Journal of Applied Analysis and Computation Website:http://www.jaac-online.com
Volume 14, Number 5, October 2024, 2523–2533 DOI:10.11948/20220549

$\bar{\partial}$-DRESSING METHOD FOR THREE-COMPONENT COUPLED NONLINEAR SCHRÖDINGER EQUATIONS*

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Abstract The dressing method based on 4×4 matrix $\bar{\partial}$-problem is extended to study the three-component coupled nonlinear Schrödinger (3DNLS) equations. The spatial and time spectral problems related to the 3DNLS equations are derived via two linear constraint equations. A 3DNLS hierarchy with source is proposed by using recursive operator. The N-solitions of the 3DNLS equations are given based on the $\bar{\partial}$-equation by selecting a spectral transformation matrix.

Keywords Three-component coupled nonlinear Schrödinger equations, $\bar{\partial}$-dressing method, lax pair, soliton solution.

MSC(2010) 37K15, 35C08.

1. Introduction

The study of multi-component nonlinear systems has attracted more and more attention, since they can describe a variety of complex physical phenomena and have richer dynamical behaviors than low-component systems. Among the various solutions of these models, the soliton plays a crucial role in illustrating some related phenomena. In recent years, many methods for solving soliton solutions have been proposed, including inverse scattering transformation (IST) [2], Darboux transformation (DT) [23], Hirota bilinear method [15, 21, 27, 28], $\bar{\partial}$-dressing method, etc. The $\bar{\partial}$-dressing method, as an extension of IST, is based on the inverse scattering theory and Lax framework. It is a powerful tool for constructing and solving integrable nonlinear equations and describing their transformations and reductions. It was first proposed by Zakharov and Shabat [35], and further developed by Beals, Coifman, Ablowitz, ManBakov, Fokas et al. [1, 4, 5, 12, 22]. So far, a large number of integrable equations have been successfully studied by the $\bar{\partial}$-dressing method [6, 7, 9–11, 13, 16–20, 24, 33, 34, 39].

The coupled nonlinear Schrödinger (NLS) equations have been widely used in nonlinear optics, deep ocean, Bose-Einstein (BE) condensation and other fields

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[3, 8, 14, 29–31, 38]. Therefore, this paper mainly considers the three-component coupled nonlinear Schrödinger (3DNLS) equations [36]:

$$\begin{aligned} &iu_{1t}+\frac{1}{2}u_{1xx}+\sigma(|u_1|^2+|u_2|^2+|u_3|^2)u_1=0,\\ &iu_{2t}+\frac{1}{2}u_{2xx}+\sigma(|u_1|^2+|u_2|^2+|u_3|^2)u_2=0,\\ &iu_{3t}+\frac{1}{2}u_{3xx}+\sigma(|u_1|^2+|u_2|^2+|u_3|^2)u_3=0. \end{aligned} \tag{1.1}$$

Here, $u_j=u_j(x,t)j=1,2,3$ are the complex functions with the temporal variable t and spatial variable x, and $\sigma>0(<0)$ stands for the attractive (repulsive) interactions. There has been increasing interest in the study of the dynamic properties of system (1.1). For example, the vector soliton solution has been derived through the Horita bilinear method [25,26], the bright-bright solitons have been obtained by Darboux transformation (DT) method from a trivial seed solution with $u_3=0$ [37], the initial-boundary value problem has been investigated by extending the Fokas unified approach [32]. However, to our knowledge, there is still no research work on system (1.1) by using $\bar{\partial}$-dressing method. For convenience, we take $\sigma=1$ for the following analysis.

The layout of this paper is organized as follows. In Section 2, starting from the $\bar{\partial}$-equation, we propose a new Lax pair with singular dispersion relation for system (1.1) using the $\bar{\partial}$-dressing method. In Section 3, based on the relationship between $\bar{\partial}$-dressing transformation matrix and potential matrix, we derive a 3DNLS hierarchy with source. In Section 4, the N-soliton solutions formula of system (1.1) are constructed. Finally, the conclusions will be drawn based on the above sections.

2. Spectral problem and Lax pair

2.1. The spatial spectra problem

We consider the 4×4 matrix $\bar{\partial}$-problem in the complex k-plane,

$$\bar{\partial}\psi=\psi R, \tag{2.1}$$

with a boundary condition $\psi(x,t,k)\to I, k\to\infty$, then a solution of the equation (2.1) can be written as

$$\psi(k)=I+\frac{1}{2\pi i}\int\int\frac{\psi(z)R(z)}{z-k}dz\wedge d\bar{z}\equiv I+\psi RC_k, \tag{2.2}$$

where C_k denotes the Cauchy-Green integral operator acting on the left. The formal solution of $\bar{\partial}$-problem (2.1) will be given from (2.2) as

$$\psi(k)=I\cdot(I-RC_k)^{-1}. \tag{2.3}$$

For convenience, we define a pairing [9]

$$\langle f,g\rangle=\frac{1}{2\pi i}\int\int f(k)g^T(k)dk\wedge d\bar{k},\quad \langle f\rangle=\langle f,I\rangle=\frac{1}{2\pi i}\int\int f(k)dk\wedge d\bar{k},$$

which can be shown to possess the following properties

$$\langle f,g\rangle^T=\langle g,f\rangle,\quad \langle fR,g\rangle=\langle f,gR^T\rangle,\quad \langle fC_k,g\rangle=-\langle f,gC_k\rangle. \tag{2.4}$$

It is easy to prove that for some matrix functions $f(k)$ and $g(k)$, the operator C_k satisfies

$$g(k)[f(k)C_k]C_k + [g(k)C_k]f(k)C_k = [g(k)C_k][f(k)C_k]. \tag{2.5}$$

It is well known that the Lax pairs of nonlinear equations play an important role in the study of integrable systems. Such as Darboux transformation, inverse scattering transformation, Riemann-Hilbert method, algebro-geometric all depend on on their Lax pairs. Here we prove that if the transform matrix $R(x,t,k)$ satisfies a simple linear equation, the spatial-time spectral problems of system (1.1) can be established from (2.1).

Proposition 2.1. *Let the transform matrix R satisfies*

$$R_x = ik[J, R], \tag{2.6}$$

where $J = diag(1, -1, -1, -1)$, then the solution ψ of the $\bar{\partial}$-equation (2.1) *satisfies the following spatial spectral problem*

$$\psi_x - ik[J, \psi] = Q\psi, \tag{2.7}$$

where

$$Q = \begin{pmatrix} 0 & -u_1^* & -u_2^* & -u_3^* \\ u_1 & 0 & 0 & 0 \\ u_2 & 0 & 0 & 0 \\ u_3 & 0 & 0 & 0 \end{pmatrix} = i[J, \langle\psi R\rangle]. \tag{2.8}$$

Proof. Please state your proof here. Using (2.3) and (2.6), we get

$$\psi_x = ik\psi R\sigma_3 C_k(I - RC_k)^{-1} - ik\psi\sigma_3 RC_k(I - RC_k)^{-1}. \tag{2.9}$$

According to the definition of C_k, we can obtain

$$ik\psi RC_k = i\langle\psi R\rangle + ik(\psi - I). \tag{2.10}$$

Since $RC_k = I - I \cdot (I - RC_K)$, then we have

$$RC_k(I - RC_k)^{-1} = (I - RC_k)^{-1} - I. \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.9), we obtain

$$\psi_x = -i\langle\psi R\rangle J\psi + iJk(I - RC_k)^{-1} - ik\psi J. \tag{2.12}$$

From (2.10), we can get

$$k(I - RC_k)^{-1} = \langle\psi R\rangle\psi + k\psi. \tag{2.13}$$

Substituting (2.13) into (2.12), we have Eq.(2.7). □

2.2. The time spectral problem

Proposition 2.2. *Suppose that R satisfies the linear equation*

$$R_t = [R, \Omega], \tag{2.14}$$

where

$$\Omega = \Omega_p + \Omega_s = -ik^2 J + \frac{1}{2\pi i}\int\int \frac{\omega(\xi)J}{\xi - k} d\xi \wedge d\bar{\xi}, \tag{2.15}$$

which comprises both a polynomial part $\Omega_p(k)$ and a singular part $\Omega_s(k)$ and $\omega(\xi)$ is a scalar function. Then the solution ψ of the $\bar{\partial}$-equation (2.1) leads to time spectral problem

$$\psi_t - ik^2[J, \psi] = \frac{i}{2}J(Q^2 - Q_x)\psi + kQ. \tag{2.16}$$

Proof. We first use the polynomial dispersion relation only$\Omega = \Omega_p = -ik^2 J$. From equations (2.2), (2.3) and (2.15), we find that

$$\psi_t = -i[k^2\psi RC_k J(I - RC_k)^{-1} - k^2\psi J(I - RC_k)^{-1}] - ik^2\psi J. \tag{2.17}$$

Through the following direct computation,

$$\begin{aligned} k^2\psi RC_k &= \langle\zeta\psi R\rangle + k\langle\psi R\rangle + k^2(\psi - I), \\ k^2(I - RC_k)^{-1} &= (\langle\zeta\psi R\rangle + \langle\psi R\rangle^2 + k\langle\psi R\rangle + k^2)\psi, \end{aligned}$$

then (2.17) is changed to

$$\begin{aligned} \psi_t &= -i[\langle\zeta\psi R\rangle, J]\psi - i[\langle\psi R\rangle, J]\langle\psi R\rangle\psi - ik[\langle\psi R\rangle, J]\psi + ik^2[J, \psi] \\ &= 2iJ\langle\zeta\psi R\rangle^{off}\psi + Q\langle\psi R\rangle\psi + kQ\psi + ik^2[J, \psi]. \end{aligned} \tag{2.18}$$

By virtue of (2.6), (2.7) and (2.8), we have

$$\begin{aligned} &\langle\psi R\rangle_x = Q\langle\psi R\rangle + i[J, \langle k\psi R\rangle], \\ &\langle\psi R\rangle_x^{off} = Q\langle\psi R\rangle^{diag} + 2iJ\langle k\psi R\rangle^{off}, \\ &\langle k\psi R\rangle^{off} = \frac{i}{2}JQ\langle\psi R\rangle^{diag} - \frac{1}{4}Q_x, \\ &\langle\psi R\rangle - \langle\psi R\rangle^{diag} = J(J\langle\psi R\rangle - J\langle\psi R\rangle^{diag}) = \frac{J}{2}[J, \langle\psi R\rangle] = -\frac{i}{2}JQ. \end{aligned} \tag{2.19}$$

Substituting (2.19) into (2.18) leads to the time-dependent linear equation

$$\begin{aligned} \psi_t &= 2iJ(\frac{i}{2}JQ\langle\psi R\rangle^{diag} - \frac{1}{4}Q_x)\psi + Q\langle\psi R\rangle\psi + kQ\psi + ik^2[J, \psi] \\ &= Q[\langle\psi R\rangle - \langle\psi R\rangle^{diag}]\psi + (kQ - \frac{i}{2}JQ_x)\psi + ik^2[J, \psi] \\ &= \frac{i}{2}J(Q^2 - Q_x)\psi + kQ + ik^2[J, \psi]. \end{aligned} \tag{2.20}$$

□

3. Recursive operators and equation hierarchy

In this section, we derive 3DNLS equations with a source. First, we define the 4×4 matrix $M = \psi J \psi^{-1}$. By using the Eq.(2.8) and definition of M, we can prove the following proposition.

Proposition 3.1. *Q defined by* (2.8) *satisfies a coupled hierarchy with a source M*

$$\begin{aligned} &Q_t + 2\alpha_n J\Lambda^n Q = i[J, \langle \omega(k) M(k) \rangle], \quad n = 1, 2, \dots, \\ &M_x - ik[J, M] = [Q, M]. \end{aligned} \tag{3.1}$$

Proof. Differentiating Q with respect to t gives

$$Q_t = i[J, \langle \psi R \rangle_t]. \tag{3.2}$$

Because of $\bar{\partial} f(k) C_k = f(k)$, then we have

$$\begin{aligned} (\psi R)_t &= \bar{\partial}\psi_t(k) \\ &= \bar{\partial}[I \cdot (I - RC_k)_t^{-1}] \\ &= \bar{\partial}[\psi R_t (I - RC_k)^{-1}] C_k \\ &= \psi R_t (I - RC_k)^{-1}. \end{aligned} \tag{3.3}$$

By using the properties (3.3), we can obtain

$$Q_t = i[J, \langle \psi R_t (I - RC_k)^{-1}, I \rangle] = i[J, \langle \psi R_t, I \cdot (I + R^T C_k)^{-1} \rangle]. \tag{3.4}$$

From the $\bar{\partial}$-equation (2.1), we have

$$\bar{\partial}\psi^{-1} = -R\psi^{-1},$$

which leads to

$$(\psi^{-1})^T = I \cdot (I + R^T C_k)^{-1}.$$

Therefore, using (2.4) and (2.14), Eq. (3.4) can be simplified to

$$Q_t = i[J, \langle (\bar{\partial}\psi)\Omega\psi^{-1} + \psi\Omega\bar{\partial}\psi^{-1} \rangle]. \tag{3.5}$$

Here we shall consider $\Omega_p = \alpha_n k^n J, \alpha_n = const$ and the fact that $\Omega_s \to 0$ as $k \to \infty$, then the above equation can be further simplified like

$$\begin{aligned} Q_t &= i[J, \langle \psi\Omega\bar{\partial}\psi^{-1} \rangle] + i[J, \langle (\bar{\partial}\psi)\Omega\psi^{-1} \rangle] \\ &= i\alpha_n [J, \langle \bar{\partial}(k^n M(k)) \rangle] + i[J, \langle \omega(k) M(k) \rangle] \\ &= 2i\alpha_n J \langle \bar{\partial}(k^n M(k)^{off}) \rangle + i[J, \langle \omega(k) M(k) \rangle]. \end{aligned} \tag{3.6}$$

By using (2.7), it can be checked that M(k) satisfies the equation

$$M_x - ik[J, M] - [Q, M] = 0. \tag{3.7}$$

From Eq.(3.7), they satisfy the following equations

$$\begin{aligned} M_x^{diag} &= [Q, M^{off}], \\ M_x^{off} &= 2ikJM^{off} + [Q, M^{diag}], \end{aligned} \tag{3.8}$$

which lead to

$$\begin{aligned} M^{diag} &= J + \partial_x^{-1}[Q, M^{off}], \\ M^{off} &= i(\Lambda - k)^{-1}Q, \end{aligned} \tag{3.9}$$

where

$$\Lambda\cdot = -\frac{i}{2}J(\partial_x \cdot -[Q, \partial_x^{-1}[Q,\cdot]]).$$

The operator Λ usually be called as recursion operator. We expand $(\Lambda - k)^{-1}$ in the series

$$(\Lambda - k)^{-1} = -\sum_{j=1}^{\infty} k^{-j}\Lambda^{j-1}.$$

By using $\bar{\partial}k^{n-j} = \pi\delta(k)\delta_{j,n+1}, j = 1, 2, \ldots$, we can derive that

$$\sum_{j=1}^{\infty}\langle\bar{\partial}k^{n-j}\rangle\Lambda^{j-1}Q = -\Lambda^n Q.$$

Substituting it into (3.6) leads to the Eq.(3.1). □

4. N-soliton solutions of cmKdV equation

In this section, we will construct the N-soliton solutions of the syetem (1.1) still based on the $\bar{\partial}$-equation (2.1),we first introduce a spectral matrix R as

$$R = \pi\sum_{j=1}^{N}\begin{pmatrix} 0 & -a_je^{2i\theta(k)}\delta(k-\bar{k}_j) & -b_je^{2i\theta(k)}\delta(k-\bar{\tilde{k}}_j) & -c_je^{2i\theta(k)}\delta(k-\bar{\hat{k}}_j) \\ a_je^{2i\theta(k)}\delta(k-k_j) & 0 & 0 & 0 \\ b_je^{2i\theta(k)}\delta(k-\tilde{k}_j) & 0 & 0 & 0 \\ c_je^{2i\theta(k)}\delta(k-\hat{k}_j) & 0 & 0 & 0 \end{pmatrix} \tag{4.1}$$

where $k_j, \tilde{k}_j, \hat{k}_j, j = 1, 2, \ldots$ are constants distinct from each other, $\theta(k) = -kx - k^2t$. Substituting (4.1) into (2.8) leads to

$$\begin{aligned} u_1 &= -2i\langle\psi R\rangle_{21} \\ &= -\frac{1}{\pi}\int\int(\psi_{22}(\zeta)R_{21}(\zeta) + \psi_{23}(\zeta)R_{31}(\zeta) + \psi_{24}(\zeta)R_{41}(\zeta))d\zeta\wedge d\bar{\zeta} \\ &= -\sum_{j=1}^{N}(a_je^{2i\theta(k_j)}\psi_{22}(k_j) + b_je^{2i\theta(\tilde{k}_j)}\psi_{23}(\tilde{k}_j) + c_je^{2i\theta(\hat{k}_j)}\psi_{24}(\hat{k}_j)). \end{aligned} \tag{4.2}$$

Substituting (4.1) into $\bar{\partial}$-equation (2.1) and resorting the properties of ŠÄ function, we can obtain

$$\psi_{22}(k) = 1 + \frac{1}{2\pi i}\int\int \frac{\psi_{21}(\zeta)R_{12}(\zeta)}{\zeta - k}d\zeta\wedge d\bar{\zeta} = 1 - \sum_{p=1}^{N}\frac{\bar{a}_p e^{-2i\theta(\bar{k}_p)}}{k-\bar{k}_p}\psi_{21}(\bar{k}_p),$$

$$\psi_{23}(k) = \frac{1}{2\pi i}\int\int \frac{\psi_{21}(\zeta)R_{13}(\zeta)}{\zeta - k}d\zeta\wedge d\bar{\zeta} = -\sum_{l=1}^{N}\frac{\bar{b}_j e^{-2i\theta(\bar{\tilde{k}}_l)}}{k-\bar{\tilde{k}}_l}\psi_{21}(\bar{\tilde{k}}_l), \tag{4.3}$$

$$\psi_{24}(k) = \frac{1}{2\pi i}\int\int \frac{\psi_{21}(\zeta)R_{14}(\zeta)}{\zeta - k}d\zeta\wedge d\bar{\zeta} = -\sum_{m=1}^{N}\frac{\bar{c}_m e^{-2i\theta(\bar{\hat{k}}_m)}}{k-\bar{\hat{k}}_m}\psi_{21}(\bar{\hat{k}}_m),$$

then introducing notation A_p, B_l, C_m written as

$$A_p(k) = \frac{a_p}{k-k_p}e^{-2i\theta(k_p)},\quad B_l(k) = \frac{b_l}{k-\tilde{k}_l}e^{-2i\theta(\tilde{k}_l)},\quad C_m(k) = \frac{c_m}{k-\hat{k}_m}e^{-2i\theta(\hat{k}_m)}. \tag{4.4}$$

From(4.3), we have

$$\psi_{22}(k) = 1 - \sum_{p,j=1}^{N}\overline{A_p(\bar{k})}[A_j(\bar{k}_p)\psi_{22}(k_j) + B_j(\bar{k}_p)\psi_{23}(\tilde{k}_j) + C_j(\bar{k}_p)\psi_{22}(\hat{k}_j)],$$

$$\psi_{23}(k) = -\sum_{j,l=1}^{N}\overline{B_l(\bar{k})}[A_j(\bar{\tilde{k}}_l)\psi_{22}(k_j) + B_j(\bar{\tilde{k}}_l)\psi_{23}(\tilde{k}_j) + C_j(\bar{\tilde{k}}_l)\psi_{24}(\hat{k}_j)], \tag{4.5}$$

$$\psi_{24}(k) = -\sum_{j,m=1}^{N}\overline{C_m(\bar{k})}[A_j(\bar{\hat{k}}_m)\psi_{22}(k_j) + B_j(\bar{\hat{k}}_m)\psi_{23}(\tilde{k}_j) + C_j(\bar{\hat{k}}_m)\psi_{24}(\hat{k}_j)],$$

taking $z = z_j$, $z = \tilde{z}_j$ and $z = \hat{z}_j$ respectively, we have

$$\begin{aligned}
&(I+M)\psi_{22}(k) + N\psi_{23}(\tilde{k}) + P\psi_{24}(\hat{k}) = E,\\
&\widetilde{M}\psi_{22}(k) + (I+\widetilde{N})\psi_{23}(\tilde{k}) + \widetilde{P}\psi_{24}(\hat{k}) = 0,\\
&\hat{M}\psi_{22}(k) + \hat{N}\psi_{23}(\tilde{k}) + (I+\hat{P})\psi_{24}(\hat{k}) = 0,
\end{aligned} \tag{4.6}$$

where $E = (1,\cdots,1)^T$, and M, N, P are $N\times N$ matrix

$$M_{n,p} = \sum_{j=1}^{N}\overline{A_j(\bar{k}_n)}A_p(\bar{k}_j),\quad N_{n,p} = \sum_{j=1}^{N}\overline{A_j(\bar{k}_n)}B_p(\bar{k}_j),\quad P_{n,p} = \sum_{j=1}^{N}\overline{A_j(\bar{k}_n)}C_p(\bar{k}_j),$$

$$\widetilde{M}_{n,p} = \sum_{j=1}^{N}\overline{B_j(\bar{k}_n)}A_p(\bar{\tilde{k}}_j),\quad \widetilde{N}_{n,p} = \sum_{j=1}^{N}\overline{B_j(\bar{k}_n)}B_p(\bar{\tilde{k}}_j),\quad \widetilde{P}_{n,p} = \sum_{j=1}^{N}\overline{B_j(\bar{k}_n)}C_p(\bar{\tilde{k}}_j),$$

$$\widehat{M}_{n,p} = \sum_{j=1}^{N}\overline{C_j(\bar{k}_n)}A_p(\bar{\hat{k}}_j),\quad \widehat{N}_{n,p} = \sum_{j=1}^{N}\overline{C_j(\bar{k}_n)}B_p(\bar{\hat{k}}_j),\quad \widehat{P}_{n,p} = \sum_{j=1}^{N}\overline{C_j(\bar{k}_n)}C_p(\bar{\hat{k}}_j),$$

then we can solve $\psi_{24}(\hat{k}), \psi_{22}(k)$ and $\psi_{23}(\tilde{k})$

$$\begin{aligned}\psi_{24}(\hat{k}) &= (I+X_1)^{-1}Y_1,\\ \psi_{22}(k) &= (I+X_2)^{-1}Y_2,\\ \psi_{23}(\tilde{k}) &= (I+X_3)^{-1}Y_3,\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}X_1 &= [(I+M)^{-1}P - \widetilde{M}^{-1}P]^{-1}[(I+M)^{-1}N - \widetilde{M}^{-1}(I+\widetilde{N})]\\ &\quad\times[\widetilde{M}^{-1}(I+\widetilde{N}) - \widehat{M}^{-1}\widehat{N}]^{-1}[\widetilde{M}^{-1}\widetilde{P} - \widehat{M}^{-1}(I+\widehat{P})],\\ X_2 &= [N^{-1}(I+M) - (I+\widetilde{N})^{-1}\widetilde{M}]^{-1}[N^{-1}P - (I+\widetilde{N})^{-1}\widetilde{P}]\\ &\quad\times[(I+\widetilde{N})^{-1}\widetilde{P} - \widehat{N}^{-1}(I+\widehat{P})]^{-1}[(I+\widetilde{N})^{-1}\widetilde{M} - \widehat{N}^{-1}\widehat{M}],\\ X_3 &= [P^{-1}N - \widetilde{P}^{-1}(I+\widetilde{N})]^{-1}[P^{-1}(I+M) - \widetilde{P}^{-1}\widetilde{M}]\\ &\quad\times[(I+\widehat{P})^{-1}\widehat{M} - \widetilde{P}^{-1}\widetilde{M}]^{-1}[(I+\widehat{P})^{-1}\widehat{N} - \widetilde{P}^{-1}(I+\widetilde{N})],\\ Y_1 &= [(I+M)^{-1}P - \widetilde{M}^{-1}P]^{-1}(I+M^{-1})E,\\ Y_2 &= [N^{-1}(I+M) - (I+\widetilde{N})^{-1}\widetilde{M}]^{-1}N^{-1}E,\\ Y_3 &= [P^{-1}N - \widetilde{P}^{-1}(I+\widetilde{N})]P^{-1}E.\end{aligned}$$

Hence, the N-soliton solutions of the system (1.1) take the form

$$\begin{aligned}u_1 &= 2i(h_1\psi_{22}(k) + h_2\psi_{23}(\tilde{k}) + h_3\psi_{24}(\hat{k})) \qquad (4.8)\\ &= 2i[h_1(I+X_1)^{-1}Y_1 + h_2(I+X_2)^{-1}Y_2 + h_3(I+X_3)^{-1}Y_3]\\ &= 2i\mathrm{tr}[(I+X_1)^{-1}Y_1h_1 + (I+X_2)^{-1}Y_2h_2 + (I+X_3)^{-1}Y_3h_3]\\ &= 2i\left[\frac{\det(I+X_1+Y_1h_1)}{\det(I+X_1)} + \frac{\det(I+X_2+Y_2h_2)}{\det(I+X_2)} + \frac{\det(I+X_3+Y_3h_3)}{\det(I+X_3)} - 3\right],\end{aligned}$$

$$u_2 = 2i\left[\frac{\det(I+X_1+Z_1h_1)}{\det(I+X_1)} + \frac{\det(I+X_2+Z_2h_2)}{\det(I+X_2)} + \frac{\det(I+X_3+Z_3h_3)}{\det(I+X_3)} - 3\right], \tag{4.9}$$

$$u_3 = 2i\left[\frac{\det(I+X_1+L_1h_1)}{\det(I+X_1)} + \frac{\det(I+X_2+L_2h_2)}{\det(I+X_2)} + \frac{\det(I+X_3+L_3h_3)}{\det(I+X_3)} - 3\right], \tag{4.10}$$

where

$$\begin{aligned}h_1 &= (h_{1,1}, h_{1,2}, \cdots, h_{1,N}), \quad h_{1,j} = a_je^{2i\theta(k_j)},\\ h_2 &= (h_{2,1}, h_{2,2}, \cdots, h_{2,N}), \quad h_{2,j} = b_je^{2i\theta(\tilde{k}_j)},\\ h_3 &= (h_{3,1}, h_{3,2}, \cdots, h_{3,N}), \quad h_{3,j} = c_je^{2i\theta(\hat{k}_j)}.\end{aligned}$$

5. Conclusion

In this paper, we have presented the dressing method based on the $\bar{\partial}$-problem to study the three-component coupled nonlinear Schrödinger (3DNLS) equations (1.1). By means of the $\bar{\partial}$-dressing method, we have obtained the spatial and time spectral problems associated with the 3DNLS equations. Then we proposed a 3DNLS hierarchy with source by using recursive operator. Finally, the N-soliton solutions of the 3DNLS equations have been constructed based on the $\bar{\partial}$-equation by selecting a special spectral transformation matrix. It is hoped that our results can help enrich the nonlinear dynamics of the NLS-type equations.

Acknowledgements

This work is supported by National Natural Science Foundation of China under Grant Nos. 12175111 and 11975131, and K. C. Wong Magna Fund in Ningbo University.

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