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BOUNDARY VALUE PROBLEMS FOR AN ITERATIVE DIFFERENTIAL EQUATION*

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Abstract This paper discusses the solutions of an iterative differential equation under general boundary value conditions. Using an auxiliary integral equation without the help of Green's functions usually being constructed in higher order equations, we prove the existence and uniqueness of solutions by the fixed point theorems of Schauder and Banach, respectively. Our theorems generalize and revise the related results.

Keywords Existence, uniqueness, iterate, differential equation, fixed point theorems.

MSC(2010) 34B15, 34K10, 39B05.

1. Introduction

Let X be a Banach space equipped with a norm $\|\cdot\|$, the n-th iterate of a self-mapping $f : X \to X$ is defined by $f^n(x) = f(f^{n-1}(x))$ and $f^0(x) = x$ for all $x \in X$ inductively. Iterative differential equations, a class of differential equations with state-dependent delays, have attracted considerable attention in the later 20th century (smoothness [11], convexity [12], analyticity [13,20], monotonicity [14], equivariance [17], periodicity [22] and so on).

Most of known results on initial value problems for those equations were related to the following equation or its special cases

$$x'(t) = f(t, x(t), x^2(t), ..., x^n(t)).$$

Among them, the works [3–5, 7, 10, 16] discussed on the initial value condition

$$x(t_0) = t_0,$$

called Return condition (the "fixed point" condition), and [1, 6, 18, 19] considered the non-Return condition

$$x(t_0) = x_0$$

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and proved the existence of nonlocal solutions. There are several results on higher-order iterative differential equations (periodicity [2, 8], Hyers-Ulam-Rassias stability [15]). It is worth mentioning that Green's function is an important tool for establishing the equivalent integral equations and localizing the solutions for higher-order differential equations.

Using Green's function, in 2018 E. R. Kaufmann [9] investigated the second order equation involving 2-th iterates

$$x^{''}(t) = f(t, x(t), x^2(t)), \quad t \in [a, b]$$

associated with the boundary value conditions

$$x(a) = a, \quad x(b) = b \quad \text{or} \quad x(a) = b, \quad x(b) = a.$$

He gave the sufficient conditions for the existence and uniqueness of solutions by fixed point theorems. However, no results on the general boundary value condition

$$x(a),\ x(b) \in (a, b)$$

was given yet in [9]. In this paper we study the equation involving n-th iterates, together with the general boundary value condition, that is, the two-point boundary value problems

$$\begin{cases} x^{''}(t) = f(t, x(t), x^2(t), ..., x^n(t)), & t \in [a, b], \\ x(a) = x_1, \quad x(b) = x_2 \end{cases} \tag{1.1}$$

is discussed, where the different points $x_1,\ x_2 \in [a, b]$. Using an auxiliary integral equation without the help of Green's functions, we prove the existence and uniqueness of solutions by the fixed point theorems of Schauder and Banach, respectively. Our theorems generalize and revise the related results.

2. Lemmas

Let $C([a, b], \mathbb{R})$ be the Banach space of all C^0 mappings from a closed interval $[a, b]$ into $\mathbb{R}$ equipped with the norm $\|\cdot\|$, defined by

$$\|x\| = \max_{t\in[a,b]} |x(t)|.$$

We have the following lemmas.

Lemma 2.1. ([21]) *Suppose that $x,\ y : [a, b] \to [a, b]$ are two C^0 mappings and*

$$|x(t_1) - x(t_2)| \le M|t_1 - t_2|$$

for $M > 0$. Then

$$\|x^j - y^j\| \le \sum_{i=0}^{j-1} M^i \|x - y\|,\ j = 1, 2, ...$$

Lemma 2.2. *The boundary value problem (1.1) is equivalent to the C^0 solution of the integral equation*

$$\begin{aligned} x(t) &= \frac{bx_1 - ax_2}{b-a} + \frac{a}{b-a} \cdot \int_a^b (b-s) f(s, x(s), x^2(s), ..., x^n(s)) ds \\ &+ \frac{x_2 - x_1}{b-a} \cdot t - \frac{t}{b-a} \cdot \int_a^b (b-s) f(s, x(s), x^2(s), ..., x^n(s)) ds \\ &+ \int_a^t (t-s) f(s, x(s), x^2(s), ..., x^n(s)) ds. \end{aligned} \tag{2.1}$$

Proof. By integrating the equation (1.1) twice, together with $x(a) = x_1$ we have

$$x(t) = x_1 + x^{'}(a)(t-a) + \int_a^t (t-s) f(s, x(s), x^2(s), ..., x^n(s)) ds. \tag{2.2}$$

Note that $x(b) = x_2$, from (2.2) we get

$$x^{'}(a) = \frac{x_2 - x_1}{b-a} - \frac{1}{b-a} \cdot \int_a^b (b-s) f(s, x(s), x^2(s), ..., x^n(s)) ds. \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$\begin{aligned} x(t) &= x_1 + \left[\frac{x_2 - x_1}{b-a} - \frac{1}{b-a} \cdot \int_a^b (b-s) f(s, x(s), x^2(s), ..., x^n(s)) ds \right] (t-a) \\ &+ \int_a^t (t-s) f(s, x(s), x^2(s), ..., x^n(s)) ds \\ &= \frac{bx_1 - ax_2}{b-a} + \frac{a}{b-a} \cdot \int_a^b (b-s) f(s, x(s), x^2(s), ..., x^n(s)) ds \\ &+ \frac{x_2 - x_1}{b-a} \cdot t - \frac{t}{b-a} \cdot \int_a^b (b-s) f(s, x(s), x^2(s), ..., x^n(s)) ds \\ &+ \int_a^t (t-s) f(s, x(s), x^2(s), ..., x^n(s)) ds. \end{aligned}$$

This proves (2.1).

Conversely, from (2.1) we get $x(a) = x_1,\ x(b) = x_2$ and the derivation

$$\begin{aligned} x^{'}(t) &= \frac{x_2 - x_1}{b-a} - \frac{1}{b-a} \cdot \int_a^b (b-s) f(s, x(s), x^2(s), ..., x^n(s)) ds \\ &+ \int_a^t f(s, x(s), x^2(s), ..., x^n(s)) ds \\ &= \frac{x_2 - x_1}{b-a} + \frac{1}{b-a} \cdot \int_a^t (s-a) f(s, x(s), x^2(s), ..., x^n(s)) ds \\ &- \frac{1}{b-a} \int_t^b (b-s) f(s, x(s), x^2(s), ..., x^n(s)) ds, \end{aligned}$$

then

$$x^{''}(t) = f(t, x(t), x^2(t), ..., x^n(t))$$

and the proof is completed. $\square$

3. Boundary value problems (1.1)

Theorem 3.1. *Suppose that $f : [a,b]^{n+1} \to \mathbb{R}$ is a C^0 function and satisfies*

$$|f(t,u_1,...,u_n) - f(t,v_1,...,v_n)| \le \sum_{i=1}^{n} L_i |u_i - v_i|$$

for some constants $L_i \ge 0$. If

$$L < \frac{|x_2 - x_1|}{(b-a)^2}, \tag{3.1}$$

where

$$L := \|f\|_{[a,b]^{n+1}} = \max_{(u_0,u_1,...,u_n)\in[a,b]^{n+1}} |f(u_0,u_1,...,u_n)|,$$

then the boundary value problem (1.1) has at least a solution $x(t)$ on $[a,b]$.

Proof. Consider the set

$$\begin{aligned}\Theta := \ &\{x \in C([a,b],\ [a,b]) :\ x(a) = x_1, \quad x(b) = x_2,\\ &|x(t_1) - x(t_2)| \le M|t_1 - t_2|,\ t_1,\ t_2 \in [a,b]\},\end{aligned}$$

where

$$M := \frac{|x_2 - x_1|}{b-a} + 2L\cdot(b-a). \tag{3.2}$$

Next, we apply the Schauder's fixed point theorem to prove the existence of a C^0 solution $x \in \Theta$ of (2.1).

Define the operator $\mathcal{T}:\ [a,b] \to \mathbb{R}$ by

$$\begin{aligned}(\mathcal{T}x)(t) \ :=\ & \frac{bx_1 - ax_2}{b-a} + \frac{a}{b-a}\cdot\int_a^b (b-s)f(s,x(s),x^2(s),...,x^n(s))ds\\ &+\frac{x_2 - x_1}{b-a}\cdot t - \frac{t}{b-a}\cdot\int_a^b (b-s)f(s,x(s),x^2(s),...,x^n(s))ds\\ &+\int_a^t (t-s)f(s,x(s),x^2(s),...,x^n(s))ds.\end{aligned}$$

We first show that $\mathcal{T}x \in \Theta$ for any $x \in \Theta$. In fact, one can easily check that

$$(\mathcal{T}x)(a) = x_1, \quad (\mathcal{T}x)(b) = x_2 \tag{3.3}$$

and

$$\begin{aligned}(\mathcal{T}x)^{'}(t) \ =\ & \frac{x_2 - x_1}{b-a} + \frac{1}{b-a}\cdot\int_a^t (s-a)f(s,x(s),x^2(s),...,x^n(s))ds\\ &-\frac{1}{b-a}\int_t^b (b-s)f(s,x(s),x^2(s),...,x^n(s))ds.\end{aligned}$$

Since

$$(\mathcal{T}x)^{'}(t) \ge \frac{x_2 - x_1}{b-a} - \frac{L}{b-a}\cdot\frac{(b-a)^2}{2} - \frac{L}{b-a}\cdot\frac{(b-a)^2}{2}$$

$$
\begin{aligned}
&= \frac{x_2 - x_1}{b-a} - (b-a)\cdot L \\
&> 0 \qquad (3.4)
\end{aligned}
$$

for $x_2 > x_1$ and

$$
\begin{aligned}
(\mathcal{T}x)'(t) &\le \frac{x_2 - x_1}{b-a} + \frac{L}{b-a}\cdot\frac{(b-a)^2}{2} + \frac{L}{b-a}\cdot\frac{(b-a)^2}{2} \\
&= \frac{x_2 - x_1}{b-a} + (b-a)\cdot L \\
&< 0 \qquad (3.5)
\end{aligned}
$$

for $x_2 < x_1$, it follows from (3.3), (3.4) and (3.5) that

$$\mathcal{T}x:\ [a,b] \to [a,b].$$

Moreover, for any $t_1, t_2 \in [a,b]$, by calculation we have

$$
\begin{aligned}
&|(\mathcal{T}x)(t_1) - (\mathcal{T}x)(t_2)| \\
&= \bigg| \frac{x_2 - x_1}{b-a}\cdot(t_1 - t_2) - \frac{t_1 - t_2}{b-a}\cdot \int_a^b (b-s) f(s, x(s), x^2(s), ..., x^m(s))ds \\
&\quad + \int_a^{t_1} (t_1 - s) f(s, x(s), x^2(s), ..., x^m(s))ds \\
&\quad - \int_a^{t_2} (t_2 - s) f(s, x(s), x^2(s), ..., x^m(s))ds \bigg| \\
&\le \frac{|x_2 - x_1|}{b-a}\cdot|t_1 - t_2| + \frac{L}{b-a}\cdot\frac{(b-a)^2}{2}\cdot|t_1 - t_2| \\
&\quad + \left| \int_{t_2}^{t_1} (t_1 - s) f(s, x(s), x^2(s), ..., x^m(s))ds \right| \\
&\quad + \left| \int_a^{t_2} (t_1 - t_2) f(s, x(s), x^2(s), ..., x^m(s))ds \right| \\
&\le \frac{|x_2 - x_1|}{b-a}\cdot|t_1 - t_2| + L\cdot\frac{b-a}{2}\cdot|t_1 - t_2| + L\cdot\frac{(t_1 - t_2)^2}{2} + L\cdot|t_1 - t_2|\cdot|t_2 - a| \\
&\le \frac{|x_2 - x_1|}{b-a}\cdot|t_1 - t_2| + L\cdot\frac{b-a}{2}\cdot|t_1 - t_2| + L\cdot\frac{b-a}{2}\cdot|t_1 - t_2| \\
&\quad + L\cdot(b-a)\cdot|t_1 - t_2| \\
&= \left(\frac{|x_2 - x_1|}{b-a} + 2L\cdot(b-a) \right)\cdot|t_1 - t_2| \\
&= M\cdot|t_1 - t_2|.
\end{aligned}
$$

Those relations imply that $\mathcal{T}x \in \Theta$, i.e., $\mathcal{T}$ is a self-mapping.

For any $x_1,\ x_2 \in \Theta$, using Lemma 2.1 we have

$$
\begin{aligned}
&\|\mathcal{T}x_1 - \mathcal{T}x_2\| \\
&= \max_{t\in[a,b]} \bigg| \frac{a-t}{b-a}\cdot \int_a^b (b-s)(f(s, x_1(s), x_1^2(s), ..., x_1^n(s) \\
&\quad - f(s, x_2(s), x_2^2(s), ..., x_2^n(s))ds
\end{aligned}
$$

$$
\begin{aligned}
& + \int_a^t (t-s)(f(s,x_1(s),x_1^2(s),...,x_1^n(s) - f(s,x_2(s),x_2^2(s),...,x_2^n(s))ds\Big| \\
\leq\ & \max_{t\in[a,b]} \left| \int_a^b (b-s)(f(s,x_1(s),x_1^2(s),...,x_1^n(s) - f(s,x_2(s),x_2^2(s),...,x_2^n(s))ds \right| \\
& + \max_{t\in[a,b]} \left| \int_a^t (t-s)(f(s,x_1(s),x_1^2(s),...,x_1^n(s) - f(s,x_2(s),x_2^2(s),...,x_2^n(s))ds \right| \\
\leq\ & (b-a)^2 \cdot \max_{t\in[a,b]} \left| f(t,x_1(t),x_1^2(t),...,x_1^n(t) - f(t,x_2(t),x_2^2(t),...,x_2^n(t)) \right| \\
\leq\ & (b-a)^2 \cdot \sum_{i=1}^n L_i \cdot (\sum_{j=0}^{i-1} M^j) \cdot \|x_1 - x_2\|, \qquad (3.6)
\end{aligned}
$$

implying $\mathcal{T}$ is continuous.

In view that

$$\|x\| \leq \max\{|a|,|b|\} \quad \text{and} \quad |x(t_1) - x(t_2)| \leq M|t_1 - t_2|,$$

Θ is uniformly bounded and equicontinuous and is relatively compact by the Arzelà-Ascoli theorem.

Therefore, Θ is a closed, convex and relatively compact subset of the Banach space $C([a,b],\mathbb{R})$ and $\mathcal{T}$ is a continuous operator. By Schauder's fixed point theorem $\mathcal{T}$ has a fixed point $x \in \Theta$, which is a solution $x \in \Theta$ of (1.1) and the proof is completed. □

Remark 3.1. Let $n = 2$ and $x_1 = a,\ x_2 = b$ (resp. $x_1 = b,\ x_2 = a$), our Theorem 3.1 is reduced to Theorem 3.1 (resp. Theorem 3.2) of [9], in which (3.1) is reduced to the main condition $(H1)$ in [9]. Thus, their results are improved in our Theorem 3.1.

Theorem 3.2. *Suppose that all conditions of Theorem 3.1 hold. If*

$$(b-a)^2 \cdot \sum_{i=1}^n L_i \cdot (\sum_{j=0}^{i-1} M^j) < 1, \qquad (3.7)$$

where M is defined by (3.2). Then the boundary value problem (1.1) has a unique solution $x(t)$ on $[a,b]$.

Proof. It is known from (3.6) and (3.7) that $\mathcal{T}$ is a contractive operator, and the remainder is same as that of Theorem 3.1. Then, the problem (1.1) has a unique solution $x \in \Theta$ from Banach fixed point theorem and the proof is completed. □

Remark 3.2. From the assumption

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \leq L_1|u_1 - u_2| + L_2|v_1 - v_2|,$$

we can get

$$
\begin{aligned}
& |f(s,x_1(s),{x_1}^2(s)) - f(s,x_2(s),{x_2}^2(s))| \\
\leq\ & L_1|x_1(s) - x_2(s)| + L_2|{x_1}^2(s) - {x_2}^2(s)| \\
\leq\ & L_1|x_1(s) - x_2(s)| + L_2(|{x_1}^2(s) - x_1 \circ x_2(s)| + |x_1 \circ x_2(s) - {x_2}^2(s)|) \\
\leq\ & L_1\|x_1 - x_2\| + L_2|{x_1}^2(s) - x_1 \circ x_2(s)| + L_2\|x_1 - x_2\|
\end{aligned}
$$

$$\begin{aligned}
&\le L_1\|x_1 - x_2\| + L_2\|{x_1}^2 - x_1 \circ x_2\| + L_2\|x_1 - x_2\| \\
&\le L_1\|x_1 - x_2\| + L_1L_2\|x_1 - x_2\| + L_2\|x_1 - x_2\| \\
&= L_1\|x_1 - x_2\| + (L_1 + 1)L_2\|x_1 - x_2\|,
\end{aligned}$$

but not

$$\begin{aligned}
&|f(s, x_1(s), x_1^2(s)) - f(s, x_2(s), x_2^2(s))| \\
\le\ & L_1|x_1(s) - x_2(s)| + L_2|{x_1}^2(s) - {x_2}^2(s)| \\
\le\ & L_1\|x_1 - x_2\| + L_2\|x_1 - x_2\|
\end{aligned}$$

appeared in [9, 15]. Hence, their condition

$$\frac{1}{6}(L_1 + L_2)(b - a)^2 < 1$$

should be replaced by (3.7) with $n = 2$.

4. Examples

We give examples to illustrate our main results.

Example 4.1. Consider the problem

$$\begin{cases} x^{''}(t) = kcos(c_1x(t) + c_2x^2(t) + c_3x^3(t)), \\ x(0) = x_1, \quad x(\pi) = x_2, \end{cases} \tag{4.1}$$

where k, c_i $(i = 1, 2, 3) \in \mathbb{R}$ and $x_1, x_2 \in [0, \pi]$.

For

$$f(t, x(t), x^2(t), x^3(t)) := kcos(c_1x(t) + c_2x^2(t) + c_3x^3(t)),$$

note that $\|f\| \le |k|$ and

$$\begin{aligned}
&|f(t, x_1(t), x_2(t), x_3(t)) - f(t, y_1(t), y_2(t), y_3(t))| \\
\le\ & |c_1k| \cdot |x_1(t) - y_1(t)| + |c_2k| \cdot |x_2(t) - y_2(t)| + |c_3k| \cdot |x_3(t) - y_3(t)|,
\end{aligned}$$

we say from Theorem 3.1 that the problem (4.1) has at least a solution $x(t)$ defined on $[0, \pi]$ if

$$k < \frac{|x_2 - x_1|}{\pi^2}.$$

Clearly, when

$$x_1 = 0,\ x_2 = \pi,\ c_1 = 0,\ c_2 = 1,\ c_3 = 0, \tag{4.2}$$

then example 4.1 is reduced to Example 3.3 of [9], i.e.,

$$\begin{cases} x^{''}(t) = kcos(x^2(t)), \quad t \in [0, \pi], \\ x(0) = 0,\ x(\pi) = \pi. \end{cases} \tag{4.3}$$

Example 4.2. Consider the problem (4.1) again.

From the left side of (3.7) we have

$$
\begin{aligned}
&(b-a)^2 \cdot \sum_{i=1}^{3} L_i \cdot (\sum_{j=0}^{i-1} M^j) \\
&= (b-a)^2 \cdot (L_1 + L_2(1+M) + L_3(1+M+M^2)) \\
&= \pi^2 \cdot (|c_1 k| + |c_2 k|(1+M) + |c_3 k|(1+M+M^2)) \\
&= \pi^2 \cdot |k| \cdot \sum_{i=1}^{3} |c_i| (\sum_{j=0}^{i-1} M^j),
\end{aligned}
$$

where

$$
M := \frac{|x_2 - x_1|}{b-a} + 2L \cdot (b-a) = \frac{|x_2 - x_1|}{\pi} + 2|k| \cdot \pi = \frac{|x_2 - x_1| + 2|k|\pi^2}{\pi}.
$$

By using Theorem 3.2, we know that the problem (4.1) has a unique solution $x(t)$ on $[0, \pi]$ if

$$
\pi^2 \cdot |k| \cdot \sum_{i=1}^{3} |c_i| (\sum_{j=0}^{i-1} M^j) < 1. \tag{4.4}
$$

For instance, choose

$$
k = -\frac{1}{10},\ c_1 = -c_2 = c_3 = \frac{1}{3\pi^2},\ x_1 = 0,\ x_2 = \pi,
$$

note that

$$
\begin{aligned}
&\pi^2 \cdot |k| \cdot \sum_{i=1}^{3} |c_i| (\sum_{j=0}^{i-1} M^j) \\
&= \frac{1}{10}[1 + \pi \cdot \frac{2}{3\pi^2}(\pi + 2 \cdot \frac{1}{10} \cdot \pi^2) + \frac{1}{3\pi^2}(\pi + 2 \cdot \frac{1}{10} \cdot \pi^2)^2] \\
&= \frac{1}{10}[1 + \cdot \frac{2}{3}(1 + \frac{\pi}{5}) + \frac{1}{3}(1 + \frac{\pi}{5})^2] \\
&< \frac{1}{10}[1 + \cdot \frac{2}{3} \cdot 2 + \frac{1}{3} \cdot 4] \\
&= \frac{11}{30} \\
&< 1,
\end{aligned}
$$

so the problem

$$
\begin{cases}
x''(t) = -\frac{1}{10} \cos \frac{1}{3\pi^2}(x(t) - x^2(t) + x^3(t)), \\
x(0) = 0, \quad x(\pi) = \pi
\end{cases}
$$

has a unique solution $x(t)$ on $[0, \pi]$.

Next, we choose the data (4.2). Since

$$
\pi^2 \cdot |k| \cdot \sum_{i=1}^{3} |c_i| (\sum_{j=0}^{i-1} M^j) = 2|k|\pi^2(1 + |k|\pi) < 1
$$

if

$$|k| < \frac{-\pi + \sqrt{\pi^2 + 2\pi}}{2\pi^2}. \tag{4.5}$$

Thus, the problem (4.3) has a unique solution $x(t)$ on $[0, \pi]$, which is the Example 3.6 in [9] and our condition (4.5) replaces the main condition

$$|k| < \frac{6}{\pi^2}.$$

Applying our Theorem 3.1 and Theorem 3.2, the examples in [15] can also be easily settled.

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