Journal of Applied Analysis and Computation
Volume 14, Number 4, August 2024, 2411–2430
Website:http://www.jaac-online.com
DOI:10.11948/20230399

QUASI-PROJECTIVE SYNCHRONIZATION ANALYSIS FOR DELAYED STOCHASTIC QUATERNION-VALUED NEURAL NETWORKS VIA STATE-FEEDBACK CONTROL STRATEGY*

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Abstract In this paper, we explore the complete synchronization and quasi-projective synchronization in a class of stochastic delayed quaternion-valued neural networks, utilizing a state-feedback control scheme. The studied neural networks into real-valued networks are short of known decomposing, by designing a very general nonlinear controller, according to the quaternion form Itô formula with a number of inequality techniques in the configuration of quaternion domain, we obtained a quasi-projective synchronization criterion for drive-response networks. Moreover, we estimate the error margin for quasi-projective synchronization. At last, the theoretical results are confirmed by a numerical simulation.

Keywords Stochastic neural network, quaternion, quasi-projective synchronization, time delays.

MSC(2010) 34C27, 34D06, 92B20.

1. Introduction

Stochastic neural networks (SNNs) is a class of neural network model that incorporates randomness into its structure or training process, which is set up by introducing random variations into the network, or by giving random transfer functions to the neurons of the network, or giving them random weights. When simulating the real nervous system and artificial neural networks, the presence of noise is inevitable, as highlighted in previous studies [1, 23, 24]. Hence, the exploration of

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[*]The authors were supported by the Yunnan Fundamental Research Projects under Grant 202201AU070170, and the Science Research Fund Projects of Yunnan University of Finance and Economics under Grant 2023D40, and the Yunnan Province XingDian Talent Support Program (YNWR-YLXZ-2018-020), and the Key Laboratory of Complex Dynamics System and Application Analysis of Department of Education of Yunnan Province.
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SNNs holds significant practical importance [9, 10, 16, 20, 36, 39, 42]. Furthermore, time delays are inevitably introduced in artificial neural networks and ecosystems due to the limited switching rate of neurons and amplifiers.

A natural progression from the traditional neural networks, the complex-valued neural networks (CNNs) and quaternion-valued neural networks (QVNNs) have found extensive applications in various domains, including robotics, satellite attitude control, image recognition, integrated control, and many other fields [12,21,29]. Currently, many researches highlight the greater versatility and practicality of non-autonomous neural networks compared to autonomous ones. Nonetheless, there remains an inadequate focus on the dynamic analysis of QVNNs with time delay [3, 34, 38]. Furthermore, in the quaternion domain, processing information has made more complex properties and resilient execution than complex-valued one. Quaternion-valued model in neural networks has quite complicated competencies as compared to the complex-valued neural networks(CVNNs) models since that can be solved in QVNNs but cannot be solved for the CVNN models [14].

Since Pecora and Carroll proposed the idea of achieving synchronization between the driving and responding components in coupled chaos models [31], chaos synchronization generates significant interest and attention. This is due to its wide-ranging applications in fields such as automatic control, biosystems science, information technology, and more [11, 22, 27]. So far, various classes of synchronization like complete synchronization [6, 13, 30], phase synchronization [33, 37], phase synchronization [40], lag synchronization [47], pinning synchronization and clusters synchronization have been investigated [41]. Later, scholars also proposed the concept of quasi-synchronization [26]. Besides, in contrast to the aforementioned synchronization types, projective synchronization can achieve more faster and faster communications due to its proportional feature [2, 7, 28]. Recently, projective synchronization has been extended to quasi-projective synchronization in [17]. The instability of dynamic neural network systems is often induced by random disturbances arising from environmental uncertainties. As a result, the synchronization problem in stochastic neural network systems has garnered considerable research attention for the past few years [8, 19, 46, 48]. Due to the difficulty in dealing with random perturbations, and complex-valued neural network systems, there are few results considering the quasi-synchronization issue of complex-valued stochastic neural networks (CVSNNs) [18, 25]. In fact, complex-valued model in neural networks has quite complicated competencies as compared to the real valued stochastic neural networks(RVSNNs) models since that can be solved in CVSNNs but cannot be solved for the RVSNNs models.

Recently, scholars have been investigating synchronization in QVNNs [5, 15, 32, 43–45], as well as examining the stability of quaternion-valued stochastic neural networks (QVSNNs) [4, 35]. However, it is still an open challenge regarding how to explore the quasi-projective synchronization of the stochastic delayed quaternion-valued neural networks (SDQVNNs) using state-feedback control strategy. Therefore, delving into the quasi-projective synchronization of SDQVNNs with state-feedback control strategy holds significant value.

To address the above discussion, we utilize a non-decomposition approach to analyze the synchronization dynamics of SDQVNNs through state feedback control. All in all, the primary contributions of this dissertation are highlighted as follows. Firstly, in this paper, the quasi-projective synchronization of the SDQVNNs with state feedback control has been considered. Secondly, by incorporating the quaternionic version of Itô formula, we derive a quasi-projective synchronization criterion along with several specific scenarios. Furthermore, we assess the synchronization error limit and elucidate its connection with the controller parameters. Thirdly, without dividing the SDQVNNs model into four real-valued models, the examined SDQVNNs is implemented as a whole one. Finally, the proposed method can be applied for investigating the quasi-projective synchronization of other sorts of SDQVNNs with state-feedback control.

The primary contributions of this paper are structured as follows: Section 2 presents the problem formulation. In Section 3, we derive criteria to guarantee quasi-projective synchronization in the networks under consideration. Section 4 offers a numerical example to validate our theoretical findings. We conclude with discussions and outline future research directions in the last section.

2. Problem formulation

In the present work, we investigate the following SDQVNNs with time delays:

$$\begin{aligned}\mathrm{d}y_p(t) = \Big[&- a_p(t)y_p(t) + \sum_{q=1}^{m} b_{pq}(t)f_q\big(y_q(t)\big) + \sum_{q=1}^{m} c_{pq}(t)g_q\big(y_q(t-\eta(t))\big)\\ &+U_p(t)\Big]\mathrm{d}t + \sum_{q=1}^{m} \sigma_{pq}\big(y_q(t-\eta(t))\big)\mathrm{d}w_q(t),\end{aligned} \tag{2.1}$$

where $p \in \{1, 2, \ldots, m\} := \Lambda, n$ denotes the number of neurons in each layer, while other relevant variables and parameters of the neural networks (2.1) are explained in Table 1.

Table 1. Parameters values for the SDQVNNs (2.1)

Symbols	Meaning
$y_p(t)$	State of the p-th neuron at time t
$a_p(t)$	Self-feedback connection weight
$b_{pq}(t), c_{pq}(t)$	The synaptic weights of delayed feedback
$U_p(t)$	External input on the p-th unit at time t
f_q, g_q	The activation functions of signal transmission
$w(t) = \big(w_1(t), w_2(t), \ldots, w_m(t)\big)^T$	m-dimensional Brownian motion
σ_{pq}	Borel measurable function
$\sigma = (\sigma_{pq})_{m\times m}$	Diffusion coefficient matrix
$\eta(t)$	Time-varying, satisfying $0 \le \eta(t) \le \eta$, and η is a constant

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathrm{P})$ stand for the complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is right continuous, and $\mathcal{F}_0$ contains all P-null sets), and E represents the expectation operator with respect to probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathrm{P})$. $C^b_{\mathcal{F}_0}([-\eta, 0], \mathbb{Q}^m)$ denotes the family of all bounded, $\mathcal{F}_0$-measurable, $BC([-\eta, 0], \mathbb{Q}^m)$-valued random variables ϕ.

We represent the skew field of quaternion number q as

$$\mathbb{Q} := \{q = q^R + iq^I + jq^J + kq^K\},$$

where q^R, q^I, q^J, $q^K \in R$, i, j and k represent imaginary units, and the quaternion satisfies the following Hamilton rules:

$$\begin{cases} ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \\ i^2 = j^2 = k^2 = ijk = -1, \end{cases}$$

$\forall y \in \mathbb{Q}$, let $y^* = y^R - iy^I - jy^J - ky^K$, and with the norm

$$\|y\|_{\mathbb{Q}} = \sqrt{yy^*} = \sqrt{(y^R)^2 + (y^I)^2 + (y^J)^2 + (y^K)^2}.$$

For every $y = (y_1, y_2, \ldots, y_m)^T \in \mathbb{Q}^m$, endowed with the norm $\|y\|_0 = \max_{p\in\Lambda}\{\|y_p\|_{\mathbb{Q}}\}$.

Set

$$\underline{a}_p = \inf_{t\in\mathbb{R}} |a_p(t)|, \quad \bar{b}_{pq} = \sup_{t\in\mathbb{R}} \|b_{pq}(t)\|_{\mathbb{Q}}, \quad \bar{c}_{pq} = \sup_{t\in\mathbb{R}} \|c_{pq}(t)\|_{\mathbb{Q}}, \quad \bar{U}_p = \sup_{t\in\mathbb{R}} \|U_p(t)\|_{\mathbb{Q}}.$$

The initial values of the networks (2.1) are described by

$$y_p(s) = \phi_p(s), \quad s \in [-\eta, 0], \ p \in \Lambda,$$

where $\phi_p(s) \in C^b_{\mathcal{F}_0}([-\eta, 0], \mathbb{Q})$.

In this article, we make the following assumptions about activation functions.

(S_1) $f_q(0) = g_q(0) = \sigma_{pq}(0) = l_q(0) = 0$, $\forall u, v \in \mathbb{Q}$ and there exist positive constants L^f_q, L^g_q, L^σ_{pq} and L^l_q such that

$$\begin{aligned} &\big\|f_q(u) - f_q(v)\big\|_{\mathbb{Q}} \le L^f_q \big\|u - v\big\|_{\mathbb{Q}}, \quad \big\|g_q(u) - g_q(v)\big\|_{\mathbb{Q}} \le L^g_q \big\|u - v\big\|_{\mathbb{Q}}, \\ &\big\|\sigma_{pq}(u) - \sigma_{pq}(v)\big\|_{\mathbb{Q}} \le L^\sigma_{pq} \big\|u - v\big\|_{\mathbb{Q}}, \quad \big\|l_q(u) - l_q(v)\big\|_{\mathbb{Q}} \le L^l_q \big\|u - v\big\|_{\mathbb{Q}}, \end{aligned}$$

where $p, q \in \Lambda$.

3. Synchronization control of stochastic neural networks

In the present section, by considering a very general nonlinear state-feedback controller, using stochastic analysis theory, Itô formula and the construction of appropriate Lyapunov functions. We will discuss the quasi-projective synchronization problem for network (2.1).

For this purpose, we consider the network (2.1) as a driver one, and the corresponding response network as follows:

$$\begin{aligned} \mathrm{d}z_p(t) = \bigg[&- a_p(t) z_p(t) + \sum_{q=1}^m b_{pq}(t) f_q\big(z_q(t)\big) + \sum_{q=1}^m c_{pq}(t) g_q\big(z_q(t - \eta(t))\big) \\ &+ U_p(t) + E_p(t)\bigg] \mathrm{d}t + \sum_{q=1}^m \sigma_{pq}\big(z_q(t - \eta(t))\big) \mathrm{d}w_q(t), \end{aligned} \tag{3.1}$$

where $p \in \Lambda$, $z_p(t) \in \mathbb{Q}$ indicates the state of the response network, and $E_p(t) \in \mathbb{Q}$ is a state-feedback controller, the other symbols are the same as network (2.1).

The initial values of network (3.1) are described by

$$z_p(s) = \psi_p(s), \quad s \in [-\eta, 0],\ p \in \Lambda,$$

where $\psi_p(s) \in C^b_{\mathcal{F}_0}\big([-\eta, 0], \mathbb{Q}\big)$. Through feedback control, the controller E_p can be described as

$$E_p(t) = -\theta_p(t)h_p(t) + \sum_{q=1}^{m} d_{pq}(t) l_q\big(h_q(t-\eta(t))\big), \tag{3.2}$$

where $\theta_p \in C(\mathbb{R}, \mathbb{R}^+)$, $d_{pq} \in C(\mathbb{R}, \mathbb{Q})$, $l_q : \mathbb{Q} \to \mathbb{Q}$, $p, q \in \Lambda$.

Let $h_p(t) = z_p(t) - \lambda y_p(t)$ define as synchronization error, in which λ denoted by projective parameters, then the error network between (2.1) and (3.1) can be expressed as

$$\begin{aligned}
\mathrm{d}h_p(t) = &\bigg[-\big(a_p(t) + \theta_p(t)\big)h_p(t) + \sum_{q=1}^{m} b_{pq}(t)\Big(f_q\big(z_q(t)\big) - \lambda f_q\big(y_q(t)\big)\Big) \\
&+ \sum_{q=1}^{n} b_{pq}(t)\Big(g_q\big(z_q(t-\eta(t))\big) - \lambda g_q\big(y_q(t-\eta(t))\big)\Big) \\
&+ \sum_{q=1}^{m} d_{pq}(t)\Big(l_q\big(z_q(t-\eta(t))\big) - \lambda l_q\big(y_q(t-\eta(t))\big)\Big) + (1-\lambda)U_p(t)\bigg]\mathrm{d}t \\
&+ \sum_{q=1}^{m} \Big(\sigma_{pq}\big(z_q(t-\eta(t))\big) - \lambda \sigma_{pq}\big(y_q(t-\eta(t))\big)\Big)\mathrm{d}w_q(t) \\
= &\bigg[-\big(a_p(t) + \theta_p(t)\big)h_p(t) + \sum_{q=1}^{m} b_{pq}(t) f_q\big(h_q(t)\big) \\
&+ \sum_{q=1}^{m} c_{pq}(t) g_q\big(h_q(t-\eta(t))\big) + \sum_{q=1}^{m} d_{pq}(t) l_q\big(h_q(t-\eta(t))\big) \\
&+ \sum_{q=1}^{m} b_{pq}(t)\Big(f_q\big(\lambda y_q(t)\big) - \lambda f_q\big(y_q(t)\big)\Big) \\
&+ \sum_{q=1}^{m} c_{pq}(t)\Big(g_q\big(\lambda y_q(t-\eta(t))\big) - \lambda g_q\big(y_q(t-\eta(t))\big)\Big) \\
&+ \sum_{q=1}^{m} d_{pq}(t)\Big(l_q\big(\lambda y_q(t-\eta(t))\big) - \lambda l_q\big(y_q(t-\eta(t))\big)\Big) \\
&+ (1-\lambda)U_p(t)\bigg]\mathrm{d}t + \sum_{q=1}^{m} \sigma_{pq}\big(h_q(t-\eta(t))\big)\mathrm{d}w_q(t) \\
&+ \sum_{q=1}^{m} \Big(\sigma_{pq}\big(\lambda y_q(t-\eta(t))\big) - \lambda \sigma_{pq}\big(y_q(t-\eta(t))\big)\Big)\mathrm{d}w_q(t),
\end{aligned} \tag{3.3}$$

where

$$f_q\big(h_q(t)\big) := f_q\big(z_q(t)\big) - f_q\big(\lambda y_q(t)\big),$$

$$g_q\big(h_q(t-\eta(t))\big) := g_q\big(z_q(t-\eta(t))\big) - g_q(\lambda y_q(t-\eta(t))),$$
$$l_q\big(h_q(t-\eta(t))\big) := l_q\big(z_q(t-\eta(t))\big) - l_q\big(\lambda y_q(t-\eta(t))\big),$$
$$\sigma_{pq}\big(h_q(t-\eta(t))\big) := \sigma_{pq}\big(z_q(t-\eta(t))\big) - \sigma_{pq}\big(\lambda y_q(t-\eta(t))\big).$$

Definition 3.1. [4] Consider an m-dimensional quaternion-valued stochastic differential equation:

$$\mathrm{d}H(t) = \mathcal{F}\big(t, H(t), H(t-\eta(t))\big)\mathrm{d}t + \mathcal{G}\big(t, H(t), H(t-\eta(t))\big)\mathrm{d}W(t),$$

where $H(t) = (H_1(t), H_2(t), \ldots, H_m(t))^T \in \mathbb{Q}^m$. For $V(t,H) : \mathbb{R}^+ \times \mathbb{Q}^m \to \mathbb{R}^+$(in reality, we can represent $V(t,H) = V(t,H,H^*)$, $\mathbb{R}$-derivative of V can be defined as

$$\frac{\partial V(t,H)}{\partial H}\bigg|_{H^*=\mathrm{const}} = \left(\frac{\partial V(t,H(t))}{\partial H_1}, \ldots, \frac{\partial V(t,H(t))}{\partial H_m}\right)\bigg|_{H^*=\mathrm{const}}$$

and

$$\frac{\partial V(t,H)}{\partial H^*}\bigg|_{H=\mathrm{const}} = \left(\frac{\partial V(t,H(t))}{\partial H_1^*}, \ldots, \frac{\partial V(t,H(t))}{\partial H_m^*}\right)\bigg|_{H=\mathrm{const}},$$

where const is the constant. Denote by $C^{1,2}(\mathbb{R}^+ \times \mathbb{Q}^m, \mathbb{R}^+)$ the family of all nonnegative functions $V(t,H)$ on $\mathbb{R}^+ \times \mathbb{Q}^m$, which are once continuously differentiable in t and twice differentiable in H and H^*. Thus, for $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{Q}^m, \mathbb{R}^+)$, according to Itô's formula, the quaternion form is as follows:

$$\begin{aligned}
\mathrm{d}V(t,H) &= \frac{\partial V(t,H)}{\partial t}\mathrm{d}t + \frac{\partial V(t,H)}{\partial H}\mathrm{d}H + \frac{\partial V(t,H)}{\partial H^*}\mathrm{d}H^* \\
&\quad + \frac{1}{2}\sum_{p,q=1}^{m} \frac{\partial^2 V(t,H)}{\partial H_p \partial H_q}\mathrm{d}H_p \mathrm{d}H_q + \frac{1}{2}\sum_{p,q=1}^{m} \frac{\partial^2 V(t,H)}{\partial H_p^* \partial H_q^*}\mathrm{d}H_p^* \mathrm{d}H_q^* \\
&\quad + \sum_{p,q=1}^{m} \frac{\partial^2 V(t,H)}{\partial H_p \partial H_q^*}\mathrm{d}H_p \mathrm{d}H_q^* \\
&= \mathcal{L}V(t,H)\mathrm{d}t + \left[\frac{\partial V(t,H)}{\partial H}\mathcal{G}(t) + \frac{\partial V(t,H)}{\partial H^*}\mathcal{G}^*(t)\right]\mathrm{d}W(t),
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{F}(t) = \mathcal{F}(t, H(t), H(t-\eta(t))), \quad \mathcal{G}(t) = \mathcal{G}(t, H(t), H(t-\eta(t))), \\
&\frac{\partial^2 V(t,H)}{\partial H^2} = \left(\frac{\partial^2 V(t,H)}{\partial H_p \partial H_q}\right)_{m\times m}, \quad \frac{\partial^2 V(t,H)}{\partial (H^*)^2} = \left(\frac{\partial^2 V(t,H)}{\partial H_p^* \partial H_q^*}\right)_{m\times m}, \\
&\frac{\partial^2 V(t,H)}{\partial H \partial H^*} = \left(\frac{\partial^2 V(t,H)}{\partial H_p \partial H_q^*}\right)_{m\times m}, \quad \mathrm{d}W(t)\mathrm{d}W(t) = \mathrm{d}t, \\
&\mathrm{d}W(t)\mathrm{d}t = \mathrm{d}t\mathrm{d}W(t) = \mathrm{d}t\mathrm{d}t = 0,
\end{aligned}$$

and operator $\mathcal{L}V(t,H)$ are described by

$$\begin{aligned}
\mathcal{L}V(t,H) &= \frac{\partial V(t,H)}{\partial t} + \frac{\partial V(t,H)}{\partial Y}\mathcal{F}(t) + \frac{\partial V(t,H)}{\partial Y^*}\mathcal{F}^*(t) \\
&\quad + \frac{1}{2}\mathcal{G}^T(t)\frac{\partial^2 V(t,H)}{\partial (H)^2}\mathcal{G}(t) + \frac{1}{2}(\mathcal{G}^*(t))^T\frac{\partial^2 V(t,H)}{\partial (H^*)^2}\mathcal{G}^*(t) \\
&\quad + \mathcal{G}^T(t)\frac{\partial^2 V(t,H)}{\partial H \partial H^*}\mathcal{G}^*(t).
\end{aligned}$$

Definition 3.2. [8] Let W represent the area where the network (2.1) shows chaotic behavior. Drive-response networks (2.1) and (3.1) can achieve quasi-projective synchronization in mean square via an error bound $\kappa > 0$, if $\lim_{t\to\infty} \mathrm{E}\|z(t)-\lambda y(t)\|_0^2 \leq \kappa$ is satisfied for any initial conditions, in which λ is stand for the projective parameters. Especially, networks (2.1) and (3.1) can achieve quasi-synchronized in mean square if $\lambda = 1$. Moreover, networks (2.1) and (3.1) can achieve complete synchronization if $\lim_{t\to\infty} \mathrm{E}\|z(t) - y(t)\|_0^2 = 0$.

Lemma 3.1. [8] *Supposing that function $h(t) : [t_0 - \eta, +\infty) \to \mathbb{R}$ is continuous and the following condition hold.*

$$\frac{\mathrm{d}h(t)}{\mathrm{d}t} \leq -\rho h(t) + \varrho h(t - \eta(t)) + \vartheta$$

for $t \geq t_0$, where $\rho > \varrho > 0$, $\vartheta > 0$, $\eta(t) \leq \eta$, it gains

$$h(t) \leq \sup_{s\in[-\eta,0]} h(s)e^{-\xi t} + \frac{\vartheta}{\xi},$$

where $\xi > 0$ is the unique solution to algebra equation $\rho - \varrho e^{\xi\eta} - \xi = 0$.

Theorem 3.1. *Assumption (S_1) holds, network (2.1) is quasi-projectively synchronized with (3.1) in mean square under the linear controller (3.2), if there exist constants $\rho, \varrho > 0$, such that*

$$\rho = \min_{p\in\Lambda} \left\{ 2\underline{a}_p + 2\underline{\theta}_p - 7 - \sum_{q=1}^{m} \left(\bar{b}_{qp} L_q^f\right)^2 \right\} > 0,$$

$$\varrho = \max_{p\in\Lambda} \left\{ \sum_{q=1}^{m} \left(\bar{c}_{qp} L_q^g\right)^2 + \sum_{q=1}^{m} \left(\bar{d}_{qp} L_q^l\right)^2 + \sum_{q=1}^{m} \left(L_{qp}^{\sigma}\right)^2 \right\} > 0$$

and $\rho - \varrho > 0$. Simultaneously, the stochastic synchronization error networks converges exponentially in mean square to the following region

$$W = \left\{ h(t) \in \mathbb{Q}^m \middle| \mathrm{E}\|h(t)\|_0^2 \leq \frac{\vartheta}{\xi} \right\},$$

where

$$\begin{aligned} \vartheta = \max_{p\in\Lambda} \Bigg\{ & (1-\lambda)^*(1-\lambda)\left(\bar{U}_p\right)^2 + \sum_{q=1}^{m} 2\left[1 + \lambda^*\lambda\right] \\ & \times \left[\left(\bar{b}_{pq} L_q^f\right)^2 + \left(\bar{c}_{pq} L_q^g\right)^2 + \left(\bar{d}_{pq} L_q^l\right)^2 + \left(L_{pq}^{\sigma}\right)^2 \right] \Bigg\} > 0 \end{aligned}$$

and $\xi > 0$ is the unique solution to algebra equation $\rho - \varrho e^{\xi\eta} - \xi = 0$.

Proof. Set $\sigma(t) = (\sigma_{pq}(t))_{m\times m}$, where $\sigma_{pq}(t) = \sigma_{pq}(h_q(t - \eta(t)))$. Choose the following Lyapunov function:

$$V(t, h(t)) = \max_{p\in\Lambda} \left\{ h_p^*(t) h_p(t) \right\}.$$

According to Itô formula, next we will consider the following stochastic differential:

$$\mathrm{d}V(t, h(t)) = \mathcal{L}V(t, h(t))\mathrm{d}t + V_h(t, h(t))\sigma(t)\mathrm{d}w(t), \tag{3.4}$$

where $V(t,h(t)) = \big(\frac{\partial V(t,h(t))}{\partial h_1},\dots,\frac{\partial V(t,h(t))}{\partial h_m}\big)$, $\mathcal{L}$ is the differential operator, then according to Definition 3.1, we have

$$\frac{\partial^2 V(t,h(t))}{\partial h(t)h^*(t)} = 1, \quad \frac{\partial^2 V(t,h(t))}{\partial h(t)^2} = \frac{\partial^2 V(t,h(t))}{\partial (h^*(t))^2} = 0.$$

Again from Definition 3.1, based on the differential operator $\mathcal{L}$, we have

$$\begin{aligned}
&\mathcal{L}V(t,h(t))\\
&= \max_{p\in\Lambda}\Bigg\{h_p(t)\Bigg[\big(-a_p(t)-\theta_p(t)\big)h_p^*(t) + \sum_{q=1}^{m}\Big[b_{pq}(t)f_q\big(h_q(t)\big)\Big]^*\\
&\quad+\sum_{q=1}^{m}\Big[c_{pq}(t)g_q\big(h_q(t-\eta(t))\big)\Big]^* + \sum_{q=1}^{m}\Big[d_{pq}(t)l_q\big(h_q(t-\eta(t))\big)\Big]^*\\
&\quad+\sum_{q=1}^{m}\Big[b_{pq}(t)\Big(f_q\big(\lambda y_q(t)\big)-\lambda f_q\big(y_q(t)\big)\Big)\Big]^*\\
&\quad+\sum_{q=1}^{m}\Big[c_{pq}(t)\Big(g_q\big(\lambda y_q(t-\eta(t))\big)-\lambda g_q\big(y_q(t-\eta(t))\big)\Big)\Big]^*\\
&\quad+\sum_{q=1}^{m}\Big[d_{pq}(t)\Big(l_q\big(\lambda y_q(t-\eta(t))\big)-\lambda l_q\big(y_q(t-\eta(t))\big)\Big)\Big]^*\\
&\quad+(1-\lambda)U_p^*(t)\Bigg] + h_p^*(t)\Bigg[\big(-a_p(t)-\theta_p(t)\big)h_p(t) + \sum_{q=1}^{m}b_{pq}(t)f_q\big(h_q(t)\big)\\
&\quad+\sum_{q=1}^{m}c_{pq}(t)g_q\big(h_q(t-\eta(t))\big) + \sum_{q=1}^{m}d_{pq}(t)l_q\big(h_q(t-\eta(t))\big)\\
&\quad+\sum_{q=1}^{m}b_{pq}(t)\Big(f_q\big(\lambda y_q(t)\big)-\lambda f_q\big(y_q(t)\big)\Big)\\
&\quad+\sum_{q=1}^{m}c_{pq}(t)\Big(g_q\big(\lambda y_q(t-\eta(t))\big)-\lambda g_q\big(y_q(t-\eta(t))\big)\Big)\\
&\quad+\sum_{q=1}^{m}d_{pq}(t)\Big(l_q\big(\lambda y_q(t-\eta(t))\big)-\lambda l_q\big(y_q(t-\eta(t))\big)\Big)\\
&\quad+(1-\lambda)U_p(t)\Bigg] + \sum_{q=1}^{m}\Big[\sigma_{pq}\big(h_q(t-\eta(t))\big)\Big]^*\Big[\sigma_{pq}\big(h_q(t-\eta(t))\big)\Big]\\
&\quad+\sum_{q=1}^{m}\Big[\sigma_{pq}\big(\lambda y_q(t-\eta(t))\big)-\lambda\sigma_{pq}\big(y_q(t-\eta(t))\big)\Big]^*\\
&\quad\times\Big[\sigma_{pq}\big(\lambda y_q(t-\eta(t))\big)-\lambda\sigma_{pq}\big(y_q(t-\eta(t))\big)\Big]\Bigg\}\\
&= \max_{p\in\Lambda}\Bigg\{\Bigg[\big(-a_p(t)-\theta_p(t)\big)h_p^*(t)h_p(t) + \sum_{q=1}^{m}h_p(t)\Big[b_{pq}(t)f_q\big(h_q(t)\big)\Big]^*
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{q=1}^{m} h_p(t)\Big[c_{pq}(t)g_q\big(h_q(t-\eta(t))\big)\Big]^* + \sum_{q=1}^{m} h_p(t)\Big[d_{pq}(t)l_q\big(h_q(t-\eta(t))\big)\Big]^* \\
&+\sum_{q=1}^{m} h_p(t)\Big[b_{pq}(t)\Big(f_q\big(\lambda y_q(t)\big) - \lambda f_q\big(y_q(t)\big)\Big)\Big]^* \\
&+\sum_{q=1}^{m} h_p(t)\Big[c_{pq}(t)\Big(g_q\big(\lambda y_q(t-\eta(t))\big) - \lambda g_q\big(y_q(t-\eta(t))\big)\Big)\Big]^* \\
&+\sum_{q=1}^{m} h_p(t)\Big[d_{pq}(t)\Big(l_q\big(\lambda y_q(t-\eta(t))\big) - \lambda l_q\big(y_q(t-\eta(t))\big)\Big)\Big]^* \\
&+(1-\lambda)^* U_p^*(t)h_p(t)\Bigg] + \Bigg[\big(-a_p(t)-\theta_p(t)\big)h_p^*(t)h_p(t) \\
&+\sum_{q=1}^{m} h_p^*(t)b_{pq}(t)f_q\big(h_q(t)\big) + \sum_{q=1}^{m} h_p^*(t)c_{pq}(t)g_q\big(h_q(t-\eta(t))\big) \\
&+\sum_{q=1}^{m} h_p^*(t)d_{pq}(t)l_q\big(h_q(t-\eta(t))\big) + \sum_{q=1}^{m} h_p^*(t)b_{pq}(t)\Big(f_q\big(\lambda y_q(t)\big) \\
&-\lambda f_q\big(y_q(t)\big)\Big) + \sum_{q=1}^{m} h_p^*(t)c_{pq}(t)\Big(g_q\big(\lambda y_q(t-\eta(t))\big) - \lambda g_q\big(y_q(t-\eta(t))\big)\Big) \\
&+\sum_{q=1}^{m} h_p^*(t)d_{pq}(t)\Big(l_q\big(\lambda y_q(t-\eta(t))\big) - \lambda l_q\big(y_q(t-\eta(t))\big)\Big) \\
&+(1-\lambda)U_p(t)h_p^*(t)\Bigg] + \sum_{q=1}^{m}\Big[\sigma_{pq}\big(h_q(t-\eta(t))\big)\Big]^*\Big[\sigma_{pq}\big(h_q(t-\eta(t))\big)\Big] \\
&+\sum_{q=1}^{m}\Big[\sigma_{pq}\big(\lambda y_q(t-\eta(t))\big) - \lambda\sigma_{pq}\big(y_q(t-\eta(t))\big)\Big]^* \\
&\times\Big[\sigma_{pq}\big(\lambda y_q(t-\eta(t))\big) - \lambda\sigma_{pq}\big(y_q(t-\eta(t))\big)\Big]\Bigg\} \\
\le\, &\max_{p\in\Lambda}\Bigg\{\big(-2a_p(t)-2\theta_p(t)+6\big)h_p^*(t)h_p(t) + \sum_{q=1}^{m}\Big[b_{pq}(t)f_q\big(h_q(t)\big)\Big]^* \\
&\times\Big[b_{pq}(t)f_q\big(h_q(t)\big)\Big] + \sum_{q=1}^{m}\Big[c_{pq}(t)g_q\big(h_q(t-\eta(t))\big)\Big]^* \\
&\times\Big[c_{pq}(t)g_q\big(h_q(t-\eta(t))\big)\Big] + \sum_{q=1}^{m}\Big[d_{pq}(t)l_q\big(h_q(t-\eta(t))\big)\Big]^* \\
&\times\Big[d_{pq}(t)l_q\big(h_q(t-\eta(t))\big)\Big] + \big[h_p^*(t)h_p(t) + (1-\lambda)^*(1-\lambda)U_p^*(t)U_p(t)\big] \\
&+\sum_{q=1}^{m}\Big[b_{pq}(t)\Big(f_q\big(\lambda y_q(t)\big) - \lambda f_q\big(y_q(t)\big)\Big)\Big]^*\Big[b_{pq}(t)\Big(f_q\big(\lambda y_q(t)\big) \\
&-\lambda f_q\big(y_q(t)\big)\Big)\Big] + \sum_{q=1}^{m}\Big[c_{pq}(t)\Big(g_q\big(\lambda y_q(t-\eta(t))\big) - \lambda g_q\big(y_q(t-\eta(t))\big)\Big)\Big]^*
\end{aligned}
$$

$$
\begin{aligned}
&\times\Big[c_{pq}(t)\Big(g_q\big(\lambda y_q(t-\eta(t))\big)-\lambda g_q\big(y_q(t-\eta(t))\big)\Big)\Big]\\
&+\sum_{q=1}^{m}\Big[d_{pq}(t)\Big(l_q\big(\lambda y_q(t-\eta(t))\big)-\lambda l_q\big(y_q(t-\eta(t))\big)\Big)\Big]^*\\
&\times\Big[d_{pq}(t)\Big(l_q\big(\lambda y_q(t-\eta(t))\big)-\lambda l_q\big(y_q(t-\eta(t))\big)\Big)\Big]\\
&+\sum_{q=1}^{m}\Big[\sigma_{pq}\big(h_q(t-\eta(t))\big)\Big]^*\Big[\sigma_{pq}\big(h_q(t-\eta(t))\big)\Big]\\
&+\sum_{q=1}^{m}\Big[\sigma_{pq}\big(\lambda y_q(t-\eta(t))\big)-\lambda\sigma_{pq}\big(y_q(t-\eta(t))\big)\Big]^*\\
&\times\Big[\sigma_{pq}\big(\lambda y_q(t-\eta(t))\big)-\lambda\sigma_{pq}\big(y_q(t-\eta(t))\big)\Big]\Big\}\\
\le{}&\max_{p\in\Lambda}\Big\{\big(-2a_p(t)-2\theta_p(t)+7\big)h_p^*(t)h_p(t)+\sum_{q=1}^{m}\big(\bar{b}_{pq}L_q^f\big)^2\\
&\times\big(h_q(t)\big)^*\big(h_q(t)\big)+\sum_{q=1}^{m}\big(\bar{c}_{pq}L_q^g\big)^2\big(h_q(t-\eta(t))\big)^*\big(h_q(t-\eta(t))\big)\\
&+\sum_{q=1}^{m}\big(\bar{d}_{pq}L_q^l\big)^2\big(h_q(t-\eta(t))\big)^*\big(h_q(t-\eta(t))\big)+(1-\lambda)^*(1-\lambda)\\
&\times U_p^*(t)U_p(t)+\sum_{q=1}^{m}\big(L_{pq}^\sigma\big)^2\big(h_q(t-\eta(t))\big)^*\big(h_q(t-\eta(t))\big)\\
&+\sum_{q=1}^{m}2\big(\bar{b}_{pq}\big)^2\Big[\Big(f_q\big(\lambda y_q(t)\big)\Big)^*\Big(f_q\big(\lambda y_q(t)\big)\Big)+\lambda^*\lambda\Big(f_q\big(y_q(t)\big)\Big)^*\Big(f_q\big(y_q(t)\big)\Big)\Big]\\
&+\sum_{q=1}^{m}2\big(\bar{c}_{pq}\big)^2\Big[\Big(g_q\big(\lambda y_q(t-\eta(t))\big)\Big)^*\Big(g_q\big(\lambda y_q(t-\eta(t))\big)\Big)\\
&+\lambda^*\lambda\Big(g_q\big(y_q(t-\eta(t))\big)\Big)^*\Big(g_q\big(y_q(t-\eta(t))\big)\Big)\Big]\\
&+\sum_{q=1}^{m}2\big(\bar{d}_{pq}\big)^2\Big[\Big(l_q\big(\lambda y_q(t-\eta(t))\big)\Big)^*\Big(l_q\big(\lambda y_q(t-\eta(t))\big)\Big)\\
&+\lambda^*\lambda\Big(l_q\big(y_q(t-\eta(t))\big)\Big)^*\Big(l_q\big(y_q(t-\eta(t))\big)\Big)\Big]\\
&+\sum_{q=1}^{m}2\Big[\Big(\sigma_{pq}\big(\lambda y_q(t-\eta(t))\big)\Big)^*\Big(\sigma_{pq}\big(\lambda y_q(t-\eta(t))\big)\Big)\\
&+\lambda^*\lambda\Big(\sigma_{pq}\big(y_q(t-\eta(t))\big)\Big)^*\Big(\sigma_{pq}\big(y_q(t-\eta(t))\big)\Big)\Big]\\
\le{}&\max_{p\in\Lambda}\Big\{-\Big[2\underline{a}_p+2\underline{\theta}_p-7\Big]h_p^*(t)h_p(t)+\sum_{q=1}^{m}\big(\bar{b}_{pq}L_q^f\big)^2\big(h_q(t)\big)^*\big(h_q(t)\big)\\
&+\sum_{q=1}^{m}\big(\bar{c}_{pq}L_q^g\big)^2\big(h_q(t-\eta(t))\big)^*\big(h_q(t-\eta(t))\big)+\sum_{q=1}^{m}\big(\bar{d}_{pq}L_q^l\big)^2
\end{aligned}
$$

$$
\begin{aligned}
&\times\big(h_q(t-\eta(t))\big)^*\big(h_q(t-\eta(t))\big)+(1-\lambda)^*(1-\lambda)\big(\bar{U}_p\big)^2\\
&+\sum_{q=1}^{m}\big(L_{pq}^{\sigma}\big)^2\big(h_q(t-\eta(t))\big)^*\big(h_q(t-\eta(t))\big)+\sum_{q=1}^{m}2\big(\bar{b}_{pq}\big)^2\big(L_q^f\big)^2\big[1+\lambda^*\lambda\big]\\
&+\sum_{q=1}^{m}2\big(\bar{c}_{pq}\big)^2\big(L_q^g\big)^2\big[1+\lambda^*\lambda\big]+\sum_{q=1}^{m}2\big(\bar{d}_{pq}\big)^2\big(L_q^l\big)^2\big[1+\lambda^*\lambda\big]\\
&+\sum_{q=1}^{m}2\big(L_{pq}^{\sigma}\big)^2\big[1+\lambda^*\lambda\big]\Bigg\}\\
\leq{}& \max_{p\in\Lambda}\Bigg\{-\Big[2\underline{a}_p+2\underline{\theta}_p-7-\sum_{q=1}^{n}\big(\bar{b}_{qp}L_q^f\big)^2\Big]h_p^*(t)h_p(t)\\
&+\Big[\sum_{q=1}^{n}\big(\bar{c}_{qp}L_q^g\big)^2+\sum_{q=1}^{m}\big(\bar{d}_{qp}L_q^l\big)^2+\sum_{q=1}^{m}\big(L_{qp}^{\sigma}\big)^2\Big]h_p^*(t-\eta(t))h_p(t-\eta(t))\\
&+(1-\lambda)^*(1-\lambda)\big(\bar{U}_p\big)^2+\sum_{q=1}^{m}2\big[1+\lambda^*\lambda\big]\Big[\big(\bar{b}_{pq}L_q^f\big)^2+\big(\bar{c}_{pq}L_q^g\big)^2\\
&+\big(\bar{d}_{pq}L_q^l\big)^2+\big(L_{pq}^{\sigma}\big)^2\Big]\Bigg\}\\
\leq{}& -\rho V(t,h(t))+\varrho V(t,h(t-\eta(t)))+\vartheta,
\end{aligned}
\tag{3.5}
$$

where

$$
\begin{aligned}
\rho&=\min_{p\in\Lambda}\Bigg\{2\underline{a}_p+2\underline{\theta}_p-7-\sum_{q=1}^{m}\big(\bar{b}_{qp}L_q^f\big)^2\Bigg\},\\
\varrho&=\max_{p\in\Lambda}\Bigg\{\sum_{q=1}^{m}\big(\bar{c}_{qp}L_q^g\big)^2+\sum_{q=1}^{m}\big(\bar{d}_{qp}L_q^l\big)^2+\sum_{q=1}^{m}\big(L_{qp}^{\sigma}\big)^2\Bigg\},\\
\vartheta&=\max_{p\in\Lambda}\Bigg\{(1-\lambda)^*(1-\lambda)\big(\bar{U}_p\big)^2+\sum_{q=1}^{m}2\big[1+\lambda^*\lambda\big]\\
&\qquad\times\Big[\big(\bar{b}_{pq}L_q^f\big)^2+\big(\bar{c}_{pq}L_q^g\big)^2+\big(\bar{d}_{pq}L_q^l\big)^2+\big(L_{pq}^{\sigma}\big)^2\Big]\Bigg\}.
\end{aligned}
$$

Next, choosing the mathematical expectation of both sides of (3.4), we get

$$
\frac{\mathrm{d}\mathrm{E}V(t,h(t))}{\mathrm{d}t}=-\rho\mathrm{E}V(t,h(t))+\varrho\mathrm{E}V(t,h(t-\eta(t)))+\vartheta. \tag{3.6}
$$

Since $\rho-\varrho>0$, by Lemma 3.1, it yields

$$
\mathrm{E}V(t,h(t))=\sup_{s\in[-\eta,0]}\mathrm{E}V(s,h(s))e^{-\xi t}+\frac{\vartheta}{\xi},
$$

if $\rho-\varrho e^{\xi\eta}-\xi=0$, the unique solution is $\xi>0$, which implies that

$$
\mathrm{E}\|h(t)\|_0^2=\sup_{s\in[-\eta,0]}\mathrm{E}V(s,h(s))e^{-\xi t}+\frac{\vartheta}{\xi}.
$$

Hence, we have that the solution of system (3.4) converges to the region $W = \{h(t) \big| \mathrm{E}\|h(t)\|_0^2 \leq \frac{\vartheta}{\xi}\}$ in mean square. Therfore, from Definition 3.2, we get that quasi-projective synchronization in mean square of networks (2.1) and (3.1). This completes the proof. □

Remark 3.1. In fact, when $\lambda = 1$, we have $\vartheta = 0$, then the problem to consider becomes the mean square synchronization between the networks (2.1) and (3.1).

Corollary 3.1. *suppose* (S_1) *is correct for the drive network* (2.1) *and response network* (3.1) *are synchronized in mean square through the linear controller* (3.2), *if there exist two constants* $\rho, \varrho > 0$, *such that*

$$\rho = \min_{p\in\Lambda}\left\{2\underline{a}_p + 2\underline{\theta}_p - 3 - \sum_{q=1}^{m}\left(\bar{b}_{qp}L_q^f\right)^2\right\} > 0,$$

$$\varrho = \max_{p\in\Lambda}\left\{\sum_{q=1}^{m}\left(\bar{c}_{qp}L_q^g\right)^2 + \sum_{q=1}^{m}\left(\bar{d}_{qp}L_q^l\right)^2 + \sum_{q=1}^{m}\left(L_{qp}^{\sigma}\right)^2\right\} > 0$$

and $\rho - \varrho > 0$.

Remark 3.2. So far, there are few studies on quasi-projective synchronization of QVNNs with time delays and state-feedback control scheme. Therefore, our theoretical results are an extension and supplement to the study of QVSNNs with time delays and state-feedback control. Moreover, considering similar networks (2.1) and (3.1), but no stochastic perturbations, we can analyze it in the same way. The proof is omitted here.

4. Illustrative example

In this section, we present a numerical example to validate the practicality of the main results derived for the stochastic quaternion-valued neural network with time delays and state-feedback control, as discussed in previous sections of this paper.

Example 4.1. Set $m = 2$, let us consider the following stochastic quaternion-valued neural network with time delays:

$$\begin{aligned}\mathrm{d}y_p(t) = \Bigg[&-a_p(t)y_p(t) + \sum_{q=1}^{2} b_{pq}(t)f_q\big(y_q(t)\big) + \sum_{q=1}^{2} c_{pq}(t)g_q\big(y_q(t-\eta(t))\big)\\ &+U_p(t)\Bigg]\mathrm{d}t + \sum_{q=1}^{2}\sigma_{pq}\big(y_q(t-\eta(t))\big)\mathrm{d}w_q(t),\end{aligned} \tag{4.1}$$

the given response network corresponds to

$$\begin{aligned}\mathrm{d}z_p(t) = \Bigg[&-a_p(t)z_p(t) + \sum_{q=1}^{2} b_{pq}(t)f_q\big(z_q(t)\big) + \sum_{q=1}^{2} c_{pq}(t)g_q\big(z_q(t-\eta(t))\big)\\ &+U_p(t) + E_p(t)\Bigg]\mathrm{d}t + \sum_{q=1}^{2}\sigma_{pq}\big(z_q(t-\eta(t))\big)\mathrm{d}w_q(t),\end{aligned} \tag{4.2}$$

and the state-feedback controller is as follow:

$$E_p(t) = -\theta_p(t)h_p(t) + \sum_{q=1}^{2} d_{pq}(t)l_q\big(h_q(t-\eta(t))\big), \tag{4.3}$$

in which $p = 1, 2$, and select the network parameters:

$$\begin{aligned}
&f_q(y_q) = \frac{1}{4}\sin y_q^R + i\frac{1}{5}\sin y_q^I + j\frac{1}{4}\tanh y_q^J + k\frac{1}{8}\sin y_q^K,\\
&g_q(y_q) = \frac{1}{10}\tanh y_q^R + i\frac{1}{5}\big|y_q^I\big| + j\frac{1}{8}\sin y_q^J + k\frac{1}{5}\sin y_q^K,\\
&\sigma_{pq}(y_q) = \frac{1}{20}\big|y_q^R\big| + i\frac{1}{10}\sin y_q^I + j\frac{1}{5}\big|y_q^J\big| + k\frac{1}{25}\arctan y_q^K,\\
&l_q(h_q) = \frac{1}{10}\arctan h_q^R + i\frac{1}{5}\tanh h_q^I + j\frac{1}{20}\sin h_q^J + k\frac{1}{10}\big|h_q^K\big|,\\
&a_1(t) = 3 + |\sin(\sqrt{3}t)|, \quad a_2(t) = 6 - 2.5\cos(\sqrt{5}t), \quad \eta(t) = \frac{1}{2}|\cos t|,\\
&b_{11}(t) = b_{12}(t) = 0.2\sin(\sqrt{2}t) + i0.2\cos(3t) + j0.4\cos(\sqrt{3}t) + k0.3\cos(3t),\\
&b_{21}(t) = b_{22}(t) = 0.4\cos(3t) + i0.2\cos(3t) + j0.5\sin(\sqrt{2}t) + k0.5\sin(\sqrt{2}t),\\
&c_{11}(t) = c_{12}(t) = 0.1\cos(\sqrt{3}t) + i0.3\cos(2t) + j0.2\sin(2t) + k0.1\sin(3t),\\
&c_{21}(t) = c_{22}(t) = 0.2\sin t + i0.4\cos(3t) + j0.5\cos t + k0.3\sin(\sqrt{3}t),\\
&\theta_1(t) = 1 + |\sin(\sqrt{3}t)|, \quad \theta_2(t) = 2 - 0.5\sin(\sqrt{3}t),\\
&d_{11}(t) = d_{21}(t) = 0.5\sin(3t) + i0.2\cos t + j0.3\sin(\sqrt{3}t) + k0.2\cos(\sqrt{3}t),\\
&d_{12}(t) = d_{22}(t) = 0.3\cos t + i0.5\sin t + j0.4\sin(\sqrt{3}t) + 0.45k\cos t,\\
&U_1(t) = 0.25\cos(\sqrt{2}t) + i0.45\sin(3t) + j0.35\cos(\sqrt{2}t) + k0.35\sin(\sqrt{2}t),\\
&U_2(t) = 0.35\sin(\sqrt{3}t) + i0.45\cos(\sqrt{3}t) + j0.55\sin t + k0.25\cos(\sqrt{5}t).
\end{aligned}$$

In this case, by a relative simplify calculation, one has

$$\begin{aligned}
&\underline{a}_1 = 3, \quad \underline{a}_2 = 4, \quad \underline{\theta}_1 = 1, \quad \underline{\theta}_2 = 1.5, \quad \eta(t) \le \frac{1}{2},\\
&L_q^f \le 0.425, \quad L_q^g \le 0.325, \quad L_{pq}^\sigma \le 0.2326, \quad L_q^l \le 0.25,\\
&\bar{b}_{11} = \bar{b}_{12} \le 0.5745, \quad \bar{b}_{21} = \bar{b}_{22} \le 0.8367,\\
&\bar{c}_{11} = \bar{c}_{12} \le 0.3873, \quad \bar{c}_{21} = \bar{c}_{22} \le 0.7348,\\
&\bar{d}_{11} = \bar{d}_{21} \le 0.6481, \quad \bar{d}_{12} = \bar{d}_{22} \le 0.8124,\\
&\bar{U}_1 \le 0.6164, \quad \bar{U}_2 \le 0.7348.
\end{aligned}$$

Then we have

$$\rho = \min_{p=1,2}\left\{2\underline{a}_p + 2\underline{\theta}_p - 7 - \sum_{q=1}^{2}\left(\bar{b}_{qp}L_q^f\right)^2\right\} \approx 0.8139 > 0,$$

$$\varrho = \max_{p=1,2}\left\{\sum_{q=1}^{2}\left[\left(\bar{c}_{qp}L_q^g\right)^2 + \left(\bar{d}_{qp}L_q^l\right)^2 + \left(L_{qp}^\sigma\right)^2\right]\right\} \approx 0.2486 > 0$$

and $\rho - \varrho = 0.5653 > 0$.

Here, choose $\lambda = 0.2 + 0.3i + 0.1j + 0.4k$, then

$$\vartheta = \max_{p=1,2} \Big\{ (1-\lambda)^*(1-\lambda)\big(\bar{U}_p\big)^2 + \sum_{q=1}^{n} 2\big[1+\lambda^*\lambda\big] \times \Big[\big(\bar{b}_{pq}L_q^f\big)^2 + \big(\bar{c}_{pq}L_q^g\big)^2 + \big(\bar{d}_{pq}L_q^l\big)^2 + \big(L_{pq}^{\sigma}\big)^2 \Big] \Big\} \approx 2.6729 > 0$$

and $\xi \approx 0.4954 > 0$ is the unique solution of algebra equation $\rho - \varrho e^{\xi\eta} - \xi = 0$. So, the conditions of Theorem 3.1 all holds. Therefore, according to Theorem 3.1, networks (4.1) and (4.2) is quasi-projectively synchronization by convergence region

$$W = \Big\{ h(t) \in \mathbb{Q}^m \big| \mathrm{E}\|h(t)\|_0^2 \leq \frac{2.6729}{0.4954} \approx 5.3954 \Big\},$$

which is verified by Fig. 6. Figs. 1 and 2 show the phase trajectories of four parts of network (4.1) with initial condition $(y_1^R(0), y_2^R(0))^T = (0.19, -0.19)^T$, $(0.15, -0.05)^T$, $(y_1^I(0), y_2^I(0))^T = (-0.25, 0.35)^T$, $(0.15, -0.15)^T$, $(y_1^J(0), y_2^J(0))^T = (-0.05, -0.25)^T$, $(0.35, 0.15)^T$, $(y_1^K(0), y_2^K(0))^T = (0.35, -0.15)^T$, $(-0.35, 0.05)^T$. Figs. 3 and 4 show the phase trajectories of four parts of netwrok (4.2) with initial condition $(z_1^R(0), z_2^R(0))^T = (0.25, -0.25)^T$, $(0.15, -0.05)^T$, $(z_1^I(0), z_2^I(0))^T = (-0.25, 0.35)^T$, $(0.15, -0.15)^T$, $(z_1^J(0), z_2^J(0))^T = (-0.05, -0.25)^T$, $(0.35, 0.15)^T$, $(z_1^K(0), z_2^K(0))^T = (0.35, -0.15)^T$, $(-0.35, 0.05)^T$. Fig. 5 illustrates the progression of synchronization error without a controller, demonstrating the failure to achieve quasi-projection synchronization.

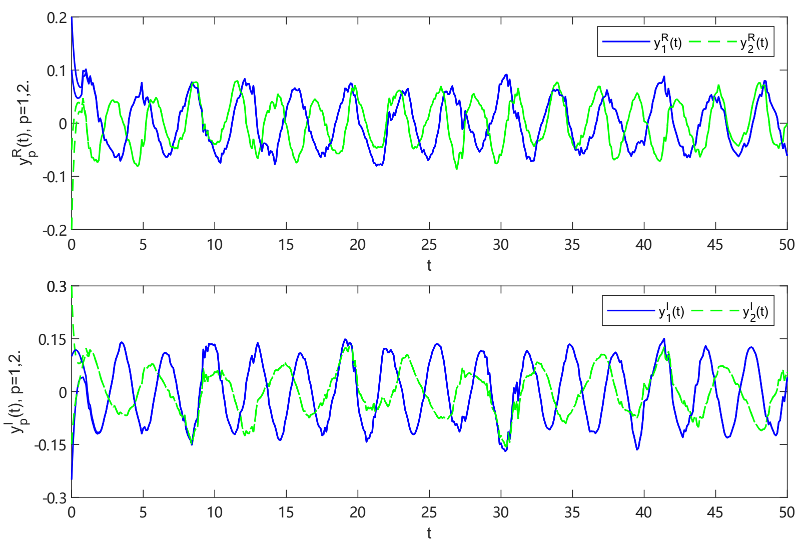

Figure 1. The trajectories of drive system states y_p^R and y_p^I, $p = 1, 2$.

5. Conclusions and future works

In this article, the quasi-projective synchronization in QVSNNs, with the consideration of time delays and a state-feedback control scheme, has been examined. By

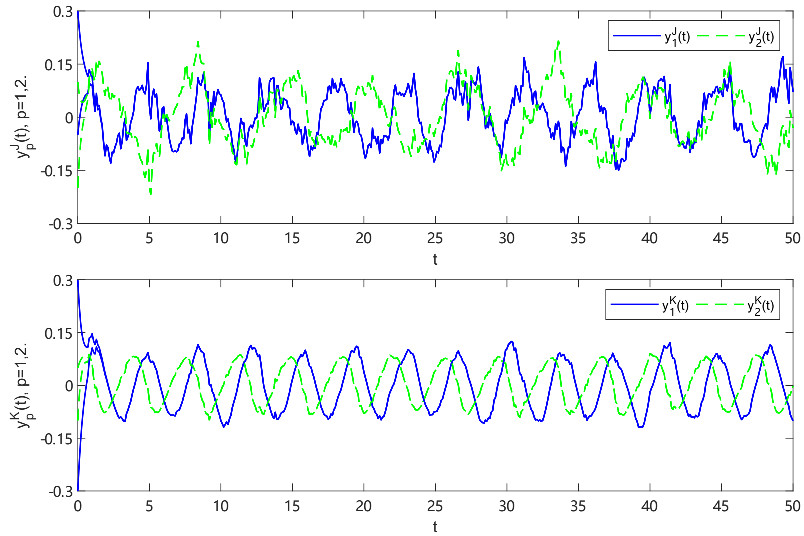

Figure 2. The trajectories of drive system states y_p^J and y_p^K, $p = 1, 2$.

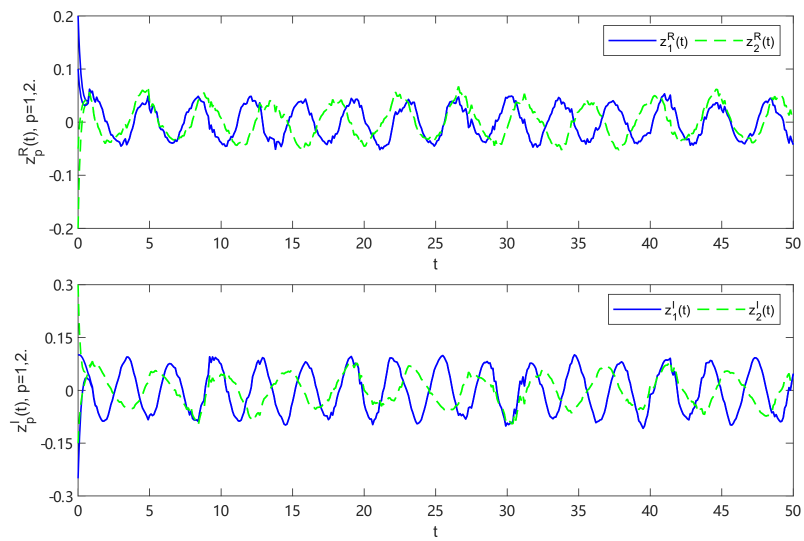

Figure 3. The trajectories of response system states z_p^R and z_p^I, $p = 1, 2$.

employing the quaternionic form of the Itô formula, along with several inequality techniques within the quaternionic range framework, a criterion for quasi-projective synchronization in the error network has been established. Finally, we provide an example and conduct computer simulations to demonstrate the practicality and validity of our research results. These results are not only applicable for addressing quasi-projective synchronization in QVNNs with time-varying delays but also provide an enhancement over several previous outcomes. The state-feedback control methods are still valid for designing the quasi-projective synchronization of Clifford-

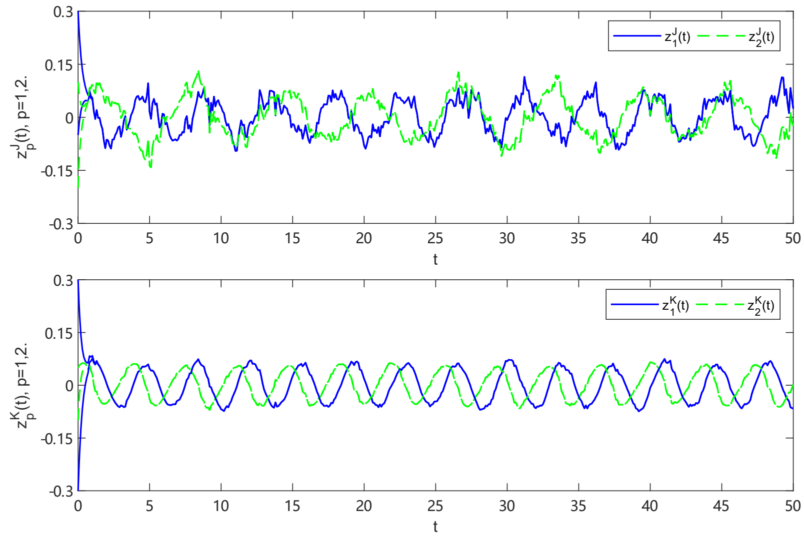

Figure 4. The trajectories of response system states z_p^J and z_p^K, $p = 1, 2$.

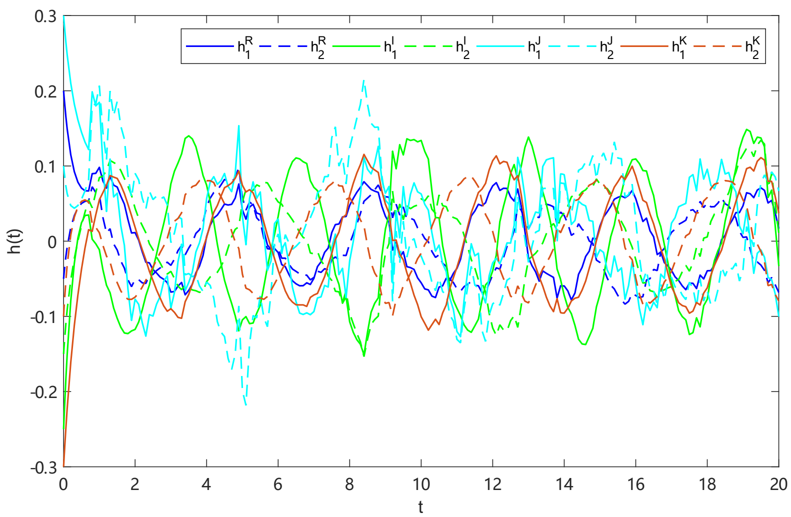

Figure 5. The evolution curves of quasi-projective synchronization error $h(t)$ without controller $E_p(t)$.

valued neural networks with time delays. This is something we will continue to explore and study in our future work, and we will concentrate on addressing quasi-projective synchronization challenges in delayed fractional-order stochastic QVNNs, considering a state-feedback control scheme, and exploring alternative synchronization control techniques. Further investigations of quasi-projective synchronization in the real applications of image recognition.

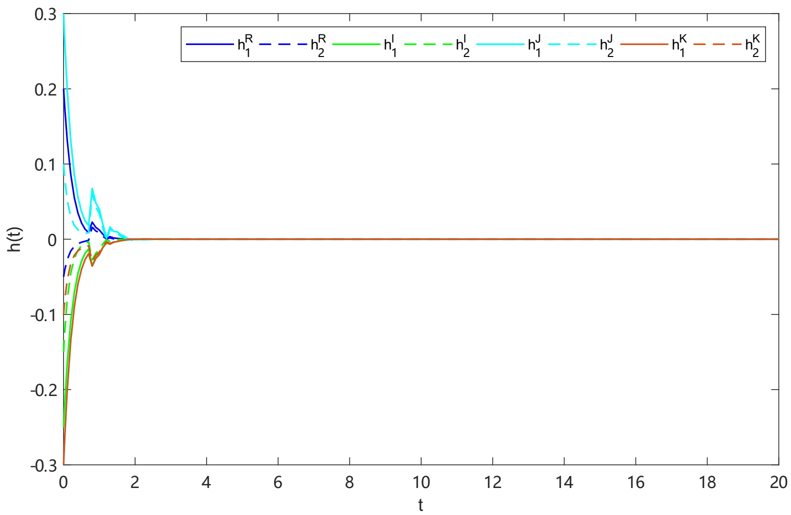

Figure 6. The evolution curves of quasi-projective synchronization error $h(t)$ with state-feedback controller $E_p(t)$.

Acknowledgements

The authors are grateful to the editor and anonymous referees for their useful comments and suggestions.

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