{tt}) \in C_r([0,T]; V_0 \times V_0),\\
&Au(t) + Du_t(t) \in C_r([0,T]; V_0^{'}), \quad A^{\frac{1}{2}} v(t) + Dv_t(t) \in C_r([0,T]; V_0^{'}),
\end{aligned}
$$

where C_r represents the space of right continuous functions, and the solution of Eq.(1.5) *satisfies the energy relation*

$$
E(t) + \int_0^t (\|u_t\|^p u_t, u_t) d\sigma + \int_0^t (\|v_t\|^p v_t, v_t) d\sigma = E(0), \tag{2.4}
$$

where

$$
\begin{aligned}
E(t) =& \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\triangle u\|^2 + \frac{1}{2}\|v_t\|^2 + \frac{1}{2}\|\nabla v\|^2\\
&+ \frac{1}{2}k^2\|(u-v)^+\|^2 + \frac{1}{q+2}\|u\|^{q+2} + \frac{1}{q+2}\|v\|^{q+2}\\
&- \int_\Omega h_B(x) u(t) dx - \int_\Omega h_S(x) v(t) dx,
\end{aligned} \tag{2.5}
$$

$$
\begin{aligned}
E_0(t) =& \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\triangle u\|^2 + \frac{1}{2}\|v_t\|^2 + \frac{1}{2}\|\nabla v\|^2 + \frac{1}{2}k^2\|(u-v)^+\|^2\\
&+ \frac{1}{q+2}\|u\|^{q+2} + \frac{1}{q+2}\|v\|^{q+2}.
\end{aligned} \tag{2.6}
$$

(ii) For any initial data $(u_0, u_1, v_0, v_1) \in V_2 \times V_0 \times V_1 \times V_0$ there exists a unique generalized solution such that

$$
(u, u_t, v, v_t) \in C([0,T]; V_2 \times V_0 \times V_1 \times V_0). \tag{2.7}
$$

Theorem 2.2. [24] *Assume that the damping operator D maps V_0 into $V_0^{'}$ and is a monotone semicontinuous operator which is bounded on bounded sets, that is, for any ρ, there have*

$$
\sup\left\{|D(u)|_{V_0'} : \ u \in V_0, \ \|u\| \le \rho\right\} < \infty. \tag{2.8}
$$

Then every generalized solution is also weak, i.e., the relation

$$
\begin{aligned}
(u_t(t), \omega) =& (u_1, \omega) - \int_0^t \Big(\big(Au(\sigma), \omega\big) - \big(Du_t(\sigma), \omega\big)\\
&+ \big((h_B - k^2(u-v)^+ - \|u\|^q u), \omega\big)\Big) d\sigma,\\
(v_t(t), \nu) =& (v_1, \nu) - \int_0^t \Big(\big(A^{\frac{1}{2}} v(\sigma), \nu\big) - \big(Dv_t(\sigma), \nu\big)\\
&+ \big((h_S + k^2(u-v)^+ - \|v\|^q v), \nu\big)\Big) d\sigma,
\end{aligned}
$$

hold for $\forall \omega \in V_2$, $\forall \nu \in V_1$ and for almost all $t \in [0, T]$.

Remark 2.2. The proof of Theorem 2.1 is similar to the proof of [24, Theorem 2.3], so we only give the above conclusion.

Corollary 2.2. *Problem* (1.5)-(1.6) *generates a dynamical system* $(\mathcal{H}, S(t))$ *in the space* $\mathcal{H}$*, the corresponding evolution operator* $S(t)$ *is given by the formula*

$$S(t)(u_0, u_1, v_0, v_1) = (u(t), u_t(t), v(t), v_t(t)), \tag{2.9}$$

where $(u(t), v(t))$ *solve* (1.5) *with the initial data* (u_0, u_1, v_0, v_1).

In order to obtain the main result for our problem, we also need the following definitions and abstract results from the book of Chueshov and Lasiecka (see [7, Chapter. 7]).

Definition 2.2. [6] A dynamical system $(X, S(t))$ is said to be asymptotically smooth if and only if for any bounded set D such that $S(t)D \subset D$ for $t > 0$, there exists a compact set $K \subset \bar{D}$ in the closure $\bar{D}$ of D, such that

$$\lim_{t\to+\infty} dist_X\{S(t)D, K\} = 0, \tag{2.10}$$

where $dist_X\{A, B\}$ is the Hausdorff semidistance between sets A and B.

Definition 2.3. [6] A bounded closed set $A \subset X$ is said to be a global attractor of the dynamical system $(X, S(t))$ if and only if

(i) A is an invariant set, i.e. $S(t)A = A$ for $\forall t \geq 0$;

(ii) A is uniformly attracting, i.e. $\lim\limits_{t\to+\infty} dist_X\{S(t)M, A\} = 0$ for all bounded set $M \subset X$.

Theorem 2.3. [6] *Let* $(X, S(t))$ *be a dynamical system on a complete metric space* X *endowed with a metric* d*. Assume that for any bounded positively invariant set* $B \subset X$ *there exist* $T > 0$*, a continuous non-decreasing function* $r : \mathbb{R}^+ \to \mathbb{R}^+$ *and a pseudometric* $\varrho_B^T \in C(0, T; X)$ *such that*

(i) $r(0) = 0$ *and* $r(s) < s$ *for every* $s > 0$*;*

(ii) the pseudometric ϱ_B^T *is precompact (with respect to* X*), i.e. for any sequence* $\{x_n\} \subset B$ *has a subsequence* $\{x_{n_k}\}$ *such that the sequence* $\{y_k\} \subset C(0, T; X)$ *of elements* $y_k(\tau) = S(\tau)x_{n_k}$ *is Cauchy with respect to* ϱ_B^T*;*

(iii) the following inequality holds

$$\begin{aligned} &d(S(T)y_1, S(T)y_2)\\ \leq& r\Big(d(y_1, y_2) + \varrho_B^T\big(\{S(\tau)y_1\},\ \{S(\tau)y_2\}\big)\Big), \quad \forall y_1, y_2 \in B, \end{aligned} \tag{2.11}$$

where we denote by $\{S(\cdot)y_i\}$ *the element in the space* $C(0, T; X)$ *given by function* $y_i(\cdot) = S(\cdot)y_i, i = 1, 2$*. Then* $(X, S(t))$ *is an asymptotically smooth dynamical system.*

Definition 2.4. [7] Let $(X, S(t))$ be a dissipative dynamical system in a complete metric space X. Then the dynamical system $(X, S(t))$ possesses a compact global attractor if and only if $(X, S(t))$ is asymptotically smooth.

Definition 2.5. [7] Let X and Y be two reflexive Banach spaces with $X \hookrightarrow\hookrightarrow Y$ and put $H = X \times Y$. If there exist a compact semi-norm n_X on X and two locally bounded nonnegative functions $a(t)$ and $c(t)$ satisfying

$$b(t) \in L^1(\mathbb{R}^+) \quad \text{with} \quad \lim_{t\to\infty} b(t) = 0, \tag{2.12}$$

$$\|S(t)y_1 - S(t)y_2\|_H^2 \le a(t)\|y_1 - y_2\|_H^2, \tag{2.13}$$

and

$$\|S(t)y_1 - S(t)y_2\|_H^2 \le b(t)\|y_1 - y_2\|_H^2 + c(t) \sup_{s\in[0,1]} [n_X(u(s) - v(s))]^2, \tag{2.14}$$

for every $y_i \in B$, $i = 1,\ 2$, where $B \subset H$ is the bounded positively invariant set, and $S(t)y_i = y_i(t)$, $t > 0$. Then $(H, S(t))$ is called quasi-stable on B.

Theorem 2.4. [7] *Let $(X, S(t))$ be a dynamical system. If $(X, S(t))$ possesses a compact global attractor A and is quasi-stable on A, then the attractor A has finite fractal dimension.*

Theorem 2.5. [7] *Let B be a bounded positively invariant absorbing set on the dynamical system $(X, S(t))$ which is quasi-stable. If there exists a larger space $\widetilde{X} \supseteq X$ such that for any $T > 0$, it holds*

$$\|S(t_1)y - S(t_2)y\|_{\widetilde{X}} \le C_B|t_1 - t_2|^\tau, \quad t_1,\ t_2 \in [0, T],\ y \in B, \tag{2.15}$$

where $C_B > 0$ depends on B, $\tau \in (0, 1]$. Then the dynamical system $(X, S(t))$ possesses a generalized exponential attractor $A^{exp} \subset X$ whose dimension is finite in $\widetilde{X}$.

3. Global attractor

In this section, we will prove the dissipativity of the semigroup $\{S(t)\}_{t\ge0}$ corresponding to (1.5)-(1.6), and verify the asymptotic smoothness of the dynamical system $(\mathcal{H}, S(t))$ by means of a priori estimates and the energy reconstruction method. Finally, the existence of global attractor is obtained.

Theorem 3.1. *The dynamical system $(\mathcal{H}, S(t))$ generated by problem (1.5)-(1.6) is dissipative in the space $\mathcal{H}$, namely, for any bounded set $B \subset \mathcal{H}$, there exist a positive constant $R > 0$ and $t_0 = t_0(B) > 0$, such that*

$$\|S(t)y\|_{\mathcal{H}} = \|\,(u(t), u_t(t), v(t), v_t(t))\|_{\mathcal{H}} \le R,$$

for all $y \in B$ and $t \ge t_0$.

Proof. Multiplying (1.5) by $u_t + \epsilon u$ and $v_t + \epsilon v$, and integrating over Ω, respectively, we obtain that

$$\begin{aligned}
&\frac{d}{dt}\Big(\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\triangle u\|^2 + \frac{1}{2}\|v_t\|^2 + \frac{1}{2}\|\nabla v\|^2 + \frac{1}{2}k^2\|(u-v)^+\|^2 \\
&+ \frac{1}{q+2}\|u\|^{q+2} + \frac{1}{q+2}\|v\|^{q+2} + (u_t, \epsilon u) + (v_t, \epsilon v) - \int_\Omega h_B u dx \\
&- \int_\Omega h_S v dx\Big) - \epsilon\|u_t\|^2 + \epsilon\|\triangle u\|^2 - \epsilon\|v_t\|^2 + \epsilon\|\nabla v\|^2 + \epsilon k^2\|(u-v)^+\|^2
\end{aligned}$$

$$+\epsilon\|u\|^{q+2}+\epsilon\|v\|^{q+2}+(\|u_t\|^p u_t, u_t+\epsilon u)+(\|v_t\|^p v_t, v_t+\epsilon v)$$
$$-\epsilon\int_\Omega h_B u dx-\epsilon\int_\Omega h_S v dx=0. \tag{3.1}$$

According to Hölder and Young inequalities and (2.1), it follows that

$$\Big|\int_\Omega h_B u dx\Big|\leq\frac{1}{\lambda_1}\|h_B\|^2+\frac{1}{4}\|\triangle u\|^2, \tag{3.2}$$
$$\Big|\int_\Omega h_S v dx\Big|\leq\frac{1}{\sqrt{\lambda_1}}\|h_S\|^2+\frac{1}{4}\|\nabla v\|^2. \tag{3.3}$$

Combining with (2.5)-(2.6) and (3.2)-(3.3), we have

$$E(t)\geq c_0E_0(t)-C_0,\qquad 0<c_0<1. \tag{3.4}$$

Denote $W(t)=E(t)+(u_t,\epsilon u)+(v_t,\epsilon v)$, using Hölder and Young inequalities, there holds

$$\epsilon|(u_t,u)|\leq\frac{1}{4}\|u_t\|^2+\frac{\epsilon^2}{\lambda_1}\|\triangle u\|^2, \tag{3.5}$$

and

$$\epsilon|(v_t,v)|\leq\frac{1}{4}\|v_t\|^2+\frac{\epsilon^2}{\sqrt{\lambda_1}}\|\nabla v\|^2, \tag{3.6}$$

exploiting (3.4)-(3.6), there exists $\epsilon_0>0$ with $0<\epsilon<\epsilon_0$ such that

$$W(t)\geq c_1E_0(t)-C_1,\quad 0<c_1<1. \tag{3.7}$$

Next, we rewrite (3.1) as follows

$$\frac{d}{dt}W(t)+\epsilon W(t)+Y(t)=0, \tag{3.8}$$

where

$$\begin{aligned}Y(t)=&(\|u_t\|^p u_t, u_t+\epsilon u)+(\|v_t\|^p v_t, v_t+\epsilon v)-\frac{3\epsilon}{2}\|u_t\|^2-\frac{3\epsilon}{2}\|v_t\|^2\\&+\frac{\epsilon}{2}\|\triangle u\|^2+\frac{\epsilon}{2}\|\nabla v\|^2+\frac{\epsilon k^2}{2}\|(u-v)^+\|^2+\frac{(q+1)\epsilon}{q+2}\|u\|^{q+2}\\&+\frac{(q+1)\epsilon}{q+2}\|v\|^{q+2}-\epsilon^2(u_t,u)-\epsilon^2(v_t,v).\end{aligned} \tag{3.9}$$

Applying Young inequality, we get that there exist constants $c_2,c_3>0$ such that

$$(u_t,u_t)=\|u_t\|^2\leq c_2+c_3\|u_t\|^{p+2}. \tag{3.10}$$

Combining with (2.4) and (3.4), there exists $C_B>0$ such that

$$E_0(t)\leq C(1+E(t))\leq C(1+E(0))\leq C_B. \tag{3.11}$$

By Cauchy and Young inequalities, (2.1) and (3.11), we have

$$|(\|u_t\|^p u_t,\epsilon u)|\leq\epsilon\|u_t\|^p(\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|u\|^2)$$

$$
\begin{aligned}
&\le \frac{\epsilon}{2}\|u_t\|^{p+2}+\frac{\epsilon}{2}\|u_t\|^{p}\|u\|^2\\
&\le \frac{\epsilon}{2}\|u_t\|^{p+2}+\frac{\epsilon}{2}(C_\sigma\|u_t\|^{p+2}+\sigma)\|u\|^2\\
&\le \frac{\epsilon}{2}\|u_t\|^{p+2}+\frac{\epsilon C_\sigma}{2\lambda_1}\|\triangle u\|^2\cdot\|u_t\|^{p+2}+\frac{\epsilon\sigma}{2\lambda_1}\|\triangle u\|^2\\
&\le \frac{\epsilon}{2}\|u_t\|^{p+2}+\frac{\epsilon C_\sigma}{2\lambda_1}E_0(t)\cdot\|u_t\|^{p+2}+\frac{\epsilon\sigma}{\lambda_1}E_0(t)\\
&\le \frac{\epsilon}{2}\|u_t\|^{p+2}+\frac{\epsilon C_\sigma C_B}{2\lambda_1}\|u_t\|^{p+2}+\epsilon C_2,
\end{aligned}
\tag{3.12}
$$

similarly, the following inequality holds

$$
|(\|v_t\|^p v_t,\epsilon v)|\le \frac{\epsilon}{2}\|v_t\|^{p+2}+\frac{\epsilon C_\sigma C_B}{2\sqrt{\lambda_1}}\|v_t\|^{p+2}+\epsilon C_3. \tag{3.13}
$$

Together with (2.1) and (3.12)-(3.13), we achieve

$$
(\|u_t\|^p u_t,u_t+\epsilon u)\ge\Big(1-\frac{\epsilon}{2}-\frac{\epsilon C_\sigma C_B}{2\lambda_1}\Big)\|u_t\|^{p+2}-\epsilon C_2, \tag{3.14}
$$

and

$$
(\|v_t\|^p v_t,v_t+\epsilon v)\ge\Big(1-\frac{\epsilon}{2}-\frac{\epsilon C_\sigma C_B}{2\sqrt{\lambda_1}}\Big)\|v_t\|^{p+2}-\epsilon C_3. \tag{3.15}
$$

Now, by virtue of (3.9)-(3.10) and (3.14)-(3.15), it follows that

$$
\begin{aligned}
Y(t)\ge&\bigg(\frac{1}{c_3}\Big(1-\frac{\epsilon}{2}-\frac{\epsilon C_\sigma C_B}{2\lambda_1}\Big)-\frac{3\epsilon}{2}-\frac{\epsilon}{4}\bigg)\|u_t\|^2\\
&+\bigg(\frac{1}{c_3}\Big(1-\frac{\epsilon}{2}-\frac{\epsilon C_\sigma C_B}{2\sqrt{\lambda_1}}\Big)-\frac{3\epsilon}{2}-\frac{\epsilon}{4}\bigg)\|v_t\|^2\\
&+\bigg(\frac{\epsilon}{2}-\frac{\epsilon^3}{\lambda_1}\bigg)\|\triangle u\|^2+\bigg(\frac{\epsilon}{2}-\frac{\epsilon^3}{\sqrt{\lambda_1}}\bigg)\|\nabla v\|^2\\
&-\frac{c_2}{c_3}\Big(2-\epsilon-\frac{\epsilon C_\sigma C_B}{2\lambda_1}-\frac{\epsilon C_\sigma C_B}{2\sqrt{\lambda_1}}\Big)-\epsilon C_2-\epsilon C_3,
\end{aligned}
\tag{3.16}
$$

choose $\epsilon>0$ small enough, such that

$$
\frac{1}{c_3}\Big(1-\frac{\epsilon}{2}-\frac{\epsilon C_\sigma C_B}{2\lambda_1}\Big)-\frac{3\epsilon}{2}-\frac{\epsilon}{4}>0,\quad \frac{\epsilon}{2}-\frac{\epsilon^3}{\lambda_1}>0,\quad \frac{\epsilon}{2}-\frac{\epsilon^3}{\sqrt{\lambda_1}}>0, \tag{3.17}
$$

$$
\frac{1}{c_3}\Big(1-\frac{\epsilon}{2}-\frac{\epsilon C_\sigma C_B}{2\sqrt{\lambda_1}}\Big)-\frac{3\epsilon}{2}-\frac{\epsilon}{4}>0,\qquad 2-\epsilon-\frac{\epsilon C_\sigma C_B}{2\lambda_1}-\frac{\epsilon C_\sigma C_B}{2\sqrt{\lambda_1}}>0. \tag{3.18}
$$

We obtain that $Y(t)\ge-\epsilon C_4$, then it yields from (3.8) that

$$
\frac{d}{dt}W(t)+\epsilon W(t)\le\epsilon C_4. \tag{3.19}
$$

Applying the Gronwall lemma, we conclude that

$$
W(t)\le W(0)e^{-\epsilon t}+C_4(1-e^{-\epsilon t}). \tag{3.20}
$$

Therefore, there exists $t_0 = t_0(B) = \frac{1}{\epsilon}\ln\frac{W(0)}{C_4}$ such that

$$W(t) \le 2C_4, \qquad \forall t \ge t_0. \tag{3.21}$$

We claim from (3.7) that

$$\|(u, u_t, v, v_t)\|_{\mathcal{H}} \le \frac{2C_4 + C_1}{c_1} = R. \tag{3.22}$$

This is the complete proof of the dissipativity. □

Remark 3.1. Theorem 3.1 implies that

$$\mathcal{B}_0 = \{(u(t), u_t(t), v(t), v_t(t)) \in \mathcal{H} : \|(u(t), u_t(t), v(t), v_t(t))\|_{\mathcal{H}} \le R\}$$

is a bounded absorbing set of semigroup $\{S(t)\}_{t\ge 0}$ corresponding to problem (1.5)-(1.6). From the above proof, it is easy to see that dissipativity of the semigroup is independent of p and q.

In order to prove the asymptotic smoothness of the dynamical system $(\mathcal{H}, S(t))$, we need first to establish the following estimates.

Theorem 3.2. *There exist $T_0 > 0$ and a constant $C > 0$ independent of T such that for any pair (u_1, v_1) and (u_2, v_2) of strong solutions for problem* (1.5)-(1.6), *we have the following relation for $T \ge T_0$,*

$$\begin{aligned}
&TE_m(t) + \int_0^T E_m(t)dt \\
\le& C(R)\Big\{ \int_0^T \|\xi_t\|^2 dt + \int_0^T \|\zeta_t\|^2 dt + \int_0^T (D(t,\xi_t), \xi_t)dt \\
&+ \int_0^T (D(t,\zeta_t), \zeta_t)dt + \int_0^T |(D(t,\xi_t), \xi_t)|dt + \int_0^T |(D(t,\zeta_t), \zeta_t)|dt \\
&+ \int_0^T \|\triangle\xi\| \cdot \|\xi_t\| dt + \int_0^T \|\nabla\zeta\| \cdot \|\zeta_t\| dt + \int_0^T \|\triangle\xi\|^2 dt \\
&+ \int_0^T \|\nabla\zeta\|^2 dt + \int_0^T dt \int_t^T \|\triangle\xi\| \cdot \|\xi_t\| d\tau + \int_0^T dt \int_t^T \|\nabla\zeta\| \cdot \|\zeta_t\| d\tau \\
&+ \Big|\int_0^T (\|u_1\|^q u_1 - \|u_2\|^q u_2, \xi)dt\Big| + \Big|\int_0^T (\|v_1\|^q v_1 - \|v_2\|^q v_2, \zeta)dt\Big| \\
&+ \Big|\int_0^T (\|u_1\|^q u_1 - \|u_2\|^q u_2, \xi_t)dt\Big| + \Big|\int_0^T (\|v_1\|^q v_1 - \|v_2\|^q v_2, \zeta_t)dt\Big| \\
&+ \Big|\int_0^T dt \int_t^T (\|u_1\|^q u_1 - \|u_2\|^q u_2, \xi_t)d\tau\Big| \\
&+ \Big|\int_0^T dt \int_t^T (\|v_1\|^q v_1 - \|v_2\|^q v_2, \zeta_t)d\tau\Big|\Big\},
\end{aligned} \tag{3.23}$$

where $\xi(t) = u_1(t) - u_2(t)$, $\zeta(t) = v_1(t) - v_2(t)$ and

$$E_m(t) = \frac{1}{2}\Big(\|\xi_t\|^2 + \|\triangle\xi\|^2 + \|\zeta_t\|^2 + \|\nabla\zeta\|^2\Big), \tag{3.24}$$

$$D(t,\xi_t) = \|u_{1t}\|^p u_{1t} - \|u_{2t}\|^p u_{2t}, \quad D(t,\zeta_t) = \|v_{1t}\|^p v_{1t} - \|v_{2t}\|^p v_{2t}. \tag{3.25}$$

Proof. Note that $\xi(t) = u_1(t) - u_2(t)$ and $\zeta(t) = v_1(t) - v_2(t)$ satisfy the following equality

$$\xi_{tt} + \xi_{xxxx} + D(t, \xi_t) + k^2(u_1 - v_1)^+ - k^2(u_2 - v_2)^+ + \|u_1\|^q u_1 - \|u_2\|^q u_2 = 0, \tag{3.26}$$

$$\zeta_{tt} - \zeta_{xx} + D(t, \zeta_t) - k^2(u_1 - v_1)^+ + k^2(u_2 - v_2)^+ + \|v_1\|^q v_1 - \|v_2\|^q v_2 = 0. \tag{3.27}$$

Multiplying (3.26) and (3.27) by ξ_t and ζ_t, and integrating over Ω, respectively, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\xi_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\triangle\xi\|^2 + (D(t, \xi_t), \xi_t) + (k^2(u_1 - v_1)^+ - k^2(u_2 - v_2)^+, \xi_t)\\ &+ (\|u_1\|^q u_1 - \|u_2\|^q u_2, \xi_t) + \frac{1}{2}\frac{d}{dt}\|\zeta_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\zeta\|^2 + (D(t, \zeta_t), \zeta_t)\\ &- (k^2(u_1 - v_1)^+ - k^2(u_2 - v_2)^+, \zeta_t) + (\|v_1\|^q v_1 - \|v_2\|^q v_2, \zeta_t) = 0,\end{aligned} \tag{3.28}$$

then

$$\begin{aligned}&\frac{d}{dt}E_m(t) + (D(t, \xi_t), \xi_t) + (D(t, \zeta_t), \zeta_t)\\ =& - (k^2(u_1 - v_1)^+ - k^2(u_2 - v_2)^+, \xi_t)\\ &+ (k^2(u_1 - v_1)^+ - k^2(u_2 - v_2)^+, \zeta_t)\\ &- (\|u_1\|^q u_1 - \|u_2\|^q u_2, \xi_t) - (\|v_1\|^q v_1 - \|v_2\|^q v_2, \zeta_t).\end{aligned} \tag{3.29}$$

Integrating over $[t, T]$ to (3.29), we get that

$$\begin{aligned}&E_m(T) + \int_t^T (D(t, \xi_t), \xi_t)d\tau + \int_t^T (D(t, \zeta_t), \zeta_t)d\tau\\ =&E_m(t) - \int_t^T (k^2(u_1 - v_1)^+ - k^2(u_2 - v_2)^+, \xi_t)d\tau\\ &+ \int_t^T (k^2(u_1 - v_1)^+ - k^2(u_2 - v_2)^+, \zeta_t)d\tau\\ &- \int_t^T (\|u_1\|^q u_1 - \|u_2\|^q u_2, \xi_t)d\tau - \int_t^T (\|v_1\|^q v_1 - \|v_2\|^q v_2, \zeta_t)d\tau.\end{aligned} \tag{3.30}$$

Multiplying (3.26) and (3.27) by ξ and ζ, and integrating over Ω, respectively, we obtain

$$\begin{aligned}&\frac{d}{dt}(\xi_t, \xi) + \frac{d}{dt}(\zeta_t, \zeta) - \|\xi_t\|^2 + \|\triangle\xi\|^2 - \|\zeta_t\|^2 + \|\nabla\zeta\|^2\\ &+ (D(t, \xi_t), \xi) + (D(t, \zeta_t), \zeta)\\ =& - (k^2(u_1 - v_1)^+ - k^2(u_2 - v_2)^+, \xi) + (k^2(u_1 - v_1)^+ - k^2(u_2 - v_2)^+, \zeta)\\ &- (\|u_1\|^q u_1 - \|u_2\|^q u_2, \xi) - (\|v_1\|^q v_1 - \|v_2\|^q v_2, \zeta),\end{aligned} \tag{3.31}$$

and integrating over $[0, T]$, it leads to

$$2\int_0^T E_m(t)dt - 2\int_0^T \|\xi_t\|^2 dt - 2\int_0^T \|\zeta_t\|^2 dt + (\xi_t, \xi)|_0^T + (\zeta_t, \zeta)|_0^T$$

$$
\begin{aligned}
&+\int_0^T (D(t,\xi_t),\xi)dt+\int_0^T (D(t,\zeta_t),\zeta)dt\\
=&-\int_0^T (k^2(u_1-v_1)^+-k^2(u_2-v_2)^+,\xi)dt\\
&+\int_0^T (k^2(u_1-v_1)^+-k^2(u_2-v_2)^+,\zeta)dt\\
&-\int_0^T (\|u_1\|^q u_1-\|u_2\|^q u_2,\xi)dt-\int_0^T (\|v_1\|^q v_1-\|v_2\|^q v_2,\zeta)dt. \qquad (3.32)
\end{aligned}
$$

By using continuously embedding theorem and (2.1), we have

$$|(\xi_t,\xi)|\le \|\xi_t\|\|\xi\|\le \frac{1}{2}(\|\xi_t\|^2+\|\xi\|^2)\le CE_m(t), \qquad (3.33)$$

$$|(\zeta_t,\zeta)|\le \|\zeta_t\|\|\zeta\|\le \frac{1}{2}(\|\zeta_t\|^2+\|\zeta\|^2)\le CE_m(t). \qquad (3.34)$$

Therefore, we infer that

$$
\begin{aligned}
2\int_0^T E_m(t)dt \le& C_5(E_m(T)-E_m(0))+2\int_0^T \|\xi_t\|^2dt+2\int_0^T \|\zeta_t\|^2dt\\
&-\int_0^T (D(t,\xi_t),\xi)dt-\int_0^T (D(t,\zeta_t),\zeta)dt\\
&-\int_0^T (k^2(u_1-v_1)^+-k^2(u_2-v_2)^+,\xi)dt\\
&+\int_0^T (k^2(u_1-v_1)^+-k^2(u_2-v_2)^+,\zeta)dt\\
&-\int_0^T (\|u_1\|^q u_1-\|u_2\|^q u_2,\xi)dt\\
&-\int_0^T (\|v_1\|^q v_1-\|v_2\|^q v_2,\zeta)dt. \qquad (3.35)
\end{aligned}
$$

Setting $t=0$ in (3.30), we have

$$
\begin{aligned}
E_m(0)=&E_m(T)+\int_0^T (D(t,\xi_t),\xi_t)dt+\int_0^T (D(t,\zeta_t),\zeta_t)dt\\
&+\int_0^T (k^2(u_1-v_1)^+-k^2(u_2-v_2)^+,\xi_t)dt\\
&-\int_0^T (k^2(u_1-v_1)^+-k^2(u_2-v_2)^+,\zeta_t)dt\\
&+\int_0^T (\|u_1\|^q u_1-\|u_2\|^q u_2,\xi_t)dt+\int_0^T (\|v_1\|^q v_1-\|v_2\|^q v_2,\zeta_t)dt. \qquad (3.36)
\end{aligned}
$$

Moreover, thanks to the monotonicity of D, integrating (3.30) from 0 to T given

$$TE_m(T)-\int_0^T E_m(t)dt\le \int_0^T dt\int_t^T (k^2(u_1-v_1)^+-k^2(u_2-v_2)^+,\xi_t)d\tau$$

$$+\int_0^T dt\int_t^T (k^2(u_1-v_1)^+ - k^2(u_2-v_2)^+,\zeta_t)d\tau$$
$$+\int_0^T dt\int_t^T (\|u_1\|^q u_1 - \|u_2\|^q u_2,\xi_t)d\tau$$
$$+\int_0^T dt\int_t^T (\|v_1\|^q v_1 - \|v_2\|^q v_2,\zeta_t)d\tau. \tag{3.37}$$

According to $|(u_1-v_1)^+-(u_2-v_2)^+|\le L|(u_1-v_1)-(u_2-v_2)|$ ($L>0$ is a suitable constant), $\|(u,u_t,v,v_t)\|_{\mathcal{H}}\le R$ and (2.1), we have

$$\begin{aligned}&|(k^2(u_1-v_1)^+ - k^2(u_2-v_2)^+,\xi)|\\ \le& Lk^2\|(u_1-v_1)-(u_2-v_2)\|\cdot\|\xi\|\\ =& Lk^2\|\xi-\zeta\|\cdot\|\xi\|\le C(R)\|\triangle\xi\|^2.\end{aligned} \tag{3.38}$$

Similarly, then

$$|(k^2(u_1-v_1)^+ - k^2(u_2-v_2)^+,\zeta)|\le C(R)\|\nabla\zeta\|^2, \tag{3.39}$$

$$|(k^2(u_1-v_1)^+ - k^2(u_2-v_2)^+,\xi_t)|\le C(R)\|\triangle\xi\|\cdot\|\xi_t\|, \tag{3.40}$$

$$|(k^2(u_1-v_1)^+ - k^2(u_2-v_2)^+,\zeta_t)|\le C(R)\|\nabla\zeta\|\cdot\|\zeta_t\|. \tag{3.41}$$

Hence, combining with (3.35)-(3.36), then (3.23) holds. □

Now, we are ready to prove the main result of this section that the dynamical system $(\mathcal{H},S(t))$ corresponding to (1.5)-(1.6) in the space $\mathcal{H}$ is asymptotically smooth.

Theorem 3.3. *The dynamical system $(\mathcal{H},S(t))$ generated by problem (1.5)-(1.6) is asymptotically smooth in the space $\mathcal{H}$.*

Proof. By Theorem 3.1, we know that $\mathcal{B}_0$ is a bounded absorbing set of semigroup $S(t)$ related to (1.5)-(1.6) in the space $\mathcal{H}$. By the definition of bounded absorbing set there exists $t_0\ge 0$ such that $S(t)\mathcal{B}_0\subset\mathcal{B}_0$ for all $t\ge t_0$. Let $\mathcal{B}=\bigcup_{t\ge t_0}S(t)\mathcal{B}_0$. It is clear that $\mathcal{B}$ is a closed bounded forward invariant set for the dynamical system $(\mathcal{H},S(t))$ in the space $\mathcal{H}$. Therefore, for any bounded set B, we have $S(t)B\subset\mathcal{B}_0$ for $t\ge t(B)$, i.e., for all $t\ge t_0+t(B)$, we have $S(t)B\subset\mathcal{B}$, hence $\mathcal{B}$ is also an bounded absorbing set for this system. Let (u_1,v_1) and (u_2,v_2) be two weak solution of (1.5)-(1.6) corresponding to two different initical datas in the invariant set $\mathcal{B}$, then

$$\begin{aligned}&\Big(u_1(t),u_{1t}(t),v_1(t),v_{1t}(t)\Big)=S(t)y_0,\\ &\Big(u_2(t),u_{2t}(t),v_2(t),v_{2t}(t)\Big)=S(t)y_1.\end{aligned} \tag{3.42}$$

Since all term of (2.13) are continuous with respect to the distance d given by the energy norm $\|\cdot\|_E$, it satisfies the condition of Theorem 2.3. Let $T>0$, according to the energy equality (2.4), we have

$$\int_0^T (D(u_1),u_{1t})dt+\int_0^T (D(v_1),v_{1t})dt+\int_0^T (D(u_2),u_{2t})dt+\int_0^T (D(v_2),v_{2t})dt$$

$$+\int_0^T (\|u_1\|^q u_1, u_{1t})dt + \int_0^T (\|v_1\|^q v_1, v_{1t})dt + \int_0^T (\|u_2\|^q u_2, u_{2t})dt$$
$$+\int_0^T (\|v_2\|^q v_2, v_{2t})dt \le C_{\mathbb{B}}. \tag{3.43}$$

Step 1. Energy reconstruction.

From (3.23), we define

$$\begin{aligned}
&\Phi_T(u_1, v_1, u_2, v_2)\\
=&\int_0^T \|\triangle\xi\|\cdot\|\xi_t\|dt + \int_0^T \|\nabla\zeta\|\cdot\|\zeta_t\|dt + \int_0^T \|\triangle\xi\|^2 dt + \int_0^T \|\nabla\zeta\|^2 dt\\
&+\int_0^T dt\int_t^T \|\triangle\xi\|\cdot\|\xi_t\|d\tau + \int_0^T dt\int_t^T \|\nabla\zeta\|\cdot\|\zeta_t\|d\tau\\
&+\Big|\int_0^T (\|u_1\|^q u_1 - \|u_2\|^q u_2, \xi)dt\Big| + \Big|\int_0^T (\|v_1\|^q v_1 - \|v_2\|^q v_2, \zeta)dt\Big|\\
&+\Big|\int_0^T (\|u_1\|^q u_1 - \|u_2\|^q u_2, \xi_t)dt\Big| + \Big|\int_0^T (\|v_1\|^q v_1 - \|v_2\|^q v_2, \zeta_t)dt\Big|\\
&+\Big|\int_0^T dt\int_t^T (\|u_1\|^q u_1 - \|u_2\|^q u_2, \xi_t)d\tau\Big|\\
&+\Big|\int_0^T dt\int_t^T (\|v_1\|^q v_1 - \|v_2\|^q v_2, \zeta_t)d\tau\Big|,
\end{aligned} \tag{3.44}$$

furthermore,

$$\begin{aligned}
&\Phi_T(u_1, v_1, u_2, v_2)\\
\le& C_{T,\mathcal{B}}\Big\{\int_0^T \|\triangle\xi\|\cdot\|\xi_t\|dt + \int_0^T \|\nabla\zeta\|\cdot\|\zeta_t\|dt + \int_0^T \|\triangle\xi\|^2 dt + \int_0^T \|\nabla\zeta\|^2 dt\\
&+\int_0^T \|(\|u_1\|^q u_1 - \|u_2\|^q u_2)\|\cdot\|\xi\|dt + \int_0^T \|(\|v_1\|^q v_1 - \|v_2\|^q v_2)\|\cdot\|\zeta\|dt\\
&+\int_0^T \|(\|u_1\|^q u_1 - \|u_2\|^q u_2)\|\cdot\|\xi_t\|dt + \int_0^T \|(\|v_1\|^q v_1 - \|v_2\|^q v_2)\|\cdot\|\zeta_t\|dt\Big\}.
\end{aligned}$$

By Cauchy inequality and compact embedding theorem, we arrive at

$$\begin{aligned}
&\int_0^T \|\triangle\xi\|\cdot\|\xi_t\| + \int_0^T \|\triangle\xi\|^2 dt\\
\le& C_\kappa\int_0^T \|\triangle\xi\|^2 dt + \frac{\kappa}{2}\int_0^T \|\xi_t\|^2 dt + \int_0^T \|\triangle\xi\|^2 dt\\
\le& C_{\mathcal{B},\kappa}\int_0^T \|A^{1-\alpha}\xi\|^2 dt + \kappa\int_0^T E_m(t)dt,
\end{aligned} \tag{3.45}$$

and

$$\begin{aligned}
&\int_0^T \|\nabla\zeta\|\cdot\|\zeta_t\|dt + \int_0^T \|\nabla\zeta\|^2 dt\\
\le& C_\kappa\int_0^T \|\nabla\zeta\|^2 dt + \frac{\kappa}{2}\int_0^T \|\zeta_t\|^2 dt + \int_0^T \|\nabla\zeta\|^2 dt
\end{aligned}$$

$$\leq C_{\mathcal{B},\kappa}\int_0^T \|A^{\frac{1}{2}-\beta}\zeta\|^2 dt + \kappa\int_0^T E_m(t)dt, \tag{3.46}$$

where $0<\alpha<\frac{1}{2}$, $0<\beta<\frac{1}{4}$.

According to the Sobolev and Hölder inequalities, it yields

$$\begin{aligned}\big|(\|u_1\|^q u_1 - \|u_2\|^q u_2)\big| &= \big|(\|u_1\|^q u_1 - \|u_1\|^q u_2 + \|u_1\|^q u_2 - \|u_2\|^q u_2)\big| \\ &\leq \|u_1\|^q|u_1-u_2| + \big|\|u_1\|^q - \|u_2\|^q\big||u_2|, \end{aligned}\tag{3.47}$$

then

$$\begin{aligned}\|(\|u_1\|^q u_1 - \|u_2\|^q u_2)\|^2 &= \int_\Omega \big|(\|u_1\|^q u_1 - \|u_2\|^q u_2)\big|^2 dx \\ &\leq \int_\Omega \big|\|u_1\|^q|u_1-u_2| + \big|\|u_1\|^q - \|u_2\|^q\big||u_2|\big|^2 dx \\ &\leq 2(\|u_1\|^{2q}\|u_1-u_2\|^2 + \|u_2\|^2\big|\|u_1\|^q - \|u_2\|^q\big|^2) \\ &\leq C(R)\|u_1-u_2\|^2 \\ &\leq C(R)\|A^{1-\tilde{\alpha}}\xi\|^2, \end{aligned}\tag{3.48}$$

similarly, we have

$$\|(\|v_1\|^q v_1 - \|v_2\|^q v_2)\|^2 \leq C(R)\|v_1-v_2\|^2 \leq C(R)\|A^{\frac{1}{2}-\tilde{\beta}}\zeta\|^2. \tag{3.49}$$

By Hölder inequality and compact embedding theorem, we infer to

$$\begin{aligned}&\int_0^T \|(\|u_1\|^q u_1 - \|u_2\|^q u_2)\|\cdot\|\xi\|dt + \int_0^T \|(\|u_1\|^q u_1 - \|u_2\|^q u_2)\|\cdot\|\xi_t\|dt \\ \leq& C_\kappa\int_0^T \|(\|u_1\|^q u_1 - \|u_2\|^q u_2)\|^2 dt + \kappa\int_0^T E_m(t)dt \\ \leq& C_{\mathcal{B},\kappa}\int_0^T \|A^{1-\tilde{\alpha}}\xi\|^2 dt + \kappa\int_0^T E_m(t)dt, \end{aligned}\tag{3.50}$$

similarly,

$$\begin{aligned}&\int_0^T \|(\|v_1\|^q v_1 - \|v_2\|^q v_2)\|\cdot\|\zeta\|dt + \int_0^T \|(\|v_1\|^q v_1 - \|v_2\|^q v_2)\|\cdot\|\zeta_t\|dt \\ \leq& C_{\mathcal{B},\kappa}\int_0^T \|A^{\frac{1}{2}-\tilde{\beta}}\zeta\|^2 dt + \kappa\int_0^T E_m(t)dt. \end{aligned}\tag{3.51}$$

Therefore, combining with (3.45)-(3.46) and (3.50)-(3.51), for any $\kappa>0$, choosing $\delta=\min\{\alpha,\tilde{\alpha}\}$, $\tilde{\delta}=\min\{\beta,\tilde{\beta}\}$, we have

$$\begin{aligned}\Phi_T(u_1,v_1,u_2,v_2) \leq& C_{\mathcal{B},\kappa}(T)\int_0^T \|A^{1-\delta}\xi\|^2 dt + C_{\mathcal{B},\kappa}(T)\int_0^T \|A^{\frac{1}{2}-\tilde{\delta}}\zeta\|^2 dt \\ &+ 4\kappa\int_0^T E_m(t)dt. \end{aligned}\tag{3.52}$$

In line with Lemma 2.2, let $H_0(s)=C_p^{-\frac{2}{p+2}}s^{\frac{2}{p+2}}$, $p\geq 0$, it is a strictly increasing, concave function, and $H_0\in C(\mathbb{R}^+)$ with the property $H_0(0)=0$ such that

$$H_0\Big((\|u+v\|^p(u+v)-\|u\|^p u, v)\Big)$$

$$\geq H_0(C_p\|v\|^{p+2}) = \|v\|^2, \quad \forall u, v \in V_2 \times V_1. \tag{3.53}$$

Hence, from Jensen inequality, it follows that

$$\begin{aligned}\int_0^T \|\xi_t\|^2 dt \leq & \int_0^T H_0(D(t,\xi_t),\xi_t)dt \\ & \leq TH_0\Big(\frac{1}{T}\int_0^T (D(t,\xi_t),\xi_t)dt\Big) \\ & = \mathcal{H}_0\Big(\int_0^T (D(t,\xi_t),\xi_t)dt\Big), \end{aligned} \tag{3.54}$$

where $\mathcal{H}_0(s) = TH_0(\frac{s}{T})$. Similarly, we have

$$\int_0^T \|\zeta_t\|^2 dt \leq \mathcal{H}_0\Big(\int_0^T (D(t,\zeta_t),\zeta_t)dt\Big). \tag{3.55}$$

By virtue of Cauchy's inequality and Sobolev's embedding theorem, there exists a positive constant η with $0 < \eta < \frac{1}{2}$ such that

$$\begin{aligned}|(D(t,\xi_t),\xi)| \leq & \|\xi\|\Big(\int_\Omega (\|u_{1t}\|^p u_{1t} - \|u_{2t}\|^p u_{2t})^2 dx\Big)^{\frac{1}{2}} \\ & \leq C\|\xi\|\Big(\|u_{1t}\|^{2p}u_{1t}^2 - \|u_{2t}\|^{2p}u_{2t}^2)^2\Big)^{\frac{1}{2}} \\ & \leq C_{\mathcal{B}}\|\xi\| \leq C_{\mathcal{B}}\|A^{1-\eta}\xi\|. \end{aligned} \tag{3.56}$$

Similarly,

$$|(D(t,\zeta_t),\zeta)| \leq C_{\mathcal{B}}\|A^{\frac{1}{2}-\tilde{\eta}}\zeta\|, \quad 0 < \tilde{\eta} < \frac{1}{2}. \tag{3.57}$$

Therefore, combining with (3.52)-(3.57) and Theorem 3.2, we have

$$\begin{aligned} & TE_m(T) + \frac{1}{2}\int_0^T E_m(t)dt \\ \leq & C_{\mathcal{B}}\Big\{(\mathcal{H}_0 + I)\big(\int_0^T (D(t,\xi_t),\xi_t)dt + \int_0^T (D(t,\zeta_t),\zeta_t)dt\big) \\ & + \int_0^T \|A^{1-\eta}\xi\|dt + \int_0^T \|A^{\frac{1}{2}-\tilde{\eta}}\zeta\|dt \\ & + C_{\mathcal{B},T}\int_0^T \|A^{1-\delta}\xi\|^2 dt + C_{\mathcal{B},T}\int_0^T \|A^{\frac{1}{2}-\tilde{\delta}}\zeta\|^2 dt\Big\}. \end{aligned} \tag{3.58}$$

Step 2. Handling of the damping.

Denote $\omega = \min\{\eta, \delta\}$, $\tilde{\omega} = \min\{\tilde{\eta}, \tilde{\delta}\}$, from (3.58) we get that

$$\begin{aligned} E_m(T) \leq & C_{\mathcal{B},T}(\mathcal{H}_0 + I)\Big(\int_0^T (D(t,\xi_t),\xi_t)dt + \int_0^T (D(t,\zeta_t),\zeta_t)dt\Big) \\ & + C_{\mathcal{B},T}\int_0^T \|A^{1-\omega}\xi\|dt + C_{\mathcal{B},T}\int_0^T \|A^{\frac{1}{2}-\tilde{\omega}}\zeta\|dt \end{aligned}$$

$$\le C_{\mathcal{B},T}(\mathcal{H}_0+I)\Big(\int_0^T(D(t,\xi_t),\xi_t)dt+\int_0^T(D(t,\zeta_t),\zeta_t)dt\Big)$$
$$+C_{\mathcal{B},T}\Big(\sup_{t\in[0,T]}\|A^{1-\omega}\xi(t)\|+\sup_{t\in[0,T]}\|A^{\frac{1}{2}-\tilde{\omega}}\zeta(t)\|\Big). \tag{3.59}$$

Let $Q_0(s)=(\mathcal{H}_0+I)^{-1}\big(\frac{s}{2C_{\mathcal{B},T}}\big)$ be a strictly increasing, convex function and $(\mathcal{H}_0+I)^{-1}(s)\le s$ for any $s\ge 0$. By (3.58) we infer that

$$\begin{aligned}Q_0(E_m(T))=&(\mathcal{H}_0+I)^{-1}\Big(\frac{E_m(T)}{2C_{\mathcal{B},T}}\Big)\\ \le&(\mathcal{H}_0+I)^{-1}\Big\{\frac{1}{2}(\mathcal{H}_0+I)\big(\int_0^T(D(t,\xi_t),\xi_t)dt+\int_0^T(D(t,\zeta_t),\zeta_t)dt\big)\\ &+\frac{1}{2}\big(\sup_{t\in[0,T]}\|A^{1-\omega}\xi(t)\|+\sup_{t\in[0,T]}\|A^{\frac{1}{2}-\tilde{\omega}}\zeta(t)\|\big)\Big\}\\ \le&\frac{1}{2}\Big\{\int_0^T(D(t,\xi_t),\xi_t)dt+\int_0^T(D(t,\zeta_t),\zeta_t)dt\Big\}\\ &+\frac{1}{2}\Big\{\sup_{t\in[0,T]}\|A^{1-\omega}\xi(t)\|+\sup_{t\in[0,T]}\|A^{\frac{1}{2}-\tilde{\omega}}\zeta(t)\|\Big\}.\end{aligned} \tag{3.60}$$

Setting $t=0$ in (3.30), and combining with (3.40)-(3.41) and (3.48)-(3.49), we achieve

$$\begin{aligned}&\int_0^T(D(t,\xi_t),\xi_t)d\tau+\int_0^T(D(t,\zeta_t),\zeta_t)d\tau\\ =&E_m(0)-E_m(T)-\int_0^T(k^2(u_1-v_1)^+-k^2(u_2-v_2)^+,\xi_t)d\tau\\ &+\int_0^T(k^2(u_1-v_1)^+-k^2(u_2-v_2)^+,\zeta_t)d\tau\\ &-\int_0^T(\|u_1\|^qu_1-\|u_2\|^qu_2,\xi_t)d\tau-\int_0^T(\|v_1\|^qv_1-\|v_2\|^qv_2,\zeta_t)d\tau\\ \le&E_m(0)-E_m(T)+C_{\mathcal{B},T}\Big(\sup_{t\in[0,T]}\|A^{1-\omega}\xi(t)\|+\sup_{t\in[0,T]}\|A^{\frac{1}{2}-\tilde{\omega}}\zeta(t)\|\Big),\end{aligned} \tag{3.61}$$

then

$$\begin{aligned}&E_m(T)+2Q_0(E_m(T))\\ \le&E_m(0)-E_m(T)+C_{\mathcal{B},T}\Big(\sup_{t\in[0,T]}\|A^{1-\omega}\xi(t)\|+\sup_{t\in[0,T]}\|A^{\frac{1}{2}-\tilde{\omega}}\zeta(t)\|\Big).\end{aligned} \tag{3.62}$$

Since $\xi(t)\in D(A)$, $\zeta(t)\in D(A^{\frac{1}{2}})$ are uniformly bounded, and the embedding $D(A)\hookrightarrow D(A^{1-\omega})\hookrightarrow V_0$, $D(A^{\frac{1}{2}})\hookrightarrow D(A^{\frac{1}{2}-\tilde{\omega}})\hookrightarrow V_0$ are compact. Thus, exploiting the interpolation inequality we obtain

$$\|A^{1-\omega}\xi(t)\|\le\|\xi(t)\|_{D(A)}^{\eta_1}\cdot\|\xi(t)\|^{1-\eta_1}\le C_R\|\xi(t)\|^{1-\eta_1},\qquad 0<\eta_1<1, \tag{3.63}$$

and

$$\|A^{\frac{1}{2}-\tilde{\omega}}\zeta(t)\|\le\|\zeta(t)\|_{D(A^{\frac{1}{2}})}^{\eta_2}\cdot\|\zeta(t)\|^{1-\eta_2}\le C_R\|\zeta(t)\|^{1-\eta_2},\qquad 0<\eta_2<1. \tag{3.64}$$

Therefore

$$\begin{aligned}&E_m(T)+2Q_0(E_m(T))\\ \le& E_m(0)+C_{\mathcal{B},T}\Big(\sup_{t\in[0,T]}\|\xi(t)\|^{\varsigma}+\sup_{t\in[0,T]}\|\zeta(t)\|^{\varsigma}\Big),\end{aligned} \tag{3.65}$$

for any $\varsigma\in(0,1]$. This implies that

$$\begin{aligned}&\|S(T)y_1-S(T)y_2\|_{\mathcal{H}}^2\\ \le& 2[I+2Q_0]^{-1}\Big\{\frac{1}{2}\|y_1-y_2\|^2+C_{\mathcal{B},T}\big(\sup_{t\in[0,T]}\|\xi(t)\|^{\varsigma}+\sup_{t\in[0,T]}\|\zeta(t)\|^{\varsigma}\big)\Big\}\\ \le& 2[I+2Q_0]^{-1}\Big\{\frac{1}{2}\big(\|y_1-y_2\|+C_{\mathcal{B},T}(\sup_{t\in[0,T]}\|\xi(t)\|^{\varsigma}+\sup_{t\in[0,T]}\|\zeta(t)\|^{\varsigma})^{\frac{1}{2}}\big)^2\Big\}.\end{aligned} \tag{3.66}$$

Choosing $\varsigma^{'}\in(0,\frac{1}{2}]$, we have

$$\begin{aligned}&\|S(T)y_1-S(T)y_2\|_{\mathcal{H}}\\ \le& \sqrt{2}\bigg[[I+2Q_0]^{-1}\Big\{\frac{1}{2}\big(\|y_1-y_2\|+C_{\mathcal{B},T}(\sup_{t\in[0,T]}\|\xi(t)\|^{\varsigma^{'}}+\sup_{t\in[0,T]}\|\zeta(t)\|^{\varsigma^{'}})\big)^2\Big\}\bigg]^{\frac{1}{2}}.\end{aligned}$$

Let $r(s)=\sqrt{2}\Big((I+2Q_0)^{-1}(\frac{s^2}{2})\Big)^{\frac{1}{2}}$, and

$$\varrho_{\mathcal{B}}^T\big(\{S(\tau)y_1\},\{S(\tau)y_2\}\big)=C_{\mathcal{B},T}(\sup_{t\in[0,T]}\|u_1(t)-u_2(t)\|^{\varsigma^{'}}+\sup_{t\in[0,T]}\|v_1(t)-v_2(t)\|^{\varsigma^{'}}),$$

so we conclude that

$$\|S(T)y_1-S(T)y_2\|_{\mathcal{H}}\le r\big(\|y_1-y_2\|+\varrho_{\mathcal{B}}^T(\{S(\tau)y_1\},\{S(\tau)y_2\})\big). \tag{3.67}$$

It is clear that the function r satisfies all the requirements of Theorem 2.3. Finally, according to the similar proof in [24] we get that the pseudometric $\varrho_{\mathcal{B}}^T$ of all solution for (1.5)-(1.6) is precompact in the interval $[0,T]$. Hence, the dynamical system $(\mathcal{H},S(t))$ is asymptotically smooth. □

Thanks to Theorem 3.1 and Theorem 3.3, we deduce the main result of this section as the following theorem.

Theorem 3.4. *The dynamical system* $(\mathcal{H},S(t))$ *generated by problem* (1.5)-(1.6) *possesses a compact global attractor* $\mathcal{A}$ *in the space* $\mathcal{H}$.

4. Fractal dimension and generalized exponential attractor

In this section, we mainly prove the quasi-stability of the dynamical system $(\mathcal{H},S(t))$ associated to (1.5)-(1.6) to give the finite fractal dimension of attractors, and further obtain the existence of the generalized exponential attractor $\mathcal{A}^{exp}$ with finite fractal dimension in a larger space $\widetilde{\mathcal{H}}\supseteq\mathcal{H}$. Firstly, we are going to prove that the dynamical system $(\mathcal{H},S(t))$ is quasi-stable on any bounded positively invariant set in $\mathcal{H}$.

Lemma 4.1. *The dynamical system* $(\mathcal{H}, S(t))$ *generated by problem* (1.5)-(1.6) *is quasi-stable in a bounded positively invariant set* $\mathcal{B} \subset \mathcal{H}$.

Proof. According to Definition 2.5, we only need to verify inequalities (2.13) and (2.14). From (3.29), we have

$$\begin{aligned}&\frac{d}{dt}E_m(t) + (D(t,\xi_t),\xi_t) + (D(t,\zeta_t),\zeta_t)\\ =&-(k^2(u_1-v_1)^+ - k^2(u_2-v_2)^+,\xi_t) - (\|u_1\|^q u_1 - \|u_2\|^q u_2,\xi_t)\\ &+(k^2(u_1-v_1)^+ - k^2(u_2-v_2)^+,\zeta_t) - (\|v_1\|^q v_1 - \|v_2\|^q v_2,\zeta_t),\end{aligned} \tag{4.1}$$

where $\xi(t) = u_1(t) - u_2(t)$, $\zeta(t) = v_1(t) - v_2(t)$, by virtue of (3.40)-(3.41), we achieve at

$$\begin{aligned}&|-(k^2(u_1-v_1)^+ - k^2(u_2-v_2)^+,\xi_t)|\\ \le& C(R)\|\triangle\xi\|\|\xi_t\| \le C(R)(\frac{1}{2}\|\triangle\xi\|^2 + \frac{1}{2}\|\xi_t\|^2)\\ \le& C(R)E_m(t),\end{aligned} \tag{4.2}$$

similarly, we have

$$|(k^2(u_1-v_1)^+ - k^2(u_2-v_2)^+,\zeta_t)| \le C(R)E_m(t). \tag{4.3}$$

According to (2.3), it follows that

$$(D(t,\xi_t),\xi_t) + (D(t,\zeta_t),\zeta_t) \ge C_p\|\xi_t\|^{P+2} + C_p\|\zeta_t\|^{P+2} \ge 0, \tag{4.4}$$

using Hölder inequality, the embedding $V_2 \hookrightarrow L^{2(q+1)}(\Omega)$ and $V_2 \hookrightarrow L^{p+2}(\Omega)$, it yields

$$\begin{aligned}&|-(\|u_1\|^q u_1 - \|u_2\|^q u_2,\xi_t)|\\ =&(q+1)\int_\Omega |\theta u_1 + (1-\theta)u_2|^q|\xi||\xi_t|dx\\ \le&(q+1)2^{2q}(\|u_1\|^q_{2(q+1)} + \|u_2\|^q_{2(q+1)})\|\xi\|_{2(q+1)}\|\xi_t\|\\ \le& C_{\mathcal{B}}\|\xi\|_{2(q+1)}\|\xi_t\|_{p+2} \le C_{\mathcal{B}}(\|\xi\|^2_{2(q+1)} + \|\xi_t\|^2_{p+2})\\ \le& C_{\mathcal{B},R}E_m(t),\end{aligned} \tag{4.5}$$

similarly,

$$|-(\|v_1\|^q v_1 - \|v_2\|^q v_2,\zeta_t)| \le C_{\mathcal{B},R}E_m(t). \tag{4.6}$$

Together with (4.1)-(4.6), we conclude that

$$\frac{d}{dt}E_m(t) \le 2C(R)E_m(t) + 2C_{\mathcal{B},R}E_m(t) \le C(R,\mathcal{B})E_m(t), \tag{4.7}$$

in line with the Gronwall lemma, we get that

$$E_m(t) \le e^{C(R,\mathcal{B})t}E_m(0), \tag{4.8}$$

in addition, we see that (2.13) holds with $a(t) = e^{C(R,\mathcal{B})t}$, where $a(t)$ is locally bounded on $[0,\infty]$ because of the boundedness of $\mathcal{B} \subset \mathcal{H}$. On the other hand, by virtue of the proof of the Theorem 3.3, we claim from (3.66) that

$$\begin{aligned} &\|S(T)y_1 - S(T)y_2\|_{\mathcal{H}}^2 \\ \leq& 2[I+2Q_0]^{-1}\Big\{\frac{1}{2}\|y_1-y_2\|^2 + C_{\mathcal{B},T}(\sup_{t\in[0,T]}\|\xi(t)\|^{\varsigma} + \sup_{t\in[0,T]}\|\zeta(t)\|^{\varsigma})\Big\} \\ \leq& [I+2Q_0]^{-1}\|y_1-y_2\|^2 + 2[I+2Q_0]^{-1}C_{\mathcal{B},T}\max_{t\in[0,T]}(\|\xi(t)\|^{\varsigma} + \|\zeta(t)\|^{\varsigma}), \end{aligned} \tag{4.9}$$

where $\varsigma \in (0,1]$, Q_0 is defined in the previous section. Thus, we define the semi-norm as follows

$$n_{\mathcal{H}}(\xi,\zeta) = \|\xi(t)\|^{\varsigma} + \|\zeta(t)\|^{\varsigma}. \tag{4.10}$$

By using the compact embedding $V_2 \hookrightarrow\hookrightarrow V_0$ and $V_1 \hookrightarrow\hookrightarrow V_0$, we conclude that $n_{\mathcal{H}}$ is a compact semi-norm on $\mathcal{H}$. Then (2.14) holds with

$$b(t) = [I+2Q_0]^{-1}, \qquad c(t) = 2[I+2Q_0]^{-1}C_{\mathcal{B},T},$$

it is easy to check that

$$b(t) \in L^1(\mathbb{R}^+), \qquad \lim_{t\to\infty} b(t) = 0.$$

Since $\mathcal{B} \subset \mathcal{H}$ is bounded, so $c(t)$ is locally bounded on $[0,\infty]$. Then we conclude that the dynamical system $(\mathcal{H}, S(t))$ is quasi-stable in a bounded positively invariant set $\mathcal{B} \subset \mathcal{H}$ by Definition 2.5. Therefore the proof of the lemma is complete. □

From the above Lemma 4.1 we know that the dynamical system $(\mathcal{H}, S(t))$ is quasi-stable on the compact global attractor $\mathcal{A}$, which is a bounded positively invariant set of $\mathcal{H}$, and Theorem 3.4 ensures that $(\mathcal{H}, S(t))$ has a compact global attractor in $\mathcal{H}$. Thus we can immediately conclude the following results by Theorem 2.4.

Theorem 4.1. *The compact global attractor $\mathcal{A}$ of the dynamical system $(\mathcal{H}, S(t))$ has finite fractal dimension.*

Now, we will prove the existence of the generalized exponential attractor $\mathcal{A}^{exp}$ and it has finite fractal dimension in a larger space $\widetilde{\mathcal{H}} \supseteq \mathcal{H}$.

Theorem 4.2. *The dynamical system $(\mathcal{H}, S(t))$ generated by problem* (1.5)-(1.6) *possesses a generalized exponential attractor $\mathcal{A}^{exp}$ with finite fractal dimension on the space $\widetilde{\mathcal{H}} = L^2(\Omega) \times H^{-2}(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega) \supseteq \mathcal{H}$.*

Proof. It is easy to see that the dynamical system $(\mathcal{H}, S(t))$ is quasi-stable in a bounded positively invariant set $\mathcal{B} \subset \mathcal{H}$ by Lemma 4.1, thus we only need to prove that the mapping $t \to S(t)y$ is Hölder continuous on the space $\widetilde{\mathcal{H}}$. Indeed, we know that $S(t)y = (u(t), u_t(t), v(t), v_t(t)) = \phi(t)$ for every $y = \phi(0) = (u_0, u_1, v_0, v_1) \in \mathcal{B}$. By virtue of Theorem 3.1, there exists $R > 0$ such that $\|u_t\|_{V_2}^2 + \|u_{tt}\|^2 + \|v_t\|_{V_1}^2 + \|v_{tt}\|^2 \leq R^2$, and then

$$\begin{aligned} \|\phi_t(t)\|_{\widetilde{\mathcal{H}}}^2 &= \|u_t\|^2 + \|u_{tt}\|_{V_{-2}}^2 + \|v_t\|^2 + \|v_{tt}\|_{V_{-1}}^2 \\ &\leq C(\|u_t\|_{V_2}^2 + \|u_{tt}\|^2 + \|v_t\|_{V_1}^2 + \|v_{tt}\|^2) \end{aligned}$$

$$\leq C_{\mathcal{B}}. \tag{4.11}$$

Hence, for any $0 \leq t_1 \leq t_2 \leq T$, it follows that

$$\|S(t_1)y - S(t_2)y\|_{\widetilde{\mathcal{H}}} \leq \int_{t_1}^{t_2} \|\phi_t(s)\|_{\widetilde{\mathcal{H}}} ds \leq C_{\mathcal{B}}|t_1 - t_2|. \tag{4.12}$$

In view of Theorem 2.5, choosing $\tau = 1$, therefore we conclude that the dynamical system $(\mathcal{H}, S(t))$ has a generalized exponential attractor $\mathcal{A}^{exp} \subset \widetilde{\mathcal{H}}$ with finite fractal dimension. □

Remark 4.1. Since the problem (1.5)-(1.6) is in one-dimensional space, and $H_0^1(\Omega) \subset L^q(\Omega)(1 \leq q \leq \infty)$, the nonlocal term $\|u\|^q u$ and $\|v\|^q v$ don't bring any difficulty.

Remark 4.2. If the nonlocal functions $\|u\|^q u$ and $\|v\|^q v$ turn into the polynomial functions $|u|^q u$ and $|v|^q v (q \geq 0)$, then all results in this paper still hold because of $H_0^1(\Omega) \subset L^q(\Omega)(1 \leq q \leq \infty)$ in one-dimensional space.

Remark 4.3. When the dimension of space is bigger than 2, the exponential q of polynomial functions $|u|^q u$ and $|v|^q v(q \geq 0)$ in the equation is required to satisfy some certain condition, see [24] for details.

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