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# EXPONENTIAL STABILITY AND APPLICATIONS OF SWITCHED POSITIVE LINEAR IMPULSIVE SYSTEMS WITH TIME-VARYING DELAYS AND ALL UNSTABLE SUBSYSTEMS

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**Abstract** The global uniform exponential stability of switched positive linear impulsive systems with time-varying delays and all unstable subsystems is studied in this paper, which includes two types of distributed time-varying delays and discrete time-varying delays. Switching behaviors dominating the switched systems can be either stabilizing or destabilizing in the new designed switching sequence. We design new linear programming algorithm process to find the feasible ratio of stabilizing switching behaviors, which can be compensated by unstable subsystems, destabilizing switching behaviors, and impulses. Specifically, we add a kind of nonnegative impulses which is consistent with the switching behaviors for the systems. Employing a multiple co-positive Lyapunov–Krasovskii functional, we present several new sufficient stability criteria and design new switching sequence. Then, we apply the obtained stability criteria to the exponential consensus of linear delayed multi-agent systems, and obtain the new exponential consensus criteria. Three simulations are provided to demonstrate the proposed stability criteria.

**Keywords** Exponential stability, switched systems, impulses, distributed time-varying delays.

**MSC(2010)** 34K34, 34K45.

## 1. Introduction

Switched systems are the special dynamical systems, which consist of a number of subsystems and a switching rule among them [6, 12, 18, 23]. As a special class of hybrid systems, when the subsystems of switched systems are all positive linear systems, they are called switched positive linear systems (SPLSs). SPLSs receive a lot of attentions since they can appropriately model some complex systems, such as automotive dynamic controls [20], multi-agent systems [22], aerospace engineering [11], traffic controls [21], and so forth. On the other hand, time-varying delays are frequently encountered in complex systems. Moreover, it is well known that even small time-varying delays may affect or even destroy the stability of the systems, which make difficulties for the stability analysis. Therefore, great interests and

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efforts have been focused on SPLSs containing time-varying delay parameters [7,15], which have been an essential and appealing study topic.

For the stability of studying SPLSs with delays, there mainly exist the following methods, such as co-positive Lyapunov function [2, 28], multiple co-positive Lyapunov–Krasovskii functional [8, 25], and diagonal Lyapunov–Krasovskii functional [1]. For many control systems, the equilibrium point of the systems is usually zero, even if the equilibrium point is not zero, we can convert it to zero by translation transformation in order to better solve the problem. The exponential stability means that the systems state converges to equilibrium point over an infinite time interval, which has a better application because of its strong stability property. Therefore, many scholars devoted to study the exponential stability of switched systems with delays. For instance, Liu etc in [15] studied the exponential stability for SPLSs with delays and impulses by the multiple linear co-positive Lyapunov functional approach and average dwell time method, which presented a delay-dependent exponential stability criterion for SPLSs. Li and Xiang [9] addressed the exponential stability and $L_1$-again control for a class of SPLSs with mixed time-varying delays and impulses by using the average dwell time approach and the co-positive Lyapunov–Krasovskii function technique, and developed an interactive convex optimization approach to verify the results. With the help of constructing an appropriate co-positive type Lyapunov–Krasovskii functional approach, Li etc in [10] investigated the exponential stability and $L_1$-again controller design for SPLSs with mixed time-varying delays. According to a co-positive Lyapunov–Krasovskii functional and average dwell time technique, Liu etc in [16] considered the exponential stability of SPLSs with impulses and mixed time-varying delays.

Up to now, many scholars are interested in the fields that the subsystems of switching systems are all unstable, but the main switching systems are stable. Some significant results are presented in [17,27,29,30]. For instance, based on the theory of spherical covering and crystal point groups, Zhang etc in [27] obtained some sufficient algebraic conditions for stabilizing switched linear systems with all unstable subsystems. Furthermore, authors in [27] designed a switching law to stabilize the unstable switched behaviors. Liu etc in [17] investigated the stabilization problem of SPLSs with discrete time-varying delays by utilizing multiple co-positive Lyapunov–Krasovskii functional, where all the subsystems of the main switching systems are unstable while all switching behaviors in the new designed switching sequence are stabilizing. However, the switching behaviors may make some increment of the energy functions, i.e., be destabilizing to the switched systems [29, 30]. Specifically, Zhou etc in [29] considered the exponential stability of SPLSs with all unstable subsystems and destabilizing switching behaviors in new switching sequence, which evaluates the ratio of stabilizing switching behaviors to compensate the state divergence caused by either unstable subsystems or destabilizing switching behaviors. After that, Zhou etc in [30] investigated the global uniform exponential stability of SPLSs with all unstable subsystems and time-varying delays, which includes two types of distributed time-varying delays and discrete time-varying delays, by adjusting the ratio of the stabilizing switching behaviors, the state divergence caused by unstable subsystems and destabilizing switching behaviors can be compensated.

In some practical applications, impulsive interference is inevitable. Impulsive behavior is regarded as a dynamical process, which expresses a state converts abruptly at some instants [3, 4, 9, 14, 19, 26]. Impulsive systems have been triumphantly applied to problems in physics, mechanics, and some fields of engineering. Recently, Ju etc in [5] took into account the impulsive effects of the switched linear time-varying systems, where the impulsive jumping is limited by a linear matrix form. The results in [5] can be applied to the exponential consensus of linear multi-agent systems. Zhou etc in [30] investigated the influences of time-varying delays and the ratio of stabilizing switching behaviors for switched systems, but they did not consider the impulsive effects for switched systems. In fact, there may exist impulsive behaviors at switching instants for the switched systems in some practical situations, and the linear multi-agent systems may encounter the effects of time-varying delays. Therefore, motivated by [5, 30], we will further study the effects of impulsive behaviors for SPLSs with time-varying delays and all unstable subsystems, and consider the effects of time-varying delays of the linear multi-agent systems with impulses.

In this paper, we add the impulses to the model in [30]. We first investigate the global uniform exponential stability for switched positive linear impulsive systems (SPLISs) with distributed time-varying delays and all unstable subsystems as follows

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}\displaystyle\int_{t-d(t)}^{t} x(s)\mathrm{d}s,\ t \geq 0,\ t \neq t_m,\ m = 1, 2, 3, \cdots, \\ x(t^+) = I_{\sigma(t^+)\sigma(t^-)}(x(t^-)),\ t = t_m,\ m = 1, 2, 3, \cdots, \\ x(t_0 + \varphi) = \varpi(\varphi), \varphi \in [-\hat{d}, 0], \end{cases} \tag{1.1}$$

where $x(t) \in R^n$ stands for the state vector, the switching rule $\sigma(t) : [0, \infty) \to \langle n\rangle$, $\langle n\rangle = \{1, 2, \cdots, n\}$, $n > 1$ is an integer, $\sigma(t)$ denotes a piecewise right-continuous function and satisfies $\lim_{t\to t_m^+} \sigma(t) = \sigma(t_m)$, the matrices $A_i(t), B_i(t) \in R^{n\times n}$, $i \in \langle n\rangle$ are continuous time-varying matrix functions. The continuous switching instants fulfill $0 \leq t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} \cdots$, and $\lim_{m\to\infty} t_m = \infty$. The distributed time-varying delays function $d(t)$ fulfills $0 \leq d(t) \leq \hat{d}$ and $\dot{d}(t) \leq d < 1$, where $\hat{d}$ is the upper bound constant value of $d(t)$, $d$ represents the upper bound constant value of $\dot{d}(t)$, in other words, $d(t)$ is upper bounded and slowly varying. $I_{\sigma(t^+)\sigma(t^-)}(x) : R^n \to R^n$ are impulses, $t_m$, $m = 1, 2, \cdots$, those are not just the switching instants but the impulsive instants, satisfying $0 < t_m < t_{m+1}$ and $\lim_{m\to\infty} t_m = \infty$. When $t \in [t_m, t_{m+1})$, the $\sigma(t_m)$th subsystem is actuated, $m = 0, 1, 2, \cdots$. At the switching and impulsive instants, define $x(t) = x(t^+)$ at $t = t_m$.

Next, we consider the SPLISs with discrete time-varying delays and all unstable subsystems as following

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}x(t - d(t)),\ t \geq 0,\ t \neq t_m,\ m = 1, 2, 3, \cdots, \\ x(t^+) = I_{\sigma(t^+)\sigma(t^-)}(x(t^-)),\ t = t_m,\ m = 1, 2, 3, \cdots, \\ x(t_0 + \varphi) = \varpi(\varphi),\ \varphi \in [-\hat{d}, 0], \end{cases} \tag{1.2}$$

where the definitions of all the variables and parameters are the same as SPLISs (1.1). Then, we apply the obtained stability criteria to the exponential consensus of linear delayed multi-agent systems with impulses.

By discretizing the dwell time interval, redefining the original switching sequence, and employing a multiple co-positive Lyapunov–Krasovskill functional, the global uniform exponential stability of the SPLISs is established as a new attempt. The global uniform exponential stability criteria imply that the ratio of stabilizing switching behaviors are crucial to the global uniform exponential stability of SPLISs with time-varying delays and all unstable subsystems. Consequently, the primary contributions are highlighted as following: (1) Compared with [29, 30], we add the nonnegative impulses to the SPLSs with time-varying delays and all unstable subsystems, which includes distributed time-varying delays and discrete time-varying delays. We first study the global uniform exponential stability of SPLISs with distributed time-varying delays, then we extend the distributed time-varying delays case to discrete time-varying delays case. New sufficient global uniform exponential stability criteria of SPLISs are obtained by multiple co-positive Lyapunov–Krasovskii functional method. So the obtained stability criteria further improve the stability theory of SPLISs. (2) Compared with [5], the effects of discrete time-varying delays and all unstable subsystems are contained for the SPLISs. Moreover, both stabilizing and destabilizing switching behaviors in new switching behaviors are included for SPLISs. By finding the feasible ratio of stabilizing switching behaviors, the systems state divergence aroused by unstable subsystems, destabilizing switching behaviors, impulsive effects are compensated. (3) Different from the existing results in [9], our model is more comprehensive because of the unstable subsystems and destabilizing switching behaviors. We design new linear programming algorithm process to better verify the main stability criteria (Algorithm 1). Furthermore, our main stability criteria are used as the exponential consensus of linear impulsive multi-agent systems with switching communication topologies and discrete time-varying delays (Example 5.3).

The framework of this paper is arranged as following. Model formation and definitions are given in Section 2. Section 3 is dedicated to proving the main stability criteria. In Section 4, the obtained stability criteria are used for the exponential consensus of linear delayed multi-agent systems. Section 5 provides three simulation examples to support our theoretical results. A conclusion and future directions are discussed in Section 6.

## 2. Problem formation and preliminaries

We give the notations as follows. $R^n$ represents the group of $n$-dimensional real vectors. $R^{n\times n}$ indicates $n \times n$-dimensional real matrices. $\langle n\rangle$ implies the group $\{1, 2, \cdots, n\}$, for $n$ represents a positive integer. $im(t)$ means the impulsive signal. A vector $Z \in R^n$ is positive described $Z \succ 0$ provided that all its elements are positive. $Z^\top$ shows the transpose of the vector $Z$. $E_n$ symbolizes the $n \times n$ unit matrix with appropriate dimension. A Metzler matrix denotes a real square matrix whose off-diagonal elements are all non-negative. A vector $y = (y_1, y_2, \cdots, y_n) \in R^n$, $\|y\|_1 = \sum_{i=1}^{n} \mid y_i \mid$. For a matrix $D = \{d_{ij}\}$, $\|D\|_1 = \max_j \sum_{i=1}^{n} \mid d_{ij} \mid$. $D \succeq 0$ denotes that all elements of matrix $D$ are non-negative. Two vectors $p, q \in R^n$, $p \succ q$ signifies $p - q \succ 0$. Set $\zeta(t^+) = \lim_{d\to 0^+} \zeta(t + d)$ and $\zeta(t^-) = \lim_{d\to 0^-} \zeta(t + d)$. A function $\psi(t)$ described on $[0, \infty)$, $\psi(t^+)$ and $\psi(t^-)$ imply the right limitation and the left limitation of $\psi(t)$, respectively. The $\max[Z]$ is the maximum component of vector $Z$. Similarly, $\min[Z]$ is the minimum component of vector $Z$.

In this study, the switching sequence is redefined in a group of segments, and each switching interval is divided into several subintervals. Moreover, the increasing and decreasing behaviors of multiple co-positive Lyapunov–Krasovskii functional at the time of switches and impulses are analyzed, which will be evaluated to describe the stabilizing and destabilizing switching behaviors.

The switching sequence is redefined as $t_{l_k}, l_{k+1} = l_k + m, k = 0, 1, 2, \cdots$, set $t_{l_0} = t_0$, where $m$ represents a predetermined positive integer. Under the recombination, a range of fragments are isolated from the initial switching sequence. In the new switching sequence, time interval $[t_{l_k}, t_{l_{k+1}})$ is defined as the $(k+1)^{th}$ segment, and each segment includes $m$ switching moments called as $t_{l_k+g}$, $\forall g \in M := \{0, 1, \cdots, m-1\}$. So the new switching sequence still fulfills $0 < \bar{a}_1 \leq t_{l_k+g+1} - t_{l_k+g} \leq \bar{a}_2 < \infty$, $\forall g \in M$, $k \in N^+$.

On $k^{th}$ part, the group of stabilizing and destabilizing switching behaviors are represented by $M_k^{\downarrow}$ and $M_k^{\uparrow}$, respectively. $|M_k^{\downarrow}|$ and $|M_k^{\uparrow}|$ are the number of stabilizing and destabilizing switching behaviors on the $k^{th}$ part, respectively. Accordingly, $|M_k^{\downarrow}|/m$ and $|M_k^{\uparrow}|/m$ represent the ratios of the stabilizing and destabilizing switching behaviors, respectively. Apparently, there often exist a pair of positive integers $m_1$ and $m_2$ with $m_1 + m_2 = m$ in order that the number of the stabilizing and destabilizing switching behaviors satisfy $|M_k^{\uparrow}| = m_1$ and $|M_k^{\downarrow}| = m_2$. Figure 1 implies the specific process of adjusting the original switching sequence.

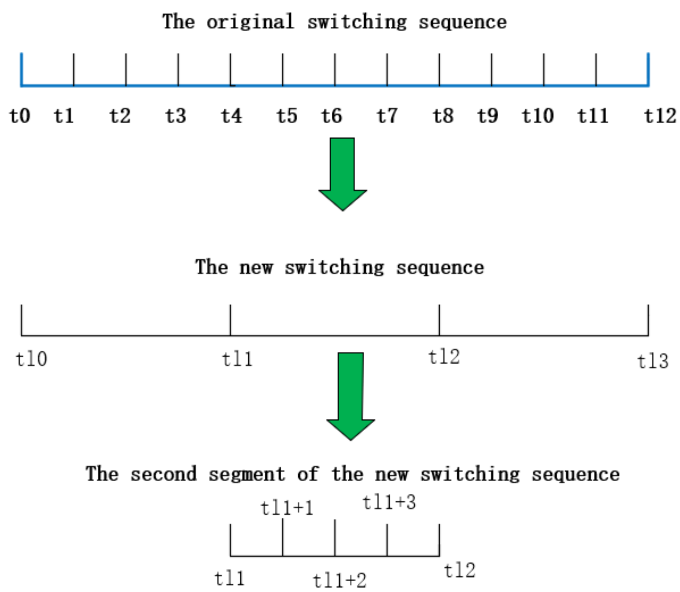



**Figure 1.** The sketch map process of adjusting the original switching sequence with $m = 4$. In time interval $[t_{l1}, t_{l2}]$, where $[t_{l1}, t_{l1+1}]$ and $[t_{l1+1}, t_{l2}]$ occur the destabilizing behaviors and stabilizing behaviors, respectively. Then, the number of the stabilizing switching and destabilizing switching behaviors satisfy $|M_k^{\uparrow}| = m_1$ and $|M_k^{\downarrow}| = m_2$, and $m_1 + m_2 = 4$, where $m_1 = 1$ and $m_2 = 3$.

The global uniform exponential stability criteria will be constructed for SPLISs (1.1) and (1.2) by multiple co-positive Lyapunov–Krasovskii functional in Section 3. Before beginning to construct stability conditions of SPLISs (1.1) and (1.2), we describe a group of subsystem couples which cause stabilizing switching behaviors as $\tilde{N} := \{(j, i);\ \exists k \in N,\ \sigma(t) = j,\ t \in [t_{l_k+g}, t_{l_k+g+1}),\ \sigma(t) = i,\ t \in [t_{l_k+g+1}, t_{l_k+g+2}),\ g + 1 \in M_{k+1}^{\downarrow}\}$. Give a definition to matrix $B$ as $B = (b_{uv}) \in R^{n \times n}$, where $(b_{uv}) = \max\{B_i{}^{(uv)}\}$, and $B_i{}^{(uv)}$ characterizes the $u$th row and $v$th column component of $B_i$. Obviously, $B \succeq B_i$.

**Remark 2.1.** As discussed above, it can be seen that the global uniform exponential stability of the SPLISs (1.1) and (1.2) in this paper is influenced by the following four aspects (see Figure 2): (1) the property of subsystems, (2) the ratio of stabilizing switching behaviors, (3) the time-varying delays, (4) the impulsive effects.

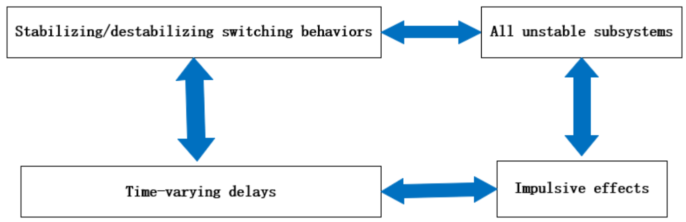



**Figure 2.** The influence factors of GUES for SPLISs (1.1) and (1.2).

In the next, the following definitions, assumption, and lemma are proposed.

**Definition 2.1** (Definition 3, [13])**.** The SPLISs (1.1) or (1.2) are called global uniform exponential stable provided that $\forall\ x(t_0+\varphi)=\varpi(\varphi)$, $\varphi\in[-\hat{d},0]$, there are two parameters $\mu>0$ and $\nu>0$ in order that every solution $x(t)$ of systems (1.1) or (1.2) fulfils $\|x(t)\|_1\le\mu e^{-\nu(t-t_0)}\|x(t)\|_{1c}$, $\forall t\ge t_0$, where $\|x(t)\|_{1c}=\sup\limits_{-\hat{d}\le\varphi\le 0}\|x(t_0+\varphi)\|_1$.

**Definition 2.2** (Definition 1, [24])**.** The SPLISs (1.1) or (1.2) are called positive provided that $\forall x(t_0+\varphi)\ge 0$, $\varphi\in[-\hat{d},0]$ and for any switching rule $\sigma(t)$, the relevant trajectory $x(t)$ fulfils $x(t)\ge 0$, $\forall t\ge t_0$.

We present the following assumption in this paper.

**Assumption 2.1.** *The impulsive function $I_{ji}(x)\ge 0$ for $x\ge 0$, and there exist a class of positive matrices $C_i\in R^{n\times n}$, satisfying $I_{ji}(x)\le C_i x$ for $x\in R^n$, $i,j\in\langle n\rangle$, $\sigma({t_m}^+)=j$, $\sigma({t_m}^-)=i$, $i\ne j$, $m=1,2,3,\cdots$.*

**Lemma 2.1** (Lemma 1, [24])**.** *The SPLISs* (1.1) *or* (1.2) *are positive if and only if $A_i$ is a Metzler matrix and $B_i\succeq 0$, $\forall i\in\langle n\rangle$.*

# 3. Main results

In this Section, we establish the new global uniform exponential stability criteria for SPLISs (1.1) and (1.2), and design the new linear programming algorithm process to better verify the obtained stability criteria.

## 3.1. Global uniform exponential stability of SPLISs (1.1)

**Theorem 3.1.** *Suppose that Assumption 2.1 holds. Given that $\bar{a}_1\le\bar{a}_2$, positive integer $T$, three positive integers $m_1$, $m_2$, $m$ satisfying $m_1+m_2=m$, if there are three constants $\zeta$, $\varepsilon$, $\eta$, fulfilling $0<\zeta<1$, $\varepsilon\ge 1$, $\eta>0$, a class of positive vectors $Z_{i,s}$, $i\in\langle n\rangle$, $s\in\{0,1,\cdots,T-1\}$, and the following inequalities*

$$\frac{(Z_{i,s+1}^T-Z_{i,s}^T)T}{\bar{a}_1}\,(1-d)\,E_n+Z_{i,s+1}^T\Big((1-d)A_i+2\hat{d}B-\eta(1-d)E_n\Big)\prec 0, \quad (3.1)$$

$$\frac{(Z_{i,s+1}^T - Z_{i,s}^T)T}{\bar{a}_1}(1-d)\,E_n + Z_{i,s}^T\Big((1-d)A_i + 2\hat{d}B - \eta(1-d)E_n\Big) \prec 0, \tag{3.2}$$

$$Z_{i,T}^T\Big((1-d)A_i + 2\hat{d}B - \eta(1-d)E_n\Big) \prec 0, \tag{3.3}$$

$$\left(\frac{(Z_{i,s+1}^T - Z_{i,s}^T)T\hat{d}}{\bar{a}_1} - Z_{i,s+1}^T\right)B \prec 0, \tag{3.4}$$

$$\left(\frac{(Z_{i,s+1}^T - Z_{i,s}^T)T\hat{d}}{\bar{a}_1} - Z_{i,s}^T\right)B \prec 0, \tag{3.5}$$

$$Z_{i,0} \preceq \begin{cases}\zeta C_j Z_{j,T}, & (j,i) \in \tilde{N},\\ \varepsilon C_j Z_{j,T}, & \textit{otherwise},\end{cases} \tag{3.6}$$

$$\frac{m_2}{m}\ln\zeta + \frac{m_1}{m}\ln\varepsilon + \bar{a}_2\eta < 0, \tag{3.7}$$

*hold, then the SPLISs* (1.1) *are global uniform exponential stable.*

**Remark 3.1.** Conditions (3.1)-(3.5) assess the evolution of each subsystem between every two switching behaviors. Condition (3.6) estimates the changes of energy at every switching and impulsive instant when stabilizing switching behaviors (by $\zeta$) and destabilizing switching behaviors (by $\varepsilon$) are activated. The ratios of all stabilizing and destabilizing switching behaviors are confined by condition (3.7).

**Remark 3.2.** Compared with [30], we add the impulses to the model in [30], and analyze the impulsive effects for the stability of the model. Then, we get the new global uniform exponential stability condition (3.6) in Theorem 3.1.

**Proof.** Firstly, each switching interval $[t_{l_k}, t_{l_k+1})$ is divided into $T+1$ segments via the next two processes: (i) The time interval $[t_{l_k}, t_{l_k+1})$ is parted into $[t_{l_k}, t_{l_k}+\bar{a}_1)$ and $[t_{l_k}+\bar{a}_1, t_{l_k+1})$. (ii) The time interval $[t_{l_k}, t_{l_k}+\bar{a}_1)$ is parted into $T$ segments called as $G_{l_k,s} = [t_{l_k}+se, t_{l_k}+(s+1)e)$, $s = 0, 1, \cdots, T-1$, with same length $e = \frac{\bar{a}_1}{T}$. Then, in terms of the switching rule $\sigma(t)$, the following vector function $Z_i(t)$ is constructed

$$Z_i(t) = \begin{cases}\varrho(t)Z_{i,s} + \tilde{\varrho}(t)Z_{i,s+1}, t \in G_{l_k,s}, & s = 0, 1, \cdots, T-1,\\ Z_{i,T}, & t \in [t_{l_k}+\bar{a}_1, t_{l_{k+1}}),\end{cases} \tag{3.8}$$

where $\varrho(t) = \dfrac{t_{l_k}+(s+1)e-t}{e}$, $t \in G_{k,s}$, $\tilde{\varrho}(t) = 1-\varrho(t) = \dfrac{t-t_{l_k}-se}{e}$.

For $t \in [t_{l_k+g}, t_{l_k+g+1})$, $Z_{\sigma(t)}(t) = Z_i(t)$ as $\sigma(t) = i$, $i \in \langle n\rangle$. From (3.8), $Z_{\sigma(t)}(t)$ is a piecewise right-continuous vector function, $\forall t \in [t_{l_k+g}, t_{l_k+g+1})$. Choose the following multiple co-positive Lyapunov–Krasovskii functional for SPLISs (1.1):

$$V_{\sigma(t)}(t) = V_{\sigma(t)}^1(t) + V_{\sigma(t)}^2(t) + V_{\sigma(t)}^3(t), \tag{3.9}$$

where

$$V_{\sigma(t)}^1(t) = (1-d)Z_{\sigma(t)}^T(t)x(t), \tag{3.10}$$

$$V_{\sigma(t)}^2(t) = \int_{t-d(t)}^{t}(s-(t-d(t)))e^{\eta(t-s)}Z_{\sigma(t)}^T(t)Bx(s)\mathrm{d}s, \tag{3.11}$$

$$V_{\sigma(t)}^3(t) = \int_{-\hat{d}}^{0}\int_{t+\varphi}^{t}e^{\eta(t-s)}Z_{\sigma(t)}^T(t)Bx(s)\mathrm{d}s\mathrm{d}\varphi. \tag{3.12}$$

For $t \in [t_{l_k+g}, t_{l_k+g+1})$, as $\sigma(t) = i$, $i \in \langle n \rangle$, by (3.10)-(3.12), we obtain the time derivative of $V_i(x)$ along the trajectory of the SPLISs (1.1) as follows

$$\begin{aligned}\dot{V}_i^1(t) &= V_{11} + V_{12} + V_{13}\\ &= (1-d)\dot{Z}_i^\top(t)x(t) + (1-d)Z_i^T(t)A_i x(t)\\ &\quad +(1-d)Z_i^T(t)B_i\int_{t-d(t)}^{t} x(s)\mathrm{d}s, \qquad (3.13)\\ \dot{V}_i^2(t) &= V_{21} + V_{22} + V_{23} - V_{24}\\ &= \eta V_i^2 + \int_{t-d(t)}^{t}(s-(s-d(t)))e^{\eta(t-s)}\dot{Z}_i^\top(t)Bx(s)\mathrm{d}s\\ &\quad +d(t)Z_i^T(t)Bx(t) - (1-\dot{d}(t))\int_{t-d(t)}^{t} e^{\eta(t-s)}Z_i^T(t)Bx(s)\mathrm{d}s, \qquad (3.14)\end{aligned}$$

$$\dot{V}_i^3(t) = V_{31} + V_{32} - V_{33} = \eta V_i^3 + \hat{d}Z_i^T(t)Bx(t) - \int_{t-\hat{d}}^{t} e^{\eta(t-s)}Z_i^T(s)Bx(s)\mathrm{d}s. \qquad (3.15)$$

Due to $\dot{d}(t) \le d < 1$, $\eta > 0$, it is acquired that $1-d < 1-\dot{d}(t)$ and $e^{\eta(t-s)} \ge 1$. Hence, $V_{13} - V_{24} \le 0$. According to the condition $0 \le d(t) \le \hat{d}$, we have

$$V_{23} + V_{32} \le 2\hat{d}Z_i^T(t)Bx(t).$$

Then, in terms of the integral formula, we obtain

$$V_{22} - V_{33} \le \int_{t-d(t)}^{t} e^{\eta(t-s)}\Big((s-(t-d(t)))\dot{Z}_i^\top(t) - Z_i^\top(s)\Big)Bx(s)\mathrm{d}s.$$

Afterwards, we gain

$$\begin{aligned}\dot{V}_i(t) &\le V_{11} + V_{12} + \eta V_i^2 + \eta V_i^3 + 2\hat{d}Z_i^T(t)Bx(t)\\ &\quad + \int_{t-d(t)}^{t} e^{\eta(t-s)}\Big((s-(t-d(t)))\dot{Z}_i^\top(t) - Z_i^\top(s)\Big)Bx(s)\mathrm{d}s. \qquad (3.16)\end{aligned}$$

According to (3.16), we acquire that

$$\begin{aligned}\dot{V}_i(t) - \eta V_i(t) &\le (1-d)\dot{Z}_i^T(t)x(t) + (1-d)Z_i^T(t)A_i x(t)\\ &\quad + \int_{t-d(t)}^{t} e^{\eta(t-s)}\Big((s-(t-d(t)))\dot{Z}_i^\top(t) - Z_i^\top(s)\Big)Bx(s)\mathrm{d}s\\ &\quad +2\hat{d}Z_i^T(t)Bx(t) - \eta(1-d)Z_i^\top(t)x(t)\\ &= (1-d)\dot{Z}_i^T(t)x(t) + Z_i^T(t)\Big((1-d)A_i + 2\hat{d}B - \eta(1-d)E_n\Big)x(t)\\ &\quad + \int_{t-d(t)}^{t} e^{\eta(t-s)}\Big((s-(t-d(t)))\dot{Z}_i^T(t) - Z_i^T(s)\Big)Bx(s)\mathrm{d}s. \qquad (3.17)\end{aligned}$$

Then, when $t \in [t_{l_k+g}, t_{l_k+g} + \bar{a}_1)$,

$$\begin{aligned}&\dot{V}_i(t) - \eta V_i(t)\\ &\le \varrho(t)\Big((1-d)\dot{Z}_i^T(t) + Z_{i,s}^T\Big((1-d)A_i + 2\hat{d}B - \eta(1-d)E_n\Big)\Big)x(t)\end{aligned}$$

$$
\begin{aligned}
&+\tilde{\varrho}(t)\left((1-d)\dot{Z}_i^T(t)+Z_{i,s+1}^T\left((1-d)A_i+2\hat{d}B-\eta(1-d)E_n\right)\right)x(t)\\
&+\int_{t-d(t)}^{t} e^{\eta(t-s)}(\varrho(s)(\hat{d}\dot{Z}_i^T(t)-Z_{i,s}^T)\\
&+\tilde{\varrho}(s)(\hat{d}\dot{Z}_i^T(t)-Z_{i,s+1}^T))Bx(s)\mathrm{d}s. \qquad (3.18)
\end{aligned}
$$

By the definition of $Z_i(t)$, we gain that

$$
\dot{Z}_i^T(t)=\frac{(Z_{i,s+1}^T-Z_{i,s}^T)T}{\bar{a}_1}, t\in[t_{l_k}+g+se, t_{l_k}+g+(s+1)e), s=0,1,\cdots,T-1.
$$

Consequently, by (3.18) and conditions (3.1), (3.2), (3.4), (3.5), we get that $\dot{V}_i(t)<\eta V_i(t)$.

For $t\in[t_{l_k+g}+\bar{a}_1, t_{l_k+g+1})$, it is apparent that $Z_i^\top(t)=Z_{i,T}^\top$. So it concludes from (3.17) and condition (3.3) that

$$
\dot{V}_i(t)<\eta V_i(t),\ t\in[t_{l_k+g}+\bar{a}_1, t_{l_k+g+1}).
$$

Subsequently, we obtain

$$
V_{\sigma(t_{l_k+g})}(t)<e^{\eta(t-t_{l_k+g})}V_{\sigma(t_{l_k+g})}(t_{l_k+g}^+),\ t\in[t_{l_k+g},\ t_{l_k+g+1}),\ \forall k\in N,\ g\in M.
$$

Since at every switching and impulsive instant $t_{l_k+g}$, denote $\sigma(t_{l_k+g}^-)=j$, $\sigma(t_{l_k+g}^+)=i,\ j,i\in\langle n\rangle,\ j\neq i,\ Z_{j,T}^\top$ is replaced by $Z_{j,T}^\top C_j^\top$, from condition (3.6) and the definition of multiple co-positive Lyapunov–Krasovskii functional. Then, we calculate the value of multiple co-positive Lyapunov–Krasovskii functional at $t_{l_k+g}$ in the following two cases:
(i) when $g\in M_k^\downarrow$,

$$
\begin{aligned}
&V(t_{l_{k+g}}{}^+, x(t_{l_k+g}{}^+))-\zeta V(t_{l_k+g}{}^-, x(t_{l_k+g}{}^-))\\
=&(1-d)\left(Z_{i,0}^T-\zeta Z_{j,T}^T C_j^\top\right)x(t)\\
&+\int_{t-d(t)}^{t}(s-(t-d(t)))\,e^{\eta(t-s)}\left(Z_{i,0}^T-\zeta Z_{j,T}^T C_j^\top\right)Bx(s)\mathrm{d}s\\
&+\int_{-\hat{d}}^{0}\int_{t+\varphi}^{t} e^{\eta(t-s)}\left(Z_{i,0}^T-\zeta Z_{j,T}^T C_j^\top\right)Bx(s)\mathrm{d}s\mathrm{d}\varphi\\
\leq&0.
\end{aligned}
$$

(ii) when $g\in M_k^\uparrow$,

$$
\begin{aligned}
&V(t_{l_{k+g}}{}^+, x(t_{l_k+g}{}^+))-\varepsilon V(t_{l_k+g}{}^-, x(t_{l_k+g}{}^-))\\
=&(1-d)\left(Z_{i,0}^T-\varepsilon Z_{j,T}^T C_j^\top\right)x(t)\\
&+\int_{t-d(t)}^{t}(s-(t-d(t)))\,e^{\eta(t-s)}\left(Z_{i,0}^T-\varepsilon Z_{j,T}^T C_j^\top\right)Bx(s)\mathrm{d}s\\
&+\int_{-\hat{d}}^{0}\int_{t+\varphi}^{t} e^{\eta(t-s)}\left(Z_{i,0}^T-\varepsilon Z_{j,T}^T C_j^\top\right)Bx(s)\mathrm{d}s\mathrm{d}\varphi\\
\leq&0.
\end{aligned}
$$

Therefore,

$$V_{\sigma(t_{l_k+g})}(t) \leq \begin{cases} \varepsilon V_{\sigma(t_{l_k+g-1})}(t^-_{l_k+g}), \ g \in M_k^{\uparrow}, \\ \zeta V_{\sigma(t_{l_k+g-1})}(t^-_{l_k+g}), \ g \in M_k^{\downarrow}. \end{cases} \tag{3.19}$$

By the mathematical induction, when $t \in [t_0, t_1)$, we get

$$V_{\sigma(t_0)}(t) < e^{\eta(t-t_0)} V_{\sigma(t_0)}(t_0).$$

In the first segment $[t_{l_0}, t_{l_1})$, according to (3.19), we obtain

$$\begin{aligned} V_{\sigma(t_{l_0})}(t^-_{l_1}) &\leq \zeta^{|M_k^{\downarrow}|} \varepsilon^{|M_k^{\uparrow}|} e^{\eta(t_{l_1}-t_{l_0})} V_{\sigma(t_{l_0})}(t^+_{l_0}) \\ &\leq \zeta^{m_2} \varepsilon^{m_1} e^{\eta(t_{l_1}-t_{l_0})} V_{\sigma(t_0)}(t_0). \end{aligned}$$

From induction, the following conclusion can be drawn: in $(k+1)^{th}$ part, when $t \in [t_{l_k+g}, t_{l_k+g+1})$,

$$V_{\sigma(t_{l_k+g})}(t) \leq \varepsilon^{m_1} (\zeta^{m_2} \varepsilon^{m_1})^k e^{\eta(t-t_0)} V_{\sigma(t_0)}(t_0). \tag{3.20}$$

When $t \in [t_{l_k+g}, t_{l_k+g+1})$, $(k+1)m\bar{a}_2 + \bar{a}_2 \geq t - t_0$, thus, $k \geq \dfrac{t - t_0 - \bar{a}_2}{m\bar{a}_2} - 1$. Furthermore, let $\gamma = m_2 \ln \zeta + m_1 \ln \varepsilon$. From condition (3.7), we get $\gamma < -m\bar{a}_2\eta < 0$.

According to (3.20), we acquire that when $t \in [t_{l_k+g}, t_{l_k+g+1})$,

$$\begin{aligned} V_{\sigma(t_{l_k+g})}(t) &\leq \varepsilon^{m_1} e^{\eta(t-t_0)} e^{\gamma k} V_{\sigma(t_0)}(t_0) \\ &= \varepsilon^{m_1} e^{\eta(t-t_0)+\gamma k} V_{\sigma(t_0)}(t_0) \\ &\leq \varepsilon^{m_1} e^{\eta(t-t_0)+(\frac{t-t_0-\bar{a}_2}{m\bar{a}_2}-1)\gamma} V_{\sigma_{(t_0)}}(t_0). \end{aligned}$$

Through condition (3.7), there exits an adequate minor positive constant $\omega$ in order that $m_2 \ln \zeta + m_1 \ln \varepsilon + m\bar{a}_2\eta + \omega < 0$, $\gamma + m\bar{a}_2\eta + \omega < 0$. Therefore,

$$V_{\sigma(t_{l_k+g})}(t) \leq \kappa e^{-\frac{\omega}{m\bar{a}_2}(t-t_0)} V_{\sigma_{(t_0)}}(t_0),$$

and $\kappa = \frac{\varepsilon^{m_1}}{e^{\gamma}(1+1/m)}$. In terms of the definition of $V_{\sigma(t)}(t)$, we get that

$$V_{\sigma(t)}(t) \geq (1-d) Z^{\top}_{\sigma(t)}(t) x(t) \geq (1-d) \min_{i,T}\{\min[Z_{i,T}]\} \|x(t)\|_1,$$

and

$$\begin{aligned} V_{\sigma(t_0)}(t_0) \leq (1-d) &\max_{i,T}\{\max[Z_{i,T}]\} \|x(t_0)\|_1 \\ &+ \hat{d} e^{\eta \hat{d}} \max_{i,T}\{\max[Z_{i,T}]\} \|B\|_1 \int_{t_0-\hat{d}}^{t_0} \|x(s)\|_1 \mathrm{d}s \\ &+ \hat{d}^2 \max_{i,T}\{\max[Z_{i,T}]\} \|B\|_1 \int_{t_0-\hat{d}}^{t_0} \|x(s)\|_1 \mathrm{d}s, \end{aligned}$$

where $T \in \{0, 1, 2, \cdots, T\}$. Therefore, we have

$$\|x(t)\|_1 \leq \frac{1}{(1-d)\min_{i,T}\{\min[Z_{i,T}]\}} \kappa e^{-\frac{\omega}{m\bar{a}_2}(t-t_0)} \{(1-d) \max_{i,T}\{\max[Z_{i,T}]\} \|x(t_0)\|_1$$

$$
\begin{aligned}
&+\hat{d}e^{\eta\hat{d}}\max_{i,T}\{\max[Z_{i,T}]\}\|B\|_1\int_{t_0-\hat{d}}^{t_0}\|x(s)\|_1\mathrm{d}s\\
&+\hat{d}^2\max_{i,T}\{\max[Z_{i,T}]\}\|B\|_1\int_{t_0-\hat{d}}^{t_0}\|x(s)\|_1\mathrm{d}s\}\\
\le\ &\mu e^{-\frac{\omega}{m\bar{a}_2}(t-t_0)}\sup_{-\hat{d}\le\varphi\le 0}\|x(t_0+\varphi)\|_1,
\end{aligned}
\tag{3.21}
$$

where $\mu=\frac{\max_{i,T}\{\max[Z_{i,T}]\}}{\min_{i,T}\{\min[Z_{i,T}]\}}\left(1+\frac{\hat{d}e^{\eta\hat{d}}+\hat{d}^2}{1-d}\|B\|_1\right)\kappa$.

Let $\nu=\frac{\omega}{m\bar{a}_2}$. We can conclude that $\|x(t)\|_1\le\mu e^{-\nu(t-t_0)}\|x(t_0)\|_{1c}$ from (3.21), $\forall t\ge t_0$, so SPLISs (1.1) are global uniform exponential stable. □

## 3.2. Global uniform exponential stability of SPLISs (1.2)

When only a certain moment of the past states processes effect on the systems dynamics, the distributed time-varying delays are reduced to the discrete time-varying delays. In this case, SPLISs (1.1) become SPLISs (1.2), and the following Corollary 3.1 can be derived easily from Theorem 3.1.

**Corollary 3.1.** *Suppose that Assumption 2.1 holds. Given that $\bar{a}_1\le\bar{a}_2$, positive integer $T$, three positive integers $m_1$, $m_2$, $m$ are satisfying $m_1+m_2=m$, if there are three constants $\zeta$, $\varepsilon$, $\eta$, fulfilling $0<\zeta<1$, $\varepsilon\ge 1$, $\eta>0$ and a class of positive vectors $Z_{i,s}$, $i\in\langle n\rangle$, $s\in\{0,1,\cdots,T-1\}$, and the following inequalities*

$$
\frac{(Z_{i,s+1}^T-Z_{i,s}^T)T}{\bar{a}_1}(1-d)E_n+Z_{i,s+1}^T\Big((1-d)A_i+(1+\hat{d})B-\eta(1-d)E_n\Big)\prec 0,
\tag{3.22}
$$

$$
\frac{(Z_{i,s+1}^T-Z_{i,s}^T)T}{\bar{a}_1}(1-d)E_n+Z_{i,s}^T\Big((1-d)A_i+(1+\hat{d})B-\eta(1-d)E_n\Big)\prec 0,
\tag{3.23}
$$

$$
Z_{i,T}^T\Big((1-d)A_i+(1+\hat{d})B-\eta(1-d)E_n\Big)\prec 0,
\tag{3.24}
$$

$$
\left(\frac{(Z_{i,s+1}^T-Z_{i,s}^T)T\hat{d}}{\bar{a}_1}-Z_{i,s+1}^T\right)B\prec 0,
\tag{3.25}
$$

$$
\left(\frac{(Z_{i,s+1}^T-Z_{i,s}^T)T\hat{d}}{\bar{a}_1}-Z_{i,s}^T\right)B\prec 0,
\tag{3.26}
$$

$$
Z_{i,0}\preceq\begin{cases}\zeta C_jZ_{j,T}, & (j,i)\in\tilde{N},\\ \varepsilon C_jZ_{j,T}, & \textit{otherwise},\end{cases}
\tag{3.27}
$$

$$
\frac{m_2}{m}\ln\zeta+\frac{m_1}{m}\ln\varepsilon+\bar{a}_2\eta<0,
\tag{3.28}
$$

*hold, then the SPLISs* (1.2) *are global uniform exponential stable.*

**Remark 3.3.** Compared to [30], we add the impulses to the model, and we get the new global uniform exponential stability condition (3.27) in Corollary 3.1. Condition (3.27) is also the constraint of vector at every switching and impulsive instant.

**Proof.** Choose the following multiple co-positive Lyapunov–Krasovskii functional for SPLISs (1.2):

$$V_{\sigma(t)}(t) = V^1_{\sigma(t)}(t) + V^2_{\sigma(t)}(t) + V^3_{\sigma(t)}(t), \tag{3.29}$$

where

$$V^1_{\sigma(t)}(t) = (1-d)Z^T_{\sigma(t)}(t)x(t), \tag{3.30}$$

$$V^2_{\sigma(t)}(t) = \int_{t-d(t)}^{t} e^{\eta(t-s)} Z^T_{\sigma(t)}(t)Bx(s)\mathrm{d}s, \tag{3.31}$$

$$V^3_{\sigma(t)}(t) = \int_{-\hat{d}}^{0}\int_{t+\varphi}^{t} e^{\eta(t-s)} Z^T_{\sigma(t)}(t)Bx(s)\mathrm{d}s\mathrm{d}\varphi. \tag{3.32}$$

For $t \in [t_{l_k+g}, t_{l_k+g+1})$, defining $\sigma(t) = i$, $i \in \langle n\rangle$, from (3.29)-(3.32), we obtain the time derivative of $V_i(x)$ along the trajectory of the SPLISs (1.2) as following

$$\begin{aligned}
\dot{V}^1_i(t) &= V_{11} + V_{12} + V_{13} \\
&= (1-d)\dot{Z}^\top_i(t)x(t) + (1-d)Z^T_i(t)A_i x(t) \\
&\quad + (1-d)Z^T_i(t)B_i x(t-d(t)), \qquad (3.33)\\
\dot{V}^2_i(t) &= V_{21} + V_{22} + V_{23} - V_{24} \\
&= \eta V^2_i + \int_{t-d(t)}^{t} e^{\eta(t-s)}\dot{Z}^\top_i(t)Bx(s)\mathrm{d}s \\
&\quad + Z^T_i(t)Bx(t) - (1-\dot{d}(t))e^{\eta d(t)}Z^T_i(t)Bx(t-d(t)), \qquad (3.34)
\end{aligned}$$

$$\dot{V}^3_i(t) = V_{31} + V_{32} - V_{33} = \eta V^3_i + \hat{d}Z^T_i(t)Bx(t) - \int_{t-\hat{d}}^{t} e^{\eta(t-s)}Z^T_i(s)Bx(s)\mathrm{d}s. \tag{3.35}$$

Then, we acquire that

$$\begin{aligned}
\dot{V}_i(t) - \eta V_i(t) = {} & V_{11} + V_{12} + V_{13} + V_{22} + V_{23} - V_{24} + V_{32} - V_{33} \\
& - \eta(1-d)Z^\top_i(t)x(t).
\end{aligned}$$

Due to $d(t) \in [0, \hat{d}]$, $d \in [\dot{d}(t), 1)$, $\int_{t-\hat{d}}^{t-d(t)} e^{\eta(t-s)}Z^T_i(s)Bx(s)\mathrm{d}s \geq 0$, and from (3.33)-(3.35), we have $V_{13} - V_{24} \leq 0$, $V_{22} - V_{33} \leq \int_{t-d(t)}^{t} e^{\eta(t-s)}\left(\dot{Z}^T_i(t) - Z^\top_i(s)\right) \times Bx(s)\mathrm{d}s$. Therefore, we gain the following

$$\begin{aligned}
&\dot{V}_i(t) - \eta V_i(t) \\
&= (1-d)\dot{Z}^T_i(t)x(t) \\
&\quad + Z^T_i(t)\left((1-d)A_i + (1+\hat{d})B - \eta(1-d)E_n\right)x(t) \\
&\quad + \int_{t-d(t)}^{t} e^{\eta(t-s)}\left(\dot{Z}^T_i(t) - Z^T_i(s)\right)Bx(s)\mathrm{d}s \\
&\leq \varrho(t)\left((1-d)\dot{Z}^T_i(t) + Z^T_{i,s}\left((1-d)A_i + (1+\hat{d})B - \eta(1-d)E_n\right)\right)x(t) \\
&\quad + \tilde{\varrho}(t)\left((1-d)\dot{Z}^T_i(t) + Z^T_{i,s+1}\left((1-d)A_i + (1+\hat{d})B - \eta(1-d)E_n\right)\right)x(t)
\end{aligned}$$

$$+\int_{t-d(t)}^{t} e^{\eta(t-s)}\left(\varrho(s)\left(\dot{Z}_i^T(t)-Z_{i,s}^T\right)+\tilde{\varrho}(s)\left(\dot{Z}_i^T(t)-Z_{i,s+1}^T\right)\right)Bx(s)\mathrm{d}s.$$

From conditions (3.22)-(3.26), we give that

$$\dot{V}_{\sigma(t)}-\eta V_{\sigma(t)}\leq 0.$$

The following procedure is the same as Theorem 3.1. Then we get the conclusion that SPLISs (1.2) are global uniform exponential stable. □

Furthermore, for the circumstance as all switching behaviors are all stabilizing, Corollary 3.1 is reduced to Corollary 3.2:

**Corollary 3.2.** *Suppose that Assumption 2.1 holds. Given that $\bar{a}_1\leq\bar{a}_2$, positive integer $T$, two positive integers $m_2$, $m$ satisfying $m_2=m$, if there are two constants $\zeta$, $\eta$, fulfilling $0<\zeta<1$, $\eta>0$, and a class of positive vectors $Z_{i,s}$, $i\in\langle n\rangle$, $s\in\{0,1,\cdots,T-1\}$, such that* (3.22)-(3.26) *and the following inequalities*

$$Z_{i,0}\leq\zeta C_j Z_{j,T},\ \ (j,i)\in\tilde{N}, \tag{3.36}$$

$$\ln\zeta+\bar{a}_2\eta<0, \tag{3.37}$$

*hold, then the SPLISs* (1.2) *are global uniform exponential stable.*

**Remark 3.4.** Compared to [30], we add the impulses to the model, and we get the new global uniform exponential stability condition (3.36) in Corollary 3.2. Condition (3.36) is also the constraint of vector at every switching and impulsive instant. Furthermore, because all switching behaviors in the new designed sequence are stabilizing, then condition (3.27) can reduce to condition (3.36).

## 3.3. The algorithm to find feasible solution

We present the following algorithm to verify Theorem 3.1. Then the similar algorithm can be easily obtained for Corollary 3.1, Corollary 3.2, and the Theorems 4.1-4.2 in Section 4.

**Algorithm 1**

**Step 1**. Choose a group of constants $0<\zeta<1$, $\varepsilon\geq 1$, $\bar{a}_2>0$, $\eta>0$, $m_1$, $m_2$, $m$ are three positive integers satisfying $m_1+m_2=m$, all parameters satisfying (3.7).
**Step 2**. Choose the lower bound of switching interval $\bar{a}_1$ satisfying $\bar{a}_1\leq\bar{a}_2$, $T>0$, $T\in N^+$, then $Z_{i,s}$ satisfying (3.1)-(3.6) can be obtained by linear programming; otherwise, go back to **Step 1**.
**Step 3**. With the feasible parameters $\zeta$, $\varepsilon$, $\eta$, $\bar{a}_2>0$, find the feasible value of $\frac{m_2}{m}$ for designing the new switching sequence.

**Remark 3.5.** Compared with [30], we add the impulsive effects to the SPLSs with distributed time-varying delays and all unstable subsystems. The number of linear matrix inequalities in this paper are more than [30], we still find the feasible solutions to the corresponding matrix inequalities, so our results are better than [30]. Furthermore, compared with [9], our model is comprehensive because of the existences of unstable subsystems and destabilizing switching behaviors. We design new linear programming algorithm process to better verify the global uniform exponential stability criteria for SPLISs (1.1) and (1.2).

# 4. Applications in the consensus of linear delayed multi-agent systems

In this Section, we apply the obtained stability criteria in Section 3.2 to the exponential consensus of linear delayed multi-agent systems. We consider the linear delayed multi-agent systems with switching topologies and impulses

$$\begin{cases} \dot{u}_i(t) = \sum_{j=0}^{n} a_{ij}^{(\sigma(t))}(u_j(t) - u_i(t)) + \sum_{j=0}^{n} b_{ij}^{(\sigma(t))}(u_j(t-d(t)) - u_i(t-d(t))), \\ \qquad t \geq 0,\ t \neq t_m,\ m = 1,2,3,\cdots, \\ u(t^+) = \sum_{j=0}^{n} c_{ij}^{(\sigma(t^+))} u_j(t^-),\ t = t_m,\ m = 1,2,3,\cdots, \end{cases} \tag{4.1}$$

where $u_i(t) \in R$ denotes the state of the $i$th agent for $i \in \langle n \rangle$, $\sigma(t) : [0,\infty) \to \langle r \rangle$ represents the switching rule, $u_0 \in R$ is the state of leader with dynamic equation $\dot{u}_0 = 0$, $a_{ij}^{(s)}$ is nonnegative weights satisfying $a_{ii}^{(s)} = 0$, $i,j \in \langle n \rangle$, $s \in \langle r \rangle$, $b_{ij}^{(s)} \geq 0$, impulses coefficients $c_{ij}^{(s)}$, where $i \in \langle n \rangle$ and $s \in \langle r \rangle$, the discrete time-varying delays function $d(t)$ satisfies $0 \leq d(t) \leq \hat{d}$ and $\dot{d}(t) \leq d < 1$, where $\hat{d}$ is the upper bound constant value of $d(t)$, $d$ represents the upper bound constant value of $\dot{d}(t)$, in other words, $d(t)$ is upper bounded and slowly varying.

Do the transformation $v_i(t) = u_i(t) - u_0(t)$. Systems (4.1) convert into the next matrix form

$$\begin{cases} \dot{v}(t) = A_{\sigma(t)}v(t) + B_{\sigma(t)}v(t-d(t)),\ t \geq 0,\ t \neq t_m,\ m = 1,2,3,\cdots, \\ v(t^+) = C_{\sigma(t^+)}v(t^-),\ t = t_m,\ m = 1,2,3,\cdots, \end{cases} \tag{4.2}$$

where $A_s = D_s - L_s$, $D_s = -\mathrm{diag}\{a_{10}^{(s)}, a_{20}^{(s)}, \ldots, a_{n0}^{(s)}\}$, $B_s = [b_{ij}^{(s)}]_{n\times n}$ and $B_s \geq 0$, $C_s = [C_{ij}^{(s)}]_{n\times n}$ and $L_s = (l_{ij}^{(s)})$ denotes the Laplace matrix with $l_{ii}^{(s)} = \sum_{j=1}^{n} a_{ij}^{(s)}$ and $l_{ij}^{(s)} = -a_{ij}^{(s)}$ for $i \neq j$, $i,j \in \langle n \rangle$ and $s \in \langle r \rangle$. When $\lim_{t\to\infty}[u_i(t) - u_0(t)] = 0$, and $i \in \langle n \rangle$, we call systems (4.1) reach exponential consensus, which is equal to the exponential stability of systems (4.2). Thus, we obtain the following exponential consensus criteria according to Corollary 3.1 and Corollary 3.2.

**Theorem 4.1.** *Suppose that Assumption 2.1 holds. Given that* $\bar{a}_1 \leq \bar{a}_2$*, positive integer* $T$*, three positive integers* $m_1$*,* $m_2$*,* $m$ *are satisfying* $m_1 + m_2 = m$*, if there are three constants* $\zeta$*,* $\varepsilon$*,* $\eta$*, fulfilling* $0 < \zeta < 1$*,* $\varepsilon \geq 1$*,* $\eta > 0$ *and a class of positive vectors* $Z_{i,s}$*,* $i \in \langle n \rangle$*,* $s \in \{0,1,\cdots,T-1\}$*, in order that inequalities* (3.22)-(3.28) *hold, then the linear delayed multi-agent systems* (4.1) *achieve exponential consensus.*

**Theorem 4.2.** *Suppose that Assumption 2.1 holds. Given that* $\bar{a}_1 \leq \bar{a}_2$*, positive integer* $T$*, two positive integers* $m$*,* $m_2$*, satisfying* $m_2 = m$*, if there are two constants* $\zeta$*,* $\eta$*, fulfilling* $0 < \zeta < 1$*,* $\eta > 0$*, and a class of positive vectors* $Z_{i,s}$*,* $i \in \langle n \rangle$*,* $s \in \{0,1,\cdots,T-1\}$*, in order that inequalities* (3.22)-(3.26)*,* (3.36)-(3.37) *hold, then the linear delayed multi-agent systems* (4.1) *achieve exponential consensus.*

**Remark 4.1.** When there is no time delay, the consensus of the systems (4.1) was investigated in [5] by utilizing discrete absolute value Lyapunov function. We consider the effects of discrete time-varying delays for systems (4.1), so the obtained stability criteria enhance the relevant stability criteria in [5].

# 5. Simulation examples

The theoretical results are verified by three simulation examples.

**Example 5.1.** Consider continuous-time SPLISs (1.1) with the following parameters:

$$A_1 = \begin{bmatrix} 0.45 & 0.45 \\ 0.9 & -15.4 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0 & 0.015 \\ 0.015 & 0.015 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.49 & 0.98 \\ 0.78 & -9.9 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0.015 & 0 \\ 0.015 & 0.015 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -9.2 & 0.46 \\ 0.09 & 0.08 \end{bmatrix}, \ B_3 = \begin{bmatrix} 0.015 & 0.015 \\ 0.015 & 0 \end{bmatrix}.$$

Impulsive matrices take the form of

$$C_1 = \begin{bmatrix} 1.05 & 0 \\ 0 & 1.04 \end{bmatrix}, \ C_2 = \begin{bmatrix} 1.1 & 0 \\ 0 & 1.2 \end{bmatrix}, \ C_3 = \begin{bmatrix} 1.3 & 0 \\ 0 & 1.6 \end{bmatrix}.$$

$d(t) = 0.15 - 0.15\cos(t)$. According to above, we acquire that $\hat{d} = 0.3$, $0.15 \leq d < 1$, $B = [0.015, 0.015; 0.015, 0.015]$.

Choosing the initial state $x(0) = (3,7)^T$, the three subsystems solution trajectories of SPLIS (1.1) are displayed in Figure 3, which indicates that the three subsystems are all unstable. Let $\eta = 1.15$, $\bar{a}_1 = 0.12$, $\bar{a}_2 = 0.15$, $\zeta = 0.7$, $\varepsilon = 1.2$, $m_1 = 1$, $m_2 = 8$, $m = 9$ and $T = 1$. It can be gained that the conditions in Theorem 3.1 hold with $Z_i^j$ given as follows

$$Z_{1,0} = \begin{bmatrix} 80.4001 \\ 169.1412 \end{bmatrix}, \ Z_{1,1} = \begin{bmatrix} 59.1225 \\ 236.9019 \end{bmatrix},$$
$$Z_{2,0} = \begin{bmatrix} 72.4943 \\ 152.5653 \end{bmatrix}, \ Z_{2,1} = \begin{bmatrix} 56.4351 \\ 205.3150 \end{bmatrix},$$
$$Z_{3,0} = \begin{bmatrix} 41.4550 \\ 170.4646 \end{bmatrix}, \ Z_{3,1} = \begin{bmatrix} 90.5496 \\ 152.8046 \end{bmatrix}.$$

It is observed that $\tilde{N} = \{(2,3),(3,2),(1,3),(3,1)\}$, i.e., the stabilizing switching behaviors occur between subsystems couples of 2 and 3 and subsystems couples of 1 and 3. Furthermore, the feasible value of $\frac{m_2}{m}$ is 0.88889. The new switching sequence is designed as follows

$$\underbrace{2 \to 3 \to 2 \to 3 \to 1 \to 3 \to 1 \to 2 \to 3.}_{\text{Loop this sequence}}$$

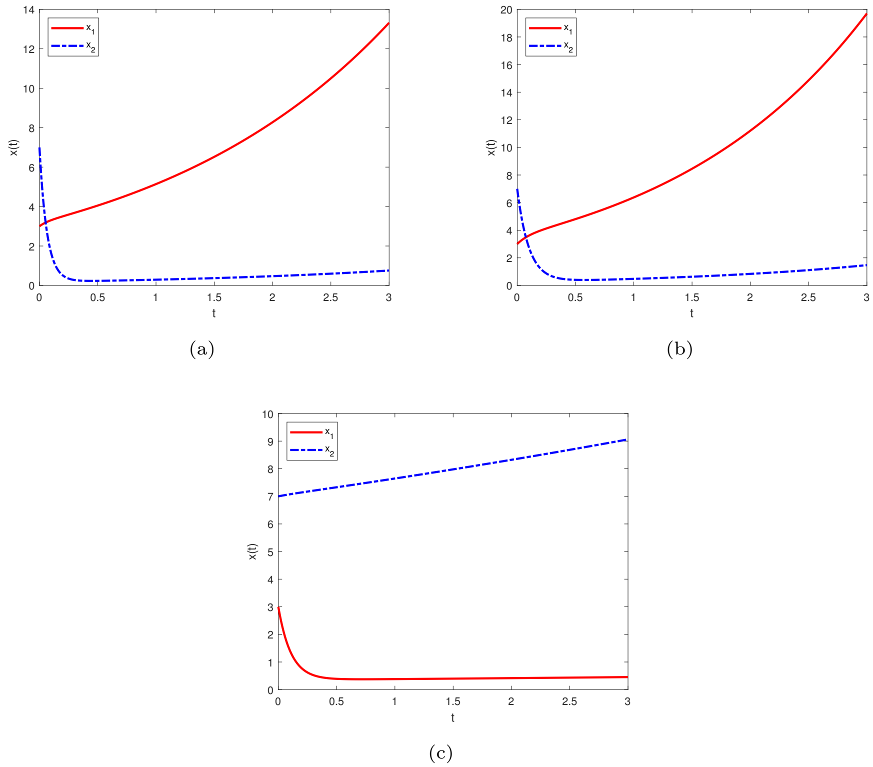



**Figure 3.** The state trajectories of three subsystems of SPLISs (1.1).

Choosing initial condition $x(0) = (3, 7)^T$, the switching and impulsive signals are given in Figure 4(a) . The solution trajectories of SPLISs (1.1) are displayed in Figure 4(b), which implies that the solution trajectories of SPLISs (1.1) are global uniform exponential stable. Furthermore, SPLISs (1.1) contain the impulsive effects, so that the main results in [30] cannot be applied to the example.

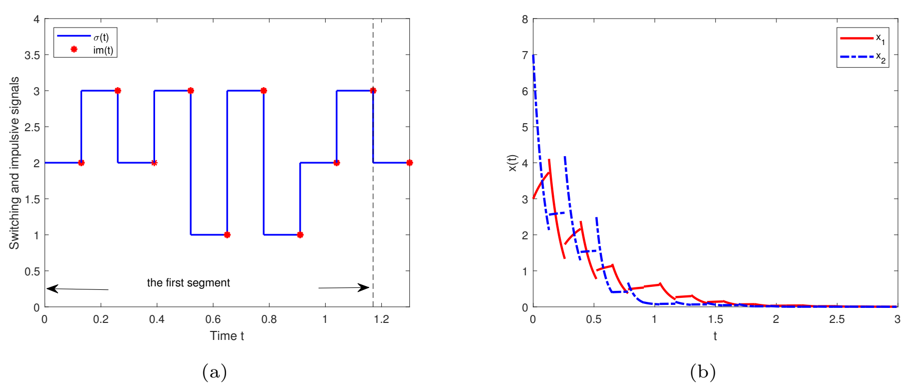



**Figure 4.** The signals and state trajectories of SPLISs (1.1).

**Remark 5.1.** In Example 5.1, there are nine switching parts on the first segment from Figure 4(a). According to the definitions of the designed sequence and stabilizing switching behaviors, only $1 \to 2$(the 8th part) indicates destabilizing switching behavior in the first segment. Therefore, the number of destabilizing and stabilizing switching behaviors is “1” and “8”, respectively. Examples 5.2-5.3 have similar explanations as Example 5.1.

**Example 5.2.** Consider continuous-time SPLISs (1.2) with the following parameters:

$$A_1 = \begin{bmatrix} 0.06 & 0.42 \\ 0.68 & -9.9 \end{bmatrix},\ B_1 = \begin{bmatrix} 0.012 & 0.012 \\ 0 & 0.012 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.09 & 1.9 \\ 0.27 & -4.4 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.012 & 0.012 \\ 0.012 & 0 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -8.5 & 0.47 \\ 1.58 & 0.09 \end{bmatrix},\ B_3 = \begin{bmatrix} 0 & 0.012 \\ 0.012 & 0.012 \end{bmatrix}.$$

Impulsive matrices take the form of

$$C_1 = \begin{bmatrix} 1.02 & 0 \\ 0 & 1.035 \end{bmatrix},\ C_2 = \begin{bmatrix} 1.08 & 0 \\ 0 & 1.014 \end{bmatrix},\ C_3 = \begin{bmatrix} 1.038 & 0 \\ 0 & 1.059 \end{bmatrix}.$$

$d(t) = 0.18 - 0.18\cos(t)$. Therefore, we gain that $\hat{d} = 0.36$, $0.18 \le d < 1$, $B = [0.012, 0.012; 0.012, 0.012]$.

Choosing the initial state $x(0) = (2, 5)^T$, the three subsystems state trajectories are shown in Figure 5, which shows that the three subsystems are all unstable.

Let $\eta = 0.76$, $\bar{a}_1 = 0.09$, $\bar{a}_2 = 0.0955$, $\zeta = 0.86$, $\varepsilon = 1.02$, $m_1 = 1$, $m_2 = 10$, $m = 11$ and $T = 1$. It can be acquired that conditions in Corollary 3.1 hold with $Z_i^j$ given as follows

$$Z_{1,0} = \begin{bmatrix} 155.7359 \\ 197.1935 \end{bmatrix},\ Z_{1,1} = \begin{bmatrix} 149.2218 \\ 250.2733 \end{bmatrix},$$
$$Z_{2,0} = \begin{bmatrix} 153.2504 \\ 197.3689 \end{bmatrix},\ Z_{2,1} = \begin{bmatrix} 140.9317 \\ 255.4565 \end{bmatrix},$$
$$Z_{3,0} = \begin{bmatrix} 128.8974 \\ 220.7683 \end{bmatrix},\ Z_{3,1} = \begin{bmatrix} 176.6993 \\ 218.7161 \end{bmatrix}.$$

It can be observed that group $\tilde{N} = \{(2, 3), (3, 2), (1, 3), (3, 1)\}$, i.e., the stabilizing switching behaviors occur between subsystems couples of 2 and 3 and subsystem couples of 1 and 3. Furthermore, the feasible limit of $\frac{m_2}{m}$ is counted as 0.90909. The new switching sequence is designed as follows

$$\underbrace{2 \to 3 \to 2 \to 3 \to 1 \to 3 \to 1 \to 2 \to 3 \to 1 \to 3.}_{\text{Loop this sequence}}$$

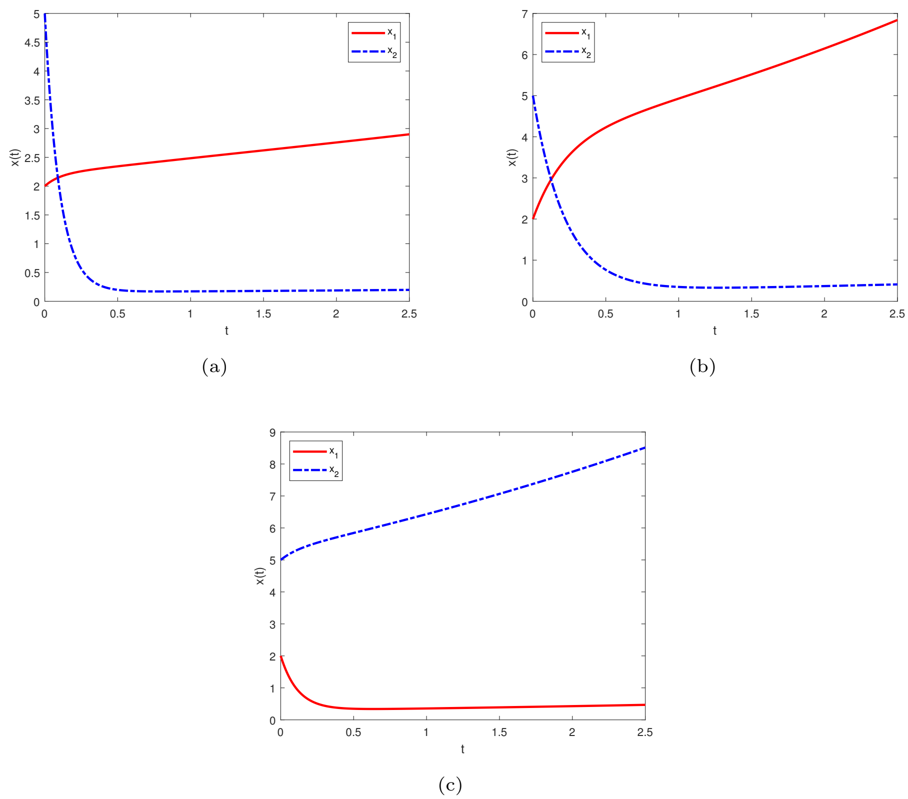



**Figure 5.** The state trajectories of three subsystems of SPLISs (1.2).

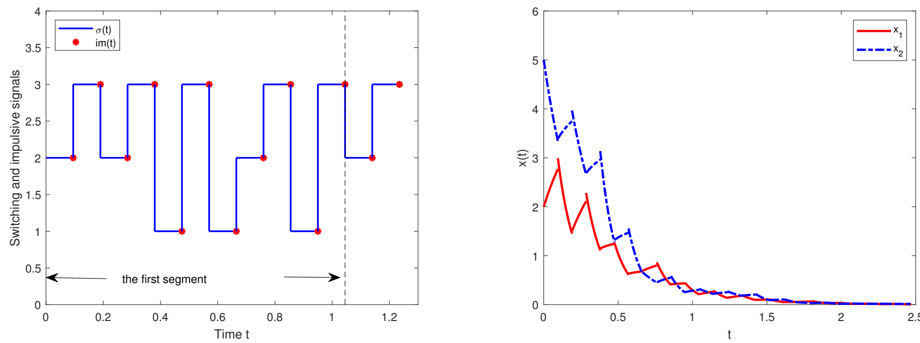
Choosing the initial condition $x(0) = (2, 5)^T$, the switching and impulsive signals are exhibited in Figure 6(a). The solution trajectories of SPLISs (1.2) are shown in Figure 6(b), which implies that the solution trajectories of SPLISs (1.2) are global uniform exponential stable. Moreover, SPLISs (1.2) contain the impulsive effects, the results in Zhou etc [30] are invalid.

**Example 5.3.** Consider the linear delayed multi-agent systems (4.1) with $r = 2$, $n = 3$, $d(t) = 0.12 - 0.12\cos(t)$, $D_1 = \mathrm{diag}\{0, -1.02, 0\}$, $D_2 = \mathrm{diag}\{-1.04, 0, 0\}$, and Laplace matrices fulfill

$$L_1 = \begin{bmatrix} 0.08 & -0.08 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; L_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -0.07 & 0.07 \end{bmatrix},$$

and

$$B_1 = \begin{bmatrix} 0.002 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.002 & 0 \\ 0 & 0 & 0.002 \end{bmatrix}.$$




(a) (b)

**Figure 6.** The signals and state trajectories of SPLISs (1.2).

Impulsive matrices take the form of

$$C_1 = \begin{bmatrix} 1.03 & 0 & 0 \\ 0 & 1.02 & 0 \\ 0 & 0 & 1.04 \end{bmatrix}, \; C_2 = \begin{bmatrix} 1.003 & 0 & 0 \\ 0 & 1.002 & 0 \\ 0 & 0 & 1.004 \end{bmatrix},$$

which infers from $A_1 = D_1 - L_1, A_2 = D_2 - L_2$, we have

$$A_1 = \begin{bmatrix} -0.08 & 0.08 & 0 \\ 0 & -1.02 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} -1.04 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.07 & -0.07 \end{bmatrix}, \; B = \begin{bmatrix} 0.002 & 0 & 0 \\ 0 & 0.002 & 0 \\ 0 & 0 & 0.002 \end{bmatrix}.$$

Since there exist the discrete time-varying delays effects, Corollary 3 in [5] cannot be applied to this example. $\hat{d} = 0.24$, $0.12 \leq d < 1$. Let $\eta = 1.09$, $\bar{a}_1 = 0.42$, $\bar{a}_2 = 0.47$, $\zeta = 0.58$, $T = 1$, $m_2 = m = 2$. It is obtained that conditions in Theorem 4.2 hold for

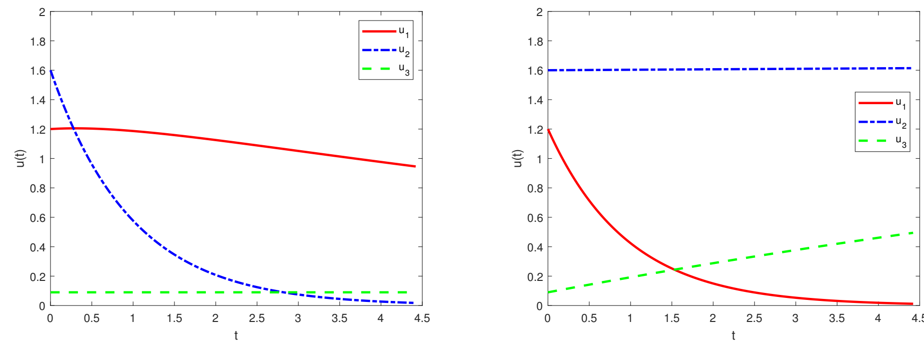
$$Z_{10} = \begin{bmatrix} 623.1397 \\ 626.1294 \\ 620.2074 \end{bmatrix}, \; Z_{11} = \begin{bmatrix} 838.6341 \\ 846.8560 \\ 830.5703 \end{bmatrix}, \; Z_{20} = \begin{bmatrix} 500 \\ 500 \\ 500 \end{bmatrix}, \; Z_{21} = \begin{bmatrix} 500 \\ 500 \\ 500 \end{bmatrix}.$$

It can be observed that group $\tilde{N} = \{(1, 2), (2, 1)\}$, so all switching behaviors are stabilizing. The switching sequence is designed as follows

$$\underbrace{1 \rightarrow 2.}_{\text{Loop this sequence}}$$

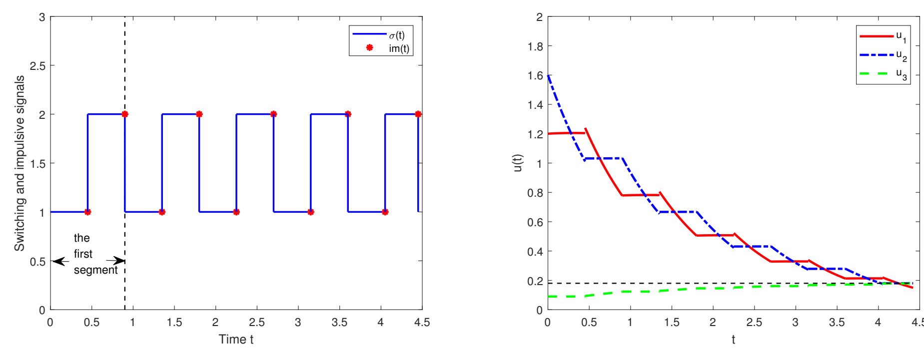
Choosing the initial state $u(0) = [1.2, 1.6, 0.09]^\top$, two subsystems of systems (4.1) are shown in Figure 7, which shows that the two subsystems are all unstable.

Therefore, the systems (4.1) reach exponential consensus. Choosing initial condition $u_0(0) = 0.18$, $u_1(0) = 1.2$, $u_2(0) = 1.6$, $u_3(0) = 0.09$ and the signals are shown in Figure 8(a). The solution trajectories of the systems (4.1) are shown in Figure 8(b), which implies that the systems (4.1) are exponential consensus.




(a) (b)

**Figure 7.** The state trajectories of two subsystems of systems (4.1).




(a) (b)

**Figure 8.** The signals and state trajectories of systems (4.1).

**Remark 5.2.** It should be noted that the recent results in [5,9,29,30] cannot be applied to Examples 5.1, 5.2 and 5.3 due to the coexistence of distributed time-varying delays, discrete time-varying delays, unstable subsystems, stabilizing/destabilizing switching behaviors, and impulsive effects. To some extent, the proposed results in this paper are more effective than previous ones.

# 6. Discussions

The global uniform exponential stability of SPLISs with time-varying delays and all unstable subsystems is investigated in this paper. By utilizing multiple co-positive

Lyapunov–Krasovskii functional, new specific global uniform exponential stability criteria for the SPLISs in the case of switching-impulse signals are obtained in the fields of linear matrix inequalities, which are solved by the linear programming algorithm. After that, the main stability criteria are applied to the exponential consensus of linear delayed multi-agent systems with switching communication topologies.

There exist some limitations to our work. It is worth noting that the impulsive instants and switching instants may not be synchronous. We deal with the impulsive effects for the SPLISs in this paper, which are synchronous with the switching signals. Then, we will gradually explore the case of global uniform exponential stability for SPLISs (1.1) and SPLISs (1.2) with asynchronous switching impulsive signals in the future. Furthermore, the method in this paper are not directly generalized to the case of two types of time-varying delays. In fact, there exist the corresponding results of two types of time-varying delays case in theory. The main difficulty is how to define Lyapunov–Krasovskii functional. This will be left for our future study.

## Declaration of competing interest

There is no competing interest.

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