Journal of Applied Analysis and Computation
Volume 14, Number 5, October 2024, 2558–2571
Website:http://www.jaac-online.com
DOI:10.11948/20230032

SOME COMMON FIXED-POINT RESULTS IN GENERALIZED $\mathcal{F}$-METRIC SPACES

Yiquan Li[1], Chuanxi Zhu[2,1,†] and Yingying Xiao[3,†]

Abstract In this paper, we establish a new common fixed-point theorem for multivalued mappings with the greatest lower bound property in generalized $\mathcal{F}$-metric spaces. Also, we propose some new theorems via more general contractions.

Keywords Generalized $\mathcal{F}$-metric space, multivalued mappings, fixed-point.

MSC(2010) 47H10, 54H25, 54D99, 54E99.

1. Introduction

In 2013, Ahmad et al. proposed the concept of a complex-valued metric space, and obtained common fixed-point results for multivalued mappings with the greatest lower bound property [3]. As a generalization of the b-metric spaces [8], the notion of a complex-valued double-controlled metric space was presented in [17]; After that, Amiri et al. have established common fixed-point theorems for multivalued mappings with the greatest lower bound property in this space [6]. Recently, with the establishment of the concept of a $\mathcal{F}$-metric space [11], there are also many interesting results appeared. For instance, by using orbital α-admissibility, Aydi et al. have improved the fixed-point theorem for α-ψ-contractive mappings [7], or several generalizations of the fixed-point results of Reich and Jungck were given in [15]. Furthermore, numerous authors aim to extend and innovate many known results in the corresponding papers, such as Zhu et al. introduced the concept of a generalized $\mathcal{F}$-metric space [20], and proved some fixed-point theorems satisfying Geraghty contraction or JS-contraction, etc, which generalized many fixed-point results in $\mathcal{F}$-metric spaces. For more details, see [1, 2, 4, 5, 9, 10, 12–14, 16, 18, 19]. Inspired by the above results, we have some new opinions with generalized $\mathcal{F}$-metric space, some examples and corollaries are used to enrich our results, and we apply one of the results to solve a class of linear algebraic equation problems, which satisfies all the conditions of Corollary 3.5.

†The corresponding author.
[1]School of Mathematics and Computer Sciences, Nanchang University, 330031 Nanchang, China
[2]School of Mathematics, Dalian University of Technology, 116024 Dalian, China
[3]School of Mathematical Science, Jiangxi Science and Technology Normal University, 330038 Nanchang, China
Email: 907915614@qq.com(Y. Li), chuanxizhu@126.com(C. Zhu), wittes163@163.com(Y. Xiao)

2. Preliminaries

Firstly, let F be the family of all functions $f : (0, +\infty) \to \mathbb{R}$, satisfying

(F_1): f is nondecreasing;

(F_2): $\lim_{n\to\infty} S_n = 0$ if and only if $\lim_{n\to\infty} f(S_n) = -\infty$, where $\{S_n\} \subseteq (0, +\infty)$.

Definition 2.1 ([20]). Let X be a non-empty set, consider the mapping $D : X \times X \to [0, +\infty)$. For all ξ_1, ξ_2, ξ_3 in X, suppose that there exist $(\delta,\, f) \in [0, +\infty) \times F$, such that

(D_1) $D(\xi_1,\, \xi_2) = 0$ if and only if $\xi_1 = \xi_2$;

(D_2) $D(\xi_1,\, \xi_2) = D(\xi_2,\, \xi_1)$;

(D_3) $f(D(\xi_1,\, \xi_2)) \leq f(D(\xi_1,\, \xi_3) + D(\xi_3,\, \xi_2)) + \delta$, if $D(\xi_1,\, \xi_2) > 0$,

then the function D is called a generalized $\mathcal{F}$-metric on X, and (X, D) is called a generalized $\mathcal{F}$-metric space.

Example 2.1. Let $X = \mathbb{R}$, $f(\xi) = -\frac{1}{2\xi}$, $\delta = \frac{1}{2}$, and

$$D(\xi_1, \xi_2) = \begin{cases} \dfrac{1}{2} e^{|\xi_1 - \xi_2|}, & \xi_1 \neq \xi_2, \\ 0, & \xi_1 = \xi_2. \end{cases}$$

Definition 2.2 ([20]). Let (X, D) be a generalized $\mathcal{F}$-metric space and $\{\xi_n\}$ be a sequence in X.

(1) For any $\epsilon > 0$, if there exists a positive integer N such that $D(\zeta, \xi_n) < \epsilon$ for all $n \geq N$, then $\{\xi_n\}$ is called $\mathcal{F}$-convergent to $\zeta \in X$;

(2) For any $\epsilon > 0$, if there exists a positive integer N such that $D(\xi_n, \xi_m) < \epsilon$ for all $n, m \geq N$, then $\{\xi_n\}$ is called a $\mathcal{F}$-Cauchy sequence;

(3) A generalized $\mathcal{F}$-metric space (X, D) is called $\mathcal{F}$-complete if any $\mathcal{F}$-Cauchy sequence in (X, D) is $\mathcal{F}$-convergent.

From [3], we investigate the multivalued mappings with the greatest lower bound property in generalized $\mathcal{F}$-metric spaces, some similar definitions are given as follows:

Definition 2.3. Let (X, D) be a generalized $\mathcal{F}$-metric space and $NCB(X)$ be the set of non-empty, bounded and closed subsets of X. For each ξ_1 in $\mathbb{R}$, we denote $\Delta(\xi_1) = \{\xi_2 \in \mathbb{R} : \xi_1 \leq \xi_2\}$.

In addition, for each $x \in X$ and $A, B \in NCB(X)$,

(i) $\Delta(x, B) = \bigcup_{b \in B} \Delta(D(x, b)) = \bigcup_{b \in B} \{u \in \mathbb{R} : D(x, b) \leq u\}$;

(ii) $\Delta(A, B) = (\bigcap_{a \in A} \Delta(a, B)) \bigcap (\bigcap_{b \in B} \Delta(b, A))$.

Definition 2.4. Let (X, D) be a generalized $\mathcal{F}$-metric space and $R : X \to NCB(X)$ be a multi-valued mapping. For all $x, y \in X$, define $A_x(Ry) = \{u \in \mathbb{R} | u = D(x, z) : z \in Ry\}$.

(i) The multi-valued mapping R has the lower bound on (X, D) if for all $x, y \in X$ there exists $u_0 \in \mathbb{R}$ such that $u_0 \leq u$ for all $u \in A_x(Ry)$;

(ii) The multi-valued mapping R has the greatest lower bound on (X, D) if there exists a greatest lower bound of $A_x(Ry)$ in $\mathbb{R}$, and we write $D(x, Ry) = \inf\{D(x, z) : z \in Ry\}$.

3. Main results

In this section, we introduce a new common fixed-point theorem for the multivalued mappings. In addition, we obtain other fixed-point results in this space, which satisfy more general contractive conditions.

In [20], let Φ be the set of all functions $\phi : [0, \infty) \to [0, \infty)$, where ϕ satisfies:
(1) continuous and nondecreasing;
(2) for any $t > 0$, $\lim\limits_{n\to\infty} \phi^n(t) = 0$.

Obviously, $\phi(0) = 0$, and $\phi(t) < t$ for any $t > 0$.

Theorem 3.1. *Let (X, D) be a $\mathcal{F}$-complete generalized $\mathcal{F}$-metric space and $R, S : X \to NCB(X)$ be multi-valued mappings. Suppose that there exists $\phi \in \Phi$, R and S have the greatest lower bound on (X, D) such that*

$$\phi(\theta(x, y)) \in \Delta(Rx, Sy) \tag{3.1}$$

for all x, y in X, where $\theta(x, y) = \frac{\lambda}{a^b} D(x, y) + \frac{\mu}{a^b} \frac{D(x,Rx)D(y,Sy)}{1+D(x,Rx)}$, $a, b > 1$ and λ, $\mu > 0$ with $\lambda + \mu < 1$. Then R and S have a common fixed-point.

Proof. By selecting any $x_0 \in X$, from (3.1), there exists $x_1 \in Rx_0$ such that

$$\phi(\theta(x_0, x_1)) \in \Delta(Rx_0, Sx_1).$$

Thus for all $a \in Rx_0, b \in Sx_1$, we have

$$\phi(\theta(x_0, x_1)) \in \Delta(a, Sx_1) = \bigcup_{b\in Sx_1} \{u \in \mathbb{R} : D(a, b) \leq u\},$$

and

$$\phi(\theta(x_0, x_1)) \in \Delta(Rx_0, b) = \bigcup_{a\in Rx_0} \{u \in \mathbb{R} : D(a, b) \leq u\}.$$

Since $x_1 \in Rx_0$, then there exists $x_2 \in Sx_1$ such that $D(x_1, x_2) \leq \phi(\theta(x_0, x_1))$. In addition, by using (3.1), we obtain

$$\phi(\theta(x_2, x_1)) \in \Delta(Rx_2, Sx_1).$$

Similarly, owing to $x_2 \in Sx_1$, thus there exists $x_3 \in Rx_2$ such that $D(x_2, x_3) \leq \phi(\theta(x_2, x_1))$. By repeating the above process, we can construct a sequence $\{x_n\}$, where $x_{2n+1} \in Rx_{2n}$, $x_{2n+2} \in Sx_{2n+1}$ for all $n \in \mathbb{N}$, thus we have

$$D(x_{2n+1}, x_{2n+2}) \leq \phi(\theta(x_{2n}, x_{2n+1})), \tag{3.2}$$

and

$$D(x_{2n+2}, x_{2n+3}) \leq \phi(\theta(x_{2n+2}, x_{2n+1})), \tag{3.3}$$

where

$$\theta(x_{2n}, x_{2n+1}) = \frac{\lambda}{a^b} D(x_{2n}, x_{2n+1}) + \frac{\mu}{a^b} \frac{D(x_{2n}, Rx_{2n})D(x_{2n+1}, Sx_{2n+1})}{1 + D(x_{2n}, Rx_{2n})}, \tag{3.4}$$

and

$$\theta(x_{2n+2}, x_{2n+1}) = \frac{\lambda}{a^b} D(x_{2n+2}, x_{2n+1}) + \frac{\mu}{a^b} \frac{D(x_{2n+2}, Rx_{2n+2})D(x_{2n+1}, Sx_{2n+1})}{1 + D(x_{2n+2}, Rx_{2n+2})}. \tag{3.5}$$

Suppose that there exists $n \in \mathbb{N}$ such that $x_{2n} = x_{2n+1}$, it can be proved that $x_{2n+1} = x_{2n+2}$. If not, consider Definition 2.2, we have

$$\begin{aligned}
0 &< D(x_{2n+1}, x_{2n+2}) \\
&\leq \phi(\theta(x_{2n}, x_{2n+1})) \\
&\leq \phi(\frac{\mu}{a^b} D(x_{2n+1}, Sx_{2n+1})) \\
&< \frac{\mu}{a^b} D(x_{2n+1}, x_{2n+2}),
\end{aligned}$$

contradiction. As a result, x_{2n+1} is a common fixed-point of R and S.

On the other hand, if $x_{2n+1} = x_{2n+2}$ for some $n \in \mathbb{N}$, then $x_{2n+2} = x_{2n+3}$ and x_{2n+2} is a common fixed-point of R and S. If not, it can be deduced that

$$\begin{aligned}
0 &< D(x_{2n+2}, x_{2n+3}) \\
&\leq \phi(\theta(x_{2n+2}, x_{2n+1})) \\
&\leq \phi(\frac{\mu}{a^b} D(x_{2n+1}, Sx_{2n+1})) \\
&< \frac{\mu}{a^b} D(x_{2n+1}, x_{2n+2}) \\
&= 0.
\end{aligned}$$

Therefore, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. By using (3.2) and (3.4), we have

$$\begin{aligned}
D(x_{2n+1}, x_{2n+2}) &\leq \phi(\theta(x_{2n}, x_{2n+1})) \\
&\leq \phi(\frac{\lambda}{a^b} D(x_{2n}, x_{2n+1}) + \frac{\mu}{a^b} D(x_{2n+1}, Sx_{2n+1})) \\
&< \frac{\lambda}{a^b} D(x_{2n}, x_{2n+1}) + \frac{\mu}{a^b} D(x_{2n+1}, x_{2n+2}),
\end{aligned}$$

and so $D(x_{2n+1}, x_{2n+2}) < \frac{\lambda}{a^b - \mu} D(x_{2n}, x_{2n+1})$, where $0 < \frac{\lambda}{a^b-\mu} < 1$.

Similarly, using (3.3) and (3.5), we obtain

$$\begin{aligned}
D(x_{2n+2}, x_{2n+3}) &\leq \phi(\theta(x_{2n+2}, x_{2n+1})) \\
&\leq \phi(\frac{\lambda}{a^b} D(x_{2n+2}, x_{2n+1}) + \frac{\mu}{a^b} D(x_{2n+1}, Sx_{2n+1})) \\
&< \frac{\lambda}{a^b} D(x_{2n+2}, x_{2n+1}) + \frac{\mu}{a^b} D(x_{2n+1}, x_{2n+2}),
\end{aligned}$$

thus $D(x_{2n+2}, x_{2n+3}) < \frac{\lambda+\mu}{a^b} D(x_{2n+1}, x_{2n+2})$, where $0 < \frac{\lambda+\mu}{a^b} < 1$.

As a consequence, $D(x_n, x_{n+1}) < \frac{\lambda+\mu}{a^b} D(x_{n-1}, x_n)$. Then,

$$D(x_n, x_{n+1}) < \frac{\lambda+\mu}{a^b} D(x_{n-1}, x_n) < \ ... \ < \{\frac{\lambda+\mu}{a^b}\}^n D(x_0, x_1),$$

it follows that

$$\lim_{n\to\infty} D(x_n, x_{n+1}) = 0. \tag{3.6}$$

Now, it will be shown that $\{x_n\}$ is a $\mathcal{F}$-Cauchy sequence. According to mathematical induction, suppose that $\lim_{n\to\infty} D(x_n, x_{n+k}) = 0$ for some $k \in \mathbb{N}$, consider (D_3), we have

$$f(D(x_n, x_{n+k+1})) \leq \ f(D(x_n, x_{n+k}) + D(x_{n+k}, x_{n+k+1})) + \delta.$$

From (3.6), we can get $\lim_{n\to\infty} D(x_n, x_{n+k}) + D(x_{n+k}, x_{n+k+1}) = 0$. Moreover, according to (F_2), we obtain

$$\lim_{n\to\infty} f(D(x_n, x_{n+k+1})) \le \lim_{n\to\infty} f(D(x_n, x_{n+k}) + D(x_{n+k}, x_{n+k+1})) + \delta \le -\infty,$$

hence,

$$\lim_{n\to\infty} D(x_n, x_{n+k+1}) = 0.$$

As a result, $\lim_{n\to\infty} D(x_n, x_{n+k}) = 0$ for all $k \in \mathbb{N}$, thus $\{x_n\}$ is a $\mathcal{F}$-Cauchy sequence and there exists an element β in X such that $x_n \to \beta$.

Finally, we will prove β is a common fixed-point of R and S. From (3.1), we obtain

$$\phi(\theta(x_{2n}, \beta)) \in \Delta(Rx_{2n}, S\beta) \subseteq \Delta(x_{2n+1}, S\beta),$$

and

$$\phi(\theta(\beta, x_{2n+1})) \in \Delta(R\beta, Sx_{2n+1}) \subseteq \Delta(R\beta, x_{2n+2}),$$

where

$$\theta(x_{2n}, \beta) = \frac{\lambda}{a^b} D(x_{2n}, \beta) + \frac{\mu}{a^b} \frac{D(x_{2n}, Rx_{2n}) D(\beta, S\beta)}{1 + D(x_{2n}, Rx_{2n})},$$

and

$$\theta(\beta, x_{2n+1}) = \frac{\lambda}{a^b} D(\beta, x_{2n+1}) + \frac{\mu}{a^b} \frac{D(\beta, R\beta) D(x_{2n+1}, Sx_{2n+1})}{1 + D(\beta, R\beta)}.$$

Therefore, there exist two sequences $\{u_n\} \subseteq R\beta$ and $\{v_n\} \subseteq S\beta$, such that

$$D(x_{2n+1}, v_n) \le \phi(\frac{\lambda}{a^b} D(x_{2n}, \beta) + \frac{\mu}{a^b} \frac{D(x_{2n}, Rx_{2n}) D(\beta, S\beta)}{1 + D(x_{2n}, Rx_{2n})}),$$

and

$$D(u_n, x_{2n+2}) \le \phi(\frac{\lambda}{a^b} D(\beta, x_{2n+1}) + \frac{\mu}{a^b} \frac{D(x_{2n+1}, Sx_{2n+1}) D(\beta, R\beta)}{1 + D(\beta, R\beta)}).$$

It follows that

$$\lim_{n\to\infty} D(x_{2n+1}, v_n) \le \lim_{n\to\infty} \phi(\frac{\lambda}{a^b} D(x_{2n}, \beta) + \frac{\mu}{a^b} \frac{D(x_{2n}, x_{2n+1}) D(\beta, S\beta)}{1 + D(x_{2n}, x_{2n+1})}) = \phi(0) = 0,$$

and

$$\begin{aligned}\lim_{n\to\infty} D(u_n, x_{2n+2}) &\le \lim_{n\to\infty} \phi(\frac{\lambda}{a^b} D(\beta, x_{2n+1}) + \frac{\mu}{a^b} \frac{D(x_{2n+1}, x_{2n+2}) D(\beta, R\beta)}{1 + D(\beta, R\beta)}) \\ &= \phi(0) \\ &= 0,\end{aligned}$$

i.e.

$$\lim_{n\to\infty} D(x_{2n+1}, v_n) = 0, \tag{3.7}$$

and

$$\lim_{n\to\infty} D(u_n, x_{2n+2}) = 0. \tag{3.8}$$

According to (D_3), we have

$$f(D(\beta, v_n)) \le f(D(\beta, x_{2n+1}) + D(x_{2n+1}, v_n)) + \delta,$$

from (3.7), we get $\lim\limits_{n\to\infty} f(D(\beta,v_n)) \leq \lim\limits_{n\to\infty} f(D(\beta,x_{2n+1})+D(x_{2n+1},v_n))+\delta \leq -\infty$, then

$$\lim_{n\to\infty} D(\beta,v_n)=0,$$

i.e. $v_n \to \beta$.

Similarly,

$$f(D(u_n,\beta)) \leq f(D(u_n,x_{2n+2})+D(x_{2n+2},\beta))+\delta,$$

from (3.8), we get

$$\lim_{n\to\infty} D(u_n,\beta)=0,$$

i.e. $u_n \to \beta$.

Owing to $R\beta$ and $S\beta$ are closed subsets, it follows that $\beta \in (R\beta \bigcap S\beta)$, thus the proof is completed. □

Example 3.1. Let $X=[0,1]$, $D(\xi_1,\xi_2)=(\xi_1-\xi_2)^2$ for all ξ_1,ξ_2 in X, $f(x)=\ln x$ and $\delta=\ln 2$.

In addition, let $R\xi_1=[0,\frac{\xi_1}{10}]$, $S\xi_2=[0,\frac{\xi_2}{20}]$, $\phi(\xi)=\frac{\xi}{2}$, $a=\sqrt{2}$, $b=2$ and $\lambda=\mu=\frac{1}{100}$. Anyone can easily check that $D(\xi_1,R\xi_1)=\frac{81}{100}\xi_1^2$, $D(\xi_2,S\xi_2)=\frac{361}{400}\xi_2^2$ and $\Delta(R\xi_1,S\xi_2)=\Delta((\frac{\xi_1}{10}-\frac{\xi_2}{20})^2)$.

Suppose that $2\xi_1<\xi_2$, we have

$$\begin{aligned}
(\frac{\xi_1}{10}-\frac{\xi_2}{20})^2 &\leq \frac{1}{400}(\xi_1-\xi_2)^2\\
&\leq \frac{1}{400}(\xi_1-\xi_2)^2+\frac{1}{400}\frac{\frac{81}{100}\xi_1^2}{1+\frac{81}{100}\xi_1^2}\frac{361}{400}\xi_2^2\\
&=\frac{1}{2}(\frac{\frac{1}{100}}{(\sqrt{2})^2}D(\xi_1,\xi_2)+\frac{\frac{1}{100}}{(\sqrt{2})^2}\frac{D(\xi_1,R\xi_1)}{1+D(\xi_1,R\xi_1)}D(\xi_2,S\xi_2))\\
&=\phi(\theta(\xi_1,\xi_2)),
\end{aligned}$$

then $\phi(\theta(\xi_1,\xi_2))\in\Delta(R\xi_1,S\xi_2)$.

Therefore, the conditions of Theorem 3.1 are satisfied, R and S have a common fixed-point $\xi=0$.

If two multi-valued mappings R and S are supposed to be equal, then Theorem 3.1 reduces to below corollary.

Corollary 3.1. *Let (X,D) be a $\mathcal{F}$-complete generalized $\mathcal{F}$-metric space and $R: X\to NCB(X)$ be a multi-valued mapping. Suppose that there exists $\phi\in\Phi$, and R has the greatest lower bound on (X,D), such that*

$$\phi(\theta(x,y))\in\Delta(Rx,Ry)$$

for all x,y in X, where $\theta(x,y)=\frac{\lambda}{a^b}D(x,y)+\frac{\mu}{a^b}\frac{D(x,Rx)D(y,Ry)}{1+D(x,Rx)}$, $a,b>1$ and λ, $\mu>0$ with $\lambda+\mu<1$. Then R has a fixed-point.

Obviously, if two multi-valued mappings R and S are supposed to be self-mappings, then the following corollary holds.

Corollary 3.2. *Let (X, D) be a $\mathcal{F}$-complete generalized $\mathcal{F}$-metric space and $R, S : X \to X$ be self-mappings. Suppose that there exists $\phi \in \Phi$ such that*

$$D(Rx, Sy) \leq \phi(\theta(x, y))$$

for all x, y in X, where $\theta(x,y) = \frac{\lambda}{a^b} D(x,y) + \frac{\mu}{a^b} \frac{D(x,Rx)D(y,Sy)}{1+D(x,Rx)}$, $a, b > 1$ and λ, $\mu > 0$ with $\lambda + \mu < 1$. Then R and S have a common fixed-point.

In [12], let L be the family of all continuous and nondecreasing functions $\omega : [0, +\infty) \to [0, +\infty)$, where ω satisfies:
(1) $\omega(0) = 0$, and $\omega(x) > 0$ for each $x > 0$;
(2) $\lim\limits_{n\to\infty} x_n = 0$ if and only if $\lim\limits_{n\to\infty} \omega(x_n) = 0$, where $\{x_n\} \subseteq (0, +\infty)$.

Theorem 3.2. *Let (X, D) be a $\mathcal{F}$-complete generalized $\mathcal{F}$-metric space and $R, S : X \to X$ be self-mappings. Suppose that there exists $\omega \in L$ such that*

$$D(Rx, Sy) \leq M(x, y) - \omega(M(x, y)) \tag{3.9}$$

for all x, y in X, where

$$M(x, y) = \max\{D(x, Rx), D(y, Sy), D(x, y)\}.$$

Then R and S have a unique common fixed-point.

Proof. By selecting any $x_0 \in X$, we can construct a sequence $\{x_n\}$ such that $x_{2n+1} = Rx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$. If $x_{2n} = x_{2n+1}$ for some $n \in \mathbb{N}$, then $x_{2n+1} = x_{2n+2}$, and so x_{2n+1} is a common fixed-point of R and S.

In fact, if $x_{2n+1} \neq x_{2n+2}$, from (3.9), we have

$$D(x_{2n+1}, x_{2n+2}) = D(Rx_{2n}, Sx_{2n+1}) \leq M(x_{2n}, x_{2n+1}) - \omega(M(x_{2n}, x_{2n+1})),$$

where

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\{D(x_{2n}, x_{2n+1}), D(x_{2n+1}, x_{2n+2}), D(x_{2n}, x_{2n+1})\} \\ &= D(x_{2n+1}, x_{2n+2}). \end{aligned}$$

It follows that

$$0 < D(x_{2n+1}, x_{2n+2}) \leq D(x_{2n+1}, x_{2n+2}) - \omega(D(x_{2n+1}, x_{2n+2})) < D(x_{2n+1}, x_{2n+2}),$$

contradiction. Similarly, if $x_{2n+1} = x_{2n+2}$ for some $n \in \mathbb{N}$ and $x_{2n+2} \neq x_{2n+3}$, then

$$\begin{aligned} D(x_{2n+2}, x_{2n+3}) &= D(x_{2n+3}, x_{2n+2}) \\ &= D(Rx_{2n+2}, Sx_{2n+1}) \\ &\leq M(x_{2n+2}, x_{2n+1}) - \omega(M(x_{2n+2}, x_{2n+1})), \end{aligned}$$

where

$$\begin{aligned} M(x_{2n+2}, x_{2n+1}) &= \max\{D(x_{2n+2}, x_{2n+3}), D(x_{2n+1}, x_{2n+2}), D(x_{2n+2}, x_{2n+1})\} \\ &= D(x_{2n+2}, x_{2n+3}). \end{aligned}$$

So we have

$$0 < D(x_{2n+2}, x_{2n+3}) \leq D(x_{2n+2}, x_{2n+3}) - \omega(D(x_{2n+2}, x_{2n+3})) < D(x_{2n+2}, x_{2n+3}),$$

contradiction, thus we get $x_{2n+2} = x_{2n+3}$, and x_{2n+2} is a common fixed-point of R and S.

As a consequence, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, and it can be easily deduced that

$$D(x_{2n+1}, x_{2n+2}) \leq D(x_{2n}, x_{2n+1}), \tag{3.10}$$

and

$$D(x_{2n+2}, x_{2n+3}) \leq D(x_{2n+1}, x_{2n+2}). \tag{3.11}$$

Owing to (3.10) and (3.11), we obtain $D(x_n, x_{n+1}) \leq D(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. Moreover, $\{D(x_n, x_{n+1})\}$ is a monotonous nonincreasing sequence and we assume that $\lim\limits_{n\to\infty} D(x_n, x_{n+1}) = e \geq 0$.

If $e > 0$, so by letting $n \to \infty$ at both sides of (3.10) and (3.11), we get

$$e < e - \omega(e) < e,$$

contradiction. It can be easily shown that $\lim\limits_{n\to\infty} D(x_n, x_{n+1}) = 0$, following the proof process of Theorem 3.1, we get $\{x_n\}$ is a $\mathcal{F}$-Cauchy sequence and there exists an element β in X such that $x_n \to \beta$.

Now, we will prove β is a unique common fixed-point of R and S. If $\beta \neq R\beta$, then

$$\begin{aligned} 0 &< \inf\{D(x, \beta) + D(x, Rx) : x \in X\} \\ &\leq \inf\{D(x_{2n}, \beta) + D(x_{2n}, Rx_{2n}) : n \in \mathbb{N}\} \\ &\leq 0 \text{ as } n \to \infty, \end{aligned}$$

contradiction, thus $\beta = R\beta$.

Uniqueness: Suppose that β and η are two common fixed-points of R and S, $\beta \neq \eta$, so that

$$0 < D(\beta, \eta) = D(R\beta, S\eta) \leq M(\beta, \eta) - \omega(M(\beta, \eta)),$$

where

$$M(\beta, \eta) = \max\{D(\beta, R\beta), D(\eta, S\eta), D(\beta, \eta)\} = D(\beta, \eta).$$

Hence,

$$0 < D(\beta, \eta) \leq D(\beta, \eta) - \omega(D(\beta, \eta)) < D(\beta, \eta),$$

contradiction.

It follows that $D(\beta, \eta) = 0$, i.e. $\beta = \eta$, the proof is completed. □

Example 3.2. Let $X = \mathbb{R}$, for all x, y in $\mathbb{R}$, $Rx = \frac{(x-1)}{4} + x$, $Sy = \frac{(y-1)}{4} + 1$, and $\omega(t) = 1 - \frac{1}{e^t}$ for all $t \geq 0$.

Furthermore, let $f(t) = -\frac{1}{t}$, $\delta = 1$, and

$$D(x, y) = \begin{cases} e^{|x-y|}, & x \neq y, \\ 0, & x = y. \end{cases}$$

It can be proved that (X, D) is a generalized $\mathcal{F}$-metric space (see [11]), and

$$D(Rx, Sy) = \begin{cases} e^{|\frac{x-y}{4} + x - 1|}, & Rx \neq Sy, \\ 0, & Rx = Sy. \end{cases}$$

Suppose that $2 \leq 4|x-1| \leq |x-y|$, we have

$$e^{|\frac{x-y}{4}+x-1|} \leq e^{\frac{|x-y|}{2}},$$

and

$$e^{\frac{|x-y|}{2}} + 1 \leq e^{|x-y|}.$$

Thus,

$$D(Rx, Sy) \leq D(x, y) - \omega(M(x, y)) \leq M(x, y) - \omega(M(x, y)),$$

where $M(x, y) = \max\{D(x, Rx), D(y, Sy), D(x, y)\}$.

Obviously, the conditions of Theorem 3.2 are satisfied, R and S have a unique common fixed-point $\xi = 1$.

It is clear that if R and S be equal, then Theorem 3.2 reduces to the following corollary.

Corollary 3.3. *Let (X, D) be a $\mathcal{F}$-complete generalized $\mathcal{F}$-metric space and $R : X \to X$ be a self-mapping. Suppose that there exist $\phi \in \Phi$ and $\omega \in L$, such that*

$$D(Rx, Ry) \leq \phi(M(x, y)) - \omega(M(x, y))$$

for all x, y in X, where

$$M(x, y) = \max\{D(x, Rx), D(y, Sy), D(x, y)\}.$$

Then R has a unique fixed-point.

Theorem 3.3. *Let (X, D) be a $\mathcal{F}$-complete generalized $\mathcal{F}$-metric space $R, S : X \to X$ be nondecreasing mappings. Suppose that there exists a continuous and nonincreasing mapping $T : X \to [0, 1)$, such that*

$$D(Rx, Sy) \leq (Tx - Ty)M(x, y) \tag{3.12}$$

for all x, y in X, where

$$M(x, y) = \max\{D(x, y), D(x, Rx), D(y, Sy)\}.$$

If $f \in F$ is an invertible function, then R and S have a unique common fixed-point.

Proof. By selecting any $x_0 \in X$, according to the property of R and S, we can construct a nondecreasing sequence $\{x_n\}$, such that $x_{2n+1} = Rx_{2n} \geq x_{2n}$ and $x_{2n+2} = Sx_{2n+1} \geq x_{2n+1}$ for all $n \in \mathbb{N}$. Inspired by the proof process of Theorem 3.1 and Theorem 3.2, we also assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. From (3.12), we get

$$D(x_{2n+1}, x_{2n+2}) = D(Rx_{2n}, Sx_{2n+1}) \leq (Tx_{2n} - Tx_{2n+1})M(x_{2n}, x_{2n+1}),$$

where

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\{D(x_{2n}, x_{2n+1}), D(x_{2n}, x_{2n+1}), D(x_{2n+1}, x_{2n+2})\} \\ &= \max\{D(x_{2n}, x_{2n+1}), D(x_{2n+1}, x_{2n+2})\}. \end{aligned}$$

If $M(x_{2n}, x_{2n+1}) = D(x_{2n+1}, x_{2n+2})$, then

$$D(x_{2n+1}, x_{2n+2}) \leq (Tx_{2n} - Tx_{2n+1})D(x_{2n+1}, x_{2n+2}),$$

contradiction. Hence,

$$D(x_{2n+1}, x_{2n+2}) \leq (Tx_{2n} - Tx_{2n+1})D(x_{2n}, x_{2n+1}).$$

Similarly, we have

$$D(x_{2n+3}, x_{2n+2}) = D(Rx_{2n+2}, Sx_{2n+1}) \leq (Tx_{2n+2} - Tx_{2n+1})M(x_{2n+2}, x_{2n+1}),$$

where

$$\begin{aligned} M(x_{2n+2}, x_{2n+1}) &= \max\{D(x_{2n+2}, x_{2n+1}), D(x_{2n+2}, x_{2n+3}), D(x_{2n+1}, x_{2n+2})\} \\ &= \max\{D(x_{2n+2}, x_{2n+1}), D(x_{2n+2}, x_{2n+3})\}. \end{aligned}$$

Obviously, $M(x_{2n+2}, x_{2n+1}) = D(x_{2n+2}, x_{2n+1})$, hence,

$$D(x_{2n+3}, x_{2n+2}) \leq (Tx_{2n+2} - Tx_{2n+1})D(x_{2n+2}, x_{2n+1}).$$

As a result, $D(x_n, x_{n+1}) \leq (Tx_{n-1} - Tx_n)D(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$, we obtain

$$\frac{D(x_n, x_{n+1})}{D(x_{n-1}, x_n)} \leq Tx_{n-1} - Tx_n,$$

and so

$$\sum_{k=1}^{n} \frac{D(x_k, x_{k+1})}{D(x_{k-1}, x_k)} \leq \sum_{k=1}^{n}(Tx_{k-1} - Tx_k) = Tx_0 - Tx_n < \infty,$$

then

$$\lim_{k\to\infty} \frac{D(x_k, x_{k+1})}{D(x_{k-1}, x_k)} = 0,$$

which implies there exist $\kappa \in (0,1)$ and $n_0 \in \mathbb{N}$ such that $D(x_n, x_{n+1}) \leq \kappa\, D(x_{n-1}, x_n)$ for all $n \geq n_0$, it can be easily proved that

$$\lim_{n\to\infty} D(x_{n-1}, x_n) = 0.$$

By continuing the proof process of Theorem 3.1, we get $\{x_n\}$ is a $\mathcal{F}$-Cauchy sequence and there exists an element β in X such that $x_n \to \beta$.

Now, we will prove $R\beta = \beta$, if not, owing to (D_3), we have

$$\begin{aligned} f(D(\beta, R\beta)) &\leq f(D(\beta, x_{2n+2}) + D(x_{2n+2}, R\beta)) + \delta \\ &= f(D(\beta, x_{2n+2}) + D(R\beta, Sx_{2n+1})) + \delta \\ &\leq f(D(\beta, x_{2n+2}) + (T\beta - Tx_{2n+1})M(\beta, x_{2n+1})) + \delta, \end{aligned}$$

where

$$\begin{aligned} M(\beta, x_{2n+1}) &= \max\{D(\beta, x_{2n+1}), D(\beta, R\beta), D(x_{2n+1}, x_{2n+2})\} \\ &= D(\beta, R\beta) \text{ as } n \to \infty. \end{aligned}$$

It yields that

$$f^{-1}[f(D(\beta, R\beta)) - \delta] \leq D(\beta, x_{2n+1}) + (T\beta - Tx_{2n+1})D(\beta, R\beta),$$

by using the property of T, we obtain $\lim_{n\to\infty} f^{-1}[f(D(\beta, R\beta)) - \delta] \leq 0$, then

$$\lim_{n\to\infty} f(D(\beta, R\beta)) \leq -\infty,$$

which is contradict with $D(\beta, R\beta) > 0$. Therefore, $\beta = R\beta$.

Similarly, owing to the continuity of T, we have

$$\begin{aligned} f(D(\beta, S\beta)) &\leq f(D(\beta, x_{2n+1}) + D(x_{2n+1}, S\beta)) + \delta \\ &= f(D(\beta, x_{2n+1}) + D(Rx_{2n}, S\beta)) + \delta \\ &\leq f(D(\beta, x_{2n+1}) + (Tx_{2n} - T\beta)M(x_{2n}, \beta)) + \delta \\ &= -\infty \text{ as } n \to \infty, \end{aligned}$$

as a result, $\beta = S\beta$.

Uniqueness: Assume that β and γ are two common fixed-points of R and S, $\gamma \neq \beta$, and so

$$0 < D(\beta, \gamma) = D(R\beta, S\gamma) \leq (T\beta - T\gamma)M(\beta, \gamma),$$

clearly, $M(\beta, \gamma) = D(\beta, \gamma)$, thus,

$$0 < D(\beta, \gamma) \leq (T\beta - T\gamma)D(\beta, \gamma), \tag{3.13}$$

and

$$0 < D(\gamma, \beta) \leq (T\gamma - T\beta)D(\gamma, \beta). \tag{3.14}$$

According to (D_2), if $T\beta \neq T\gamma$, it can be proved that (3.13) and (3.14) cannot be established at the same time. As a consequence, $D(\beta, \gamma) = 0$, i.e. $\beta = \gamma$, the proof is completed.

Using Theorem 3.3 with $R = S$, we can easily obtain the following corollary. □

Corollary 3.4. *Let (X, D) be a $\mathcal{F}$-complete generalized $\mathcal{F}$-metric space and $R : X \to X$ be a nondecreasing mapping. Suppose that there exists a continuous and nonincreasing mapping $T : X \to [0, 1)$, such that*

$$D(Rx, Ry) \leq (Tx - Ty)M(x, y)$$

for all x, y in X, where

$$M(x, y) = \max\{D(x, y), D(x, Rx), D(y, Ry)\},$$

then R has a unique fixed-point.

Since $0 \leq T(x) < 1$, thus Theorem 3.3 can reduce below corollary.

Corollary 3.5. *Let (X, D) be a $\mathcal{F}$-complete generalized $\mathcal{F}$-metric space and $R, S : X \to X$ be self-mappings. Suppose that there exists $k \in [0, 1)$ such that*

$$D(Rx, Sy) \leq kM(x, y)$$

for all x, y in X, where

$$M(x, y) = \max\{D(x, y), D(x, Rx), D(y, Sy)\},$$

then R and S have a unique common fixed-point.

Let $R = S$ in Corollary 3.5, it follows that

Corollary 3.6 ([20]). *Let (X, D) be a $\mathcal{F}$-complete generalized $\mathcal{F}$-metric space and $R : X \to X$ be a self-mapping. Suppose that there exists $k \in [0, 1)$ such that*

$$D(Rx, Ry) \leq kD(x, y)$$

for all x, y in X, then R has a unique fixed-point.

4. Application

In this section, we will apply Corollary 3.5 to solve a system of linear algebraic equations as follows:

Consider the following linear algebraic equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n + \beta_1 = 0, \\ ... \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n + \beta_n = 0, \end{cases} \tag{4.1}$$

and

$$\begin{cases} c_{11}y_1 + c_{12}y_2 + ... + c_{1n}y_n + \beta_1 = 0, \\ ... \\ c_{n1}y_1 + c_{n2}y_2 + ... + c_{nn}y_n + \beta_n = 0. \end{cases} \tag{4.2}$$

Then (4.1) can be written as $Ax+\beta = O$, where $A = (a_{ij})_{n\times n}$, $O = (0, 0, ..., 0)^T$, $x = (x_1, x_2, ..., x_n)^T$ and $\beta = (\beta_1, \beta_2, ..., \beta_n)^T$. Similarly, (4.2) can be written as $Cy + \beta = O$, where $y = (y_1, y_2, ..., y_n)^T$ and $C = (c_{ij})_{n\times n}$.

Let $X = \mathbb{R}^n$, for all x, y in X, $D(x, y) = \max\limits_{1\leq i\leq n} |x_i - y_i|^2$, $f(x) = \ln x$ and $\delta = \ln 2$. Moreover, we define two self-mappings $R, S : \mathbb{R}^n \to \mathbb{R}^n$ as

$$Rx = Bx + \beta, \tag{4.3}$$

and

$$Sy = Dy + \beta, \tag{4.4}$$

where $B = (b_{ij})_{n\times n}$, $b_{ii} = a_{ii} + 1$ and $b_{ij} = a_{ij}$ if $i \neq j$, and also $D = (d_{ij})_{n\times n}$, $d_{ii} = c_{ii} + 1$ and $d_{ij} = c_{ij}$ if $i \neq j$.

Clearly, the linear algebraic equations (4.1) and (4.2) have a common solution x^* in X if and only if x^* is a common fixed-point of R and S. For all $1 \leq i \leq n$, suppose that

$$\sum_{j=1}^{n} (b_{ij}x_j - d_{ij}y_j) \leq \max_{1\leq j\leq n} \sqrt{k}(x_j - y_j), \tag{4.5}$$

where $0 \leq k < 1$. From (4.3)-(4.5), we get

$$D(Rx, Sy) = \max_{1\leq i\leq n} [\sum_{j=1}^{n} (b_{ij}x_j - d_{ij}y_j)]^2 \leq \max_{1\leq j\leq n} k(x_j - y_j)^2 \leq kM(x, y).$$

Obviously, all conditions of Corollary 3.5 are satisfied, R and S have a common fixed-point x^*, and so x^* is a common solution of the linear algebraic equations (4.1) and (4.2).

5. Conclusion

In short, we have obtained some interesting and latest fixed-point results in generalized $\mathcal{F}$-metric spaces, and also an application for solving the linear algebraic equations. Applying these results to the field of integral equation or differential equation is worth spending more time to study.

Acknowledgments

The authors would like to express their sincere gratitude to the referee for careful reading of the manuscript and corrections.

References

[1] H. Afshari, M. Atapour and E. Karapınar, *A discussion on a generalized geraghty multi-valued mappings and applications*, Adv. Difference Equ., 2020, 2020(356), 14 pp.

[2] J. Ahmad, A. S. Al-Rawashdeh and A. E. Al-Mazrooei, *Fixed point results for* $(\alpha, \perp_{\mathcal{F}})$*-contractions in orthogonal* $\mathcal{F}$*-metric spaces with applications*, J. Funct. Space., 2022. DOI: 10.1155/2022/8532797.

[3] J. Ahmad, C. Klin-Eam and A. Azam1, *Common fixed points for multivalued mappings in complex valued metric spaces with applications*, Abstr. Appl. Anal., 2013, Art ID 854965, 12 pp.

[4] A. E. Al-Mazrooei and J. Ahmad, *Fixed point theorems for rational contractions in* $\mathcal{F}$*-metric spaces*, J. Math. Anal., 2019, 10, 79–86.

[5] S. A. Al-Mezel, J. Ahmad and G. Marino, *Fixed point theorems for generalized* $(\alpha\beta$-$\psi)$*-contractions in* $\mathcal{F}$*-metric spaces with applications*, Mathematics-Basel., 2020, 8, 584. DOI: 10.3390/math8040584.

[6] P. Amiri and H. Afshari, *Common fixed point results for multi-valued mappings in complex-valued double controlled metric spaces and their applications to the existence of solution of fractional integral inclusion systems*, Chaos. Soliton. Fract., 2022, 154, 15 pp.

[7] H. Aydi, E. Karapınar, Z. D. Mitrović and T. Rashid, *A remark on "Existence and uniqueness for a neutral differential problem with unbounded delay via fixed point results* $\mathcal{F}$*-metric spaces"*, RACSAM., 2019, 113(4), 3197–3206.

[8] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Semin. Mat. Fis. Univ. Modena., 1998, 46, 263–276.

[9] O. Ege, C. Park and A. H. Ansari, *A different approach to complex valued* G_b*-metric spaces*, Adv. Differ. Equ., 2020, 2020(152), 13 pp.

[10] Y. F. Fan, C. X. Zhu and Z. Q. Wu, *Some* φ*-coupled fixed point results via modified F-control function's concept in metric spaces and its applications*, J. Comput. Appl. Math., 2019, 349, 70–81.

[11] M. Jleli and B. Samet, *On a new generalization of metric spaces*, J. Fix Point Theory A, 2018. DOI: 10.1007/s11784-018-0606-6.

[12] H. Lakzian and B. E. Rhoades, *Some fixed point theorems using weaker Meir-Keeler function in metric spaces with w-distance*, Appl. Math. Comput., 2019, 342, 18–25.

[13] Y. Q. Li, C. X. Zhu, Y. Y. Xiao and L. Zhou, *A note on three different contractions in partially ordered complex valued G_b-metric spaces*, AIMS. Math., 2022, 7, 12322–12341.

[14] M. Liang, C. X. Zhu, C. F. Chen and Z. Q. Wu, *Some new theorems for cyclic contractions in G_b-metric spaces and some applications*, Appl. Math. Comput., 2019, 346, 545–558.

[15] Z. D. Mitrović, H. Aydi, N. Hussain and A. Mukheimer, *Reich, Jungck, and Berinde common fixed point results on $\mathcal{F}$-metric spaces and an application*, Mathematics-Basel., 2019, 7(5), 387.

[16] Z. Mustafa, R. J. Shahkoohi, V. Parvaneh, Z. Kadelburg and M. M. M. Jaradat, *Ordered S_p-metric spaces and some fixed point theorems for contractive mappings with application to periodic boundary value problems*, Fixed Point Theory A, 2019, 2019, 16.

[17] S. K. Panda, T. Abdeljawad and C. Ravichandran, *A complex valued approach to the solutions of Riemann-Liouville integral, Atangana-Baleanu integral operator and non-linear telegraph equation via fixed point method*, Chaos. Soliton. Fract., 2020, 130, 11 pp.

[18] T. Senapati, L. K. Dey, A. Chanda and H. P. Huang, *Some non-unique fixed point or periodic point results in JS-metric spaces*, J. Fix Point Theory Appl., 2019, 21(51), 15 pp.

[19] A. Shoaib, Z. Nisar, A. Hussain, O. Ozer and M. Arshad, *Modified Banach fixed point results for locally contractive mappings in complete G_d-metric like space*, Electr. J. Math. Anal. Appl., 2018, 6, 144–155.

[20] C. X. Zhu, J. Chen, J. H. Chen, C. F. Chen and H. P. Huang, *A new generalization of $\mathcal{F}$-metric spaces and some fixed point theorems and an application*, J. Appl. Anal. Comput., 2021, 11, 2649–2663.