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MULTIQUADRIC QUASI-INTERPOLATION METHOD FOR FRACTIONAL INTEGRAL-DIFFERENTIAL EQUATIONS*

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Abstract In this paper, Multiquadric quasi-interpolation method is used to approximate fractional integral equations and fractional differential equations. Firstly, we construct two operators for approximating the Hadamard integral-differential equation based on quasi interpolators, and verify their properties and order of convergence. Secondly, we obtain that the approximation order of the numerical integral scheme is 3, and the approximation order of the numerical scheme is $3-\mu$ for $\mu(0<\mu<1)$ order fractional Hadamard derivative. Finally, the results of numerical experiments show that the numerical results are in agreement with the theoretical analysis.

Keywords Multiquadric quasi-interpolation, fractional integral-differential equations, Hadamard derivative and integral, error analysis.

MSC(2010) 65R20, 65D30.

1. Introduction

Fractional integral equations have significant applications in various fields of applied science and engineering, such as fluid mechanics, viscoelasticity, bioengineering and etc [23]. In recent years, these equations have become increasingly attractive in applied science, and many numerical methods have been proposed to solve these equations. Radial basis functions (RBFs) are known as a powerful tool in approximation theory for reconstructing functions from scattered values. In [3], it was entered into the field of numerical solution of partial differential equations. In [24], they constructed a new numerical scheme for spatial fractional diffusion equation by quasi-interpolation operators. Based on the method of RBFs, they proposed

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a procedure for approximating fractional derivatives values from one-dimensional scattered noisy data in [14]. In [6], the Lagrange's form of RBFs interpolation with zero-degree algebraic precision was used to construct high order order's finite difference for differential equations. Multiquadric quasi-interpolation has been extensively studied in approximation to integral functionals in [9]. They applied a new non-uniform mesh of points based on modified Legendre polynomial zeros in order to computationally solve partial integro-differential equation in [21]. In [10], they present a new reduced order model based on RBFs and proper orthogonal decomposition methods for fractional advection diffusion equations with a Caputo fractional derivative in time. In [26] and [27], the meshless method were constructed based on spatial trial spaces spanned by the RBFs for the numerical solution of a class of initial-boundary value fractional diffusion equations with variable coefficients on a finite domain. In [25], they constructed Spectral approximation method for generalized fractional ordinary differential equation and Hadamard-type integral equations by a variable transform technique and α-th$(\alpha > 0)$ order fractional derivative of Jacobi polynomials. In [4], three kinds of numerical formulas were proposed for approximating the Caputo-Hadamard fractional derivatives, which are called $L1$-2 formula, $L2$-1_σ formula, and $H2N2$ formula, respectively. They construct and analyze a high-order time-stepping scheme for $\alpha(0 < \alpha < 1)$ order Caputo derivative in [1] with $3+\alpha$ order convergence based on the block-by-block method. In [2], the finite difference/iterative method for the fractional telegraph equation with Hadamard derivatives was constructed. For more research, readers can refer to [7,11,12,17–19] further. The advantages of the multiquadric quasi-interpolation method lie in several aspects, such as good shape preserving properties, very smooth, filter noise and more stable,etc. Recently, multiquadric quasi-interpolation method becomes increasingly popular in many fields of applied mathematics. For more details, readers can refer to [5, 8, 20–22]. Considering the advantages of quasi interpolation algorithms, this paper constructs a log-type Multiquadric quasi-interpolation method for solving the Hadamard fractional integral-differential equations with high convergence order based on the idea of [15, 16, 24].

The outline of this paper is as follows. In Section 2, we introduce a log-type quasi-interpolation operator. In Section 3, we introduce two operators for approximating the Hadamard integral-differential equation based on the operator $\hat{L}_{\log}(u(x))$, and verify their properties. In Section 4, the convergence order is verified by five examples, and the validity of the scheme is verified again. Finally, some conclusion are given in Section 5.

2. Log-type multiquadric quasi-interpolation

In this section, we will construct a log-type quasi-interpolation operator $\hat{L}_{\log}(u(x))$ based on the idea of [7]. Denoted the function $\Phi_k(x) = \frac{1}{3}[(\log\frac{x}{x_k})^2 + (\log(1+\delta))^2]^{\frac{3}{2}}$ as basis functions and $\log A = \log x_0 < \log x_1 < \cdots < \log x_M = \log B, \tau = \max_{0\le i\le M-1}(x_{i+1} - x_i)$.

Similar to [7], we assume that $\hat{L}_{\log}(u(x))$ has the following form

$$\begin{aligned}\hat{L}_{\log}(u(x)) =& u(\log x_0)\hat{\alpha}_0(x) + u(\log x_1)\hat{\alpha}_1(x) + u(\log x_2)\hat{\alpha}_2(x) \\ &+ \sum_{k=3}^{M-3} u(\log x_k)\hat{\alpha}_k(x) + u(\log x_{M-2})\hat{\alpha}_{M-2}(x)\end{aligned} \tag{2.1}$$

$$+ u(\log x_{M-1})\hat{\alpha}_{M-1}(x) + u(\log x_M)\hat{\alpha}_M(x),$$

where

$$\begin{aligned}
\hat{\alpha}_k(x) &= \frac{\theta_k(x)-\theta_{k+1}(x)}{2\log\frac{x_{k+2}}{x_k}\log\frac{x_{k+1}}{x_k}} - \frac{\theta_{k-1}(x)-\theta_k(x)}{2\log\frac{x_{k+1}}{x_k}\log\frac{x_{k+1}}{x_{k-1}}} - \frac{\theta_{k-1}(x)-\theta_k(x)}{2\log\frac{x_{k+1}}{x_{k-1}}\log\frac{x_k}{x_{k-1}}} \\
&\quad + \frac{\theta_{k-2}(x)-\theta_{k-1}(x)}{2\log\frac{x_k}{x_{k-1}}\log\frac{x_k}{x_{k-2}}}, \quad 3\le k\le M-3, \\
\hat{\alpha}_0(x) &= \frac{1}{2} + \frac{(\log\frac{x}{x_0})^2-\theta_1(x)}{2\log\frac{x_2}{x_0}\log\frac{x_1}{x_0}} - \frac{\log\frac{x}{x_0}}{2\log\frac{x_2}{x_0}} - \frac{\log\frac{x}{x_0}}{2\log\frac{x_1}{x_0}}, \\
\hat{\alpha}_1(x) &= \frac{\theta_1(x)-\theta_2(x)}{2\log\frac{x_3}{x_1}\log\frac{x_2}{x_1}} - \frac{(\log\frac{x}{x_0})^2-\theta_1(x)}{2\log\frac{x_2}{x_0}\log\frac{x_1}{x_0}} + \frac{\log\frac{x}{x_0}}{2\log\frac{x_2}{x_0}} + \frac{\log\frac{x}{x_0}}{2\log\frac{x_1}{x_0}} \\
&\quad - \frac{(\log\frac{x}{x_0})^2-\log\frac{x_1}{x_0}\log\frac{x}{x_0}-\theta_1(x)}{2\log\frac{x_2}{x_0}\log\frac{x_2}{x_1}}, \\
\hat{\alpha}_2(x) &= \frac{\theta_2(x)-\theta_3(x)}{2\log\frac{x_4}{x_2}\log\frac{x_3}{x_2}} - \frac{\theta_1(x)-\theta_2(x)}{2\log\frac{x_3}{x_2}\log\frac{x_3}{x_1}} - \frac{\theta_1(x)-\theta_2(x)}{2\log\frac{x_3}{x_1}\log\frac{x_2}{x_1}} \\
&\quad + \frac{(\log\frac{x}{x_0})^2-\log\frac{x_1}{x_0}\log\frac{x}{x_0}-\theta_1(x)}{2\log\frac{x_2}{x_0}\log\frac{x_2}{x_1}}, \qquad (2.2) \\
\hat{\alpha}_{M-2}(x) &= \frac{(\log\frac{x_M}{x})^2-\log\frac{x_M}{x_{M-1}}\log\frac{x_M}{x}+\theta_{M-2}(x)}{2\log\frac{x_M}{x_{M-2}}\log\frac{x_{M-1}}{x_{M-2}}} + \frac{\theta_{M-4}(x)-\theta_{M-3}(x)}{2\log\frac{x_{M-2}}{x_{M-3}}\log\frac{x_{M-2}}{x_{M-4}}} \\
&\quad - \frac{\theta_{M-3}(x)-\theta_{M-2}(x)}{2\log\frac{x_{M-1}}{x_{M-2}}\log\frac{x_{M-1}}{x_{M-3}}} - \frac{\theta_{M-3}(x)-\theta_{M-2}(x)}{2\log\frac{x_{M-1}}{x_{M-3}}\log\frac{x_{M-2}}{x_{M-3}}}, \\
\hat{\alpha}_{M-1}(x) &= \frac{\log\frac{x_M}{x}}{2\log\frac{x_M}{x_{M-2}}} - \frac{(\log\frac{x_M}{x})^2+\theta_{M-2}(x)}{2\log\frac{x_M}{x_{M-1}}\log\frac{x_M}{x_{M-2}}} + \frac{\theta_{M-3}(x)-\theta_{M-2}(x)}{2\log\frac{x_{M-1}}{x_{M-3}}\log\frac{x_{M-1}}{x_{M-2}}} \\
&\quad + \frac{\log\frac{x_M}{x}}{2\log\frac{x_M}{x_{M-1}}} - \frac{(\log\frac{x_M}{x})^2-\log\frac{x_M}{x_{M-1}}\log\frac{x_M}{x}+\theta_{M-2}(x)}{2\log\frac{x_M}{x_{M-2}}\log\frac{x_{M-1}}{x_{M-2}}}, \\
\hat{\alpha}_M(x) &= \frac{1}{2} + \frac{(\log\frac{x_M}{x})^2+\theta_{M-2}(x)}{2\log\frac{x_M}{x_{M-1}}\log\frac{x_M}{x_{M-2}}} - \frac{\log\frac{x_M}{x}}{2\log\frac{x_M}{x_{M-1}}} - \frac{\log\frac{x_M}{x}}{2\log\frac{x_M}{x_{M-2}}},
\end{aligned}$$

and $\theta_k(x), 1\le k\le M-2$, is defined as

$$\theta_k(x) = \frac{\Phi_k(x)-\Phi_{k+1}(x)}{\log x_{k+1}-\log x_k}. \qquad (2.3)$$

In order to obtain some properties and error estimates of (2.1), we can rewrite it as follows

$$\begin{aligned}
&\hat{L}_{\log}(u(x)) \\
&= \frac{1}{2}\sum_{k=1}^{M-2}\{u[\log x_{k+2},\log x_{k+1},\log x_k] - u[\log x_{k+1},\log x_k,\log x_{k-1}]\}\theta_k(x) \\
&\quad + \frac{1}{2}\{u(\log x_0) + u[\log x_1,\log x_0](\log x-\log x_0) \\
&\quad + u[\log x_2,\log x_1,\log x_0](\log x-\log x_0)^2\}
\end{aligned}$$

$$\begin{aligned}
&+\frac{1}{2}\{u(\log x_M)+u[\log x_M,\log x_{M-1}](\log x-\log x_M) \qquad (2.4)\\
&+u[\log x_M,\log x_{M-1},\log x_{M-2}](\log x-\log x_M)^2\}\\
&-\frac{1}{2}u[\log x_2,\log x_1,\log x_0](\log x_1-\log x_0)(\log x-\log x_0)\\
&-\frac{1}{2}u[\log x_M,\log x_{M-1},\log x_{M-2}](\log x_M-\log x_{M-1})(\log x_M-\log x),
\end{aligned}$$

where $u[\log x_{k+1},\log x_k,\log x_{k-1}]$ is defined by

$$u[\log x_{k+1},\log x_k,\log x_{k-1}]=\frac{u[\log x_k,\log x_{k-1}]-u[\log x_{k+1},\log x_k]}{\log x_{k-1}-\log x_{k+1}}. \qquad (2.5)$$

Based on (2.4) and the idea of [7], it is easy to prove the following lemmas.

Lemma 2.1. *Quasi-interpolation $\hat{L}_{\log}(u(x))$ satisfies the quadric polynomial reproduction property, i.e.*

$$\sum_{k=0}^{M}[a_0(\log x_k)^2+a_1\log x_k+a_2]\hat{\alpha}_k(x)=a_0(\log x)^2+a_1\log x+a_2,\forall a_0,a_1,a_2\in R,$$

where $\hat{\alpha}_k(x)$ is defined by (2.2).

Lemma 2.2. *If data $\{u(\log x_k)\}_{k=0}^M$ are from a convex function $u(\log x)\in C[\log x_0,\log x_M]$, then the quasi-interpolation $\hat{L}_{\log}(u(x))$ is also a convex function.*

Lemma 2.3. *If $u''(\log x)$ is Lipschitz continuous, then the approximation capacity of $\hat{L}_{\log}(u(x))$ satisfies*

$$\begin{aligned}
&\|\hat{L}_{\log}(u(x))-u(\log x)\|_\infty\\
&\le O(\tau^3)+O(\log(1+\delta)\tau^2)+O((\log(1+\delta))^2\tau)+O((\log(1+\delta))^2).
\end{aligned}$$

Proof. Denote $\hat{\tau}=\max_i(\log x_{i+1}-\log x_i)$, it is easy to prove that $\hat{\tau}\le\frac{\tau}{A}$.

For any $x\in[A,B]$, denote the first three terms of local Taylor polynomial expression of $u(t)$ at point x, i.e.,

$$y(t)=u(x)+u'(x)(t-x)+\frac{1}{2!}u''(x)(t-x)^2. \qquad (2.6)$$

According to Lemma 2.1, we have

$$\sum_{k=0}^{M}(\log x-\log x_k)^r\hat{\alpha}_k(x)=0,r=1,2, \qquad (2.7)$$

$$\sum_{k=0}^{M}\hat{\alpha}_k(x)=1. \qquad (2.8)$$

Therefore, according to (2.7) and (2.8), we have

$$\sum_{k=0}^{M}y(\log x_k)\hat{\alpha}_k(x)=\sum_{k=0}^{M}[u(\log x)+u'(\log x)(\log x_k-\log x)$$

$$
\begin{aligned}
&+\frac{1}{2!}u''(\log x)(\log x_k-\log x)^2]\hat{\alpha}_k(x)\\
&=u(\log x)\sum_{k=0}^{M}\hat{\alpha}_k(x)+u'(\log x)\sum_{k=0}^{M}(\log x_k-\log x)\hat{\alpha}_k(x)\\
&+\frac{1}{2!}u''(\log x)\sum_{k=0}^{M}(\log x_k-\log x)^2\hat{\alpha}_k(x)=u(\log x).
\end{aligned}
$$

Because $u''(\log x)$ is Lipschitz continuous, then for any $\xi_1,\xi_2\in[A,B]$,

$$|u''(\log\xi_1)-u''(\log\xi_2)|\le C_0|\log\xi_1-\log\xi_2|,$$

where $C_0=\underset{A\le x\le B}{esssup}|u'''(\log x)|$.

Therefore, we obtain

$$
\begin{aligned}
&|\hat{L}_{\log}(u(x))-u(\log x)|\\
&=|\sum_{k=0}^{M}[u(\log x_k)-y(\log x_k)]\hat{\alpha}_k(x)|\\
&=\frac{1}{2}|\sum_{k=1}^{M-2}u[\log x_{k+2},\log x_{k+1},\log x_k,\log x_{k-1}](\log x_{k+2}-\log x_{k-1})\hat{\theta}_k(x)\\
&+\frac{u'''(\log\hat{\xi}_1)}{3!}(\log x_0-\log x)^3+\frac{u'''(\log\hat{\xi}_2)}{3!}(\log x_M-\log x)^3|\\
&=\frac{1}{2}|\sum_{k=1}^{M-2}\frac{u'''(\log\xi_k)}{3!}(\log x_{k+2}-\log x_{k-1})\hat{\theta}_k(x)\\
&+\frac{u'''(\log\hat{\xi}_1)}{3!}(\log x_0-\log x)^3+\frac{u'''(\log\hat{\xi}_2)}{3!}(\log x_M-\log x)^3|,
\end{aligned}
$$

where $\hat{\xi}_1\in(x_0,x_2),\hat{\xi}_2\in(x_{M-2},x_M)$ and $\xi_k\in(x_{k-1},x_{k-2})$.

Furthermore, we have

$$
\begin{aligned}
&|\hat{L}_{\log}(u(x))-u(\log x)|\\
&\le\frac{C_0}{12}|\sum_{k=1}^{M-2}(\log x_{k+2}-\log x_{k-1})\hat{\theta}_k(x)+(\log x_0-\log x)^3+(\log x_M-\log x)^3|\\
&\le\frac{C_0C_1}{12}\{|\sum_{|\log x-\log x_k|\le\hat{\tau}}(\log x_{k+2}-\log x_{k-1})\hat{\theta}_k(x)|\\
&+|\sum_{\log x-\log x_k\ge\hat{\tau}}(\log x_{k+2}-\log x_{k-1})\hat{\theta}_k(x)+(\log x_0-\log x)^3|\\
&+|\sum_{\log x_k-\log x\ge\hat{\tau}}(\log x_{k+2}-\log x_{k-1})\hat{\theta}_k(x)+(\log x_M-\log x)^3|\}\\
&\le\frac{C_0C_1}{12}\{3\hat{\tau}\sum_{|\log x-\log x_k|\le\hat{\tau}}|\hat{\theta}_k(x)|+C_2|\sum_{\log x-\log x_k\ge\hat{\tau}}3\hat{\tau}\hat{\theta}_k(x)+(\log x_0-\log x)^3|\\
&+C_3|\sum_{\log x_k-\log x\ge\hat{\tau}}3\hat{\tau}\hat{\theta}_k(x)+(\log x_M-\log x)^3|\}\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{C_0C_1}{12}\{3\hat{\tau}\sum_{|\log x-\log x_k|\leq\hat{\tau}}|\log x-\log x_k|\sqrt{(\log x-\log x_k)^2+(\log(1+\delta))^2}\\
&\quad +C_2|3\int_{\log x-t\geq\hat{\tau}}(\log x-t)\sqrt{(\log x-t)^2+(\log(1+\delta))^2}dt+(\log x_0-\log x)^3|\\
&\quad +C_3|3\int_{t-\log x\geq\hat{\tau}}(\log x-t)\sqrt{(\log x-t)^2+(\log(1+\delta))^2}dt+(\log x_M-\log x)^3|\}\\
&\leq \frac{C_0C_1}{12}\{3\hat{\tau}^2(\hat{\tau}+\log(1+\delta))+C_2|[(\log x-\log x_0)^2+(\log(1+\delta))^2]^{\frac{3}{2}}\\
&\quad -[(\log x-(\log x-\hat{\tau}))^2+(\log(1+\delta))^2]^{\frac{3}{2}}+(\log x_0-\log x)^3|\\
&\quad +C_3|[(\log x-(\log x+\hat{\tau}))^2+(\log(1+\delta))^2]^{\frac{3}{2}}\\
&\quad -[(\log x-\log x_M)^2+(\log(1+\delta))^2]^{\frac{3}{2}}+(\log x_M-\log x)^3|\}\\
&\leq C(\hat{\tau}^3+\hat{\tau}^2\log(1+\delta)+(\log(1+\delta))^2\hat{\tau}+(\log(1+\delta))^2)\\
&\leq C((\frac{\tau}{A})^3+(\frac{\tau}{A})^2\log(1+\delta)+(\log(1+\delta))^2(\frac{\tau}{A})+(\log(1+\delta))^2)\\
&\leq O(\tau^3)+O(\log(1+\delta)\tau^2)+O((\log(1+\delta))^2\tau)+O((\log(1+\delta))^2),
\end{aligned}
$$

where C_1, C_2 and C_3 are positive constants independent of τ and δ. Then complete the proof of Lemma 2.3. □

3. Quasi-interpolation operators for Hadamard fractional derivatives and integral based on $\hat{L}_{\log}(u(x))$

In this section, we will use the quasi-interpolator $\hat{L}_\lambda(\log x)$ to construct two quasi-interpolation operators ${}_AD_x^\mu\hat{L}_{\log}(u(x))$ and ${}_AH_x^\mu\hat{L}_{\log}(u(x))$ to approximate Hadamard fractional derivatives and integral, respectively.

3.1. The quasi-interpolation operator ${}_AD_x^\mu\hat{L}_{\log}(u(x))$

Let $\sigma=\omega\frac{d}{d\omega}$, the left-sided Caputo-Hadamard fractional derivatives of order $\mu(\mu>0)$ on (A,B) in [13] are defined by

$$
{}_AD_x^\mu u(x)=\frac{1}{\Gamma(1-\mu)}\int_A^x(\log\frac{x}{\omega})^{-\mu}\sigma u(\omega)\frac{d\omega}{\omega}.
$$

Base on (2.1), we construct an operator ${}_AD_x^\mu\hat{L}_{\log}(u(x))$ for the Hadamard fractional derivatives as following

$$
\begin{aligned}
{}_AD_x^\mu\hat{L}_{\log}(u(x)) &= u(\log x_0)\gamma_0(x)+u(\log x_1)\gamma_1(x)+u(\log x_2)\gamma_2(x)\\
&\quad+\sum_{k=3}^{M-3}u(\log x_k)\gamma_k(x)+u(\log x_{M-2})\gamma_{M-2}(x)\\
&\quad+u(\log x_{M-1})\gamma_{M-1}(x)+u(\log x_M)\gamma_M(x),
\end{aligned}\tag{3.1}
$$

where

$$\begin{aligned}
\gamma_k(x) &= \frac{\hat{\theta}_k(x)-\hat{\theta}_{k+1}(x)}{2\log\frac{x_{k+2}}{x_k}\log\frac{x_{k+1}}{x_k}} - \frac{\hat{\theta}_{k-1}(x)-\hat{\theta}_k(x)}{2\log\frac{x_{k+1}}{x_k}\log\frac{x_{k+1}}{x_{k-1}}} \\
&\quad - \frac{\hat{\theta}_{k-1}(x)-\hat{\theta}_k(x)}{2\log\frac{x_{k+1}}{x_{k-1}}\log\frac{x_k}{x_{k-1}}} + \frac{\hat{\theta}_{k-2}(x)-\hat{\theta}_{k-1}(x)}{2\log\frac{x_k}{x_{k-1}}\log\frac{x_k}{x_{k-2}}}, \quad 3 \le k \le M-3, \\
\gamma_0(x) &= -\frac{(\log\frac{x}{A})^{1-\mu}}{2\Gamma(2-\mu)\log\frac{x_1}{x_0}} + \frac{\log\frac{A}{x_0}(\log\frac{x}{A})^{1-\mu}}{\Gamma(2-\mu)\log\frac{x_1}{x_0}\log\frac{x_2}{x_0}} - \frac{\hat{\theta}_1(x)}{2\log\frac{x_1}{x_0}\log\frac{x_2}{x_0}} \\
&\quad + \frac{(\log\frac{x}{A})^{2-\mu}}{\Gamma(3-\mu)\log\frac{x_1}{x_0}\log\frac{x_2}{x_0}} - \frac{(\log\frac{x}{A})^{1-\mu}}{2\Gamma(2-\mu)\log\frac{x_2}{x_0}}, \\
\gamma_1(x) &= \frac{(\log\frac{x}{A})^{1-\mu}}{2\Gamma(2-\mu)\log\frac{x_1}{x_0}} - \frac{\log\frac{A}{x_0}(\log\frac{x}{A})^{1-\mu}}{\Gamma(2-\mu)\log\frac{x_1}{x_0}\log\frac{x_2}{x_0}} + \frac{(\log\frac{x}{A})^{1-\mu}}{2\Gamma(2-\mu)\log\frac{x_2}{x_0}} \\
&\quad - \frac{(\log\frac{x}{A})^{2-\mu}}{\Gamma(3-\mu)\log\frac{x_1}{x_0}\log\frac{x_2}{x_0}} + \frac{\hat{\theta}_1(x)}{2\log\frac{x_1}{x_0}\log\frac{x_2}{x_0}} + \frac{\hat{\theta}_1(x)-\hat{\theta}_2(x)}{2\log\frac{x_2}{x_1}\log\frac{x_3}{x_1}} \\
&\quad - \frac{\log\frac{A}{x_0}(\log\frac{x}{A})^{1-\mu}}{\Gamma(2-\mu)\log\frac{x_2}{x_1}\log\frac{x_2}{x_0}} + \frac{\log\frac{x_1}{x_0}(\log\frac{x}{A})^{1-\mu}+\hat{\theta}_1(x)\Gamma(2-\mu)}{2\Gamma(2-\mu)\log\frac{x_2}{x_1}\log\frac{x_2}{x_0}} \\
&\quad - \frac{(\log\frac{x}{A})^{2-\mu}}{\Gamma(3-\mu)\log\frac{x_2}{x_1}\log\frac{x_2}{x_0}}, \\
\gamma_2(x) &= \frac{\log\frac{A}{x_0}(\log\frac{x}{A})^{1-\mu}}{\Gamma(2-\mu)\log\frac{x_2}{x_1}\log\frac{x_2}{x_0}} + \frac{(\log\frac{x}{A})^{2-\mu}}{\Gamma(3-\mu)\log\frac{x_2}{x_1}\log\frac{x_2}{x_0}} + \frac{\hat{\theta}_2(x)-\hat{\theta}_3(x)}{2\log\frac{x_3}{x_2}\log\frac{x_4}{x_2}} \\
&\quad - \frac{\log\frac{x_1}{x_0}(\log\frac{x}{A})^{1-\mu}+\hat{\theta}_1(x)\Gamma(2-\mu)}{2\Gamma(2-\mu)\log\frac{x_2}{x_1}\log\frac{x_2}{x_0}} - \frac{\hat{\theta}_1(x)-\hat{\theta}_2(x)}{2\log\frac{x_2}{x_1}\log\frac{x_3}{x_1}} - \frac{\hat{\theta}_1(x)-\hat{\theta}_2(x)}{2\log\frac{x_3}{x_2}\log\frac{x_3}{x_1}}, \\
\gamma_{M-2}(x) &= \frac{\log\frac{A}{x_M}(\log\frac{x}{A})^{1-\mu}}{\Gamma(2-\mu)\log\frac{x_{M-1}}{x_{M-2}}\log\frac{x_M}{x_{M-2}}} + \frac{(\log\frac{x}{A})^{2-\mu}}{\Gamma(3-\mu)\log\frac{x_{M-1}}{x_{M-2}}\log\frac{x_M}{x_{M-2}}} \\
&\quad + \frac{\log\frac{x_M}{x_{M-1}}(\log\frac{x}{A})^{1-\mu}}{\Gamma(2-\mu)\log\frac{x_{M-1}}{x_{M-2}}\log\frac{x_M}{x_{M-2}}} - \frac{\hat{\theta}_{M-3}(x)-\hat{\theta}_{M-2}(x)}{2\log\frac{x_{M-2}}{x_{M-3}}\log\frac{x_{M-1}}{x_{M-3}}} \qquad (3.2) \\
&\quad - \frac{\hat{\theta}_{M-3}(x)-\hat{\theta}_{M-2}(x)}{2\log\frac{x_{M-1}}{x_{M-2}}\log\frac{x_{M-1}}{x_{M-3}}} + \frac{\hat{\theta}_{M-4}(x)-\hat{\theta}_{M-3}(x)}{2\log\frac{x_{M-2}}{x_{M-3}}\log\frac{x_{M-2}}{x_{M-4}}} + \frac{\hat{\theta}_{M-2}(x)}{2\log\frac{x_{M-1}}{x_{M-2}}\log\frac{x_M}{x_{M-2}}}, \\
\gamma_{M-1}(x) &= -\frac{(\log\frac{x}{A})^{1-\mu}}{2\Gamma(2-\mu)\log\frac{x_M}{x_{M-1}}} - \frac{2\log\frac{A}{x_M}(\log\frac{x}{A})^{1-\mu}+\hat{\theta}_{M-2}(x)\Gamma(2-\mu)}{2\Gamma(2-\mu)\log\frac{x_{M-1}}{x_{M-2}}\log\frac{x_M}{x_{M-2}}} \\
&\quad - \frac{(\log\frac{x}{A})^{2-\mu}}{\Gamma(3-\mu)\log\frac{x_{M-1}}{x_{M-2}}\log\frac{x_M}{x_{M-2}}} - \frac{\log\frac{x_M}{x_{M-1}}(\log\frac{x}{A})^{1-\mu}}{2\Gamma(2-\mu)\log\frac{x_{M-1}}{x_{M-2}}\log\frac{x_M}{x_{M-2}}} \\
&\quad - \frac{2\log\frac{A}{x_M}(\log\frac{x}{A})^{1-\mu}+\hat{\theta}_{M-2}(x)\Gamma(2-\mu)}{2\Gamma(2-\mu)\log\frac{x_M}{x_{M-1}}\log\frac{x_M}{x_{M-2}}} - \frac{(\log\frac{x}{A})^{1-\mu}}{2\Gamma(2-\mu)\log\frac{x_M}{x_{M-2}}} \\
&\quad - \frac{(\log\frac{x}{A})^{2-\mu}}{\Gamma(3-\mu)\log\frac{x_M}{x_{M-1}}\log\frac{x_M}{x_{M-2}}} + \frac{\hat{\theta}_{M-3}(x)-\hat{\theta}_{M-2}(x)}{2\log\frac{x_{M-1}}{x_{M-2}}\log\frac{x_{M-1}}{x_{M-3}}},
\end{aligned}$$

$$\gamma_M(x) = \frac{(\log\frac{x}{A})^{1-\mu}}{2\Gamma(2-\mu)\log\frac{x_M}{x_{M-1}}} + \frac{2\log\frac{A}{x_M}(\log\frac{x}{A})^{1-\mu} + \hat{\theta}_{M-2}(x)\Gamma(2-\mu)}{2\Gamma(2-\mu)\log\frac{x_M}{x_{M-1}}\log\frac{x_M}{x_{M-2}}}$$
$$+ \frac{(\log\frac{x}{A})^{2-\mu}}{\Gamma(3-\mu)\log\frac{x_M}{x_{M-1}}\log\frac{x_M}{x_{M-2}}} + \frac{(\log\frac{x}{A})^{1-\mu}}{2\Gamma(2-\mu)\log\frac{x_M}{x_{M-2}}},$$

and $\hat{\theta}_k(x), 1 \le k \le M-2$, is defined as

$$\hat{\theta}_k(x) = \frac{{}_AD_x^{\mu}\Phi_k(x) - {}_AD_x^{\mu}\Phi_{k+1}(x)}{\log x_{k+1} - \log x_k}. \tag{3.3}$$

In order to avoid the singularity of the integrand function, we calculate ${}_AD_x^{\mu}\Phi_k(x), 2 \le k \le M-2$, as follows

$$\begin{aligned} {}_AD_x^{\mu}\Phi_k(x) &= \frac{1}{\Gamma(1-\mu)}\int_A^x \log\frac{\omega}{x_k}\sqrt{(\log\frac{\omega}{x_k})^2 + (\log(1+\delta))^2}(\log\frac{x}{\omega})^{-\mu}\frac{d\omega}{\omega} \\ &= \frac{1}{\Gamma(2-\mu)}\Big\{\log\frac{A}{x_k}\sqrt{(\log\frac{A}{x_k})^2 + (\log(1+\delta))^2}(\log\frac{x}{A})^{1-\mu} \\ &\quad + \int_{\log A}^{\log x}\frac{2(t-\log x_k)^2 + (\log(1+\delta))^2}{\sqrt{(t-\log x_k)^2 + (\log(1+\delta))^2}}(\log x - t)^{1-\mu}dt\Big\}. \end{aligned} \tag{3.4}$$

In order to analysis some properties and error estimates of (3.1), one can rewrite it as follows

$$\begin{aligned} &{}_AD_x^{\mu}\hat{L}_{\log}(u(x)) \\ &= \frac{1}{2}\sum_{k=1}^{M-2}\{u[\log x_{k+2}, \log x_{k+1}, \log x_k] - u[\log x_{k+1}, \log x_k, \log x_{k-1}]\}\hat{\theta}_k(x) \\ &+\frac{1}{2}\{u[\log x_1, \log x_0]{}_AD_x^{\mu}[\log\frac{x}{x_0}] + u[\log x_2, \log x_1, \log x_0]{}_AD_x^{\mu}[(\log\frac{x}{x_0})^2]\} \\ &+\frac{1}{2}\{u[\log x_M, \log x_{M-1}]{}_AD_x^{\mu}[\log\frac{x}{x_M}] \\ &+u[\log x_M, \log x_{M-1}, \log x_{M-2}]{}_AD_x^{\mu}[(\log\frac{x}{x_M})^2]\} \\ &-\frac{1}{2}u[\log x_2, \log x_1, \log x_0]\log\frac{x_1}{x_0}{}_AD_x^{\mu}[\log\frac{x}{x_0}] \\ &-\frac{1}{2}u[\log x_M, \log x_{M-1}, \log x_{M-2}]\log\frac{x_M}{x_{M-1}}{}_AD_x^{\mu}[\log\frac{x_M}{x}], \end{aligned} \tag{3.5}$$

where $u[\log x_{k+1}, \log x_k, \log x_{k-1}]$ is defined in (2.5).

Similar to Lemma 2.1, we will study the regeneration property of quadratic polynomial for ${}_AD_x^{\mu}\hat{L}_{\log}(u(x))$ as follows.

Theorem 3.1. *The quasi-interpolation operator ${}_AD_x^{\mu}\hat{L}_{\log}(u(x))$ satisfies the Hadamard fractional derivatives regeneration property of quadratic polynomial, i.e., $\forall a_0, a_1, a_2 \in R$,$u(x) = a_0x^2 + a_1x + a_2$ such that*

$${}_AD_x^{\mu}\hat{L}_{\log}(a_0x^2 + a_1x + a_2) = {}_AD_x^{\mu}[a_0(\log x)^2 + a_1\log x + a_2],$$

where $\gamma_k(x)$ is defined by (3.2).

Proof. Denote $F(x) = a_0x^2 + a_1x + a_2$, one can have

$$\begin{aligned}
&{}_AD_x^\mu \hat{L}_{\log}(F(x)) \\
&= \frac{1}{2}\sum_{k=1}^{M-2}\{F[\log x_{k+2}, \log x_{k+1}, \log x_k] - F[\log x_{k+1}, \log x_k, \log x_{k-1}]\}\hat{\theta}_k(x) \\
&\quad + \frac{1}{2}\{[a_0(\log x_0 + \log x_1) + a_1]{}_AD_x^\mu[\log \frac{x}{x_0}] + a_0{}_AD_x^\mu[(\log \frac{x}{x_0})^2] \\
&\quad - a_0 \log \frac{x_1}{x_0}{}_AD_x^\mu[\log \frac{x}{x_0}]\} \\
&\quad + \frac{1}{2}\{[a_0(\log x_{M-1} + \log x_M) + a_1]{}_AD_x^\mu[\log \frac{x}{x_M}] + a_0{}_AD_x^\mu[(\log \frac{x}{x_M})^2] \\
&\quad - a_0 \log \frac{x_M}{x_{M-1}}{}_AD_x^\mu[\log \frac{x_M}{x}]\} \\
&= \frac{[2a_0 \log A + a_1](\log \frac{x}{A})^{1-\mu}}{\Gamma(2-\mu)} + \frac{2a_0(\log \frac{x}{A})^{2-\mu}}{\Gamma(3-\mu)},
\end{aligned}$$

and

$${}_AD_x^\mu[a_0(\log x)^2 + a_1 \log x + a_2] = \frac{[2a_0 \log A + a_1](\log \frac{x}{A})^{1-\mu}}{\Gamma(2-\mu)} + \frac{2a_0(\log \frac{x}{A})^{2-\mu}}{\Gamma(3-\mu)},$$

where

$$F[\log x_2, \log x_1, \log x_0] = F[\log x_M, \log x_{M-1}, \log x_{M-2}] = a_0.$$

Based on the above analysis, one can obtain that

$${}_AD_x^\mu \hat{L}_{\log}(a_0x^2 + a_1x + a_2) = {}_AD_x^\mu[a_0(\log x)^2 + a_1 \log x + a_2].$$

Hence, we have proved ${}_AD_x^\mu \hat{L}_{\log}(u(x))$ satisfies the Hadamard fractional derivatives regeneration property of quadric polynomial and Theorem 3.1 is proved. □

Similar to Lemma 2.3, we will prove the the approximation capacity of ${}_AD_x^\mu \hat{L}_{\log}(u(x))$ as following.

Theorem 3.2. *Assumed that the second derivative of $u(\log x)$ is Lipschitz continuous, the approximation capacity of ${}_AD_x^\mu \hat{L}_{\log}(u(x))$ satisfies*

$$\begin{aligned}
&\|{}_AD_x^\mu \hat{L}_{\log}(u(x)) - {}_AD_x^\mu u(\log x)\|_\infty \\
&\le O(\tau^{3-\mu} + \log(1+\delta)\tau^{2-\mu} + (\log(1+\delta))^2\tau^{1-\mu} + (\log(1+\delta))^2\tau^{-\mu}).
\end{aligned}$$

Proof. For any fixed $x \in [A, B]$, suppose $y(t)$ be the first three items of local Taylor polynomial expansion of $u(t)$ at point x, i.e.,

$$y(t) = u(x) + u'(x)(t-x) + \frac{1}{2!}u''(x)(t-x)^2.$$

According to Theorem 3.1, we know

$$\begin{aligned}
&\sum_{k=0}^{M}(\log \frac{x}{x_k})^r \gamma_k(x) = {}_AD_x^\mu[\sum_{k=0}^{M}(\log \frac{x}{x_k})^r \hat{\alpha}_k(x)] = 0, r = 1, 2, \\
&\sum_{k=0}^{M}\gamma_k(x) = {}_AD_x^\mu[\sum_{k=0}^{M}\hat{\alpha}_k(x)] = {}_AD_x^\mu[1],
\end{aligned}$$

where $\gamma_k(x) = {}_AD_x^\mu \hat{\alpha}_k(x), k = 0, \cdots, M$, then,

$$
\begin{aligned}
&\sum_{k=0}^{M} y(\log x_k)\gamma_k(x)\\
&= {}_AD_x^\mu\Big[\sum_{k=0}^{M}\Big(u(\log x) + u'(\log x)\log\frac{x_k}{x} + \frac{u''(\log x)}{2!}\Big(\log\frac{x_k}{x}\Big)^2\Big)\hat{\alpha}_k(x)\Big]\\
&= {}_AD_x^\mu\Big[u(\log x)\sum_{k=0}^{M}\hat{\alpha}_k(x)\Big] + {}_AD_x^\mu\Big[u'(\log x)\sum_{k=0}^{M}\log\frac{x_k}{x}\hat{\alpha}_k(x)\Big]\\
&\quad + \frac{1}{2!}{}_AD_x^\mu\Big[u''(\log x)\sum_{k=0}^{M}\Big(\log\frac{x_k}{x}\Big)^2\hat{\alpha}_k(x)\Big]\\
&= {}_AD_x^\mu[u(\log x)].
\end{aligned}
$$

Using the above equation, one can rewrite $|{}_AD_x^\mu \hat{L}_{\log}(u(x)) - {}_AD_x^\mu u(\log x)|$ as follows

$$
\begin{aligned}
&|{}_AD_x^\mu \hat{L}_{\log}(u(x)) - {}_AD_x^\mu u(\log x)| \qquad (3.6)\\
&= \Big|\sum_{k=0}^{M}[u(\log x_k) - y(\log x_k)]\gamma_k(x)\Big|\\
&= {}_AD_x^\mu\Big|\sum_{k=0}^{M}[u(\log x_k) - y(\log x_k)]\hat{\alpha}_k(x)\Big|\\
&= \frac{1}{\Gamma(1-\mu)}\Big|\int_A^x\Big[\sum_{k=0}^{M}(u(\log x_k) - y(\log x_k))\hat{\alpha}_k(\omega)\Big]'\omega\Big(\log\frac{x}{\omega}\Big)^{-\mu}\frac{d\omega}{\omega}\Big|\\
&= \frac{1}{\Gamma(1-\mu)}\Big|\int_A^x\Big(\log\frac{x}{\omega}\Big)^{-\mu}d\Big[\sum_{k=0}^{M}(u(\log x_k) - y(\log x_k))\hat{\alpha}_k(\omega)\Big]\Big|\\
&= \frac{1}{\Gamma(1-\mu)}\Big|\int_{\log A}^{\log x}(\log x - t)^{-\mu}d\Big[\sum_{k=0}^{M}(u(\log x_k) - y(\log x_k))\hat{\alpha}_k(t)\Big]\Big|\\
&= \frac{1}{\Gamma(1-\mu)}\Big|\int_{\log x-\tau}^{\log x}(\log x - t)^{-\mu}d\Big[\sum_{k=0}^{M}(u(\log x_k) - y(\log x_k))\hat{\alpha}_k(t)\Big]\\
&\quad + \int_{\log A}^{\log x-\tau}(\log x - t)^{-\mu}d\Big[\sum_{k=0}^{M}(u(\log x_k) - y(\log x_k))\hat{\alpha}_k(t)\Big]\Big|\\
&\le \frac{1}{\Gamma(1-\mu)}\int_{\log x-\tau}^{\log x}|\log x - t|^{-\mu}\Big|\sum_{k=0}^{M}(u(\log x_k) - y(\log x_k))\hat{\alpha}_k'(t)\Big|dt\\
&\quad + \frac{1}{\Gamma(1-\mu)}\int_{\log A}^{\log x-\tau}|\log x - t|^{-\mu}d\Big[\sum_{k=0}^{M}(u(\log x_k) - y(\log x_k))\hat{\alpha}_k(t)\Big],
\end{aligned}
$$

where $t = \log\omega$.

Because $u''(\log x)$ is Lipschitz continuous, then for any $x_1, x_2 \in [A, B]$, there exists L_0, such that

$$|u''(\log x_1) - u''(\log x_2)| \le L_0|\log x_1 - \log x_2|.$$

Next let's start with the integral of the first part in (3.6). First, for $|\sum_{k=0}^{M}(u(\log x_k) - y(\log x_k))\hat{\alpha}'_k(t)|$, $t \in (\log x - \tau, \log x)$, according to (2.4), because $|u(\log s) - y(\log s)| \leq L_0|\log s - \log x|^3$, similar to the proof in [7], we know

$$
\begin{aligned}
&|\sum_{k=0}^{M}(u(\log x_k) - y(\log x_k))\hat{\alpha}'_k(t)| \\
\leq\ & \frac{1}{2}|\sum_{k=1}^{M-2}((u[\log x_{k+2}, \log x_{k+1}, \log x_k] - u[\log x_{k+1}, \log x_k, \log x_{k-1}]) \\
&-(y[\log x_{k+2}, \log x_{k+1}, \log x_k] - y[\log x_{k+1}, \log x_k, \log x_{k-1}]))\theta'_k(t)| \\
&+L_1[|t - \log x_0|^3]' + L_2[|\log x_M - t|^3]' \\
\leq\ & \frac{1}{4}\sum_{k=1}^{M-2}|u''(\xi) - u''(\eta)||\theta'_k(t)| + L_1[|t - \log x_0|^3]' + L_2[|\log x_M - t|^3]' \\
\leq\ & \frac{L_0}{4}\sum_{k=1}^{M-2}|\frac{\log x_{k+2} - \log x_{k-1}}{\log x_{k+1} - \log x_k}||\Phi'_k(t)| + L_1[|t - \log x_0|^3]' + L_2[|\log x_M - t|^3]' \\
\leq\ & \frac{3L_0}{4}\sum_{k=1}^{M-2}|\frac{x_{k+1}}{x_{k-1}}||\Phi'_k(t)| + L_1[|t - \log x_0|^3]' + L_2[|\log x_M - t|^3]' \\
\leq\ & \frac{3BL_0}{4A}\sum_{k=1}^{M-2}|t - \log x_k|\sqrt{(t - \log x_k)^2 + (\log(1+\delta))^2} + L_1[|t - \log x_0|^3]' \\
&+L_2[|\log x_M - t|^3]' \\
\leq\ & \frac{3BL_0}{4A}\sum_{|t - \log x_k| \leq \tau}|t - \log x_k|\sqrt{(t - \log x_k)^2 + (\log(1+\delta))^2} + L_1|t - \log x_0|^2 \\
&+L_2|\log x_M - t|^2 \\
\leq\ & \frac{3BL_0}{4A}\sum_{|t - \log x_k| \leq \tau}\tau[\tau + \log(1+\delta)] + L_1|\log x_M - \log x_0|^2 \\
&+L_2|\log x_M - \log x_0 + \tau|^2 \\
\leq\ & L_0\tau[\tau + \log(1+\delta)] + L_1(\frac{M\tau}{A})^2 + L_2(\frac{M\tau}{A} + \tau)^2 \\
\leq\ & L_0\tau[\tau + \log(1+\delta)] + L_1\tau^2 + L_2\tau^2,
\end{aligned}
$$

where $\xi \in (\log x_k, \log x_{k+2}), \eta \in (\log x_{k-1}, \log x_{k+1})$.

Bringing $|\sum_{k=0}^{M}(u(\log x_k) - y(\log x_k))\hat{\alpha}'_k(t)|$ for $t \in (\log x - \tau, \log x)$ into the first part in (3.6), one get

$$
\begin{aligned}
&\frac{1}{\Gamma(1-\mu)}\int_{\log x - \tau}^{\log x}|\log x - t|^{-\mu}|\sum_{k=0}^{M}(u(\log x_k) - y(\log x_k))\hat{\alpha}'_k(t)|dt \\
\leq&\frac{L_0\tau[\tau + \log(1+\delta)] + L_1\tau^2 + L_2\tau^2}{\Gamma(1-\mu)}\int_{\log x - \tau}^{\log x}|\log x - t|^{-\mu}dt
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
&\leq \frac{L_0\tau^{2-\mu}[\tau+\log(1+\delta)]+L_1\tau^{3-\mu}+L_2\tau^{3-\mu}}{\Gamma(1-\mu)}\\
&\leq L_0(\tau^{3-\mu}+\tau^{2-\mu}\log(1+\delta))+L_1\tau^{3-\mu}+L_2\tau^{3-\mu},
\end{aligned}
$$

here L_1, L_2 are two positive constants and independent of δ and τ.

For the last part in (3.6), using direct calculation one can be obtained that

$$
\begin{aligned}
&\frac{1}{\Gamma(1-\mu)}\int_{\log A}^{\log x-\tau}|\log x-t|^{-\mu}d[\sum_{k=0}^{M}(u(\log x_k)-y(\log x_k))\hat{\alpha}_k(t)]\\
=&|\frac{\tau^{-\mu}}{\Gamma(1-\mu)}\sum_{k=0}^{M}(u(\log x_k)-y(\log x_k))\hat{\alpha}_k(\log x-\tau)\\
&-\frac{1}{\Gamma(1-\mu)}(\log\frac{B}{A})^{-\mu}\sum_{k=0}^{M}(u(\log x_k)-y(\log x_k))\hat{\alpha}_k(A)\\
&-\frac{\mu}{\Gamma(1-\mu)}\int_{\log A}^{\log x-\tau}[\sum_{k=0}^{M}(u(\log x_k)-y(\log x_k))\hat{\alpha}_k(t)](\log x-t)^{-\mu-1}dt|\\
\leq&\frac{1}{\Gamma(1-\mu)}\{\tau^{-\mu}|\sum_{k=0}^{M}(u(\log x_k)-y(\log x_k))\hat{\alpha}_k(\log x-\tau)|\\
&+(\log\frac{B}{A})^{-\mu}|\sum_{k=0}^{M}(u(\log x_k)-y(\log x_k))\hat{\alpha}_k(A)|\\
&+\mu\tau^{-\mu}|\sum_{k=0}^{M}(u(\log x_k)-y(\log x_k))\hat{\alpha}_k(t)|\\
&+\mu(\log\frac{x_M}{A})^{-\mu}|\sum_{k=0}^{M}(u(\log x_k)-y(\log x_k))\hat{\alpha}_k(t)|\}\\
\doteq&\frac{1}{\Gamma(1-\mu)}(P_1+P_2+P_3+P_4).
\end{aligned}
\tag{3.8}
$$

For P_1, using Lemma 2.2 and Lemma 2.3 one can obtain

$$
\begin{aligned}
P_1=&\tau^{-\mu}|\sum_{k=0}^{M}(u(\log x_k)-y(\log x_k))\hat{\alpha}_k(\log x-\tau)|\\
\leq&\tau^{-\mu}|\sum_{k=0}^{M}(u(\log x_k)-y(\log x_k))(\hat{\alpha}_k(\log x-\tau)-\hat{\alpha}_k(\log x))|\\
&+\tau^{-\mu}|\sum_{k=0}^{M}u(\log x_k)-y(\log x_k)\hat{\alpha}_k(\log x)|\\
\leq&\tau^{-\mu}\sum_{k=0}^{M}|u(\log x_k)-y(\log x_k)||\hat{\alpha}_k(\log x-\tau)-\hat{\alpha}_k(\log x)|\\
&+\tau^{-\mu}|\sum_{k=0}^{M}u(\log x_k)-y(\log x_k)\hat{\alpha}_k(\log x)|
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
&\le \tau^{-\mu}\sum_{k=0}^{M}(\log\frac{x_k}{x_0})^3\tau|\hat{\alpha}'_k(\log x)| \\
&\quad +O(\tau^{3-\mu}+\log(1+\delta)\tau^{2-\mu}+(\log(1+\delta))^2\tau^{1-\mu}+(\log(1+\delta))^2\tau^{-\mu}) \\
&\le \tau^{1-\mu}\sum_{k=0}^{M}(\frac{k\tau}{A})^3|\hat{\alpha}'_k(\log x)| \\
&\quad +O(\tau^{3-\mu}+\log(1+\delta)\tau^{2-\mu}+(\log(1+\delta))^2\tau^{1-\mu}+(\log(1+\delta))^2\tau^{-\mu}) \\
&\le O(\tau^{3-\mu}+\log(1+\delta)\tau^{2-\mu}+(\log(1+\delta))^2\tau^{1-\mu}+(\log(1+\delta))^2\tau^{-\mu}).
\end{aligned}
$$

For P_2, we have

$$
\begin{aligned}
P_2 &= (\log\frac{B}{A})^{-\mu}|\sum_{k=0}^{M}(u(\log x_k)-y(\log x_k))\hat{\alpha}_k(A)| \\
&\le (\frac{B-A}{A})^{-\mu}O(\tau^3+\log(1+\delta)\tau^2+(\log(1+\delta))^2\tau+(\log(1+\delta))^2).
\end{aligned} \tag{3.10}
$$

For P_3 and P_4, we obtain

$$
\begin{aligned}
&P_3+P_4 \\
&= \mu\tau^{-\mu}|\sum_{k=0}^{M}(u(\log x_k)-y(\log x_k))\hat{\alpha}_k(t)| \\
&\quad +\mu(\log\frac{x_M}{A})^{-\mu}|\sum_{k=0}^{M}(u(\log x_k)-y(\log x_k))\hat{\alpha}_k(t)| \\
&\le O(\tau^{3-\mu})+O(\log(1+\delta)\tau^{2-\mu})+O((\log(1+\delta))^2\tau^{1-\mu})+O((\log(1+\delta))^2\tau^{-\mu}) \\
&\quad +O(\tau^3+\log(1+\delta)\tau^2+(\log(1+\delta))^2\tau+(\log(1+\delta))^2)(\frac{B-A}{A})^{-\mu} \\
&\le O(\tau^{3-\mu})+O(\log(1+\delta)\tau^{2-\mu})+O((\log(1+\delta))^2\tau^{1-\mu})+O((\log(1+\delta))^2\tau^{-\mu}).
\end{aligned} \tag{3.11}
$$

Substituting (3.9), (3.10), (3.11) into (3.8), one can obtain immediately that $\forall x\in(A,B)$,

$$
\begin{aligned}
&|{}_AD_x^{\mu}\hat{L}_{\log}(u(x))-{}_AD_x^{\mu}u(\log x)| \\
&\le O(\tau^{3-\mu})+O(\log(1+\delta)\tau^{2-\mu})+O((\log(1+\delta))^2\tau^{1-\mu})+O((\log(1+\delta))^2\tau^{-\mu}).
\end{aligned}
$$

Therefore, one can obtain that

$$
\begin{aligned}
&\|{}_AD_x^{\mu}\hat{L}_{\log}(u(x))-{}_AD_x^{\mu}u(\log x)\|_\infty \\
&\le O(\tau^{3-\mu}+\log(1+\delta)\tau^{2-\mu}+(\log(1+\delta))^2\tau^{1-\mu}+(\log(1+\delta))^2\tau^{-\mu}).
\end{aligned}
$$

The proof is completed. □

Remark 3.1. When $\delta=O(\tau^{1.5})$, we have

$$
\|{}_AD_x^{\mu}\hat{L}_{\log}(u(x))-{}_AD_x^{\mu}u(\log x)\|_\infty \le O(\tau^{3-\mu}).
$$

3.2. The quasi-interpolation operator ${}_AH_x^\mu\hat{L}_{\log}(u(x))$

The left-sided Hadamard fractional integrals of order $\mu(\mu>0)$ are given by [13] as follows

$$ {}_AH_x^\mu u(x)=\frac{1}{\Gamma(\mu)}\int_A^x u(\omega)(\log\frac{x}{\omega})^{\mu-1}\frac{d\omega}{\omega},\quad x\in(A,B). $$

Similar to (2.1), we will construct an operator ${}_AH_x^\mu\hat{L}_{\log}(u(x))$ for the Hadamard fractional integral as follows

$$ \begin{aligned} {}_AH_x^\mu\hat{L}_{\log}(u(x)) &= u(\log x_0)\beta_0(x)+u(\log x_1)\beta_1(x)+u(\log x_2)\beta_2(x)\\ &+\sum_{k=3}^{M-3}u(\log x_k)\beta_k(x)+u(\log x_{M-2})\beta_{M-2}(x) \\ &+u(\log x_{M-1})\beta_{M-1}(x)+u(\log x_M)\beta_M(x), \end{aligned} \tag{3.12} $$

where

$$ \begin{aligned} \beta_k(x) &= \frac{\bar\theta_k(x)-\bar\theta_{k+1}(x)}{2\log\frac{x_{k+2}}{x_k}\log\frac{x_{k+1}}{x_k}}-\frac{\bar\theta_{k-1}(x)-\bar\theta_k(x)}{2\log\frac{x_{k+1}}{x_k}\log\frac{x_{k+1}}{x_{k-1}}}\\ &-\frac{\bar\theta_{k-1}(x)-\bar\theta_k(x)}{2\log\frac{x_{k+1}}{x_{k-1}}\log\frac{x_k}{x_{k-1}}}+\frac{\bar\theta_{k-2}(x)-\bar\theta_{k-1}(x)}{2\log\frac{x_k}{x_{k-1}}\log\frac{x_k}{x_{k-2}}},\quad 3\le k\le M-3,\\ \beta_0(x) &= \frac{(\log\frac{x}{A})^\mu}{2\Gamma(\mu+1)}-\frac{\log\frac{A}{x_0}(\log\frac{x}{A})^\mu}{2\Gamma(\mu+1)\log\frac{x_1}{x_0}}-\frac{(\log\frac{x}{A})^{\mu+1}}{2\Gamma(\mu+2)\log\frac{x_1}{x_0}}\\ &+\frac{(\log\frac{A}{x_0})^2(\log\frac{x}{A})^\mu-\bar\theta_1(x)\Gamma(\mu+1)}{2\Gamma(\mu+1)\log\frac{x_1}{x_0}\log\frac{x_2}{x_0}}+\frac{\log\frac{A}{x_0}(\log\frac{x}{A})^{\mu+1}}{\Gamma(\mu+2)\log\frac{x_1}{x_0}\log\frac{x_2}{x_0}}\\ &+\frac{(\log\frac{x}{A})^{\mu+2}}{\Gamma(\mu+3)\log\frac{x_1}{x_0}\log\frac{x_2}{x_0}}-\frac{\log\frac{A}{x_0}(\log\frac{x}{A})^\mu}{2\Gamma(\mu+1)\log\frac{x_2}{x_0}}-\frac{(\log\frac{x}{A})^{\mu+1}}{2\Gamma(\mu+2)\log\frac{x_2}{x_0}},\\ \beta_1(x) &= \frac{\log\frac{A}{x_0}(\log\frac{x}{A})^\mu}{2\Gamma(\mu+1)\log\frac{x_1}{x_0}}+\frac{(\log\frac{x}{A})^{\mu+1}}{2\Gamma(\mu+2)\log\frac{x_1}{x_0}}-[\frac{(\log\frac{A}{x_0})^2(\log\frac{x}{A})^\mu}{2\Gamma(\mu+1)}\\ &+\frac{\log\frac{A}{x_0}(\log\frac{x}{A})^{\mu+1}}{\Gamma(\mu+2)}+\frac{(\log\frac{x}{A})^{\mu+2}}{\Gamma(\mu+3)}-\frac{\bar\theta_1(x)}{2}]\times[\frac{1}{\log\frac{x_1}{x_0}\log\frac{x_2}{x_0}}\\ &+\frac{1}{\log\frac{x_2}{x_1}\log\frac{x_2}{x_0}}]+[\frac{\log\frac{A}{x_0}(\log\frac{x}{A})^\mu}{2\Gamma(\mu+1)}+\frac{(\log\frac{x}{A})^{\mu+1}}{2\Gamma(\mu+2)}]\times[\frac{1}{\log\frac{x_2}{x_0}}\\ &+\frac{\log\frac{x_1}{x_0}}{\log\frac{x_2}{x_1}\log\frac{x_2}{x_0}}]+\frac{\bar\theta_1(x)-\bar\theta_2(x)}{2\log\frac{x_2}{x_1}\log\frac{x_3}{x_1}},\\ \beta_2(x) &= \frac{(\log\frac{A}{x_0})^2(\log\frac{x}{A})^\mu}{2\Gamma(\mu+1)\log\frac{x_2}{x_1}\log\frac{x_2}{x_0}}+\frac{\log\frac{A}{x_0}(\log\frac{x}{A})^{\mu+1}}{\Gamma(\mu+2)\log\frac{x_2}{x_1}\log\frac{x_2}{x_0}}+\frac{\bar\theta_2(x)-\bar\theta_3(x)}{2\log\frac{x_3}{x_2}\log\frac{x_4}{x_2}}\\ &+\frac{(\log\frac{x}{A})^{\mu+2}}{\Gamma(\mu+3)\log\frac{x_2}{x_1}\log\frac{x_2}{x_0}}-\frac{\log\frac{x_1}{x_0}\log\frac{A}{x_0}(\log\frac{x}{A})^\mu}{2\Gamma(\mu+1)\log\frac{x_2}{x_1}\log\frac{x_2}{x_0}}\\ &-\frac{\log\frac{x_1}{x_0}(\log\frac{x}{A})^{\mu+1}+\bar\theta_1(x)\Gamma(\mu+2)}{2\Gamma(\mu+2)\log\frac{x_2}{x_1}\log\frac{x_2}{x_0}}-\frac{\bar\theta_1(x)-\bar\theta_2(x)}{2\log\frac{x_2}{x_1}\log\frac{x_3}{x_1}}-\frac{\bar\theta_1(x)-\bar\theta_2(x)}{2\log\frac{x_3}{x_2}\log\frac{x_3}{x_1}}, \end{aligned} $$

$$
\begin{aligned}
\beta_{M-2}(x) &= \frac{(\log\frac{A}{x_M})^2(\log\frac{x}{A})^{\mu}}{2\Gamma(\mu+1)\log\frac{x_{M-1}}{x_{M-2}}\log\frac{x_M}{x_{M-2}}} + \frac{\log\frac{A}{x_M}(\log\frac{x}{A})^{\mu+1}}{\Gamma(\mu+2)\log\frac{x_{M-1}}{x_{M-2}}\log\frac{x_M}{x_{M-2}}} \\
&\quad + \frac{(\log\frac{x}{A})^{\mu+2}}{\Gamma(\mu+3)\log\frac{x_{M-1}}{x_{M-2}}\log\frac{x_M}{x_{M-2}}} - \frac{\log\frac{x_M}{x_{M-1}}\log\frac{x_M}{A}(\log\frac{x}{A})^{\mu}}{2\Gamma(\mu+1)\log\frac{x_{M-1}}{x_{M-2}}\log\frac{x_M}{x_{M-2}}} \\
&\quad + \frac{\log\frac{x_M}{x_{M-1}}(\log\frac{x}{A})^{\mu+1}}{2\Gamma(\mu+2)\log\frac{x_{M-1}}{x_{M-2}}\log\frac{x_M}{x_{M-2}}} + \frac{\bar\theta_{M-2}(x)}{2\log\frac{x_{M-1}}{x_{M-2}}\log\frac{x_M}{x_{M-2}}} \qquad (3.13) \\
&\quad - \frac{\bar\theta_{M-3}(x)-\bar\theta_{M-2}(x)}{2\log\frac{x_{M-2}}{x_{M-3}}\log\frac{x_{M-1}}{x_{M-3}}} - \frac{\bar\theta_{M-3}(x)-\bar\theta_{M-2}(x)}{2\log\frac{x_{M-1}}{x_{M-2}}\log\frac{x_{M-1}}{x_{M-3}}} \\
&\quad + \frac{\bar\theta_{M-4}(x)-\bar\theta_{M-3}(x)}{2\log\frac{x_{M-2}}{x_{M-3}}\log\frac{x_{M-2}}{x_{M-4}}},
\end{aligned}
$$

$$
\begin{aligned}
\beta_{M-1}(x) &= -\frac{\log\frac{A}{x_M}(\log\frac{x}{A})^{\mu}}{2\Gamma(\mu+1)\log\frac{x_M}{x_{M-1}}} - \frac{(\log\frac{x}{A})^{\mu+1}}{2\Gamma(\mu+2)\log\frac{x_M}{x_{M-1}}} - [\frac{\bar\theta_{M-2}(x)}{2} \\
&\quad + \frac{(\log\frac{A}{x_M})^2(\log\frac{x}{A})^{\mu}}{2\Gamma(\mu+1)} + \frac{\log\frac{A}{x_M}(\log\frac{x}{A})^{\mu+1}}{\Gamma(\mu+2)} + \frac{(\log\frac{x}{A})^{\mu+2}}{\Gamma(\mu+3)}] \\
&\quad \times [\frac{1}{\log\frac{x_{M-1}}{x_{M-2}}\log\frac{x_M}{x_{M-2}}} + \frac{1}{\log\frac{x_M}{x_{M-1}}\log\frac{x_M}{x_{M-2}}}] + [\frac{\log\frac{x_M}{A}(\log\frac{x}{A})^{\mu}}{2\Gamma(\mu+1)} \\
&\quad - \frac{(\log\frac{x}{A})^{\mu+1}}{2\Gamma(\mu+2)}] \times [\frac{\log\frac{x_M}{x_{M-1}}}{\log\frac{x_{M-1}}{x_{M-2}}\log\frac{x_M}{x_{M-2}}} + \frac{1}{\log\frac{x_M}{x_{M-2}}}] \\
&\quad + \frac{\bar\theta_{M-3}(x)-\bar\theta_{M-2}(x)}{2\log\frac{x_{M-1}}{x_{M-2}}\log\frac{x_{M-1}}{x_{M-3}}},
\end{aligned}
$$

$$
\begin{aligned}
\beta_M(x) &= \frac{(\log\frac{x}{A})^{\mu}}{2\Gamma(\mu+1)} + \frac{\log\frac{A}{x_M}(\log\frac{x}{A})^{\mu}}{2\Gamma(\mu+1)\log\frac{x_M}{x_{M-1}}} + \frac{(\log\frac{x}{A})^{\mu+1}}{2\Gamma(\mu+2)\log\frac{x_M}{x_{M-1}}} \\
&\quad + \frac{(\log\frac{A}{x_M})^2(\log\frac{x}{A})^{\mu}}{2\Gamma(\mu+1)\log\frac{x_M}{x_{M-1}}\log\frac{x_M}{x_{M-2}}} + \frac{\log\frac{A}{x_M}(\log\frac{x}{A})^{\mu+1}}{\Gamma(\mu+2)\log\frac{x_M}{x_{M-1}}\log\frac{x_M}{x_{M-2}}} \\
&\quad + \frac{(\log\frac{x}{A})^{\mu+2}}{\Gamma(\mu+3)\log\frac{x_M}{x_{M-1}}\log\frac{x_M}{x_{M-2}}} + \frac{\bar\theta_{M-2}(x)}{2\log\frac{x_M}{x_{M-1}}\log\frac{x_M}{x_{M-2}}} \\
&\quad - \frac{\log\frac{x_M}{A}(\log\frac{x}{A})^{\mu}}{2\Gamma(\mu+1)\log\frac{x_M}{x_{M-2}}} + \frac{(\log\frac{x}{A})^{\mu+1}}{2\Gamma(\mu+2)\log\frac{x_M}{x_{M-2}}},
\end{aligned}
$$

and $\bar\theta_k(x), 1 \le k \le M-2$, is defined as

$$\bar\theta_k(x) = \frac{{}_AH_x^{\mu}\Phi_k(x) - {}_AH_x^{\mu}\Phi_{k+1}(x)}{\log x_{k+1} - \log x_k}. \qquad (3.14)$$

In order to avoid the singularity of the integrand function, one can calculate

${}_AH_x^\mu\Phi_k(x), 2 \le k \le M-2$, as follows

$$\begin{aligned}
{}_AH_x^\mu\Phi_k(x) &= \frac{1}{3\Gamma(\mu)}\int_A^x [(\log\frac{\omega}{x_k})^2 + (\log(1+\delta))^2]^{\frac{3}{2}}(\log\frac{x}{\omega})^{\mu-1}\frac{d\omega}{\omega} \\
&= \frac{1}{\Gamma(\mu+1)}\{\frac{1}{3}[(\log\frac{A}{x_k})^2 + (\log(1+\delta))^2]^{\frac{3}{2}}(\log\frac{x}{A})^\mu \\
&\quad + \int_{\log A}^{\log x}(t-\log x_k)\sqrt{(t-\log x_k)^2 + (\log(1+\delta))^2}(\log x - t)^\mu dt\}.
\end{aligned} \tag{3.15}$$

In order to analysis some properties and error estimates of (3.12), one can also rewrite it as follows

$$\begin{aligned}
&{}_AH_x^\mu\hat{L}_{\log}(u(x)) \\
=& \frac{1}{2}\sum_{k=1}^{M-2}\{u[\log x_{k+2}, \log x_{k+1}, \log x_k] - u[\log x_{k+1}, \log x_k, \log x_{k-1}]\}\bar{\theta}_k(x) \\
&+\frac{1}{2}\{u(\log x_0)\,{}_AH_x^\mu[1] + u[\log x_1, \log x_0]\,{}_AH_x^\mu[\log\frac{x}{x_0}] \\
&+u[\log x_2, \log x_1, \log x_0]\,{}_AH_x^\mu[(\log\frac{x}{x_0})^2]\} \\
&+\frac{1}{2}\{u(\log x_M)\,{}_AH_x^\mu[1] + u[\log x_M, \log x_{M-1}]\,{}_AH_x^\mu[\log\frac{x}{x_M}] \\
&+u[\log x_M, \log x_{M-1}, \log x_{M-2}]\,{}_AH_x^\mu[(\log\frac{x}{x_M})^2]\} \\
&-\frac{1}{2}u[\log x_2, \log x_1, \log x_0]\log\frac{x_1}{x_0}\,{}_AH_x^\mu[\log\frac{x}{x_0}] \\
&-\frac{1}{2}u[\log x_M, \log x_{M-1}, \log x_{M-2}]\log\frac{x_M}{x_{M-1}}\,{}_AH_x^\mu[\log\frac{x_M}{x}],
\end{aligned} \tag{3.16}$$

where $u[\log x_{k+1}, \log x_k, \log x_{k-1}]$ is defined in (2.5).

Similar to Lemma 2.1, we will study the properties and approximation degree of μ-order Hadamard fractional integral of quasi-interpolator $\hat{L}_{\log}(u(x))$ in the following.

Theorem 3.3. *The quasi-interpolation operator ${}_AH_x^\mu\hat{L}_{\log}(u(x))$ satisfies the Hadamard fractional integral regeneration property of quadratic polynomial, i.e. $\forall a_0, a_1, a_2 \in R, u(x) = a_0x^2 + a_1x + a_2$, such that*

$$\sum_{k=0}^{M}[a_0(\log x_k)^2 + a_1\log x_k + a_2]\beta_k(x) = {}_AH_x^\mu[a_0(\log x)^2 + a_1\log x + a_2],$$

where $\beta_k(x)$ is defined by (3.13).

Proof. Set $G(x) = a_0x^2 + a_1x + a_2$, based on (3.16) one have

$$\begin{aligned}
&{}_AH_x^\mu\hat{L}_{\log}(u(x)) \\
=& \frac{1}{2}\sum_{k=1}^{M-2}\{G[\log x_{k+2}, \log x_{k+1}, \log x_k] - G[\log x_{k+1}, \log x_k, \log x_{k-1}]\}\theta_k(x) \\
&+\frac{1}{2}\{[a_0(\log x_0)^2 + a_1\log x_0 + a_2]\,{}_AH_x^\mu[1] + [a_0(\log x_0 + \log x_1) + a_1]\,{}_AH_x^\mu[\log\frac{x}{x_0}]
\end{aligned}$$

$$
\begin{aligned}
&+a_0\, {}_AH_x^\mu[(\log\frac{x}{x_0})^2]-a_0\log\frac{x_1}{x_0}\, {}_AH_x^\mu[\log\frac{x}{x_0}]\}\\
&+\frac{1}{2}\{[a_0(\log x_M)^2+a_1\log x_M+a_2]\, {}_AH_x^\mu[1]+[a_0(\log x_{M-1}+\log x_M)+a_1]\\
&\times {}_AH_x^\mu[\log\frac{x}{x_M}]+a_0\, {}_AH_x^\mu[(\log\frac{x}{x_M})^2]-a_0\log\frac{x_M}{x_{M-1}}\, {}_AH_x^\mu[\log\frac{x_M}{x}]\}\\
=&\ \frac{[a_0(\log A)^2+a_1\log A+a_2](\log\frac{x}{A})^\mu}{\Gamma(\mu+1)}+\frac{[2a_0\log A+a_1](\log\frac{x}{A})^{\mu+1}}{\Gamma(\mu+2)}\\
&+\frac{2a_0(\log\frac{x}{A})^{\mu+2}}{\Gamma(\mu+3)},
\end{aligned}
$$

because

$$
\begin{aligned}
&{}_AH_x^\mu[a_0(\log x)^2+a_1\log x+a_2]\\
&=\frac{[a_0(\log A)^2+a_1\log A+a_2](\log\frac{x}{A})^\mu}{\Gamma(\mu+1)}\\
&\quad+\frac{[2a_0\log A+a_1](\log\frac{x}{A})^{\mu+1}}{\Gamma(\mu+2)}+\frac{2a_0(\log\frac{x}{A})^{\mu+2}}{\Gamma(\mu+3)}.
\end{aligned}
$$

Therefore, based on $G[\log x_2,\log x_1,\log x_0]=G[\log x_M,\log x_{M-1},\log x_{M-2}]=a_0$, we have

$$
{}_AH_x^\mu\hat{L}_{\log}(u(x))={}_AH_x^\mu[a_0(\log x)^2+a_1\log x+a_2].
$$

Hence, we have proved ${}_AH_x^\mu\hat{L}_{\log}(u(x))$ satisfies the Hadamard fractional integral regeneration property of quadric polynomial. So the Theorem 3.3 is proved. □

In the following, we will study the approximation order of the quasi-interpolation operator ${}_AH_x^\mu\hat{L}_{\log}(u(x))$ based on the idea of Theorem 3.2.

Theorem 3.4. *Assumed that the second derivative of $u(\log x)$ is Lipschitz continuous, the approximation capacity of ${}_AH_x^\mu\hat{L}_{\log}(u(x))$ satisfies*

$$
\begin{aligned}
&\|{}_AH_x^\mu\hat{L}_{\log}(u(x))-{}_AH_x^\mu u(\log x)\|_\infty\\
&\le O(\tau^3+\log(1+\delta)\tau^2+(\log(1+\delta))^2\tau+(\log(1+\delta))^2).
\end{aligned}
$$

Proof. According to Theorem 3.3, one can obtain immediately that

$$
\begin{aligned}
&\sum_{k=0}^{M}(\log\frac{x}{x_k})^r\beta_k(x)={}_AH_x^\mu[\sum_{k=0}^{M}(\log\frac{x}{x_k})^r\hat{\alpha}_k(x)]=0, r=1,2,\\
&\sum_{k=0}^{M}\beta_k(x)={}_AH_x^\mu[\sum_{k=0}^{M}\hat{\alpha}_k(x)]={}_AH_x^\mu[1],
\end{aligned}
$$

where $\beta_k(x)={}_AH_x^\mu\hat{\alpha}_k(x), k=0,\cdots,M$.

After direct calculation, it can be immediately obtained that

$$
\begin{aligned}
&\sum_{k=0}^{M}y(\log x_k)\beta_k(x)\\
&={}_AH_x^\mu[\sum_{k=0}^{M}(u(\log x)+u'(\log x)\log\frac{x_k}{x}+\frac{u''(\log x)}{2!}(\log\frac{x_k}{x})^2)\hat{\alpha}_k(x)]
\end{aligned}
$$

$$
\begin{aligned}
&= {}_AH_x^\mu[u(\log x)\sum_{k=0}^{M}\hat{\alpha}_k(x)] + {}_AH_x^\mu[u'(\log x)\sum_{k=0}^{M}\log\frac{x_k}{x}\hat{\alpha}_k(x)] \\
&\quad + \frac{1}{2!}{}_AH_x^\mu[u''(\log x)\sum_{k=0}^{M}(\log\frac{x_k}{x})^2\hat{\alpha}_k(x)] \\
&= {}_AH_x^\mu[u(\log x)].
\end{aligned}
$$

Therefore, one can rewrite $|{}_AH_x^\mu\hat{L}_{\log}(u(x)) - {}_AH_x^\mu u(\log x)|$ in the form

$$
\begin{aligned}
&|{}_AH_x^\mu\hat{L}_{\log}(u(x)) - {}_AH_x^\mu u(\log x)| \\
&= |\sum_{k=0}^{M}[u(\log x_k) - y(\log x_k)]\beta_k(x)| \\
&= {}_AH_x^\mu|\sum_{k=0}^{M}[u(\log x_k) - y(\log x_k)]\hat{\alpha}_k(x)| \\
&= \frac{1}{\Gamma(\mu)}|\int_A^x\sum_{k=0}^{M}[u(\log x_k) - y(\log x_k)]\hat{\alpha}_k(\omega)(\log\frac{x}{\omega})^{\mu-1}\frac{d\omega}{\omega}| \\
&\leq \frac{1}{\Gamma(\mu)}\int_A^x|\sum_{k=0}^{M}[u(\log x_k) - y(\log x_k)]\hat{\alpha}_k(\omega)||\log\frac{x}{\omega}|^{\mu-1}\frac{d\omega}{\omega}.
\end{aligned}
\tag{3.17}
$$

Based on the Lemma 2.3, one has

$$
\begin{aligned}
&\|u(\log x) - \hat{L}_{\log}(u(x))\|_\infty \\
&\leq O(\tau^3) + O(\log(1+\delta)\tau^2) + O((\log(1+\delta))^2\tau) + O((\log(1+\delta))^2).
\end{aligned}
\tag{3.18}
$$

Bringing (3.18) into (3.17), one can obtain that

$$
\begin{aligned}
&|{}_AH_x^\mu\hat{L}_{\log}(u(x)) - {}_AH_x^\mu u(\log x)| \\
&\leq \frac{(\log\frac{x}{A})^\mu}{\Gamma(\mu+1)}|\sum_{k=0}^{M}[u(\log x_k) - y(\log x_k)]\hat{\alpha}_k(\omega)| \\
&\leq \frac{(B-A)^\mu}{A\Gamma(\mu+1)}|\sum_{k=0}^{M}[u(\log x_k) - y(\log x_k)]\hat{\alpha}_k(\omega)| \\
&\leq O(\tau^3) + O(\log(1+\delta)\tau^2) + O((\log(1+\delta))^2\tau) + O((\log(1+\delta))^2).
\end{aligned}
$$

Based on the above analysis, one can get

$$
\begin{aligned}
&\|{}_AH_x^\mu\hat{L}_{\log}(u(x)) - {}_AH_x^\mu u(\log x)\|_\infty \\
&\leq O(\tau^3 + \log(1+\delta)\tau^2 + (\log(1+\delta))^2\tau + (\log(1+\delta))^2).
\end{aligned}
$$

To sum up, the approximation order of ${}_AH_x^\mu\hat{L}_{\log}(u(x))$ has been proven. □

Remark 3.2. When $\delta = O(\tau^{1.5})$, one obtain

$$\|{}_AH_x^\mu\hat{L}_{\log}(u(x)) - {}_AH_x^\mu u(\log x)\|_\infty \leq O(\tau^3).$$

4. Numerical results

In this section, we will provide five numerical examples to demonstrate the effectiveness of using log-type MQ quasi-interpolation operators for solving the Hadamard fractional integral equations and Hadamard fractional differential equations. For simplicity, we choose equidistant partial sample points $\{\log x_k\}_{k=0}^{M}$ and take $A = 1, B = 2$.

Example 4.1. In order to test the approximation of the quasi interpolator ${}_{A}H_{x}^{\mu}\hat{L}_{\log}(u(x))$ to the function ${}_{A}H_{x}^{\mu}u(\log x)$, we choose $u(\log x) = (\log x)^3$.

In Table 1, we set $\tau = \frac{1}{10}$,$\delta = 0.01\tau, 0.1\tau, 0.2\tau, 0.5\tau, \tau, 2\tau$ to observe the accuracy of ${}_{A}H_{x}^{\mu}\hat{L}_{\log}(u(x))$ approaching ${}_{A}H_{x}^{\mu}u(\log x)$. From the Table 1, one can see that the ${}_{A}H_{x}^{\mu}\hat{L}_{\log}(u(x))$ has good accuracy to approximate ${}_{A}H_{x}^{\mu}u(\log x)$.

Table 1. The approximation capacity of ${}_{A}H_{x}^{\mu}\hat{L}_{\log}(u(x))$ as $\tau = \frac{1}{10}$ for Example 4.1.

δ	$\frac{1}{1000}$	$\frac{1}{100}$	$\frac{1}{50}$	$\frac{1}{20}$	$\frac{1}{10}$	$\frac{1}{5}$
τ	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$
$\mu = 0.3$	6.4194E-4	6.5821E-4	7.0656E-4	1.0757E-3	2.4676E-3	6.9136E-3
$\mu = 0.5$	4.1699E-4	4.2435E-4	4.4619E-4	7.3284E-4	1.8066E-3	5.2401E-3
$\mu = 0.7$	2.4640E-4	2.4740E-4	2.7506E-4	5.3772E-4	1.3806E-3	4.1174E-3

In Table 2, we set $\tau = \frac{1}{100}$,$\delta = 0.01\tau, 0.1\tau, 0.2\tau, 0.5\tau, \tau, 2\tau$ to observe the accuracy of ${}_{A}H_{x}^{\mu}\hat{L}_{\log}(u(x))$ approaching ${}_{A}H_{x}^{\mu}u(\log x)$. From the Table 2, one can see that the ${}_{A}H_{x}^{\mu}\hat{L}_{\log}(u(x))$ has high accuracy to approximate ${}_{A}H_{x}^{\mu}u(\log x)$ than Table 1 when $\tau = \frac{1}{10}$.

Table 2. The approximation capacity of ${}_{A}H_{x}^{\mu}\hat{L}_{\log}(u(x))$ as $\tau = \frac{1}{100}$ for Example 4.1.

δ	$\frac{1}{10000}$	$\frac{1}{1000}$	$\frac{1}{500}$	$\frac{1}{200}$	$\frac{1}{100}$	$\frac{1}{50}$
τ	$\frac{1}{100}$	$\frac{1}{100}$	$\frac{1}{100}$	$\frac{1}{100}$	$\frac{1}{100}$	$\frac{1}{100}$
$\mu = 0.3$	1.1582E-5	1.1846E-5	1.2645E-5	1.8216E-5	3.7928E-5	1.1527E-4
$\mu = 0.5$	8.0998E-6	8.2503E-6	8.7056E-6	1.1879E-5	2.3117E-5	6.9919E-5
$\mu = 0.7$	5.2910E-6	5.3590E-6	5.5649E-6	7.0005E-6	1.5900E-5	5.3500E-5

It can be seen from Table 3 that when $\delta = O(\tau^{1.5})$, the convergence order of the quasi interpolator approaches 3. This numerical results are consistent with the theoretical analysis results of Lemma 2.3.

Example 4.2. We consider the Hadamard fractional integral equation as follows

$$\begin{cases} {}_{A}H_{x}^{\mu}u(\log x) = \widehat{F}(\log x, u(\log x)), 1 \le x \le 2, 0 < \mu < 1, \\ u(1) = 0, \end{cases} \tag{4.1}$$

with the following right hand side function

$$\widehat{F}(\log x, u(\log x)) = \frac{6}{\Gamma(\mu+4)}(\log x)^{\mu+3} + (\log x)^3 - u(\log x),$$

Table 3. Maximum errors and decay rate as functions of τ and δ with $\mu = 0.3, 0.5, 0.7$ for Example 4.1.

τ	δ	$\mu = 0.3$	Rate	$\mu = 0.5$	Rate	$\mu = 0.7$	Rate
$\frac{1}{30}$	$\frac{60}{30^{1.5}}$	1.7718E-2	–	1.3137E-2	–	1.0258E-2	–
$\frac{1}{60}$	$\frac{60}{60^{1.5}}$	3.4903E-3	2.3437	2.3585E-3	2.4776	1.8217E-3	2.4934
$\frac{1}{120}$	$\frac{60}{120^{1.5}}$	5.2904E-4	2.7219	3.3106E-4	2.8327	2.5452E-4	2.8394
$\frac{1}{240}$	$\frac{60}{240^{1.5}}$	7.1851E-5	2.8802	4.2902E-5	2.9479	3.2903E-5	2.9515
$\frac{1}{480}$	$\frac{60}{480^{1.5}}$	9.4776E-6	2.9224	5.5694E-6	2.9454	4.1940E-6	2.9718

and the corresponding exact solution $u(\log x) = (\log x)^3$.

Table 4 shows the maximum error and corresponding convergence order when $\mu = 0.3, 0.5, 0.7$, the step size $\tau = \frac{1}{40*i}, i = 1, 2, \cdots, 8$, the shape parameter $\delta = 80\tau^{1.5}$. It can be seen from Table 4 that for all $0 < \mu < 1$, the convergence rate is close to 3. This is in a good agreement with the theoretical prediction of Theorem 3.4.

Table 4. Maximum errors and decay rate as functions of τ and δ with $\mu = 0.3, 0.5, 0.7$ for Example 4.2.

τ	δ	$\mu = 0.3$	Rate	$\mu = 0.5$	Rate	$\mu = 0.7$	Rate
$\frac{1}{40}$	$\frac{80}{40^{1.5}}$	1.6977E-2	–	1.2653E-2	–	6.7125E-3	–
$\frac{1}{80}$	$\frac{80}{80^{1.5}}$	2.3996E-3	2.8227	1.7521E-3	2.8523	1.0458E-3	2.6822
$\frac{1}{120}$	$\frac{80}{120^{1.5}}$	7.2154E-4	2.9636	5.2714E-4	2.9622	3.2798E-4	2.8598
$\frac{1}{160}$	$\frac{80}{160^{1.5}}$	3.0529E-4	2.9897	2.2370E-4	2.9795	1.4259E-4	2.8954
$\frac{1}{200}$	$\frac{80}{200^{1.5}}$	1.5646E-4	2.9957	1.1497E-4	2.9827	7.4231E-5	2.9255
$\frac{1}{240}$	$\frac{80}{240^{1.5}}$	9.0611E-5	2.9959	6.6754E-5	2.9822	4.3402E-5	2.9435
$\frac{1}{280}$	$\frac{80}{280^{1.5}}$	5.7112E-5	2.9942	4.2165E-5	2.9804	2.7523E-5	2.9547
$\frac{1}{320}$	$\frac{80}{320^{1.5}}$	3.8302E-5	2.9917	2.8330E-5	2.9780	1.8534E-5	2.9611

Example 4.3. We consider the fractional differential problem as follows

$$\begin{cases} {}_A D_x^{\mu} u(\log x) = \widehat{F}(\log x, u(\log x)), 1 \leq x \leq 2, 0 < \mu < 1, \\ u(1) = 0, \end{cases} \tag{4.2}$$

and the right hand side function is

$$\widehat{F}(\log x, u(\log x)) = \frac{6(\log x)^{3-\mu}}{\Gamma(4-\mu)} + (\log x)^3 - u(\log x),$$

it can be verified that the exact solution also is $u(\log x) = (\log x)^3$.

Table 5 show the maximum errors and corresponding convergence orders as τ, δ and μ take a series of different values. It can be seen from Table 5 that for all $0 < \mu < 1$, when $\delta = \tau^{1.5}$, the convergence rate is close to $3 - \mu$. The numerical results can well verify the validity the theory of Theorem 3.2.

Table 5. Maximum errors and decay rate as functions of τ and δ with $\mu = 0.3, 0.5, 0.7$ for Example 4.3.

τ	δ	$\mu = 0.3$	Rate	$\mu = 0.5$	Rate	$\mu = 0.7$	Rate
$\frac{1}{20}$	$\frac{1}{20^{1.5}}$	2.2058E-3	–	6.5182E-3	–	2.4414E-2	–
$\frac{1}{40}$	$\frac{1}{40^{1.5}}$	3.6978E-4	2.5765	1.2939E-3	2.3327	5.7088E-3	2.0964
$\frac{1}{60}$	$\frac{1}{60^{1.5}}$	1.2790E-4	2.6182	4.9228E-4	2.3833	2.3725E-3	2.1655
$\frac{1}{80}$	$\frac{1}{80^{1.5}}$	5.9922E-5	2.6357	2.4635E-4	2.4063	1.2611E-3	2.1967
$\frac{1}{100}$	$\frac{1}{100^{1.5}}$	3.3204E-5	2.6456	1.4356E-4	2.4200	7.6929E-4	2.2150
$\frac{1}{120}$	$\frac{1}{120^{1.5}}$	2.0474E-5	2.6520	9.2191E-5	2.4292	5.1255E-4	2.2271

Example 4.4. We consider the fractional differential problem:

$$\begin{cases} {}_A D_x^{\mu} u(\log x) = \widehat{F}(\log x, u(\log x)), 1 \le x \le 2, 0 < \mu < 1, \\ u(1) = 0, \end{cases} \tag{4.3}$$

where

$$\begin{aligned} &\widehat{F}(\log x, u(\log x)) \\ &= \frac{\Gamma(6)}{\Gamma(6-\mu)}(\log x)^{5-\mu} - \frac{\Gamma(5)}{\Gamma(5-\mu)}(\log x)^{4-\mu} \\ &\quad + \frac{2\Gamma(4)}{\Gamma(4-\mu)}(\log x)^{3-\mu} + (\log x)^5 - (\log x)^4 + 2(\log x)^3 - u(\log x), \end{aligned}$$

and the exact solution is $u(\log x) = (\log x)^5 - (\log x)^4 + 2(\log x)^3$.

Table 6 is similar to Table 5, it shows the maximum errors and corresponding convergence orders as τ, δ and μ take a series of different values. We also take $\delta = \tau^{1.5}$, from Table 6, we find the convergence rate is close to $3 - \mu$ for $0 < \mu < 1$.

Table 6. Maximum errors and decay rate as functions of τ and δ with $\mu = 0.3, 0.5, 0.7$ for Example 4.4.

τ	δ	$\mu = 0.3$	Rate	$\mu = 0.5$	Rate	$\mu = 0.7$	Rate
$\frac{1}{20}$	$\frac{1}{20^{1.5}}$	3.9250E-3	–	1.1575E-2	–	4.3227E-2	–
$\frac{1}{40}$	$\frac{1}{40^{1.5}}$	6.9360E-4	2.5005	2.4247E-3	2.2551	1.0681E-2	2.0167
$\frac{1}{60}$	$\frac{1}{60^{1.5}}$	2.4480E-4	2.5684	9.4162E-4	2.3327	4.5333E-3	2.1138
$\frac{1}{80}$	$\frac{1}{80^{1.5}}$	1.1590E-4	2.5990	4.7630E-4	2.3690	2.4362E-3	2.1586
$\frac{1}{100}$	$\frac{1}{100^{1.5}}$	6.4645E-5	2.6165	2.7939E-4	2.3905	1.4961E-3	2.1848
$\frac{1}{120}$	$\frac{1}{120^{1.5}}$	4.0036E-5	2.6279	1.8021E-4	2.4049	1.0014E-3	2.2022
$\frac{1}{140}$	$\frac{1}{140^{1.5}}$	2.6666E-5	2.6361	1.2419E-4	2.4153	7.1177E-4	2.2147

Example 4.5. We consider the Hadamard fractional integration problem as follows

$$\begin{cases} {}_A H_x^{\mu} u(\log x) + {}_A D_x^{\mu} u(\log x) = \widehat{F}(\log x, u(\log x)), 1 \le x \le 2, 0 < \mu < 1, \\ u(1) = 0, \end{cases} \tag{4.4}$$

and the right hand side function is

$$\widehat{F}(\log x, u(\log x)) = \frac{6(\log x)^{3-\mu}}{\Gamma(4-\mu)} + \frac{6(\log x)^{\mu+3}}{\Gamma(\mu+4)}.$$

It can be verified that the exact solution is $u(\log x) = (\log x)^3$.

Figure 1 shows the log-log sketches of the theoretical convergence order with $\mu = 0.3$, $\tau = \frac{1}{20}, \frac{1}{40}, \frac{1}{60}, \frac{1}{80}, \frac{1}{100}$ and shape parameter $\delta = O(\tau^{1.5})$. Figure 2 shows the log-log sketches of the theoretical convergence order with $\mu = 0.6$,$\tau = \frac{1}{30}, \frac{1}{60}, \frac{1}{90}, \frac{1}{120}, \frac{1}{150}$ and shape parameter $\delta = O(\tau^{1.5})$. As estimated by theory, the error convergence order of the scheme is close to $3 - \mu$, that is, we can find that the red line is approximately parallel to the blue line, so the error slope of the curve is 2.7 and 2.4, when $\mu = 0.3, 0.6$ in log-log coordinates.

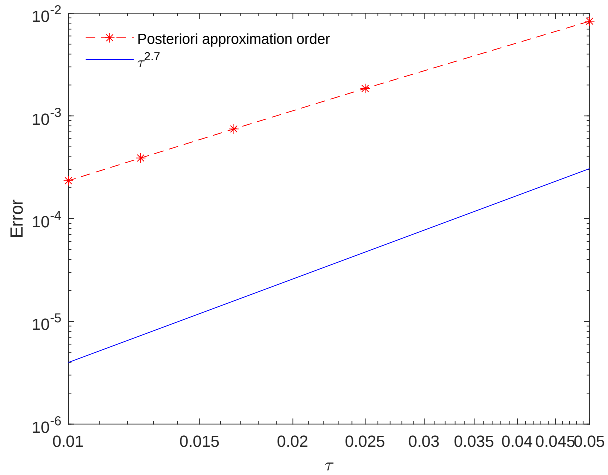

Figure 1. Log-log sketches of approximation orders with $\mu = 0.3$ for Example 4.5.

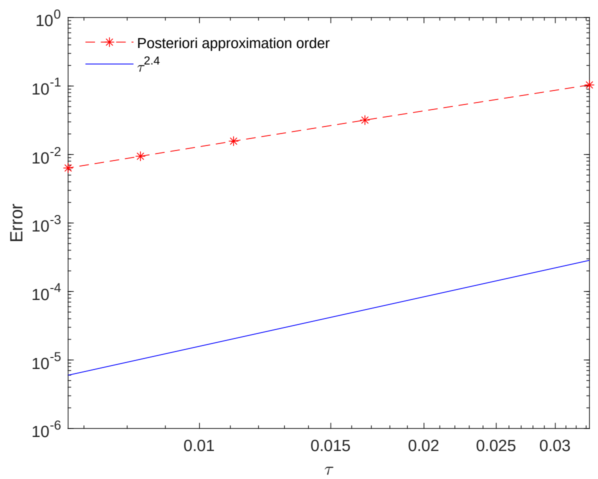

Figure 2. Log-log sketches of approximation orders with $\mu = 0.6$ for Example 4.5.

5. Conclusion

In this paper, the log-type MQ quasi-interpolation operators are constructed. And the quadric polynomial reproduction and convexity-preserving properties of log-type MQ quasi-interpolation operators are studied. Considering the log-type MQ quasi-interpolation operator with advantages of preserving quadratic polynomial and convexity, we use it to solve the Hadamard fractional integral equation and Hadamard fractional differential equation. The approximation order of the numerical scheme based on the log-type MQ quasi-interpolation operators is established. Theoretical analysis indicates that the approximation order of the integral scheme is 3, and the approximation order of the differential scheme is $3-\mu$. The correctness of the theoretical prediction is verified by the linear numerical experiments of Hadamard fractional integral equation and Hadamard fractional differential equation. The numerical results show that it is feasible to construct the numerical scheme with MQ fitting interpolation algorithm.

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