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ON THE WELL-POSEDNESS AND STABILITY FOR CARBON NANOTUBES AS COUPLED TWO TIMOSHENKO BEAMS WITH FRICTIONAL DAMPINGS

Aissa Guesmia[1]

Abstract The objective of this paper is to study the well-posedness and stability questions for double wall carbon nanotubes modeled as linear one-dimensional coupled two Timoshenko beams in a bounded domain under frictional dampings. First, we prove the well-posedness of our system by applying the semigroups theory of linear operators. Second, we show several strong, non-exponential, exponential, polynomial and non-polynomial stability results depending on the number of frictional dampings, their position and some connections between the coefficients. In some cases, the optimality of the polynomial decay rate is also proved. The proofs of these stability results are based on a combination of the energy method and the frequency domain approach.

Keywords Coupled Timoshenko beams, well-posedness, asymptotic behavior, semigroups theory, energy method, frequency domain approach.

MSC(2010) 35B40, 35L45, 74H40, 93D20, 93D15.

1. Introduction

The system under consideration in this paper is the following:

$$\begin{cases} \varphi_{tt} - k_1\left(\varphi_x + \psi\right)_x - k_0\left(w - \varphi\right) + \tau_1 a_1 \varphi_t = 0 \text{ in } (0,1)\times(0,\infty), \\ \psi_{tt} - k_2\psi_{xx} + k_1\left(\varphi_x + \psi\right) + \tau_2 a_2 \psi_t = 0 \quad \text{in } (0,1)\times(0,\infty), \\ w_{tt} - k_3\left(w_x + z\right)_x + k_0\left(w - \varphi\right) + \tau_3 a_3 w_t = 0 \text{ in } (0,1)\times(0,\infty), \\ z_{tt} - k_4 z_{xx} + k_3\left(w_x + z\right) + \tau_4 a_4 z_t = 0 \quad \text{in } (0,1)\times(0,\infty) \end{cases} \tag{1.1}$$

along with the homogeneous Dirichlet-Neumann boundary conditions

$$\begin{cases} \varphi_x(0,t) = \psi(0,t) = w_x(0,t) = z(0,t) = 0 \text{ in } (0,\infty), \\ \varphi(1,t) = \psi_x(1,t) = w(1,t) = z_x(1,t) = 0 \text{ in } (0,\infty) \end{cases} \tag{1.2}$$

[1]Institut Elie Cartan de Lorraine, UMR 7502, Université de Lorraine, 3 Rue Augustin Fresnel, BP 45112, 57073 Metz Cedex 03, France
Email: aissa.guesmia@univ-lorraine.fr(A. Guesmia)

and the initial data

$$\begin{cases} \varphi\left(x,0\right)=\varphi_0\left(x\right), \, \psi\left(x,0\right)=\psi_0\left(x\right), \, w\left(x,0\right)=w_0\left(x\right), \, z\left(x,0\right)=z_0\left(x\right) \quad \text{ in } \left(0,1\right), \\ \varphi_t\left(x,0\right)=\varphi_1\left(x\right), \, \psi_t\left(x,0\right)=\psi_1\left(x\right), \, w_t\left(x,0\right)=w_1\left(x\right), \, z_t\left(x,0\right)=z_1\left(x\right) \text{ in } \left(0,1\right), \end{cases} \tag{1.3}$$

where k_j, $j=0,1,2,3,4$, and a_j, $j=1,2,3,4$, are positive constants,

$$(\tau_1,\tau_2,\tau_3,\tau_4)\in\{0,1\}^4 \quad \text{and} \quad (\tau_1,\tau_2,\tau_3,\tau_4)\neq(0,0,0,0), \tag{1.4}$$

the functions φ_j, ψ_j, w_j and z_j, $j=0,1$, are fixed initial data,

$$(\varphi,\psi,w,z):(x,t)\in(0,1)\times(0,\infty)\mapsto(\varphi(x,t),\psi(x,t),w(x,t),z(x,t))\in\mathbb{R}^4$$

is the unknown of (1.1)-(1.3), and the subscripts t and x denote, respectively, the derivative with respect to the time variable t and the space variable x.

In the case $k_0=0$, both $(1.1)_1$-$(1.1)_2$ and $(1.1)_3$-$(1.1)_4$ are reduced to the well-known single Timoshenko beam introduced in [42], so (1.1) can be seen as coupled two Timoshenko beams thanks to the coupling terms $-k_0\left(w-\varphi\right)$ and $k_0\left(w-\varphi\right)$.

The well-posedness and stability questions for the single Timoshenko beam have been widely treated in the literature during the last few decades using various controls, like frictional or fractional dampings, memories, heat conduction and boundary feedbacks. Several stability and non-stability results have been established depending on the considered controls and some connections between the coefficients; we refer the readers to, for example, [3–5, 8, 12–14, 18–20, 28–32, 35, 38, 40] and the references therein. In the particular case of a dissipation related to frictional dampings, it was proved in [4, 31, 35, 40] (under some boundary conditions) that the following Timoshenko-type system:

$$\begin{cases} \rho_1\varphi_{tt}-k_1\left(\varphi_x+\psi\right)_x+\tau_1 a_1\varphi_t=0 \qquad\quad \text{ in } \left(0,L\right)\times\left(0,\infty\right), \\ \rho_2\psi_{tt}-k_2\psi_{xx}+k_1\left(\varphi_x+\psi\right)+\tau_2 a_2\psi_t=0 \text{ in } \left(0,L\right)\times\left(0,\infty\right), \end{cases} \tag{1.5}$$

where ρ_1,ρ_2 and L are positive constants, is exponentially stable if

$$(\tau_1,\tau_2)=(1,1) \quad \text{or} \quad \left[(\tau_1,\tau_2)\in\{(1,0),(0,1)\} \text{ and } \frac{k_1}{\rho_1}=\frac{k_2}{\rho_2}\right], \tag{1.6}$$

however, when

$$(\tau_1,\tau_2)\in\{(1,0),(0,1)\} \quad \text{and} \quad \frac{k_1}{\rho_1}\neq\frac{k_2}{\rho_2}, \tag{1.7}$$

system (1.5) is not exponentially stable but it is polynomially stable with an optimal decay rate, at infinity, of type $\frac{1}{\sqrt{t}}$ for the norm of its solution.

Similar exponential and polynomial stability results are obtained in the last few years for Bresse type systems (coupled three wave equations) and Rao-Nakra sandwish type systems (coupled two wave equations and one Euler-Bernoulli equation) under various kinds of controls; for more details, see, for example, [1, 2, 12, 24, 26, 36] and the references therein.

During the last three decades, many authors were interested by the study of finite carbon structures consisting of needle-like tubes; see, for example, [11, 23, 37, 39, 41, 43–48]. In these papers, and according to various physical considerations, several models of carbon nanotubes were descriped and classified; like single wall carbon nanotubes (SWCNT), double wall carbon nanotubes (DWCNT) and multi-wall carbon nanotubes (MWCNT). In the case of double wall carbon nanotubes, the modeling is based on the Timoshenko beam theory (as in [43–48]) by neglecting some physical properties of carbon nanotubes and/or assuming some relationships between them.

The authors of [47] proposed the following coupled two Timoshenko beam models to model the double wall carbon nanotubes:

$$\begin{cases} \rho A_1 Y_{1,tt} - kGA_1 \left(Y_{1,x} - \varphi_1\right)_x - P = 0, \\ \rho I_1 \varphi_{1,tt} - EI_1 \varphi_{1,xx} - kGA_1 \left(Y_{1,x} - \varphi_1\right) = 0, \\ \rho A_2 Y_{2,tt} - kGA_2 \left(Y_{2,x} - \varphi_2\right)_x + P = 0, \\ \rho I_2 \varphi_{2,tt} - EI_2 \varphi_{2,xx} - kGA_2 \left(Y_{2,x} - \varphi_2\right) = 0, \end{cases} \tag{1.8}$$

where the functions Y_j and φ_j, $j = 1, 2$, represent, respectively, the total deflection and the inclination due to the bending of the nanotube j, the constants I_j and A_j, $j = 1, 2$, denote, respectively, the moment of inertia and the cross-sectional area of the nanotube j, the constants ρ, E, G and k represent, respectively, the mass density of the material, the Young's modulus, the stiffness modulus and the shearn factor, and P is the Van der Waals force acting on the interaction between the two nanotubes and given by

$$P = \mathcal{L}(Y_2 - Y_1),$$

where $\mathcal{L}$ is the Van der Waals interaction coefficient for the interaction pressure.

To the best of our knowledge, the stability problem of (1.8) is new and have not been discussed earlier. Only in order to simplify the mathematical study, we replace Y_1, φ_1, Y_2 and φ_2 by $\varphi, -\psi, w$ and $-z$, respectively, replace kGA_1, EI_1, kGA_2, EI_2 and $\mathcal{L}$ by k_1, k_2, k_3, k_4 and k_0, respectively, and, without loss of generality, assume that $\rho A_j = \rho I_j = L = 1$, where L is the length of tubes. So (1.8) is reduced to (1.1) with $(\tau_1, \tau_2, \tau_3, \tau_4) = (0, 0, 0, 0)$.

Our main objective in this paper is to treat this stability problem for (1.1)-(1.3), where the dissipation is generated by the frictional dampings $\tau_1 a_1 \varphi_t, \tau_2 a_2 \psi_t, \tau_3 a_3 w_t$ and $\tau_4 a_4 z_t$. First, we will show the existence and uniqueness of solutions of (1.1)-(1.3) in a given Hilbert space, and get some of their smoothness properties depending on the fixed initial data. Second, we will provide strong, non-exponential, exponential, polynomial, non-polynomial and optimality stability results for (1.1)-(1.3) depending on the values of τ_j in (1.4) and some connections between the coefficients k_j. For strong and exponential stability results, we introduce necessary and sufficient conditions. Moreover, in some cases, we prove the optimality of polynomial decay rate.

The proof of the well-posedness results is based on the linear semigroups theory. However, the stability results are proved using the energy method combining with the frequency domain approach and some contradiction arguments by constructing judicious counter examples in order to prove the optimality and non-polynomial stability results.

The paper is organized as follows: in section 2, we prove the well-posedness of (1.1)-(1.3). Section 3 is devoted to the proof of the strong stability for (1.1)-(1.3). In sections 4, 5 and 6, we show, respectively, our non-exponential, exponential and polynomial stability results for (1.1)-(1.3). Sections 7 and 8 are devoted to the proof of our, respectively, optimal polynomial decay rate and non-polynomial stability results. Finally, we end our paper by giving some comments and issues in section 9.

2. Abstract formulation and well-posedness

We consider the Hilbert Sobolev spaces

$$V_0 = \left\{ v \in H^1(0,1) : v(0) = 0 \right\} \quad \text{and} \quad V_1 = \left\{ v \in H^1(0,1) : v(1) = 0 \right\},$$

and we introduce the space

$$\mathcal{H} = V_1 \times L^2(0,1) \times V_0 \times L^2(0,1) \times V_1 \times L^2(0,1) \times V_0 \times L^2(0,1),$$

where $L^2(0,1)$ is equipped with its standard inner product $\langle \cdot, \cdot \rangle$ and generated norm $\| \cdot \|$. For

$$\Phi_j = (\varphi_j, \tilde{\varphi}_j, \psi_j, \tilde{\psi}_j, w_j, \tilde{w}_j, z_j, \tilde{z}_j)^T, \quad j = 1, 2,$$

we consider on $\mathcal{H}$ the inner product

$$\begin{aligned} \langle \Phi_1, \Phi_2 \rangle_{\mathcal{H}} = & k_1 \langle \varphi_{1,x} + \psi_1, \varphi_{2,x} + \psi_2 \rangle + k_2 \langle \psi_{1,x}, \psi_{2,x} \rangle + k_3 \langle w_{1,x} + z_1, w_{2,x} + z_2 \rangle \\ & + k_4 \langle z_{1,x}, z_{2,x} \rangle + k_0 \langle w_1 - \varphi_1, w_2 - \varphi_2 \rangle \\ & + \langle \tilde{\varphi}_1, \tilde{\varphi}_2 \rangle + \langle \tilde{\psi}_1, \tilde{\psi}_2 \rangle + \langle \tilde{w}_1, \tilde{w}_2 \rangle + \langle \tilde{z}_1, \tilde{z}_2 \rangle . \end{aligned} \tag{2.1}$$

Using Young's inequality, we see that there exist a positive constant b_1 (depending only on k_j) such that

$$\begin{aligned} & k_1 \|\varphi_x + \psi\|^2 + k_2 \|\psi_x\|^2 + k_3 \|w_x + z\|^2 + k_4 \|z_x\|^2 + k_0 \|w - \varphi\|^2 \\ \leq & b_1 \left(\|\varphi\|^2_{H^1(0,1)} + \|\psi\|^2_{H^1(0,1)} + \|w\|^2_{H^1(0,1)} + \|z\|^2_{H^1(0,1)} \right). \end{aligned} \tag{2.2}$$

On the other hand, using Cauchy-Schwarz and Young's inequalities, we observe that, for any $\epsilon > 1$,

$$\begin{aligned} & k_1 \|\varphi_x + \psi\|^2 + k_2 \|\psi_x\|^2 + k_3 \|w_x + z\|^2 + k_4 \|z_x\|^2 + k_0 \|w - \varphi\|^2 \\ \geq & k_1 \left(\|\varphi_x\|^2 + \|\psi\|^2 + 2 \langle \varphi_x, \psi \rangle \right) + k_2 \|\psi_x\|^2 + k_3 \left(\|w_x\|^2 + \|z\|^2 + 2 \langle w_x, z \rangle \right) \\ & + k_4 \|z_x\|^2 \\ \geq & k_1 \left(1 - \tfrac{1}{\epsilon}\right) \|\varphi_x\|^2 + k_1 (1 - \epsilon) \|\psi\|^2 + k_2 \|\psi_x\|^2 + k_3 \left(1 - \tfrac{1}{\epsilon}\right) \|w_x\|^2 \\ & + k_3 (1 - \epsilon) \|z\|^2 + k_4 \|z_x\|^2 , \end{aligned}$$

therefore, because $\psi(x = 0) = z(x = 0) = 0$, one can apply Poincaré's inequality to ψ and z, and get (c_0 denotes the Poincaré's constant)

$$k_1 \|\varphi_x + \psi\|^2 + k_2 \|\psi_x\|^2 + k_3 \|w_x + z\|^2 + k_4 \|z_x\|^2 + k_0 \|w - \varphi\|^2$$

$$\geq k_1\left(1-\frac{1}{\epsilon}\right)\|\varphi_x\|^2+[k_2+k_1(1-\epsilon)c_0]\|\psi_x\|^2+k_3\left(1-\frac{1}{\epsilon}\right)\|w_x\|^2 + [k_4+k_3(1-\epsilon)c_0]\|z_x\|^2,$$

then, by choosing $1<\epsilon<1+\frac{1}{c_0}\min\left\{\frac{k_2}{k_1},\frac{k_4}{k_3}\right\}$, we observe that there exists a positive constant b_2 (depending only on k_j and c_0) such that

$$\begin{aligned} &k_1\|\varphi_x+\psi\|^2+k_2\|\psi_x\|^2+k_3\|w_x+z\|^2+k_4\|z_x\|^2+k_0\|w-\varphi\|^2 \\ \geq& b_2\left(\|\varphi\|^2_{H^1(0,1)}+\|\psi\|^2_{H^1(0,1)}+\|w\|^2_{H^1(0,1)}+\|z\|^2_{H^1(0,1)}\right). \end{aligned} \tag{2.3}$$

Consequently, we deduce from (2.2) and (2.3) that $\mathcal{H}$, endowed with the inner product $\langle,\rangle_{\mathcal{H}}$, is a Hilbert space and its norm $\|\cdot\|_{\mathcal{H}}=\sqrt{\langle\cdot,\cdot\rangle_{\mathcal{H}}}$ is equivalent to the one of $\left(H^1(0,1)\times L^2(0,1)\right)^4$.

Now, we put

$$\begin{cases} \tilde{\varphi}=\varphi_t, \quad \tilde{\psi}=\psi_t, \quad \tilde{w}=w_t, \quad \tilde{z}=z_t, \\ \Phi=\left(\varphi,\,\tilde{\varphi},\,\psi,\,\tilde{\psi},\,w,\,\tilde{w},\,z,\,\tilde{z}\right)^T, \\ \Phi_0=\left(\varphi_0,\,\varphi_1,\,\psi_0,\,\psi_1,\,w_0,\,w_1,\,z_0,\,z_1\right)^T. \end{cases}$$

System (1.1)-(1.3) can be formulated in the following first order one:

$$\begin{cases} \Phi_t=\mathcal{A}\Phi, & t\in(0,\infty), \\ \Phi(t=0)=\Phi_0, \end{cases} \tag{2.4}$$

where $\mathcal{A}$ is a linear operator defined by

$$\mathcal{A}\Phi=\begin{pmatrix} \tilde{\varphi} \\ k_1(\varphi_x+\psi)_x+k_0(w-\varphi)-\tau_1 a_1\tilde{\varphi} \\ \tilde{\psi} \\ k_2\psi_{xx}-k_1(\varphi_x+\psi)-\tau_2 a_2\tilde{\psi} \\ \tilde{w} \\ k_3(w_x+z)_x-k_0(w-\varphi)-\tau_3 a_3\tilde{w} \\ \tilde{z} \\ k_4 z_{xx}-k_3(w_x+z)-\tau_4 a_4\tilde{z} \end{pmatrix} \tag{2.5}$$

with domain given by

$$D(\mathcal{A})=\left\{\begin{array}{c} \Phi\in\mathcal{H}:\,(\varphi,\psi,w,z)\in\left(H^2(0,1)\right)^4,\,(\tilde{\varphi},\tilde{\psi},\tilde{w},\tilde{z})\in V_1\times V_0\times V_1\times V_0, \\ \varphi_x(0)=\psi_x(1)=w_x(0)=z_x(1) \end{array}\right\}.$$

Theorem 2.1. *For any $\Phi_0 \in \mathcal{H}$, system* (2.4) *admits a unique solution*

$$\Phi \in C\left(\mathbb{R}_+;\mathcal{H}\right), \tag{2.6}$$

where $\mathbb{R}_+ = [0,\infty)$. Moreover, if $\Phi_0 \in D(\mathcal{A})$, then the solution satisfies

$$\Phi \in C^1\left(\mathbb{R}_+;\mathcal{H}\right) \cap C\left(\mathbb{R}_+; D\left(\mathcal{A}\right)\right). \tag{2.7}$$

Proof. First, using (2.1) and (2.5), integrating with respect to x and using the boundary conditions (1.2), we get, for any $\Phi \in D\left(\mathcal{A}\right)$,

$$\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} = -\left(\tau_1 a_1 \|\tilde{\varphi}\|^2 + \tau_2 a_2 \|\tilde{\psi}\|^2 + \tau_3 a_3 \|\tilde{w}\|^2 + \tau_4 a_4 \|\tilde{z}\|^2\right) \leq 0, \tag{2.8}$$

hence $\mathcal{A}$ is dissipative in $\mathcal{H}$.

After, we show that $0 \in \rho\left(\mathcal{A}\right)$, where $\rho\left(\mathcal{A}\right)$ denotes the resolvent of $\mathcal{A}$; that is, for any

$$F := (f_1, \cdots, f_8)^T \in \mathcal{H},$$

there exists a unique $\Phi \in D\left(\mathcal{A}\right)$ satisfying

$$\mathcal{A}\Phi = F. \tag{2.9}$$

From (2.5), we remark that $(2.9)_1$, $(2.9)_3$, $(2.9)_5$ and $(2.9)_7$ are reduced to

$$\tilde{\varphi} = f_1, \quad \tilde{\psi} = f_3, \quad \tilde{w} = f_5 \quad \text{and} \quad \tilde{z} = f_7, \tag{2.10}$$

and then

$$(\tilde{\varphi}, \tilde{\psi}, \tilde{w}, \tilde{z}) \in V_1 \times V_0 \times V_1 \times V_0. \tag{2.11}$$

So (2.9) has a unique solution $\Phi \in D\left(\mathcal{A}\right)$ if there exists a unique

$$(\varphi, \psi, w, z) \in (H^2(0,1)\cap V_1)\times(H^2(0,1)\cap V_0)\times(H^2(0,1)\cap V_1)\times(H^2(0,1)\cap V_0) \tag{2.12}$$

satisfying

$$\varphi_x(0) = \psi_x(1) = w_x(0) = z_x(1) = 0 \tag{2.13}$$

and the equations $(2.9)_2$, $(2.9)_4$, $(2.9)_6$ and $(2.9)_8$. Assuming that such unknown (φ, ψ, w, z) exists, then, multiplying $(2.9)_2$, $(2.9)_4$, $(2.9)_6$ and $(2.9)_8$ by $(\hat{\varphi}, \hat{\psi}, \hat{w}, \hat{z}) \in V_1 \times V_0 \times V_1 \times V_0$, respectively, inegrating by parts and using (2.10) and (2.13), we remark that (φ, ψ, w, z) is a solution of the variational formulation

$$B\left((\varphi, \psi, w, z), (\hat{\varphi}, \hat{\psi}, \hat{w}, \hat{z})\right) = \hat{B}(\hat{\varphi}, \hat{\psi}, \hat{w}, \hat{z}),\ \forall (\hat{\varphi}, \hat{\psi}, \hat{w}, \hat{z}) \in V_1 \times V_0 \times V_1 \times V_0, \tag{2.14}$$

where B is a bilinear form on $\left(V_1 \times V_0 \times V_1 \times V_0\right)^2$ given by

$$\begin{aligned} & B\left((\varphi, \psi, w, z), (\hat{\varphi}, \hat{\psi}, \hat{w}, \hat{z})\right) \\ = {} & k_1 \left\langle \varphi_x + \psi, \hat{\varphi}_x + \hat{\psi}\right\rangle + k_2 \left\langle \psi_x, \hat{\psi}_x\right\rangle + k_3 \left\langle w_x + z, \hat{w}_x + \hat{z}\right\rangle \\ & + k_4 \left\langle z_x, \hat{z}_x\right\rangle + k_0 \left\langle w - \varphi, \hat{w} - \hat{\varphi}\right\rangle \end{aligned}$$

and $\hat{B}$ is a linear form on $V_1 \times V_0 \times V_1 \times V_0$ defined by

$$\hat{B}(\hat{\varphi}, \hat{\psi}, \hat{w}, \hat{z}) = -\left\langle \tau_1 a_1 f_1 + f_2, \hat{\varphi}\right\rangle - \left\langle \tau_2 a_2 f_3 + f_4, \hat{\psi}\right\rangle$$

$$- \langle \tau_3 a_3 f_5 + f_6, \hat{w} \rangle - \langle \tau_4 a_4 f_7 + f_8, \hat{z} \rangle .$$

According to the fact that $F \in \mathcal{H}$ and using (2.2) and (2.3), it is easy to see that B is continuous and coercive, and $\hat{B}$ is continuous. Then, the Lax-Milgram theorem implies that (2.14) has a unique solution

$$(\varphi, \psi, w, z) \in V_1 \times V_0 \times V_1 \times V_0. \tag{2.15}$$

By considering in (2.14) the particular test functions $(\hat{\varphi}, 0, 0, 0)$, $(0, \hat{\psi}, 0, 0)$, $(0, 0, \hat{w}, 0)$ and $(0, 0, 0, \hat{z})$, for $(\hat{\varphi}, \hat{\psi}, \hat{w}, \hat{z}) \in (C_c^\infty(0,1))^4$, integrating by parts and using (2.10) and the density of $C_c^\infty(0,1)$ in $L^2(0,1)$, we get, respectively, $(2.9)_2$, $(2.9)_4$, $(2.9)_6$ and $(2.9)_8$. Therefore, thanks to (2.11) and (2.15), we get

$$(\varphi_{xx}, \psi_{xx}, w_{xx}, z_{xx}) \in \left(L^2(0,1)\right)^4,$$

so (2.12) holds. To show (2.13), we consider in (2.14) test functions $(\hat{\varphi}, 0, 0, 0)$, $(0, \hat{\psi}, 0, 0)$, $(0, 0, \hat{w}, 0)$ and $(0, 0, 0, \hat{z})$ such that $(\hat{\varphi}, \hat{\psi}, \hat{w}, \hat{z}) \in V_1 \times V_0 \times V_1 \times V_0$ and

$$\hat{\varphi}(0) = \hat{\psi}(1) = \hat{w}(0) = \hat{z}(1) = 1,$$

integrating by parts and using $(2.9)_2$, $(2.9)_4$, $(2.9)_6$, $(2.9)_8$ and (2.10), we obtain (2.13). Consequently, we have proved that, for any $F \in \mathcal{H}$, (2.9) admits a unique solution $\Phi \in D(\mathcal{A})$. By the resolvent identity, we have $\lambda I - \mathcal{A}$ is surjective, for any $\lambda > 0$ (see [27]), where I is the identity operator. Finally, $\mathcal{A}$ is densely defined (see Theorem 4.6 of [33]) and the Lumer-Phillips theorem implies that $\mathcal{A}$ is the infinitesimal generator of linear C_0-semigroups of contractions on $\mathcal{H}$. The linear semigroups theory guarantees the results of Theorem 2.1 (see [33]). □

Remark 2.1. From the proof of the dissipativity of $\mathcal{A}$, we observe that $(2.4)_1$ and (2.8) lead to

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\|\Phi\|_{\mathcal{H}}^2\right) &= 2\,\langle \Phi_t, \Phi\rangle_{\mathcal{H}} \\
&= 2\,\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} \\
&= -2\left(\tau_1 a_1 \|\tilde{\varphi}\|^2 + \tau_2 a_2 \|\tilde{\psi}\|^2 + \tau_3 a_3 \|\tilde{w}\|^2 + \tau_4 a_4 \|\tilde{z}\|^2\right),
\end{aligned} \tag{2.16}$$

then, if $(\tau_1, \tau_2, \tau_3, \tau_4) = (0, 0, 0, 0)$, the function $t \mapsto \|\Phi(\cdot, t)\|_{\mathcal{H}}$ is constant, and so the problem in not posed. This show that, to get the strong stability of (2.4); that is

$$\forall \Phi_0 \in \mathcal{H}: \ \lim_{t \to \infty} \|\Phi\|_{\mathcal{H}} = 0, \tag{2.17}$$

at least one frictional damping must be considered; this why we are assuming (1.4).

3. Strong stability

In this section, we prove our first stability result concerning the strong stability (2.17) for (2.4) in the following three cases:

$$\begin{cases}
(\tau_1, \tau_2, \tau_3, \tau_4) = (1, 0, 0, 0), \\
\quad \left[(k_2 - k_3)\left(\frac{\pi}{2} + m\pi\right)^2 + k_1 - k_0\right]\left[(k_2 - k_4)\left(\frac{\pi}{2} + m\pi\right)^2 + k_1 - k_3\right] \\
\neq k_3^2\left(\frac{\pi}{2} + m\pi\right)^2, \ \forall m \in \mathbb{N},
\end{cases} \tag{3.1}$$

$$\begin{cases} (\tau_1, \tau_2, \tau_3, \tau_4) = (0, 0, 1, 0), \\ \quad \left[(k_4 - k_1)\left(\frac{\pi}{2} + m\pi\right)^2 + k_3 - k_0\right]\left[(k_4 - k_2)\left(\frac{\pi}{2} + m\pi\right)^2 + k_3 - k_1\right] \\ \neq k_1^2\left(\frac{\pi}{2} + m\pi\right)^2, \forall m \in \mathbb{N} \end{cases} \tag{3.2}$$

and

$$(\tau_1, \tau_2, \tau_3, \tau_4) \notin \{(0, 0, 0, 0), (1, 0, 0, 0), (0, 0, 1, 0)\}. \tag{3.3}$$

Theorem 3.1. *The strong stability* (2.17) *holds if and only if* (3.1) *or* (3.2) *or* (3.3) *is satisfied.*

Proof. A C_0 semigroup of contractions $e^{t\mathcal{A}}$ generated by an operator $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ with a compact resolvent $\rho(\mathcal{A})$ in $\mathcal{H}$ is strogly stable if and only if $\mathcal{A}$ has no imaginary eigenvalues; that is

$$\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset,$$

where $\sigma(\mathcal{A})$ is the spectrum set of $\mathcal{A}$ (see [6]). According to the fact that $0 \in \rho(\mathcal{A})$ (proved in section 2) and since $D(\mathcal{A})$ has a compact embedding into $\mathcal{H}$, the linear bounded operator $\mathcal{A}^{-1}$ is a bijection between $\mathcal{H}$ and $D(\mathcal{A})$, and $\mathcal{A}^{-1}$ is a compact operator, which implies that $\sigma(\mathcal{A})$ is discrete and has only eigenvalues. Consequently, to get the equivalence between (2.17) and (3.1)-(3.3), it is sufficient to prove that (3.1) or (3.2) or (3.3) holds if and only if

$$ker\,(i\lambda I - \mathcal{A}) = \{0\}. \tag{3.4}$$

In section 2, we have proved (3.4) for $\lambda = 0$. So let $\lambda \in \mathbb{R}^*$ and

$$\Phi = \left(\varphi, \tilde{\varphi}, \psi, \tilde{\psi}, w, \tilde{w}, z, \tilde{z}\right) \in D(\mathcal{A})$$

such that

$$i\,\lambda\,\Phi - \mathcal{A}\Phi = 0. \tag{3.5}$$

We have to prove that $\Phi = 0$ if and only if (3.1) or (3.2) or (3.3) is satisfied. From (2.8) and (3.5), we find

$$\begin{aligned} 0 &= Re\, i\lambda\, \|\Phi\|_{\mathcal{H}}^2 \\ &= Re\, \langle i\lambda\Phi, \Phi\rangle_{\mathcal{H}} \\ &= Re\, \langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} \\ &= -\left(\tau_1 a_1\|\tilde{\varphi}\|^2 + \tau_2 a_2\|\tilde{\psi}\|^2 + \tau_3 a_3\|\tilde{w}\|^2 + \tau_4 a_4\|\tilde{z}\|^2\right), \end{aligned}$$

then

$$\tau_1 a_1\|\tilde{\varphi}\|^2 + \tau_2 a_2\|\tilde{\psi}\|^2 + \tau_3 a_3\|\tilde{w}\|^2 + \tau_4 a_4\|\tilde{z}\|^2 = 0. \tag{3.6}$$

It is enough to consider the two cases

$$(\tau_1, \tau_2, \tau_3, \tau_4) \in \{(1, 0, 0, 0), (0, 1, 0, 0)\}. \tag{3.7}$$

Indeed, the proof in cases

$$(\tau_1, \tau_2, \tau_3, \tau_4) \in \{(0, 0, 1, 0), (0, 0, 0, 1)\}$$

is identical to the one that will be given in cases (3.7) because $(1.1)_1$-$(1.1)_2$ and $(1.1)_3$-$(1.1)_4$ play symmetrical roles, since, by replacing $(\varphi, \psi, k_1, k_2)$ by (w, z, k_3, k_4) and conversely, we get the same system (1.1). Then, clearly, $\Phi = 0$ holds also in the other cases, where at least two frictional dampings are present.

3.1. Case $(\tau_1, \tau_2, \tau_3, \tau_4) = (1, 0, 0, 0)$

In vertue of $(2.5)_1$, $(3.5)_1$ and (3.6), we have

$$\varphi = \tilde{\varphi} = 0. \tag{3.8}$$

Then (2.5), (3.5) and (3.8) lead to

$$\begin{cases} \tilde{\psi} = i\lambda\psi, \\ \tilde{w} = i\lambda w, \\ \tilde{z} = i\lambda z, \\ k_1\psi_x + k_0 w = 0, \\ k_2\psi_{xx} + \left(\lambda^2 - k_1\right)\psi = 0, \\ k_3(w_x + z)_x + \left(\lambda^2 - k_0\right) w = 0, \\ k_4 z_{xx} + \left(\lambda^2 - k_3\right) z - k_3 w_x = 0. \end{cases} \tag{3.9}$$

The equation $(3.9)_4$ is equivalent to

$$w = \frac{-k_1}{k_0}\psi_x. \tag{3.10}$$

Combining $(3.9)_6$ and (3.10), we obtain

$$\left[k_3(w_x + z) - \frac{k_1}{k_0}\left(\lambda^2 - k_0\right)\psi\right]_x = 0.$$

Since $h := k_3(w_x + z) - \frac{k_1}{k_0}\left(\lambda^2 - k_0\right)\psi$ satisfies $h(0) = 0$ (according to the definition of $D(\mathcal{A})$), then $h = 0$, which implies that (using (3.10))

$$z = \frac{k_1}{k_0}\psi_{xx} + \frac{k_1}{k_0 k_3}\left(\lambda^2 - k_0\right)\psi. \tag{3.11}$$

Now, to solve the equation $(3.9)_5$, we distiguish three subcases.

Subcase 1. $\lambda^2 = k_1$. Equation $(3.9)_5$ implies that, for some c_1, $c_2 \in \mathbb{C}$, $\psi(x) = c_1 x + c_2$. Therefore, the boundary conditions

$$\psi(0) = \psi_x(1) = 0 \tag{3.12}$$

lead to $c_1 = c_2 = 0$; that is $\psi = 0$. Consequently, according to (3.8), $(3.9)_1$, $(3.9)_2$, $(3.9)_3$, (3.10) and (3.11), we find $\Phi = 0$.

Subcase 2. $\lambda^2 < k_1$. Equation $(3.9)_5$ lead to, for some c_1, $c_2 \in \mathbb{C}$,

$$\psi(x) = c_1 e^{\sqrt{\frac{1}{k_2}(k_1 - \lambda^2)}x} + c_2 e^{-\sqrt{\frac{1}{k_2}(k_1 - \lambda^2)}x}.$$

Similarly, (3.12) implies that $c_1 = c_2 = 0$, which leads to $\Phi = 0$ as in subcase 1.

Subcase 3. $\lambda^2 > k_1$. From $(3.9)_5$, we have, for some c_1, $c_2 \in \mathbb{C}$,

$$\psi(x) = c_1 \cos\left(\sqrt{\frac{1}{k_2}\left(\lambda^2 - k_1\right)}x\right) + c_2 \sin\left(\sqrt{\frac{1}{k_2}\left(\lambda^2 - k_1\right)}x\right).$$

The boundary conditions (3.12) lead to $c_1 = 0$ and

$$c_2 = 0 \quad \text{or} \quad \exists\, m \in \mathbb{N} : \sqrt{\frac{1}{k_2}\left(\lambda^2 - k_1\right)} = \frac{\pi}{2} + m\pi. \tag{3.13}$$

Therefore

$$\psi(x) = c_2 \sin\left(\sqrt{\frac{1}{k_2}\left(\lambda^2 - k_1\right)}x\right), \tag{3.14}$$

and so, using (3.10) and (3.11),

$$w(x) = -\frac{c_2 k_1}{k_0}\sqrt{\frac{1}{k_2}\left(\lambda^2 - k_1\right)} \cos\left(\sqrt{\frac{1}{k_2}\left(\lambda^2 - k_1\right)}x\right) \tag{3.15}$$

and

$$z(x) = c_2 \left[\frac{k_1}{k_0 k_3}\left(\lambda^2 - k_0\right) - \frac{k_1}{k_0 k_2}\left(\lambda^2 - k_1\right)\right] \sin\left(\sqrt{\frac{1}{k_2}\left(\lambda^2 - k_1\right)}x\right), \tag{3.16}$$

then, by combining $(3.9)_7$, (3.15) and (3.16), we see that

$$\begin{aligned} &c_2 = 0 \quad \text{or} \\ &\left[(k_2 - k_3)\lambda^2 + k_1 k_3 - k_0 k_2\right]\left[(k_2 - k_4)\lambda^2 + k_1 k_4 - k_2 k_3\right] - k_2 k_3^2\left(\lambda^2 - k_1\right) = 0. \end{aligned} \tag{3.17}$$

Assume by contradiction that $c_2 \neq 0$. Then, according to (3.13), we have, for some $m \in \mathbb{N}$,

$$\lambda^2 = k_2\left(\frac{\pi}{2} + m\pi\right)^2 + k_1. \tag{3.18}$$

By combining $(3.17)_2$ and (3.18), we get a contradiction to $(3.1)_2$. Consequently, $c_2 = 0$, hence we arrive at $\Phi = 0$.

On the other hand, if $(3.1)_2$ does not hold, then there exists $\lambda \in \mathbb{R}$ defined by (3.18) such that $i\lambda$ is an eigenvalue of $\mathcal{A}$ with a corresponding eigenvector given by (3.8), $(3.9)_1$-$(3.9)_3$ and (3.14)-(3.16), for any $c_2 \in \mathbb{C}^*$.

3.2. Case $(\tau_1, \tau_2, \tau_3, \tau_4) = (0, 1, 0, 0)$

From $(2.5)_3$, $(3.5)_3$ and (3.6), we have

$$\psi = \tilde{\psi} = 0. \tag{3.19}$$

Then (2.5), (3.5) and (3.19) lead to

$$\begin{cases} \tilde{\varphi} = i\lambda\varphi, \\ \tilde{w} = i\lambda w, \\ \tilde{z} = i\lambda z, \\ k_1\varphi_{xx} + \left(\lambda^2 - k_0\right)\varphi + k_0 w = 0, \\ \varphi_x = 0, \\ k_3(w_x + z)_x + \left(\lambda^2 - k_0\right) w + k_0\varphi = 0, \\ k_4 z_{xx} + \left(\lambda^2 - k_3\right) z - k_3 w_x = 0. \end{cases} \tag{3.20}$$

The equation $(3.20)_5$ with the boundary condition $\varphi(1) = 0$ imply that $\varphi = 0$, and then, using $(3.20)_4$, we get $w = 0$. Therefore, $(3.20)_6$ and the boundary condition $z(0) = 0$ imply that $z = 0$. Consequently, using $(3.20)_1$, $(3.20)_2$ and $(3.20)_3$, we conclude that $\Phi = 0$. Finally, (3.4) holds and thus the proof of Theorem 3.1 is ended. □

4. Lack of exponential stability

The subject of this section is to show that, in the following cases:

$$(\tau_1, \tau_2, \tau_3, \tau_4) \in \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,1,0,0), (0,0,1,1)\}, \tag{4.1}$$

$$(\tau_1, \tau_2, \tau_3, \tau_4) \in \{(1,1,0,1), (1,1,1,0)\} \quad \text{and} \quad k_3 \neq k_4, \tag{4.2}$$

$$(\tau_1, \tau_2, \tau_3, \tau_4) \in \{(0,1,1,1), (1,0,1,1)\} \quad \text{and} \quad k_1 \neq k_2 \tag{4.3}$$

and

$$\begin{aligned} &(\tau_1, \tau_2, \tau_3, \tau_4) \in \{(1,0,0,1), (1,0,1,0), (0,1,0,1), (0,1,1,0)\} \\ &\text{and} \quad (k_1, k_3) \neq (k_2, k_4), \end{aligned} \tag{4.4}$$

system (2.4) is not exponentially stable; that is the following property is not satisfied:

$$\forall \Phi_0 \in \mathcal{H},\, \exists\, c_1,\, c_2 > 0 : \|\Phi(t)\|_{\mathcal{H}} \leq c_1 e^{-c_2 t}, \quad \forall t \geq 0. \tag{4.5}$$

Theorem 4.1. *In cases* (4.1)-(4.4)*, the exponential stability* (4.5) *does not hold.*

Proof. It is known that the exponential stability (4.5) is equivalent to (see [22,34])

$$i\,\mathbb{R} \subset \rho\left(\mathcal{A}\right) \quad \text{and} \quad \sup_{\lambda\in\mathbb{R}} \left\| \left(i\lambda I - \mathcal{A}\right)^{-1} \right\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{4.6}$$

It is sufficient to prove that the second condition in (4.6) does not hold. To do so, we prove that there exists a sequence $(\lambda_n)_n \subset \mathbb{R}$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \left\| \left(i\lambda_n I - \mathcal{A}\right)^{-1} \right\|_{\mathcal{L}(\mathcal{H})} = \infty.$$

This is equivalent to prove that there exists a sequence $(F_n)_n \subset \mathcal{H}$ satisfying

$$\|F_n\|_{\mathcal{H}} := \left\|(f_{1,n}, \cdots, f_{8,n})^T\right\|_{\mathcal{H}} \leq 1, \quad \forall n \in \mathbb{N} \tag{4.7}$$

and

$$\lim_{n\to\infty} \|\left(i\lambda_n I - \mathcal{A}\right)^{-1} F_n\|_{\mathcal{H}} = \infty. \tag{4.8}$$

For this purpose, let

$$\Phi_n := \left(\varphi_n, \tilde{\varphi}_n, \psi_n, \tilde{\psi}_n, w_n, \tilde{w}_n, z_n, \tilde{z}_n\right)^T = \left(i\lambda_n I - \mathcal{A}\right)^{-1} F_n, \quad \forall n \in \mathbb{N}.$$

Then, we have to prove that $(\Phi_n)_n \subset D\left(\mathcal{A}\right)$, (4.7) holds,

$$\lim_{n\to\infty} \|\Phi_n\|_{\mathcal{H}} = \infty \quad \text{and} \quad i\lambda_n\Phi_n - \mathcal{A}\Phi_n = F_n,\ \forall n \in \mathbb{N}. \tag{4.9}$$

From (2.5), we observe that the second equality in (4.9) can be presented as

$$\begin{cases} i\lambda_n\varphi_n - \tilde{\varphi}_n = f_{1,n}, \\ i\lambda_n\tilde{\varphi}_n - k_1\left(\varphi_{n,x} + \psi_n\right)_x - k_0\left(w_n - \varphi_n\right) + \tau_1 a_1\tilde{\varphi}_n = f_{2,n}, \\ i\lambda_n\psi_n - \tilde{\psi}_n = f_{3,n}, \\ i\lambda_n\tilde{\psi}_n - k_2\psi_{n,xx} + k_1\left(\varphi_{n,x} + \psi_n\right) + \tau_2 a_2\tilde{\psi}_n = f_{4,n}, \\ i\lambda_n w_n - \tilde{w}_n = f_{5,n}, \\ i\lambda_n\tilde{w}_n - k_3\left(w_{n,x} + z_n\right)_x + k_0\left(w_n - \varphi_n\right) + \tau_3 a_3\tilde{w}_n = f_{6,n}, \\ i\lambda_n z_n - \tilde{z}_n = f_{7,n}, \\ i\lambda_n\tilde{z}_n - k_4 z_{n,xx} + k_3\left(w_{n,x} + z_n\right) + \tau_4 a_4\tilde{z}_n = f_{8,n}. \end{cases} \tag{4.10}$$

We choose

$$\begin{cases} \tilde{\varphi}_n = i\lambda_n\varphi_n, \quad \tilde{\psi}_n = i\lambda_n\psi_n, \quad \tilde{w}_n = i\lambda_n w_n, \quad \tilde{z}_n = i\lambda_n z_n, \\ f_{1,n} = f_{3,n} = f_{5,n} = f_{7,n} = 0. \end{cases} \tag{4.11}$$

Then $(4.10)_1$, $(4.10)_3$, $(4.10)_5$ and $(4.10)_7$ are satisfied. On the other hand, we put

$$N = \frac{\pi}{2} + n\pi$$

(in order to simplify the computations) and choose

$$\begin{cases} \varphi_n(x) = \alpha_{1,n}\cos\left(Nx\right), \quad \psi_n(x) = \alpha_{2,n}\sin\left(Nx\right), \\ w_n(x) = \alpha_{3,n}\cos\left(Nx\right), \quad z_n(x) = \alpha_{4,n}\sin\left(Nx\right), \\ f_{2,n}(x) = -\beta_{2,n}\cos\left(Nx\right), \quad f_{4,n}(x) = -\beta_{4,n}\sin\left(Nx\right), \\ f_{6,n}(x) = -\beta_{6,n}\cos\left(Nx\right), \quad f_{8,n}(x) = -\beta_{8,n}\sin\left(Nx\right), \end{cases} \tag{4.12}$$

where $\alpha_{j,n}$, $\beta_{j,n} \in \mathbb{C}$. The choices (4.11) and (4.12) guarantee that $\Phi_n \in D(\mathcal{A})$ and $F_n \in \mathcal{H}$. Moreover, $(4.10)_2$, $(4.10)_4$, $(4.10)_6$ and $(4.10)_8$ are reduced to the following algebraic system:

$$\begin{cases} \left(\lambda_n^2 - k_1 N^2 - k_0 - i\tau_1 a_1 \lambda_n\right) \alpha_{1,n} + k_1 N \alpha_{2,n} + k_0 \alpha_{3,n} = \beta_{2,n}, \\ k_1 N \alpha_{1,n} + \left(\lambda_n^2 - k_2 N^2 - k_1 - i\tau_2 a_2 \lambda_n\right) \alpha_{2,n} = \beta_{4,n}, \\ k_0 \alpha_{1,n} + \left(\lambda_n^2 - k_3 N^2 - k_0 - i\tau_3 a_3 \lambda_n\right) \alpha_{3,n} + k_3 N \alpha_{4,n} = \beta_{6,n}, \\ k_3 N \alpha_{3,n} + \left(\lambda_n^2 - k_4 N^2 - k_3 - i\tau_4 a_4 \lambda_n\right) \alpha_{4,n} = \beta_{8,n}. \end{cases} \tag{4.13}$$

4.1. Case (4.1)

It is sufficient to treat the cases

$$(\tau_1, \tau_2, \tau_3, \tau_4) \in \{(1,0,0,0), (0,1,0,0), (1,1,0,0)\}. \tag{4.14}$$

Indeed, the proof in cases

$$(\tau_1, \tau_2, \tau_3, \tau_4) \in \{(0,0,1,0), (0,0,0,1), (0,0,1,1)\}$$

is similar to the one that will be given in cases (4.14), since $(1.1)_1$-$(1.1)_2$ and $(1.1)_3$-$(1.1)_4$ play symmetrical roles. We distinguish two subcases.

Subcase 1. (4.14) **with** $k_3 \neq k_4$. We choose

$$\begin{cases} \alpha_{1,n} = \alpha_{2,n} = \beta_{4,n} = 0, \quad \alpha_{3,n} = \dfrac{\beta_{2,n}}{k_0}, \quad \alpha_{4,n} = \dfrac{k_3 \beta_{2,n}}{k_0 (k_4 - k_3) N}, \\ \beta_{6,n} = \dfrac{k_3^2 \beta_{2,n}}{k_0 (k_4 - k_3)}, \quad \beta_{8,n} = \dfrac{k_3 (k_0 - k_3) \beta_{2,n}}{k_0 (k_4 - k_3) N}, \quad \lambda_n = \sqrt{k_3 N^2 + k_0}. \end{cases} \tag{4.15}$$

We see that (4.13) is satisfied. Moreover, according to $(4.11)_2$, $(4.12)_3$, $(4.12)_4$ and (4.15), it appears that

$$\begin{aligned} \|F_n\|_{\mathcal{H}}^2 &= \|f_{2,n}\|^2 + \|f_{4,n}\|^2 + \|f_{6,n}\|^2 + \|f_{8,n}\|^2 \\ &\leq \beta_{2,n}^2 + \beta_{4,n}^2 + \beta_{6,n}^2 + \beta_{8,n}^2 \\ &\leq \beta_{2,n}^2 \left[1 + \frac{k_3^4}{k_0^2 (k_4 - k_3)^2} + \frac{k_3^2 (k_0 - k_3)^2}{k_0^2 (k_4 - k_3)^2 N^2}\right], \end{aligned}$$

then one can choose $\beta_{2,n} = \epsilon > 0$ independent of n and small enough so that (4.7) holds. On the other hand, from $(4.12)_2$, we have

$$\begin{aligned} \|\Phi_n\|_{\mathcal{H}}^2 &\geq k_3 \|w_{n,x} + z_n\|^2 \\ &= k_3 \left(-\alpha_{3,n} N + \alpha_{4,n}\right)^2 \int_0^1 \sin^2(Nx) dx \\ &\geq \frac{k_3}{2} (-\alpha_{3,n} N + \alpha_{4,n})^2 \int_0^1 [1 - \cos\,(2Nx)]\, dx \\ &= \frac{k_3}{2} (-\alpha_{3,n} N + \alpha_{4,n})^2, \end{aligned}$$

hence (4.8), since $(4.15)_1$ implies $\lim_{n\to\infty} \alpha_{3,n}N = \infty$ and $\lim_{n\to\infty} \alpha_{4,n} = 0$, and so

$$\lim_{n\to\infty} \|\Phi_n\|_{\mathcal{H}} = \infty. \tag{4.16}$$

Subcase 2. (4.14) **with** $k_3 = k_4$. We choose

$$\begin{cases} \alpha_{1,n} = \alpha_{2,n} = \beta_{4,n} = 0, \quad \alpha_{3,n} = \dfrac{\beta_{2,n}}{k_0}, \quad \alpha_{4,n} = -\dfrac{\beta_{2,n}}{k_0}, \\ \beta_{6,n} = -\beta_{2,n}, \quad \beta_{8,n} = \dfrac{k_3\beta_{2,n}}{k_0}, \quad \lambda_n = \sqrt{k_3N^2 + k_3N}. \end{cases}$$

As in the previous subcase 1, we remark that (4.7), (4.13) and (4.16) are satisfied, by choosing $\beta_{2,n} = \epsilon > 0$ independent of n and small enough.

4.2. Case (4.2)

We distinguish two subcases.

Subcase 1. $(\tau_1, \tau_2, \tau_3, \tau_4) = (1, 1, 0, 1)$ **with** $k_3 \neq k_4$. We take

$$\begin{cases} \alpha_{1,n} = \alpha_{2,n} = \beta_{4,n} = 0, \quad \alpha_{3,n} = \dfrac{\beta_{2,n}}{k_0}, \quad \alpha_{4,n} = \dfrac{k_3\beta_{2,n}}{k_0(k_4 - k_3)N}, \\ \beta_{6,n} = \dfrac{k_3^2\beta_{2,n}}{k_0(k_4 - k_3)}, \quad \beta_{8,n} = \dfrac{k_3\left(k_0 - k_3 - ia_4\sqrt{k_3N^2 + k_0}\right)\beta_{2,n}}{k_0(k_4 - k_3)N}, \\ \lambda_n = \sqrt{k_3N^2 + k_0}. \end{cases}$$

Notice that (4.13) is satisfied and

$$\lim_{n\to\infty} \beta_{8,n} = -\frac{ik_3\sqrt{k_3}a_4\beta_{2,n}}{k_0(k_4 - k_3)}.$$

Then, by choosing $\beta_{2,n} = \epsilon > 0$ independent of n and small enough, we get (4.7) and (4.16).

Subcase 2. $(\tau_1, \tau_2, \tau_3, \tau_4) = (1, 1, 1, 0)$ **with** $k_3 \neq k_4$. We choose, for $\epsilon > 0$,

$$\begin{cases} \alpha_{1,n} = \alpha_{2,n} = \beta_{4,n} = \beta_{6,n} = 0, \\ \alpha_{3,n} = \dfrac{\epsilon}{k_0N}, \quad \alpha_{4,n} = \dfrac{\epsilon\left[(k_3 - k_4)N^2 + k_0 - k_3 + ia_3\sqrt{k_4N^2 + k_3}\right]}{k_0k_3N^2}, \\ \beta_{2,n} = \dfrac{\epsilon}{N}, \quad \beta_{8,n} = \dfrac{k_3\epsilon}{k_0}, \quad \lambda_n = \sqrt{k_4N^2 + k_3}. \end{cases}$$

We observe that (4.13) is satisfied and

$$\lim_{n\to\infty} \alpha_{4,n} = \frac{(k_3 - k_4)\epsilon}{k_0k_3} \neq 0. \tag{4.17}$$

By choosing $\epsilon > 0$ small enough, we get (4.7). Moreover, from $(4.12)_2$, we have

$$\|\Phi_n\|_{\mathcal{H}}^2 \geq k_4 \left\|z_{n,x}\right\|^2$$

$$
\begin{aligned}
&= k_4 \alpha_{4,n}^2 N^2 \int_0^1 \cos^2(Nx)dx \\
&= \frac{k_4}{2} \alpha_{4,n}^2 N^2 \int_0^1 [1+\cos\,(2Nx)]\,dx \\
&= \frac{k_4}{2} \alpha_{4,n}^2 N^2,
\end{aligned}
$$

which implies (4.16), since (4.17).

4.3. Case (4.3)

By symmetry, the proof is similar to the one given in case (4.2), where k_1 and k_2 play the roles of k_3 and k_4, respectively.

4.4. Case (4.4)

As before, by symmetry, the proof for $(\tau_1, \tau_2, \tau_3, \tau_4) = (0,1,1,0)$ is similar to the one that will be given for $(\tau_1, \tau_2, \tau_3, \tau_4) = (1,0,0,1)$. So we need to consider only the cases

$$
(\tau_1, \tau_2, \tau_3, \tau_4) \in \{(1,0,0,1), (1,0,1,0), (0,1,0,1)\}. \tag{4.18}
$$

Because we are assuming in this case that $(k_1, k_3) \neq (k_2, k_4)$, then we have $k_1 \neq k_2$ or $k_3 \neq k_4$, so we distinguish the next four subcases.

Subcase 1. $(\tau_1, \tau_2, \tau_3, \tau_4) \in \{(1,0,0,1), (0,1,0,1)\}$ **with** $k_3 \neq k_4$. The choices considered in Case (4.2) - Subcase 1 lead to the desired result.

Subcase 2. $(\tau_1, \tau_2, \tau_3, \tau_4) = (1,0,1,0)$ **with** $k_3 \neq k_4$. Using the choices considered in Case (4.2) - Subcase 2, we get the desired result.

Subcase 3. $(\tau_1, \tau_2, \tau_3, \tau_4) = (0,1,0,1)$ **with** $k_1 \neq k_2$. We choose

$$
\begin{cases}
\alpha_{3,n} = \alpha_{4,n} = \beta_{8,n} = 0, \quad \alpha_{1,n} = \dfrac{\beta_{6,n}}{k_0}, \quad \alpha_{2,n} = \dfrac{k_1 \beta_{6,n}}{k_0(k_2-k_1)N}, \\
\beta_{2,n} = \dfrac{k_1^2 \beta_{6,n}}{k_0(k_2-k_1)}, \quad \beta_{4,n} = \dfrac{k_1\left(k_0 - k_1 - i a_2 \sqrt{k_1 N^2 + k_0}\right)\beta_{6,n}}{k_0(k_2-k_1)N}, \\
\lambda_n = \sqrt{k_1 N^2 + k_0}.
\end{cases}
$$

Notice that (4.13) is satisfied and, for any $\beta_{6,n} = \epsilon > 0$ independent of n,

$$
\lim_{n\to\infty} N\alpha_{1,n} = \infty, \quad \lim_{n\to\infty} \alpha_{2,n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \beta_{4,n} = -\frac{i k_1 \sqrt{k_1} a_2 \beta_{6,n}}{k_0(k_2-k_1)}. \tag{4.19}
$$

Then, by choosing $\epsilon > 0$ small enough, we get (4.7). Moreover, from $(4.12)_1$, we see that

$$
\begin{aligned}
\|\Phi_n\|_{\mathcal{H}}^2 &\geq k_1 \left\|\varphi_{n,x} + \psi_n\right\|^2 \\
&= k_1 \left(-\alpha_{1,n} N + \alpha_{2,n}\right)^2 \int_0^1 \sin^2(Nx)dx \\
&\geq \frac{k_1}{2} (-\alpha_{1,n} N + \alpha_{2,n})^2 \int_0^1 [1-\cos\,(2Nx)]\,dx
\end{aligned}
$$

$$= \frac{k_1}{2}(-\alpha_{1,n}N + \alpha_{2,n})^2,$$

so (4.16) holds, since (4.19).

Subcase 4. $(\tau_1, \tau_2, \tau_3, \tau_4) \in \{(1,0,0,1),(1,0,1,0)\}$ **with** $k_1 \neq k_2$. We take, for $\epsilon > 0$,

$$\begin{cases} \alpha_{3,n} = \alpha_{4,n} = \beta_{2,n} = \beta_{8,n} = 0, \quad \alpha_{1,n} = \dfrac{\epsilon}{k_0 N}, \\ \alpha_{2,n} = \dfrac{\epsilon\left[(k_1 - k_2)N^2 + k_0 - k_1 + ia_1\sqrt{k_2N^2 + k_1}\right]}{k_0k_1N^2}, \\ \beta_{6,n} = \dfrac{\epsilon}{N}, \quad \beta_{4,n} = \dfrac{k_1\epsilon}{k_0}, \quad \lambda_n = \sqrt{k_2N^2 + k_1}. \end{cases}$$

We observe that (4.13) is satisfied and

$$\lim_{n\to\infty} \alpha_{2,n} = \frac{(k_1 - k_2)\epsilon}{k_0k_1} \neq 0. \tag{4.20}$$

By choosing $\epsilon > 0$ small enough, we get (4.7). Moreover, using $(4.12)_1$, we get

$$\begin{aligned} \|\Phi_n\|^2_{\mathcal{H}} &\geq k_2 \|\psi_{n,x}\|^2 \\ &= k_2\alpha^2_{2,n}N^2 \int_0^1 \cos^2(Nx)dx \\ &= \frac{k_2}{2}\alpha^2_{2,n}N^2 \int_0^1 [1 + \cos\,(2Nx)]\,dx \\ &= \frac{k_2}{2}\alpha^2_{2,n}N^2, \end{aligned}$$

which implies (4.16), since (4.20). This ends the proof of Theorem 4.1. □

5. Exponential stability

In this section, we give necessary and sufficient conditions for the exponentailly stability (4.5).

Theorem 5.1. *The exponentailly stability* (4.5) *for* (2.4) *holds if and only if*

$$(\tau_1, \tau_2, \tau_3, \tau_4) = (1,1,1,1) \tag{5.1}$$

or

$$(\tau_1, \tau_2, \tau_3, \tau_4) \in \{(1,1,0,1),(1,1,1,0)\} \quad \text{and} \quad k_3 = k_4 \tag{5.2}$$

or

$$(\tau_1, \tau_2, \tau_3, \tau_4) \in \{(0,1,1,1),(1,0,1,1)\} \quad \text{and} \quad k_1 = k_2 \tag{5.3}$$

or

$$\begin{aligned} &(\tau_1, \tau_2, \tau_3, \tau_4) \in \{(1,0,0,1),(1,0,1,0),(0,1,0,1),(0,1,1,0)\} \\ &\text{and} \quad (k_1, k_3) = (k_2, k_4). \end{aligned} \tag{5.4}$$

Proof. According to the results of section 4, (4.5) does not hold if (5.1)-(5.4) are not satisfied. On the other hand, from the results of section 3, we remark that the first condition in (4.6) holds if (5.1) or (5.2) or (5.3) or (5.4) is satisfied. Moreover, the exponential stability (4.5) is equivalent to (4.6) (see [22, 34]). So, to get Theorem 5.1, it is sufficient to prove that the second condition in (4.6) holds in cases (5.1)-(5.4).

We assume by contradiction that the second condition in (4.6) is false. Then there exist sequences $(\lambda_n)_n \subset \mathbb{R}$ and $(\Phi_n)_n \subset D(\mathcal{A})$, $n \in \mathbb{N}$, such that

$$\|\,\Phi_n\|_{\mathcal{H}} \,=\, 1, \quad \forall\, n \in \mathbb{N}, \tag{5.5}$$

$$\lim_{n\to\infty} |\lambda_n| = \infty \tag{5.6}$$

and

$$\lim_{n\to\infty} \|(i\lambda_n I - \mathcal{A})\,\Phi_n\|_{\mathcal{H}} = 0. \tag{5.7}$$

Let, as in section 4,

$$\Phi_n := \left(\varphi_n, \tilde{\varphi}_n, \psi_n, \tilde{\psi}_n, w_n, \tilde{w}_n, z_n, \tilde{z}_n\right)^T. \tag{5.8}$$

We will prove that

$$\lim_{n\to\infty} \|\Phi_n\|_{\mathcal{H}} = 0, \tag{5.9}$$

which is a contradiction with (5.5). The limit (5.7) is equivalent to the following convergences:

$$\begin{cases} i\lambda_n\varphi_n - \tilde{\varphi}_n \to 0 & \text{in } V_1, \\ i\lambda_n\tilde{\varphi}_n - k_1\left(\varphi_{n,x} + \psi_n\right)_x - k_0\left(w_n - \varphi_n\right) + \tau_1 a_1 \tilde{\varphi}_n \to 0 & \text{in } L^2(0,1), \\ i\lambda_n\psi_n - \tilde{\psi}_n \to 0 & \text{in } V_0, \\ i\lambda_n\tilde{\psi}_n - k_2\psi_{n,xx} + k_1\left(\varphi_{n,x} + \psi_n\right) + \tau_2 a_2 \tilde{\psi}_n \to 0 & \text{in } L^2(0,1), \\ i\lambda_n w_n - \tilde{w}_n \to 0 & \text{in } V_1, \\ i\lambda_n\tilde{w}_n - k_3\left(w_{n,x} + z_n\right)_x + k_0\left(w_n - \varphi_n\right) + \tau_3 a_3 \tilde{w}_n \to 0 & \text{in } L^2(0,1), \\ i\lambda_n z_n - \tilde{z}_n \to 0 & \text{in } V_0, \\ i\lambda_n\tilde{z}_n - k_4 z_{n,xx} + k_3\left(w_{n,x} + z_n\right) + \tau_4 a_4 \tilde{z}_n \to 0 & \text{in } L^2(0,1), \end{cases} \tag{5.10}$$

where "$\to 0$" means "converges to zero when n converges to ∞". Taking the inner product of $(i\,\lambda_n\, I\,-\,\mathcal{A})\,\Phi_n$ with Φ_n in $\mathcal{H}$ and using (2.8), we get

$$\begin{aligned} Re\,\langle (i\lambda_n I - \mathcal{A})\,\Phi_n, \Phi_n\rangle_{\mathcal{H}} &= Re\,\langle -\mathcal{A}\Phi_n, \Phi_n\rangle_{\mathcal{H}} \\ &= \tau_1 a_1 \|\tilde{\varphi}\|^2 + \tau_2 a_2 \|\tilde{\psi}\|^2 + \tau_3 a_3 \|\tilde{w}\|^2 + \tau_4 a_4 \|\tilde{z}\|^2, \end{aligned}$$

so, (5.5) and (5.7) imply that

$$\tau_1 a_1 \|\tilde{\varphi}_n\|^2 + \tau_2 a_2 \|\tilde{\psi}_n\|^2 + \tau_3 a_3 \|\tilde{w}_n\|^2 + \tau_4 a_4 \|\tilde{z}_n\|^2 \to 0. \tag{5.11}$$

5.1. Case (5.1)

By combining (5.1) and (5.11), we find

$$\tilde{\varphi}_n,\ \tilde{\psi}_n,\ \tilde{w}_n,\ \tilde{z}_n \to 0 \text{ in } L^2(0,1), \tag{5.12}$$

and then $(5.10)_1$, $(5.10)_3$, $(5.10)_5$ and $(5.10)_7$ imply that

$$\lambda_n\varphi_n,\ \lambda_n\psi_n,\ \lambda_n w_n,\ \lambda_n z_n \to 0 \text{ in } L^2(0,1), \tag{5.13}$$

so, from (5.6) and (5.13), we conclude that

$$\varphi_n,\ \psi_n,\ w_n,\ z_n \to 0 \text{ in } L^2(0,1). \tag{5.14}$$

Taking the inner product of $(5.10)_2$ with φ_n in $L^2(0,1)$, integrating by parts and using (5.5) and the boundary conditions, we entail

$$i\langle\tilde{\varphi}_n, \lambda_n\varphi_n\rangle - \langle k_1\psi_{n,x} + k_0(w_n - \varphi_n) - a_1\tilde{\varphi}_n, \varphi_n\rangle + k_1\|\varphi_{n,x}\|^2 \to 0, \tag{5.15}$$

then, combining (5.5), (5.13), (5.14) and (5.15), it follows that

$$\varphi_{n,x} \to 0 \text{ in } L^2(0,1). \tag{5.16}$$

Similarly, taking the inner product in $L^2(0,1)$ of $(5.10)_4$, $(5.10)_6$ and $(5.10)_8$ with ψ_n, w_n and z_n, respectively, integrating by parts, using the boundary conditions and (5.5), we find

$$i\left\langle\tilde{\psi}_n, \lambda_n\psi_n\right\rangle + \left\langle k_1(\varphi_{n,x} + \psi_n) + a_2\tilde{\psi}_n, \psi_n\right\rangle + k_2\|\psi_{n,x}\|^2 \to 0, \tag{5.17}$$

$$i\langle\tilde{w}_n, \lambda_n w_n\rangle - \langle k_3 z_{n,x} - k_0(w_n - \varphi_n) - a_3\tilde{w}_n, w_n\rangle + k_3\|w_{n,x}\|^2 \to 0 \tag{5.18}$$

and

$$i\langle\tilde{z}_n, \lambda_n z_n\rangle + \langle k_3(w_{n,x} + z_n) + a_4\tilde{z}_n, z_n\rangle + k_4\|z_{n,x}\|^2 \to 0, \tag{5.19}$$

then, by combining (5.5), (5.13), (5.14) and (5.17)-(5.19), we arrive at

$$\psi_{n,x},\ w_{n,x},\ z_{n,x} \to 0 \text{ in } L^2(0,1). \tag{5.20}$$

The limits (5.12), (5.14), (5.16) and (5.20) lead to (5.9).

5.2. Case (5.2)

We are assuming in this case that $k_3 = k_4$. We distinguish two subcases.

Subcase 1. $(\tau_1, \tau_2, \tau_3, \tau_4) = (1, 1, 0, 1)$ **and** $k_3 = k_4$. According to (5.11), we get

$$\tilde{\varphi}_n,\ \tilde{\psi}_n,\ \tilde{z}_n \to 0 \text{ in } L^2(0,1), \tag{5.21}$$

so $(5.10)_1$, $(5.10)_3$ and $(5.10)_7$ lead to

$$\lambda_n\varphi_n,\ \lambda_n\psi_n,\ \lambda_n z_n \to 0 \text{ in } L^2(0,1), \tag{5.22}$$

hence, from (5.6) and (5.22), we deduce that

$$\varphi_n,\ \psi_n,\ z_n \to 0 \text{ in } L^2(0,1). \tag{5.23}$$

As for (5.16) and (5.20) in the previous case (5.1), taking the inner product in $L^2(0,1)$ of $(5.10)_2$, $(5.10)_4$ and $(5.10)_8$ with φ_n, ψ_n and z_n, respectively, integrating by parts and using the boundary conditions, we get (5.15), (5.17) and (5.19), then, combining with (5.5), (5.22) and (5.23), it appears that

$$\varphi_{n,x},\ \psi_{n,x},\ z_{n,x} \to 0 \text{ in } L^2(0,1). \tag{5.24}$$

From (5.5) and $(5.10)_5$, we have

$$(\lambda_n w_n)_n \text{ is bounded in } L^2(0,1), \tag{5.25}$$

then, by combining (5.6) and (5.25), we find

$$w_n \to 0 \text{ in } L^2(0,1). \tag{5.26}$$

Taking the inner product of $(5.10)_6$ with $z_{n,x}$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, (5.5) and (5.24), we obtain

$$\langle i\lambda_n \tilde{w}_n, z_{n,x}\rangle - k_3 \langle w_{n,xx}, z_{n,x}\rangle \to 0. \tag{5.27}$$

Similarly, taking the inner product of $w_{n,x}$ with $(5.10)_8$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, (5.5), (5.21) and (5.23), we find

$$\langle w_{n,x}, i\lambda_n \tilde{z}_n\rangle + k_4 \langle w_{n,xx}, z_{n,x}\rangle + k_3 \|w_{n,x}\|^2 \to 0, \tag{5.28}$$

therefore, adding (5.27) and (5.28), and noticing that $k_3 = k_4$, we deduce that

$$k_3 \|w_{n,x}\|^2 + \langle i\lambda_n \tilde{w}_n, z_{n,x}\rangle + \langle w_{n,x}, i\lambda_n \tilde{z}_n\rangle \to 0. \tag{5.29}$$

But we observe that

$$\langle i\lambda_n \tilde{w}_n, z_{n,x}\rangle = -\langle \tilde{w}_n, i\lambda_n z_{n,x}\rangle = -\langle \tilde{w}_n, i\lambda_n z_{n,x} - \tilde{z}_{n,x}\rangle - \langle \tilde{w}_n, \tilde{z}_{n,x}\rangle$$

and, using also inegrating by parts,

$$\begin{aligned}\langle w_{n,x}, i\lambda_n \tilde{z}_n\rangle &= -\langle i\lambda_n w_{n,x}, \tilde{z}_n\rangle \\ &= -\langle i\lambda_n w_{n,x} - \tilde{w}_{n,x}, \tilde{z}_n\rangle - \langle \tilde{w}_{n,x}, \tilde{z}_n\rangle \\ &= -\langle i\lambda_n w_{n,x} - \tilde{w}_{n,x}, \tilde{z}_n\rangle + \langle \tilde{w}_n, \tilde{z}_{n,x}\rangle,\end{aligned}$$

so, by adding the above two identities and using (5.5) and the limits $(5.10)_5$ and $(5.10)_7$, we see that

$$\langle i\lambda_n \tilde{w}_n, z_{n,x}\rangle + \langle w_{n,x}, i\lambda_n \tilde{z}_n\rangle \to 0, \tag{5.30}$$

then, by combining (5.29) and (5.30), we conclude that

$$w_{n,x} \to 0 \text{ in } L^2(0,1). \tag{5.31}$$

Taking the inner product in $L^2(0,1)$ of $(5.10)_6$ with w_n, integrating by parts, using (5.5) and the boundary conditions and exploiting (5.26) and (5.31), it follows that

$$\langle i\lambda_n \tilde{w}_n, w_n\rangle \to 0. \tag{5.32}$$

Because

$$\langle i\lambda_n \tilde{w}_n, w_n\rangle = -\langle \tilde{w}_n, i\lambda_n w_n\rangle = -\langle \tilde{w}_n, i\lambda_n w_n - \tilde{w}_n\rangle - \|\tilde{w}_n\|^2,$$

then, by combining with $(5.10)_5$ and (5.32), we obtain

$$\tilde{w}_n \to 0 \text{ in } L^2(0,1). \tag{5.33}$$

Finally, the limits (5.21), (5.23), (5.24), (5.26), (5.31) and (5.33) imply (5.9).

Subcase 2. $(\tau_1, \tau_2, \tau_3, \tau_4) = (1,1,1,0)$ **and** $k_3 = k_4$. From (5.11), we have

$$\tilde{\varphi}_n,\ \tilde{\psi}_n,\ \tilde{w}_n \to 0 \text{ in } L^2(0,1), \tag{5.34}$$

then $(5.10)_1$, $(5.10)_3$ and $(5.10)_5$ imply that

$$\lambda_n \varphi_n,\ \lambda_n \psi_n,\ \lambda_n w_n \to 0 \text{ in } L^2(0,1), \tag{5.35}$$

then, according to (5.6) and (5.35), we deduce that

$$\varphi_n,\ \psi_n,\ w_n \to 0 \text{ in } L^2(0,1). \tag{5.36}$$

Similarly to the prrof of (5.16) and (5.20), taking the inner product in $L^2(0,1)$ of $(5.10)_2$, $(5.10)_4$ and $(5.10)_6$ with φ_n, ψ_n and w_n, respectively, integrating by parts and using (5.5) and the boundary conditions, we obtain (5.15), (5.17) and (5.18), therefore, by combining with (5.35) and (5.36), we observe that

$$\varphi_{n,x},\ \psi_{n,x},\ w_{n,x} \to 0 \text{ in } L^2(0,1). \tag{5.37}$$

Using (5.5) and $(5.10)_7$, we see that

$$(\lambda_n z_n)_n \text{ is bounded in } L^2(0,1), \tag{5.38}$$

then, by combining (5.6) and (5.38), we get

$$z_n \to 0 \text{ in } L^2(0,1). \tag{5.39}$$

Taking the inner product of $(5.10)_6$ with $z_{n,x}$ in $L^2(0,1)$, integrating by parts, using (5.5) and the boundary conditions and exploiting (5.34) and (5.36), we obtain

$$\langle i\lambda_n \tilde{w}_n, z_{n,x}\rangle - k_3 \langle w_{n,xx}, z_{n,x}\rangle - k_3 \|z_{n,x}\|^2 \to 0. \tag{5.40}$$

Similarly, taking the inner product of $w_{n,x}$ with $(5.10)_8$ in $L^2(0,1)$, integrating by parts and using (5.5), (5.37) and the boundary conditions, we find

$$\langle w_{n,x}, i\lambda_n \tilde{z}_n\rangle + k_4 \langle w_{n,xx}, z_{n,x}\rangle \to 0. \tag{5.41}$$

Therefore, adding (5.40) and (5.41), and noticing that $k_3 = k_4$, we conclude that

$$-k_3 \|z_{n,x}\|^2 + \langle i\lambda_n \tilde{w}_n, z_{n,x}\rangle + \langle w_{n,x}, i\lambda_n \tilde{z}_n\rangle \to 0. \tag{5.42}$$

As in the previous subcase 1, we remark that (5.30) holds, then, combining with (5.42), we deduce that

$$z_{n,x} \to 0 \text{ in } L^2(0,1). \tag{5.43}$$

Taking the inner product in $L^2(0,1)$ of $(5.10)_8$ with z_n, integrating by parts, using (5.5) and the boundary conditions and exploiting (5.39) and (5.43), it follows that

$$\langle i\lambda_n \tilde{z}_n, z_n\rangle \to 0. \tag{5.44}$$

But we remark that

$$\langle i\lambda_n \tilde{z}_n, z_n\rangle = -\langle \tilde{z}_n, i\lambda_n z_n\rangle = -\langle \tilde{z}_n, i\lambda_n z_n - \tilde{z}_n\rangle - \|\tilde{z}_n\|^2, \tag{5.45}$$

then, by combining with $(5.10)_7$ and (5.44), we find

$$\tilde{z}_n \to 0 \text{ in } L^2(0,1). \tag{5.46}$$

Consequently, (5.34), (5.36), (5.37), (5.39), (5.43) and (5.46) lead to (5.9).

5.3. Case (5.3)

By symmetry, the proof is similar to the one given in case (5.2), where k_1 and k_2 play the roles of k_3 and k_4, respectively.

5.4. Case (5.4)

As before, by symmetry, the proof for $(\tau_1, \tau_2, \tau_3, \tau_4) = (0, 1, 1, 0)$ is similar to the one that will be given for $(\tau_1, \tau_2, \tau_3, \tau_4) = (1, 0, 0, 1)$. So we need to consider only the three cases

$$(\tau_1, \tau_2, \tau_3, \tau_4) \in \{(1, 0, 0, 1), (1, 0, 1, 0), (0, 1, 0, 1)\} \quad \text{and} \quad (k_1, k_3) = (k_2, k_4). \tag{5.47}$$

Subcase 1. $(\tau_1, \tau_2, \tau_3, \tau_4) = (1, 0, 0, 1)$ **and** $(k_1, k_3) = (k_2, k_4)$. According to (5.11), we see that

$$\tilde{\varphi}_n,\ \tilde{z}_n \to 0 \text{ in } L^2(0,1), \tag{5.48}$$

so $(5.10)_1$ and $(5.10)_7$ lead to

$$\lambda_n \varphi_n,\ \lambda_n z_n \to 0 \text{ in } L^2(0,1), \tag{5.49}$$

then (5.6) and (5.49) imply that

$$\varphi_n,\ z_n \to 0 \text{ in } L^2(0,1). \tag{5.50}$$

Taking the inner product in $L^2(0,1)$ of $(5.10)_2$ and $(5.10)_8$ with φ_n and z_n, respectively, integrating by parts and using the boundary conditions and (5.5), we get (5.15) and (5.19), then, combining with (5.49) and (5.50), it appears that

$$\varphi_{n,x},\ z_{n,x} \to 0 \text{ in } L^2(0,1). \tag{5.51}$$

From (5.5), $(5.10)_3$ and $(5.10)_5$, we have

$$(\lambda_n \psi_n)_n,\ (\lambda_n w_n)_n \text{ are bounded in } L^2(0,1), \tag{5.52}$$

then, by combining (5.6) and (5.52), we find

$$\psi_n,\ w_n \to 0 \text{ in } L^2(0,1). \tag{5.53}$$

We observe that (5.27), (5.28), (5.29), (5.30) and (5.32) are satisfied also in this subcase 1, since $k_3 = k_4$ and $(\tau_3, \tau_4) = (0, 1)$ as in Case (5.2)-Subcase 1, so, similarly, this leads to

$$w_{n,x},\ \tilde{w}_n \to 0 \text{ in } L^2(0,1). \tag{5.54}$$

Taking the inner product of $(5.10)_2$ with $\psi_{n,x}$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, (5.5), (5.48), (5.50) and (5.53), we obtain

$$\langle i\lambda_n \tilde{\varphi}_n, \psi_{n,x} \rangle - k_1 \langle \varphi_{n,xx}, \psi_{n,x} \rangle - k_1 \|\psi_{n,x}\|^2 \to 0. \tag{5.55}$$

Similarly, taking the inner product of $\varphi_{n,x}$ with $(5.10)_4$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, (5.5) and (5.51), we find

$$\left\langle \varphi_{n,x}, i\lambda_n \tilde{\psi}_n \right\rangle + k_2 \langle \varphi_{n,xx}, \psi_{n,x} \rangle \to 0, \tag{5.56}$$

therefore, adding (5.55) and (5.56), and noticing that $k_1 = k_2$, we deduce that

$$-k_1 \|\psi_{n,x}\|^2 + \langle i\lambda_n \tilde{\varphi}_n, \psi_{n,x} \rangle + \left\langle \varphi_{n,x}, i\lambda_n \tilde{\psi}_n \right\rangle \to 0. \tag{5.57}$$

On the other hand, we have

$$\langle i\lambda_n \tilde{\varphi}_n, \psi_{n,x} \rangle = -\langle \tilde{\varphi}_n, i\lambda_n \psi_{n,x} \rangle = -\left\langle \tilde{\varphi}_n, i\lambda_n \psi_{n,x} - \tilde{\psi}_{n,x} \right\rangle - \left\langle \tilde{\varphi}_n, \tilde{\psi}_{n,x} \right\rangle$$

and, using also inegrating by parts,

$$\begin{aligned} \left\langle \varphi_{n,x}, i\lambda_n \tilde{\psi}_n \right\rangle &= -\left\langle i\lambda_n \varphi_{n,x}, \tilde{\psi}_n \right\rangle \\ &= -\left\langle i\lambda_n \varphi_{n,x} - \tilde{\varphi}_{n,x}, \tilde{\psi}_n \right\rangle - \left\langle \tilde{\varphi}_{n,x}, \tilde{\psi}_n \right\rangle \\ &= -\left\langle i\lambda_n \varphi_{n,x} - \tilde{\varphi}_{n,x}, \tilde{\psi}_n \right\rangle + \left\langle \tilde{\varphi}_n, \tilde{\psi}_{n,x} \right\rangle, \end{aligned}$$

so, by adding the above two identities and using (5.5) and the limits $(5.10)_1$ and $(5.10)_3$, we see that

$$\langle i\lambda_n \tilde{\varphi}_n, \psi_{n,x} \rangle + \left\langle \varphi_{n,x}, i\lambda_n \tilde{\psi}_n \right\rangle \to 0, \tag{5.58}$$

then, by combining (5.57) and (5.58), we conclude that

$$\psi_{n,x} \to 0 \text{ in } L^2(0,1). \tag{5.59}$$

Taking the inner product in $L^2(0,1)$ of $(5.10)_4$ with ψ_n, integrating by parts, using (5.5) and the boundary conditions and exploiting (5.53) and (5.59), it follows that

$$\left\langle i\lambda_n \tilde{\psi}_n, \psi_n \right\rangle \to 0. \tag{5.60}$$

Because

$$\left\langle i\lambda_n \tilde{\psi}_n, \psi_n \right\rangle = -\left\langle \tilde{\psi}_n, i\lambda_n \psi_n \right\rangle = -\left\langle \tilde{\psi}_n, i\lambda_n \psi_n - \tilde{\psi}_n \right\rangle - \left\| \tilde{\psi}_n \right\|^2,$$

then, by combining with $(5.10)_3$ and (5.60), we obtain

$$\tilde{\psi}_n \to 0 \text{ in } L^2(0,1). \tag{5.61}$$

Finally, the limits (5.48), (5.50), (5.51), (5.53), (5.54), (5.59) and (5.61) lead to (5.9).

Subcase 2. $(\tau_1, \tau_2, \tau_3, \tau_4) = (1, 0, 1, 0)$ **and** $(k_1, k_3) = (k_2, k_4)$. From (5.11), it appears that

$$\tilde{\varphi}_n, \ \tilde{w}_n \to 0 \text{ in } L^2(0,1), \tag{5.62}$$

so $(5.10)_1$ and $(5.10)_5$ lead to

$$\lambda_n \varphi_n, \ \lambda_n w_n \to 0 \text{ in } L^2(0,1), \tag{5.63}$$

then, using (5.6) and (5.63), we find

$$\varphi_n, \ w_n \to 0 \text{ in } L^2(0,1). \tag{5.64}$$

Taking the inner product in $L^2(0,1)$ of $(5.10)_2$ and $(5.10)_6$ with φ_n and w_n, respectively, integrating by parts and using the boundary conditions and (5.5), we get (5.15) and (5.18), then it follows from (5.63) and (5.64) that

$$\varphi_{n,x},\ w_{n,x} \to 0 \text{ in } L^2(0,1). \tag{5.65}$$

Thanks to (5.5), $(5.10)_3$ and $(5.10)_7$, we have

$$(\lambda_n \psi_n)_n\,,\ (\lambda_n z_n)_n \text{ are bounded in } L^2(0,1)\,, \tag{5.66}$$

then, by combining (5.6) and (5.66), we find

$$\psi_n,\ z_n \to 0 \text{ in } L^2(0,1). \tag{5.67}$$

We notice that (5.55), (5.56), (5.57), (5.58) and (5.60) hold also in this subcase 2, since $k_1 = k_2$ and $(\tau_1, \tau_2) = (1, 0)$ as in Case (5.4)-Subcase 1, so we get

$$\psi_{n,x},\ \tilde{\psi}_n \to 0 \text{ in } L^2(0,1). \tag{5.68}$$

On the other hand, we see that (5.40), (5.41), (5.42), (5.44) and (5.45) are still satisfied in this subcase 2 because $k_3 = k_4$ and $(\tau_3, \tau_4) = (1, 0)$ as in Case (5.2)-Subcase 2, then we arrive at

$$z_{n,x},\ \tilde{z}_n \to 0 \text{ in } L^2(0,1). \tag{5.69}$$

Consequently, the limits (5.62), (5.64), (5.65), (5.67), (5.68) and (5.69) lead to (5.9).

Subcase 3. $(\tau_1, \tau_2, \tau_3, \tau_4) = (0, 1, 0, 1)$ **and** $(k_1, k_3) = (k_2, k_4)$. The identity (5.11) implies that

$$\tilde{\psi}_n,\ \tilde{z}_n \to 0 \text{ in } L^2(0,1)\,, \tag{5.70}$$

then $(5.10)_3$ and $(5.10)_7$ lead to

$$\lambda_n \psi_n,\ \lambda_n z_n \to 0 \text{ in } L^2(0,1)\,, \tag{5.71}$$

so, using (5.6) and (5.71), we obtain

$$\psi_n,\ z_n \to 0 \text{ in } L^2(0,1). \tag{5.72}$$

Taking the inner product in $L^2(0,1)$ of $(5.10)_4$ and $(5.10)_8$ with ψ_n and z_n, respectively, integrating by parts and using the boundary conditions and (5.5), we find (5.17) and (5.19), then, combining with (5.71) and (5.72), it follows that

$$\psi_{n,x},\ z_{n,x} \to 0 \text{ in } L^2(0,1). \tag{5.73}$$

According to (5.5), $(5.10)_1$ and $(5.10)_5$, we have

$$(\lambda_n \varphi_n)_n\,,\ (\lambda_n w_n)_n \text{ are bounded in } L^2(0,1)\,, \tag{5.74}$$

then, by combining (5.6) and (5.74), we get

$$\varphi_n,\ w_n \to 0 \text{ in } L^2(0,1). \tag{5.75}$$

We remark that (5.27), (5.28), (5.29), (5.30) and (5.32) hold also in this subcase 3, since $k_3 = k_4$ and $(\tau_3, \tau_4) = (0, 1)$ as in Case (5.2)-Subcase 1, hence

$$w_{n,x},\ \tilde{w}_n \to 0 \text{ in } L^2(0,1). \tag{5.76}$$

Taking the inner product of $(5.10)_2$ with $\psi_{n,x}$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, (5.5) and (5.73), we obtain

$$\langle i\lambda_n \tilde{\varphi}_n, \psi_{n,x}\rangle - k_1 \langle \varphi_{n,xx}, \psi_{n,x}\rangle \to 0. \tag{5.77}$$

Similarly, taking the inner product of $\varphi_{n,x}$ with $(5.10)_4$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, (5.5), (5.70) and (5.72), we find

$$\left\langle \varphi_{n,x}, i\lambda_n \tilde{\psi}_n \right\rangle + k_2 \langle \varphi_{n,xx}, \psi_{n,x}\rangle + k_1 \|\varphi_{n,x}\|^2 \to 0, \tag{5.78}$$

therefore, adding (5.77) and (5.78), and exploiting the property $k_1 = k_2$, we deduce that

$$k_1 \|\varphi_{n,x}\|^2 + \langle i\lambda_n \tilde{\varphi}_n, \psi_{n,x}\rangle + \left\langle \varphi_{n,x}, i\lambda_n \tilde{\psi}_n \right\rangle \to 0. \tag{5.79}$$

On the other hand, we observe that (5.58) holds, and then, by combining with (5.79), we conclude that

$$\varphi_{n,x} \to 0 \text{ in } L^2(0,1). \tag{5.80}$$

Taking the inner product in $L^2(0,1)$ of $(5.10)_2$ with φ_n, integrating by parts, using (5.5) and the boundary conditions and exploiting (5.75) and (5.80), we get

$$\langle i\lambda_n \tilde{\varphi}_n, \varphi_n\rangle \to 0. \tag{5.81}$$

Because

$$\langle i\lambda_n \tilde{\varphi}_n, \varphi_n\rangle = -\langle \tilde{\varphi}_n, i\lambda_n \varphi_n\rangle = -\langle \tilde{\varphi}_n, i\lambda_n \varphi_n - \tilde{\varphi}_n\rangle - \|\tilde{\varphi}_n\|^2,$$

then, by combining with $(5.10)_1$ and (5.81), we obtain

$$\tilde{\varphi}_n \to 0 \text{ in } L^2(0,1). \tag{5.82}$$

Hence, the limit (5.9) holds according to the limits (5.70), (5.72), (5.73), (5.75), (5.76), (5.80) and (5.82). Finally, the proof of Theorem 5.1 is completed. □

6. Polynomial stability

In this section, we study the decay rate of solutions in the following cases:

$$(\tau_1, \tau_2, \tau_3, \tau_4) \in \{(0,1,0,0), (0,0,0,1)\}, \tag{6.1}$$

$$(\tau_1, \tau_2, \tau_3, \tau_4) \in \{(1,1,0,0), (0,0,1,1)\}, \tag{6.2}$$

$$(\tau_1, \tau_2, \tau_3, \tau_4) \in \{(1,1,0,1), (1,1,1,0)\} \quad \text{and} \quad k_3 \neq k_4, \tag{6.3}$$

$$(\tau_1, \tau_2, \tau_3, \tau_4) \in \{(0,1,1,1), (1,0,1,1)\} \quad \text{and} \quad k_1 \neq k_2 \tag{6.4}$$

and

$$(\tau_1, \tau_2, \tau_3, \tau_4) \in \{(1,0,0,1), (1,0,1,0), (0,1,0,1), (0,1,1,0)\}$$
$$\text{and} \quad (k_1, k_3) \neq (k_2, k_4), \tag{6.5}$$

where the strong stability (2.17) is satisfied but the exponential one (4.5) does not hold (see sections 3 and 4). We will prove that the decay rate of solutions in these cases is at least of polynomial type; that is, there exists $\delta > 0$ such that

$$\forall \Phi_0 \in D(\mathcal{A}), \exists\, c > 0 : \|\Phi(t)\|_{\mathcal{H}} \leq c t^{-\delta}, \quad \forall t > 0. \tag{6.6}$$

Theorem 6.1. *The polynomial decay* (6.6) *is satisfied in cases* (6.1)-(6.5) *with*

$$\delta = \begin{cases} \dfrac{1}{18} & \textit{in case } (6.1), \\ \dfrac{1}{14} & \textit{in case } (6.2), \\ \dfrac{1}{2} & \textit{in cases } (6.3)\text{-}(6.5). \end{cases} \tag{6.7}$$

Proof. It is known by now (see [7, 9, 10]) that (6.6) holds if

$$i\mathbb{R} \subset \rho(\mathcal{A}) \quad \text{and} \quad \sup_{|\lambda|\geq 1} |\lambda|^{-\frac{1}{\delta}} \left\| (i\lambda I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{6.8}$$

We have proved in section 3 that the first condition in (6.8) holds in cases (6.1)-(6.5). So we will prove that the second condition in (6.8) is also satisfied. This will be done by contradiction arguments. Let us assume that the second condition in (6.8) is false, then, there exist sequences $(\Phi_n)_n \subset D(\mathcal{A})$ and $(\lambda_n)_n \subset \mathbb{R}$, $n \in \mathbb{N}$, satisfying (5.5), (5.6) and

$$\lim_{n\to\infty} |\lambda_n|^{\frac{1}{\delta}} \left\| (i\lambda_n I - \mathcal{A})\, \Phi_n \right\|_{\mathcal{H}} = 0. \tag{6.9}$$

The contradiction will be obtained by proving (5.9). Let define Φ_n by (5.8). From (6.9), we get

$$\begin{cases} |\lambda_n|^{\frac{1}{\delta}} \left[i\lambda_n \varphi_n - \tilde{\varphi}_n \right] \to 0 & \text{in } V_1, \\ |\lambda_n|^{\frac{1}{\delta}} \left[i\lambda_n \tilde{\varphi}_n - k_1 (\varphi_{n,x} + \psi_n)_x - k_0 (w_n - \varphi_n) + \tau_1 a_1 \tilde{\varphi}_n \right] \to 0 & \text{in } L^2(0,1), \\ |\lambda_n|^{\frac{1}{\delta}} \left[i\lambda_n \psi_n - \tilde{\psi}_n \right] \to 0 & \text{in } V_0, \\ |\lambda_n|^{\frac{1}{\delta}} \left[i\lambda_n \tilde{\psi}_n - k_2 \psi_{n,xx} + k_1 (\varphi_{n,x} + \psi_n) + \tau_2 a_2 \tilde{\psi}_n \right] \to 0 & \text{in } L^2(0,1), \\ |\lambda_n|^{\frac{1}{\delta}} \left[i\lambda_n w_n - \tilde{w}_n \right] \to 0 & \text{in } V_1, \\ |\lambda_n|^{\frac{1}{\delta}} \left[i\lambda_n \tilde{w}_n - k_3 (w_{n,x} + z_n)_x + k_0 (w_n - \varphi_n) + \tau_3 a_3 \tilde{w}_n \right] \to 0 & \text{in } L^2(0,1), \\ |\lambda_n|^{\frac{1}{\delta}} \left[i\lambda_n z_n - \tilde{z}_n \right] \to 0 & \text{in } V_0, \\ |\lambda_n|^{\frac{1}{\delta}} \left[i\lambda_n \tilde{z}_n - k_4 z_{n,xx} + k_3 (w_{n,x} + z_n) + \tau_4 a_4 \tilde{z}_n \right] \to 0 & \text{in } L^2(0,1). \end{cases} \tag{6.10}$$

Taking the inner product of $|\lambda_n|^{\frac{1}{\delta}} (i\,\lambda_n I - \mathcal{A})\, \Phi_n$ with Φ_n in $\mathcal{H}$ and using (2.8), we get

$$\begin{aligned} & Re \left\langle |\lambda_n|^{\frac{1}{\delta}} (i\lambda_n I - \mathcal{A}) \Phi_n, \Phi_n \right\rangle_{\mathcal{H}} \\ = & -|\lambda_n|^{\frac{1}{\delta}} Re \left\langle \mathcal{A}\Phi_n, \Phi_n \right\rangle_{\mathcal{H}} \\ = & \; |\lambda_n|^{\frac{1}{\delta}} \left(\tau_1 a_1 \|\tilde{\varphi}\|^2 + \tau_2 a_2 \|\tilde{\psi}\|^2 + \tau_3 a_3 \|\tilde{w}\|^2 + \tau_4 a_4 \|\tilde{z}\|^2 \right), \end{aligned}$$

so, (5.5) and (6.9) imply that

$$|\lambda_n|^{\frac{1}{\delta}}\left(\tau_1 a_1\|\tilde{\varphi}_n\|^2+\tau_2 a_2\|\tilde{\psi}_n\|^2+\tau_3 a_3\|\tilde{w}_n\|^2+\tau_4 a_4\|\tilde{z}_n\|^2\right)\to 0. \tag{6.11}$$

Multiplying $(6.10)_1$, $(6.10)_3$, $(6.10)_5$ and $(6.10)_7$ by $|\lambda_n|^{-\frac{1}{\delta}-1}$ and using (5.5) and (5.6), we obtain

$$\varphi_n,\,\psi_n,\,w_n,\,z_n\to 0 \text{ in } L^2(0,1). \tag{6.12}$$

Multiplying $(6.10)_1$, $(6.10)_3$, $(6.10)_5$ and $(6.10)_7$ by $|\lambda_n|^{-\frac{1}{\delta}}$ and exploiting (5.5) and (5.6), we deduce that

$$(\lambda_n\varphi_n)_n\,,\ (\lambda_n\psi_n)_n\,,\ (\lambda_n w_n)_n\,,\ (\lambda_n z_n)_n \text{ are bounded in } L^2(0,1). \tag{6.13}$$

Multiplying $(6.10)_2$, $(6.10)_4$, $(6.10)_6$ and $(6.10)_8$ by $|\lambda_n|^{-\frac{1}{\delta}-1}$ and using (5.5) and (5.6), it appears that

$$\left(\lambda_n^{-1}\varphi_{n,xx}\right)_n,\ \left(\lambda_n^{-1}\psi_{n,xx}\right)_n,\ \left(\lambda_n^{-1}w_{n,xx}\right)_n,\ \left(\lambda_n^{-1}z_{n,xx}\right)_n \text{ are bounded in } L^2(0,1). \tag{6.14}$$

Taking the inner product of $(6.10)_2$ with $|\lambda_n|^{-\frac{1}{\delta}}\varphi_n$ in $L^2(0,1)$, using (5.5) and (5.6), integrating by parts and using the boundary conditions, we find

$$\begin{aligned}&-\langle\tilde{\varphi}_n, i\lambda_n\varphi_n-\tilde{\varphi}_n\rangle-\|\tilde{\varphi}_n\|^2+k_1\|\varphi_{n,x}\|^2\\&-\langle k_1\psi_{n,x}+k_0 w_n-k_0\varphi_n-\tau_1 a_1\tilde{\varphi}_n,\varphi_n\rangle\to 0,\end{aligned}$$

then, using (5.5), $(6.10)_1$ and (6.12), we observe that the first and last terms of this limit converge to zero, and so

$$k_1\|\varphi_{n,x}\|^2-\|\tilde{\varphi}_n\|^2\to 0. \tag{6.15}$$

Similarly to the proof of (6.15), taking the inner product of $(6.10)_4$, $(6.10)_6$ and $(6.10)_8$ with, respectively, $|\lambda_n|^{-\frac{1}{\delta}}\psi_n$, $|\lambda_n|^{-\frac{1}{\delta}}w_n$ and $|\lambda_n|^{-\frac{1}{\delta}}z_n$ in $L^2(0,1)$, using (5.5) and (5.6), integrating by parts and using the boundary conditions, it follows that

$$k_2\|\psi_{n,x}\|^2-\left\|\tilde{\psi}_n\right\|^2\to 0, \tag{6.16}$$

$$k_3\|w_{n,x}\|^2-\|\tilde{w}_n\|^2\to 0 \tag{6.17}$$

and

$$k_4\|z_{n,x}\|^2-\|\tilde{z}_n\|^2\to 0. \tag{6.18}$$

Taking the inner product of $(6.10)_1$ with $i\lambda_n\varphi_n$ in $L^2(0,1)$ and using (6.13), we find

$$|\lambda_n|^{\frac{1}{\delta}}\left[\lambda_n^2\|\varphi_n\|^2-\|\tilde{\varphi}_n\|^2\right]-\left\langle\tilde{\varphi}_n,|\lambda_n|^{\frac{1}{\delta}}(i\lambda_n\varphi_n-\tilde{\varphi}_n)\right\rangle\to 0,$$

so, according to (5.5) and $(6.10)_1$, it is clear that the last term of this limit converges to zero, hence

$$|\lambda_n|^{\frac{1}{\delta}}\left[\lambda_n^2\|\varphi_n\|^2-\|\tilde{\varphi}_n\|^2\right]\to 0. \tag{6.19}$$

Similarly to the proof of (6.19), taking the inner product of $(6.10)_3$, $(6.10)_5$ and $(6.10)_7$ with, respectively, $i\lambda_n\psi_n$, $i\lambda_n w_n$ and $i\lambda_n z_n$ in $L^2(0,1)$, we arrive at

$$|\lambda_n|^{\frac{1}{\delta}}\left[\lambda_n^2\|\psi_n\|^2-\left\|\tilde{\psi}_n\right\|^2\right]\to 0, \tag{6.20}$$

$$|\lambda_n|^{\frac{1}{\delta}}\left[\lambda_n^2\|w_n\|^2-\|\tilde{w}_n\|^2\right]\to 0 \tag{6.21}$$

and

$$|\lambda_n|^{\frac{1}{\delta}}\left[\lambda_n^2\|z_n\|^2-\|\tilde{z}_n\|^2\right]\to 0. \tag{6.22}$$

Now, we notice that we need to treat only the cases

$$\begin{cases}(\tau_1,\tau_2,\tau_3,\tau_4)=(0,1,0,0)\text{ and }\delta=\dfrac{1}{18},\\ (\tau_1,\tau_2,\tau_3,\tau_4)=(1,1,0,0)\text{ and }\delta=\dfrac{1}{14},\\ (\tau_1,\tau_2,\tau_3,\tau_4)\in\{(1,0,0,1),(1,0,1,0),(1,1,1,0),(1,1,0,1),(0,1,0,1)\}\\ \text{and }\delta=\dfrac{1}{2},\end{cases} \tag{6.23}$$

since (as in section 3), the proof in cases

$$\begin{cases}(\tau_1,\tau_2,\tau_3,\tau_4)=(0,0,0,1)\text{ and }\delta=\dfrac{1}{18},\\ (\tau_1,\tau_2,\tau_3,\tau_4)=(0,0,1,1)\text{ and }\delta=\dfrac{1}{14},\\ (\tau_1,\tau_2,\tau_3,\tau_4)\in\{(0,1,1,0),(1,0,1,1),(0,1,1,1)\}\text{ and }\delta=\dfrac{1}{2},\end{cases}$$

is, by symmetry, identical to the one that will be given in cases (6.23).

6.1. Case $(\tau_1,\tau_2,\tau_3,\tau_4)=(0,1,0,0)$ and $\delta=\frac{1}{18}$

In vertue of (6.11), it is clear that

$$\lambda_n^9\tilde{\psi}_n\to 0\text{ in }L^2(0,1), \tag{6.24}$$

and then, according to (6.20), we get

$$\lambda_n^{10}\psi_n\to 0\text{ in }L^2(0,1). \tag{6.25}$$

Taking the inner product of $(6.10)_4$ with $\lambda_n^{-8}\psi_n$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, (5.5) and (5.6), we find

$$k_2\lambda_n^{10}\|\psi_{n,x}\|^2+\left\langle i\lambda_n\tilde{\psi}_n+k_1\left(\varphi_{n,x}+\psi_n\right)+a_2\tilde{\psi}_n,\lambda_n^{10}\psi_n\right\rangle\to 0,$$

therefore, using (5.5), (6.24) and (6.25), we observe that

$$\left\langle i\lambda_n\tilde{\psi}_n+k_1\left(\varphi_{n,x}+\psi_n\right)+a_2\tilde{\psi}_n,\lambda_n^{10}\psi_n\right\rangle\to 0,$$

hence, by combining the above two limits, we arrive at

$$\lambda_n^5\psi_{n,x}\to 0\text{ in }L^2\left(0,1\right). \tag{6.26}$$

Taking the inner product of $(6.10)_4$ with $\lambda_n^{-10}\varphi_{n,x}$ in $L^2\left(0,1\right)$, integrating by parts and using (5.5), (5.6) and the boundary conditions, we arrive at

$$k_1\lambda_n^8\|\varphi_{n,x}\|^2+k_2\lambda_n^8\left\langle\psi_{n,x},\varphi_{n,xx}\right\rangle+\left\langle i\lambda_n^9\tilde{\psi}_n+k_1\lambda_n^8\psi_n+a_2\lambda_n^8\tilde{\psi}_n,\varphi_{n,x}\right\rangle\to 0,$$

therefore, exploiting (5.6), (6.24) and (6.25), we entail

$$\left\langle i\lambda_n^9\tilde{\psi}_n + k_1\lambda_n^8\psi_n + a_2\lambda_n^8\tilde{\psi}_n, \varphi_{n,x}\right\rangle \to 0,$$

so, by combining the above two limits, we get

$$k_1\lambda_n^8\|\varphi_{n,x}\|^2 + k_2\lambda_n^8\left\langle \psi_{n,x}, \varphi_{n,xx}\right\rangle \to 0. \tag{6.27}$$

Taking the inner product of $(6.10)_2$ with $\lambda_n^{-10}\psi_{n,x}$ in $L^2(0,1)$, integrating by parts and using (5.5), (5.6) and the boundary conditions, it follows that

$$-k_1\lambda_n^8\|\psi_{n,x}\|^2 + k_0\left\langle w_{n,x} - \varphi_{n,x}, \lambda_n^8\psi_n\right\rangle - \left\langle i\lambda_n\tilde{\varphi}_{n,x}, \lambda_n^8\psi_n\right\rangle - k_1\lambda_n^8\left\langle \varphi_{n,xx}, \psi_{n,x}\right\rangle \to 0. \tag{6.28}$$

On the other hand, exploiting (5.6), (6.25) and (6.26), it appears that

$$-k_1\lambda_n^8\|\psi_{n,x}\|^2 + k_0\left\langle w_{n,x} - \varphi_{n,x}, \lambda_n^8\psi_n\right\rangle \to 0. \tag{6.29}$$

Moreover, we have

$$-\left\langle i\lambda_n\tilde{\varphi}_{n,x}, \lambda_n^8\psi_n\right\rangle = \left\langle \varphi_{n,x}, \lambda_n^{10}\psi_n\right\rangle - i\left\langle \lambda_n^9\left(\tilde{\varphi}_{n,x} - i\lambda_n\varphi_{n,x}\right), \psi_n\right\rangle,$$

therefore, using $(6.10)_1$ and (6.25), we find

$$-\left\langle i\lambda_n\tilde{\varphi}_{n,x}, \lambda_n^8\psi_n\right\rangle \to 0, \tag{6.30}$$

then, from (6.28), (6.29) and (6.30), we deduce that

$$\lambda_n^8\left\langle \varphi_{n,xx}, \psi_{n,x}\right\rangle \to 0, \tag{6.31}$$

therefore, by combining (6.27) and (6.31), we obtain

$$\lambda_n^4\varphi_{n,x} \to 0 \text{ in } L^2(0,1), \tag{6.32}$$

hence, by combining (6.15) and (6.32), we see that

$$\tilde{\varphi}_n \to 0 \text{ in } L^2(0,1). \tag{6.33}$$

Taking the inner product of $(6.10)_2$ with $\lambda_n^{-16}w_{n,xx}$ in $L^2(0,1)$, integrating by parts and using (5.5), (5.6), (6.14) and the boundary conditions, it follows that

$$\begin{aligned} &k_0\lambda_n^2\|w_{n,x}\|^2 - k_1\lambda_n^2\left\langle \varphi_{n,xx}, w_{n,xx}\right\rangle - \lambda_n^2\left\langle i\lambda_n\tilde{\varphi}_{n,x}, w_{n,x}\right\rangle \\ &- k_1\left\langle \lambda_n^3\psi_{n,x}, \lambda_n^{-1}w_{n,xx}\right\rangle - k_0\left\langle \lambda_n^2\varphi_{n,x}, w_{n,x}\right\rangle \to 0. \end{aligned} \tag{6.34}$$

By exploiting (6.14), (6.26) and (6.32), we get

$$-k_1\left\langle \lambda_n^3\psi_{n,x}, \lambda_n^{-1}w_{n,xx}\right\rangle - k_0\left\langle \lambda_n^2\varphi_{n,x}, w_{n,x}\right\rangle \to 0. \tag{6.35}$$

Moreover, we see that

$$-\lambda_n^2\left\langle i\lambda_n\tilde{\varphi}_{n,x}, w_{n,x}\right\rangle = \left\langle i\lambda_n^3\left(i\lambda_n\varphi_{n,x} - \tilde{\varphi}_{n,x}\right), w_{n,x}\right\rangle + \left\langle \lambda_n^4\varphi_{n,x}, w_{n,x}\right\rangle,$$

then, according to $(6.10)_1$ and (6.32), we conclude that

$$-\lambda_n^2\left\langle i\lambda_n\tilde{\varphi}_{n,x}, w_{n,x}\right\rangle \to 0, \tag{6.36}$$

and so, by combining (6.34), (6.35) and (6.36), we obtain

$$k_0\lambda_n^2\|w_{n,x}\|^2 - k_1\lambda_n^2\left\langle \varphi_{n,xx}, w_{n,xx}\right\rangle \to 0. \tag{6.37}$$

On the other hand, taking the inner product of $(6.10)_6$ with $\lambda_n^{-16}\varphi_{n,xx}$ in $L^2(0,1)$, integrating by parts and using (5.5), (5.6), (6.14) and the boundary conditions, we entail

$$\begin{aligned} &- k_3\lambda_n^2\left\langle w_{n,xx}, \varphi_{n,xx}\right\rangle - \left\langle i\tilde{w}_{n,x}, \lambda_n^3\varphi_{n,x}\right\rangle + k_3\left\langle \lambda_n^{-1}z_{n,xx}, \lambda_n^3\varphi_{n,x}\right\rangle \\ &- k_0\left\langle w_{n,x} - \varphi_{n,x}, \lambda_n^2\varphi_{n,x}\right\rangle \to 0. \end{aligned} \tag{6.38}$$

Thanks to (6.14) and (6.32), it appears that

$$k_3\left\langle \lambda_n^{-1}z_{n,xx}, \lambda_n^3\varphi_{n,x}\right\rangle - k_0\left\langle w_{n,x} - \varphi_{n,x}, \lambda_n^2\varphi_{n,x}\right\rangle \to 0. \tag{6.39}$$

On the other hand, we have

$$-\left\langle i\tilde{w}_{n,x}, \lambda_n^3\varphi_{n,x}\right\rangle = \left\langle i\left(i\lambda_n w_{n,x} - \tilde{w}_{n,x}\right), \lambda_n^3\varphi_{n,x}\right\rangle + \left\langle w_{n,x}, \lambda_n^4\varphi_{n,x}\right\rangle,$$

so, using $(6.10)_5$ and (6.32), we find

$$-\left\langle i\tilde{w}_{n,x}, \lambda_n^3\varphi_{n,x}\right\rangle \to 0. \tag{6.40}$$

By combining (6.38), (6.39) and (6.40), we get

$$\lambda_n^2\left\langle w_{n,xx}, \varphi_{n,xx}\right\rangle \to 0, \tag{6.41}$$

hence, (6.37) and (6.41) imply that

$$\lambda_n w_{n,x} \to 0 \text{ in } L^2(0,1), \tag{6.42}$$

and then, using (6.17),

$$\tilde{w}_n \to 0 \text{ in } L^2(0,1). \tag{6.43}$$

Taking the inner product of $(6.10)_6$ with $\lambda_n^{-18}z_{n,x}$ in $L^2(0,1)$, integrating by parts and using (5.5), (5.6), and the boundary conditions, it follows that

$$\begin{aligned} &- k_3\|z_{n,x}\|^2 + k_3\left\langle \lambda_n w_{n,x}, \lambda_n^{-1}z_{n,xx}\right\rangle - k_0\left\langle w_{n,x} - \varphi_{n,x}, z_n\right\rangle \\ &- i\left\langle \tilde{w}_{n,x} - i\lambda_n w_{n,x}, \lambda_n z_n\right\rangle + \left\langle \lambda_n w_{n,x}, \lambda_n z_n\right\rangle \to 0, \end{aligned}$$

because, according to $(6.10)_5$, (6.13), (6.14), (6.32) and (6.42),

$$\begin{aligned} &k_3\left\langle \lambda_n w_{n,x}, \lambda_n^{-1}z_{n,xx}\right\rangle - k_0\left\langle w_{n,x} - \varphi_{n,x}, z_n\right\rangle - i\left\langle \tilde{w}_{n,x} - i\lambda_n w_{n,x}, \lambda_n z_n\right\rangle \\ &+ \left\langle \lambda_n w_{n,x}, \lambda_n z_n\right\rangle \to 0, \end{aligned}$$

we see that the above two limits lead to

$$z_{n,x} \to 0 \text{ in } L^2(0,1), \tag{6.44}$$

and by combining (6.18) and (6.44), we get

$$\tilde{z}_n \to 0 \text{ in } L^2(0,1). \tag{6.45}$$

Finally, the obtained limits (6.12), (6.24), (6.26), (6.32), (6.33), (6.42), (6.43), (6.44) and (6.45) imply (5.9), which is a contradiction with (5.5).

6.2. Case $(\tau_1, \tau_2, \tau_3, \tau_4) = (1, 1, 0, 0)$ and $\delta = \frac{1}{14}$

In virtue of (6.11), it is clear that

$$\lambda_n^7 \tilde{\varphi}_n,\ \lambda_n^7 \tilde{\psi}_n \to 0 \text{ in } L^2(0,1), \tag{6.46}$$

and then, according to (6.19) and (6.20), we get

$$\lambda_n^8 \varphi_n,\ \lambda_n^8 \psi_n \to 0 \text{ in } L^2(0,1). \tag{6.47}$$

Taking the inner product of $(6.10)_4$ with $\lambda_n^{-6}\psi_n$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, (5.5) and (5.6), we find

$$k_2\lambda_n^8 \|\psi_{n,x}\|^2 + \left\langle i\lambda_n \tilde{\psi}_n + k_1 (\varphi_{n,x} + \psi_n) + a_2\tilde{\psi}_n, \lambda_n^8 \psi_n \right\rangle \to 0,$$

therefore, using (6.46) and (6.47), we observe that

$$\left\langle i\lambda_n \tilde{\psi}_n + k_1 (\varphi_{n,x} + \psi_n) + a_2\tilde{\psi}_n, \lambda_n^8 \psi_n \right\rangle \to 0,$$

hence, by combining the above two limits, we arrive at

$$\lambda_n^4 \psi_{n,x} \to 0 \text{ in } L^2(0,1). \tag{6.48}$$

Similarly, taking the inner product of $(6.10)_2$ with $\lambda_n^{-6}\varphi_n$ in $L^2(0,1)$ and using the same arguments as for (6.48), we find

$$\lambda_n^4 \varphi_{n,x} \to 0 \text{ in } L^2(0,1), \tag{6.49}$$

which coincides with (6.32). Taking the inner product of $(6.10)_2$ with $\lambda_n^{-12} w_{n,xx}$ in $L^2(0,1)$ and proceeding as is subsection 6.1, we get (6.37) (using (6.48) instead of (6.26) to find (6.35)). On the other hand, taking the inner product of $(6.10)_6$ with $\lambda_n^{-12}\varphi_{n,xx}$ in $L^2(0,1)$ and following the same arguments as in subsection 6.1, we find (6.42) and (6.43). Therefore, the prrof can be completed as in subsection 6.1 by taking the inner product of $(6.10)_6$ with $\lambda_n^{-14} z_{n,x}$ in $L^2(0,1)$ to get (6.44) and (6.45). Consequently, (5.9) holds.

6.3. Case $(\tau_1, \tau_2, \tau_3, \tau_4) = (1, 0, 0, 1)$ and $\delta = \frac{1}{2}$

According to (6.11), we have

$$\lambda_n \tilde{\varphi}_n,\ \lambda_n \tilde{z}_n \to 0 \text{ in } L^2(0,1), \tag{6.50}$$

and then, thanks to (6.19) and (6.22), we find

$$\lambda_n^2 \varphi_n,\ \lambda_n^2 z_n \to 0 \text{ in } L^2(0,1). \tag{6.51}$$

Taking the inner product of $(6.10)_2$ and $(6.10)_8$, respectively, with φ_n and z_n in $L^2(0,1)$, integrating by parts and using the boundary conditions and (5.5), we obtain

$$k_1\lambda_n^2 \|\varphi_{n,x}\|^2 + \left\langle i\lambda_n \tilde{\varphi}_n - k_1\psi_{n,x} - k_0 (w_n - \varphi_n) + a_1\tilde{\varphi}_n, \lambda_n^2 \varphi_n \right\rangle \to 0$$

and

$$k_4\lambda_n^2 \|z_{n,x}\|^2 + \left\langle i\lambda_n \tilde{z}_n + k_3 (w_{n,x} + z_n) + a_4\tilde{z}_n, \lambda_n^2 z_n \right\rangle \to 0,$$

therefore, according to (5.5), (6.50) and (6.51), it is clear that

$$\left\langle i\lambda_n\tilde{\varphi}_n - k_1\psi_{n,x} - k_0\left(w_n - \varphi_n\right) + a_1\tilde{\varphi}_n, \lambda_n^2\varphi_n\right\rangle \to 0$$

and

$$\left\langle i\lambda_n\tilde{z}_n + k_3\left(w_{n,x} + z_n\right) + a_4\tilde{z}_n, \lambda_n^2 z_n\right\rangle \to 0,$$

then, from the above four limits, we deduce that

$$\lambda_n\varphi_{n,x},\ \lambda_n z_{n,x} \to 0 \text{ in } L^2\left(0,1\right). \tag{6.52}$$

Similarly, taking the inner product of $(6.10)_2$ and $(6.10)_8$, respectively, with $\lambda_n^{-2}\psi_{n,x}$ and $\lambda_n^{-2}w_{n,x}$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, (5.5) and (5.6), we arrive at

$$-k_1\|\psi_{n,x}\|^2 + \langle i\lambda_n\tilde{\varphi}_n - k_0\left(w_n - \varphi_n\right) + a_1\tilde{\varphi}_n, \psi_{n,x}\rangle + k_1\left\langle \lambda_n\varphi_{n,x}, \lambda_n^{-1}\psi_{n,xx}\right\rangle \to 0$$

and

$$k_3\|w_{n,x}\|^2 + \langle i\lambda_n\tilde{z}_n + k_3 z_n + a_4\tilde{z}_n, w_{n,x}\rangle + k_4\left\langle \lambda_n z_{n,x}, \lambda_n^{-1}w_{n,xx}\right\rangle \to 0,$$

so, according to (6.12), (6.14), (6.50) and (6.52), it is clear that

$$\langle i\lambda_n\tilde{\varphi}_n - k_0\left(w_n - \varphi_n\right) + a_1\tilde{\varphi}_n, \psi_{n,x}\rangle + k_1\left\langle \lambda_n\varphi_{n,x}, \lambda_n^{-1}\psi_{n,xx}\right\rangle \to 0$$

and

$$\langle i\lambda_n\tilde{z}_n + k_3 z_n + a_4\tilde{z}_n, w_{n,x}\rangle + k_4\left\langle \lambda_n z_{n,x}, \lambda_n^{-1}w_{n,xx}\right\rangle \to 0,$$

hence these four limits imply that

$$\psi_{n,x},\ w_{n,x} \to 0 \text{ in } L^2\left(0,1\right), \tag{6.53}$$

and by combining (6.16), (6.17) and (6.53), it follows that

$$\tilde{\psi}_n,\ \tilde{w}_n \to 0 \text{ in } L^2\left(0,1\right), \tag{6.54}$$

Finally, the obtained limits (6.12), (6.50) and (6.52)-(6.54) lead to (5.9).

6.4. Case $(\tau_1, \tau_2, \tau_3, \tau_4) = (1, 0, 1, 0)$ and $\delta = \frac{1}{2}$

From (6.11), it appears that

$$\lambda_n\tilde{\varphi}_n,\ \lambda_n\tilde{w}_n \to 0 \text{ in } L^2(0,1), \tag{6.55}$$

therefore, according to (6.19) and (6.21), we have

$$\lambda_n^2\varphi_n,\ \lambda_n^2 w_n \to 0 \text{ in } L^2(0,1). \tag{6.56}$$

The limits

$$\lambda_n\varphi_{n,x},\ \psi_{n,x} \to 0 \text{ in } L^2(0,1) \tag{6.57}$$

can be proved exactly as in subsection 6.3, and therefore, by exploiting (6.16), we find

$$\tilde{\psi}_n \to 0 \text{ in } L^2(0,1). \tag{6.58}$$

On the other hand, taking the inner product of $(6.10)_6$ with w_n in $L^2(0,1)$, integrating by parts and using the boundary conditions and (5.5), we obtain

$$k_3\lambda_n^2\|w_{n,x}\|^2 + \left\langle i\lambda_n\tilde{w}_n - k_3 z_{n,x} + k_0\left(w_n - \varphi_n\right) + a_3\tilde{w}_n, \lambda_n^2 w_n\right\rangle \to 0,$$

therefore, according to (5.5), (6.55) and (6.56), it appears that

$$\left\langle i\lambda_n\tilde{w}_n - k_3 z_{n,x} + k_0\left(w_n - \varphi_n\right) + a_3\tilde{w}_n, \lambda_n^2 w_n\right\rangle \to 0,$$

then these two limits imply that

$$\lambda_n w_{n,x} \to 0 \text{ in } L^2\left(0,1\right). \tag{6.59}$$

Similarly, taking the inner product of $(6.10)_6$ with $\lambda_n^{-2}z_{n,x}$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, (5.5) and (5.6), we get

$$-k_3\|z_{n,x}\|^2 + \langle i\lambda_n\tilde{w}_n + k_0\left(w_n - \varphi_n\right) + a_3\tilde{w}_n, z_{n,x}\rangle + k_3\left\langle \lambda_n w_{n,x}, \lambda_n^{-1}z_{n,xx}\right\rangle \to 0,$$

then, using (6.14), (6.55), (6.56) and (6.59), we obtain

$$\langle i\lambda_n\tilde{w}_n + k_0\left(w_n - \varphi_n\right) + a_3\tilde{w}_n, z_{n,x}\rangle + k_3\left\langle \lambda_n w_{n,x}, \lambda_n^{-1}z_{n,xx}\right\rangle \to 0,$$

hence

$$z_{n,x} \to 0 \text{ in } L^2\left(0,1\right), \tag{6.60}$$

and by combining (6.18) and (6.60), we find

$$\tilde{z}_n \to 0 \text{ in } L^2\left(0,1\right). \tag{6.61}$$

Consequetly, the limit (5.9) can be directly deduced from the ones (6.12), (6.55) and (6.57)-(6.61).

6.5. Case $(\tau_1, \tau_2, \tau_3, \tau_4) = (1,1,1,0)$ and $\delta = \frac{1}{2}$

The limit (6.11) implies that

$$\lambda_n\tilde{\varphi}_n,\ \lambda_n\tilde{\psi}_n,\ \lambda_n\tilde{w}_n \to 0 \text{ in } L^2(0,1), \tag{6.62}$$

which implies (6.55), so the proof can be finished as in subsection 6.4.

6.6. Case $(\tau_1, \tau_2, \tau_3, \tau_4) = (1,1,0,1)$ and $\delta = \frac{1}{2}$

We deduce from (6.11) that

$$\lambda_n\tilde{\varphi}_n,\ \lambda_n\tilde{\psi}_n,\ \lambda_n\tilde{z}_n \to 0 \text{ in } L^2(0,1), \tag{6.63}$$

which implies (6.50), then the proof can be ended as in subsection 6.3.

6.7. Case $(\tau_1, \tau_2, \tau_3, \tau_4) = (0, 1, 0, 1)$ and $\delta = \frac{1}{2}$

The limit (6.11) leads to

$$\lambda_n \tilde{\psi}_n, \ \lambda_n \tilde{z}_n \to 0 \text{ in } L^2(0,1). \tag{6.64}$$

The limits

$$\lambda_n^2 z_n, \ \lambda_n z_{n,x} \to 0 \text{ in } L^2(0,1) \tag{6.65}$$

can be proved as in subsection 6.3. Similarly, we can prove the limits

$$\lambda_n^2 \psi_n, \ \lambda_n \psi_{n,x} \to 0 \text{ in } L^2(0,1) \tag{6.66}$$

(by exploiting (6.20) and multiplying $(6.10)_4$ with ψ_n; we omit the details here). On the other hand, taking the inner product of $(6.10)_8$ with $\lambda_n^{-2} w_{n,x}$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, (5.5) and (5.6), we find

$$k_3 \|w_{n,x}\|^2 + \langle i\lambda_n \tilde{z}_n + k_3 z_n + a_4 \tilde{z}_n, w_{n,x}\rangle + k_4 \left\langle \lambda_n z_{n,x}, \lambda_n^{-1} w_{n,xx}\right\rangle \to 0,$$

then, using (6.14), (6.64) and (6.65), we find

$$\langle i\lambda_n \tilde{z}_n + k_3 z_n + a_4 \tilde{z}_n, w_{n,x}\rangle + k_4 \left\langle \lambda_n z_{n,x}, \lambda_n^{-1} w_{n,xx}\right\rangle \to 0,$$

hence

$$w_{n,x} \to 0 \text{ in } L^2\left(0,1\right), \tag{6.67}$$

and by combining (6.17) and (6.67), we deduce that

$$\tilde{w}_n \to 0 \text{ in } L^2\left(0,1\right). \tag{6.68}$$

Similarly (using $(6.10)_4$ and $\lambda_n^{-2}\varphi_{n,x}$ instead of $(6.10)_8$ and $\lambda_n^{-2} w_{n,x}$, respectively, and exploiting (6.15)), we have

$$\varphi_{n,x}, \ \tilde{\varphi}_n \to 0 \text{ in } L^2\left(0,1\right). \tag{6.69}$$

Consequently, the limit (5.9) holds. The proof of Theorem 6.1 is then completed. □

7. Optimality of the polynomial decay rate: Cases (6.3)-(6.5)

In this section, we prove that the polynomial decay rate given in Theorem 6.1 in cases (6.3)-(6.5) is optimal in the sense that there is no $\epsilon > 0$ such that

$$\forall \Phi_0 \in D(\mathcal{A}), \exists\, c > 0 : \ \|\Phi(t)\|_{\mathcal{H}} \le c t^{-\frac{1}{2}-\epsilon}, \quad \forall t > 0. \tag{7.1}$$

Theorem 7.1. *For any $\epsilon > 0$, the polynomial decay* (7.1) *does not hold in cases* (6.3)-(6.5).

Proof. To prove Theorem 7.1, it is sufficient to show that (see [9, 10])

$$\limsup_{\lambda\to\infty} \lambda^{-2} \left\|(i\lambda I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} > 0. \tag{7.2}$$

To get (7.2), it will be enough to find sequences $(\lambda_n)_n \subset \mathbb{R}$, $(F_n)_n \subset \mathcal{H}$ and $(\Phi_n)_n \subset D(\mathcal{A})$, $n \in \mathbb{N}$, satisfying

$$\begin{cases} i\lambda_n \Phi_n - \mathcal{A}\Phi_n = F_n, \\ \|F_n\|_{\mathcal{H}} \leq 1, \\ \lim\limits_{n\to\infty} \lambda_n = \infty, \\ \lim\limits_{n\to\infty} \lambda_n^{-2} \|\Phi_n\|_{\mathcal{H}} > 0. \end{cases} \tag{7.3}$$

As in section 4, let $\Phi_n := \left(\varphi_n, \tilde{\varphi}_n, \psi_n, \tilde{\psi}_n, w_n, \tilde{w}_n, z_n, \tilde{z}_n\right)^T$, $F_n := (f_{1,n}, \cdots, f_{n,8})^T$ and $N := \frac{\pi}{2} + n\pi$. Then $(7.3)_1$ is equivalent to (4.10). By considering the choices (4.11) and (4.12), we see that $(F_n)_n \subset \mathcal{H}$, $(\Phi_n)_n \subset D(\mathcal{A})$ and $(7.3)_1$ is reduced to the algebraic system (4.13). In order to simplify the computations, we put

$$\begin{cases} J_1 = \lambda_n^2 - k_1 N^2 - k_0 - i\tau_1 a_1 \lambda_n, \\ J_2 = \lambda_n^2 - k_2 N^2 - k_1 - i\tau_2 a_2 \lambda_n, \\ J_3 = \lambda_n^2 - k_3 N^2 - k_0 - i\tau_3 a_3 \lambda_n, \\ J_4 = \lambda_n^2 - k_4 N^2 - k_3 - i\tau_4 a_4 \lambda_n, \end{cases} \tag{7.4}$$

so (4.13) can be presented in the form

$$\begin{cases} J_1 \alpha_{1,n} + k_1 N \alpha_{2,n} + k_0 \alpha_{3,n} = \beta_{2,n}, \\ k_1 N \alpha_{1,n} + J_2 \alpha_{2,n} = \beta_{4,n}, \\ k_0 \alpha_{1,n} + J_3 \alpha_{3,n} + k_3 N \alpha_{4,n} = \beta_{6,n}, \\ k_3 N \alpha_{3,n} + J_4 \alpha_{4,n} = \beta_{8,n}. \end{cases} \tag{7.5}$$

Now, because we need here to prove the stronger limit $(7.3)_4$ than the one (4.16) needed in section 4, we have to consider other choices of λ_n, $\alpha_{j,n}$ and $\beta_{j,n}$. On the other hand, to cover the cases (6.3)-(6.5), we need to treat only the cases (6.3) and

$$(\tau_1, \tau_2, \tau_3, \tau_4) \in \{(1,0,0,1), (1,0,1,0), (0,1,0,1))\} \quad \text{and} \quad (k_1, k_3) \neq (k_2, k_4), \tag{7.6}$$

since, by symmetry, the proofs in cases (6.4) and $(\tau_1, \tau_2, \tau_3, \tau_4) = (0,1,1,0)$ are similar to the ones of, respectively, (6.3) and $(\tau_1, \tau_2, \tau_3, \tau_4) = (1,0,0,1)$.

7.1. Case $(\tau_1, \tau_2, \tau_3, \tau_4) = (1,1,0,1)$ and $k_3 \neq k_4$

We choose

$$\beta_{2,n} = \beta_{4,n} = \beta_{8,n} = 0, \quad \beta_{6,n} = 1 \quad \text{and} \quad \lambda_n = \sqrt{k_3 N^2 + k_0 + \frac{k_3^2}{k_3 - k_4}}, \tag{7.7}$$

for $n \in \mathbb{N}$ such that $k_3 N^2 + k_0 + \frac{k_3^2}{k_3 - k_4} > 0$. We see that $(7.3)_2$ and $(7.3)_3$ are satisfied, since, according to $(4.11)_2$, $(4.12)_3$, $(4.12)_4$ and (7.7), we have

$$\|F_n\|_{\mathcal{H}}^2 = \|f_{6,n}\|^2 = \int_0^1 \cos^2(Nx)dx \leq 1. \tag{7.8}$$

On the other hand, by a direct computations, it appears that (7.5) has the unique solution

$$\begin{cases} \alpha_{1,n} = \dfrac{-k_0 J_2 J_4}{(J_3 J_4 - k_3^2 N^2)(J_1 J_2 - k_1^2 N^2) - k_0^2 J_2 J_4}, \\ \alpha_{2,n} = \dfrac{k_0 k_1 J_4 N}{(J_3 J_4 - k_3^2 N^2)(J_1 J_2 - k_1^2 N^2) - k_0^2 J_2 J_4}, \\ \alpha_{3,n} = \dfrac{J_4 \left(J_1 J_2 - k_1^2 N^2\right)}{(J_3 J_4 - k_3^2 N^2)(J_1 J_2 - k_1^2 N^2) - k_0^2 J_2 J_4}, \\ \alpha_{4,n} = \dfrac{-k_3 N \left(J_1 J_2 - k_1^2 N^2\right)}{(J_3 J_4 - k_3^2 N^2)(J_1 J_2 - k_1^2 N^2) - k_0^2 J_2 J_4}. \end{cases} \tag{7.9}$$

We have

$$\begin{aligned} &\left(J_3 J_4 - k_3^2 N^2\right)\left(J_1 J_2 - k_1^2 N^2\right) - k_0^2 J_2 J_4 \\ =& \frac{k_3^2}{k_4 - k_3}\left(i a_4 \lambda_n - k_0 + \frac{k_3 k_4}{k_4 - k_3}\right)\left[\left((k_3 - k_1)N^2 - i a_1 \lambda_n + \frac{k_3^2}{k_3 - k_4}\right)\right. \\ &\times \left.\left((k_3 - k_2)N^2 - i a_2 \lambda_n + k_0 - k_1 + \frac{k_3^2}{k_3 - k_4}\right) - k_1^2 N^2\right] \\ &- k_0^2 \left[(k_3 - k_2)N^2 - i a_2 \lambda_n + k_0 - k_1 + \frac{k_3^2}{k_3 - k_4}\right] \\ &\times \left[(k_3 - k_4)N^2 - i a_4 \lambda_n + k_0 + \frac{k_3 k_4}{k_3 - k_4}\right], \end{aligned}$$

then, we denote by "$\sim$" the "asymptotic equivalence when n goes to infinity" and we find

$$J_1 J_2 - k_1^2 N^2 \sim \begin{cases} (k_3 - k_1)(k_3 - k_2)N^4 & \text{if } k_3 \notin \{k_1, k_2\}, \\ i a_1 \sqrt{k_3}(k_2 - k_3)N^3 & \text{if } k_3 = k_1 \text{ and } k_3 \neq k_2, \\ i a_2 \sqrt{k_3}(k_1 - k_3)N^3 & \text{if } k_3 \neq k_1 \text{ and } k_3 = k_2, \\ -\left(a_1 a_2 k_3 + k_1^2\right) N^2 & \text{if } k_3 = k_1 = k_2 \end{cases} \tag{7.10}$$

and

$$\left(J_3 J_4 - k_3^2 N^2\right)\left(J_1 J_2 - k_1^2 N^2\right) - k_0^2 J_2 J_4 \tag{7.11}$$

$$\sim \begin{cases} \dfrac{ia_4k_3^2\sqrt{k_3}(k_3-k_1)(k_3-k_2)}{k_4-k_3}N^5 & \text{if } k_3 \notin \{k_1,k_2\}, \\ \dfrac{\left[a_1a_4k_3^3+k_0^2(k_3-k_4)^2\right](k_3-k_2)}{k_4-k_3}N^4 & \text{if } k_3 = k_1 \text{ and } k_3 \neq k_2, \\ \dfrac{a_2a_4k_3^3(k_3-k_1)}{k_4-k_3}N^4 & \text{if } k_3 \neq k_1 \text{ and } k_3 = k_2, \\ \dfrac{i\sqrt{k_3}\left[k_3^2a_4\left(a_1a_2k_3+k_1^2\right)+k_0^2a_2(k_3-k_4)^2\right]}{k_3-k_4}N^3 & \text{if } k_3 = k_1 = k_2, \end{cases}$$

therefore, by combining (7.10) and (7.11), we deduce from $(7.9)_3$ and $(7.9)_4$ that

$$(\alpha_{3,n},\alpha_{4,n}) \sim \begin{cases} \dfrac{i(k_3-k_4)}{a_4k_3^2\sqrt{k_3}}\left((k_3-k_4)N,-k_3\right) \text{ if } k_3 \notin \{k_1,k_2\}, \\ \dfrac{ia_1\sqrt{k_3}(k_3-k_4)}{a_1a_4k_3^3+k_0^2(k_3-k_4)^2}\left((k_3-k_4)N,-k_3\right) \text{ if } k_3 = k_1 \text{ and } k_3 \neq k_2, \\ \dfrac{i(k_3-k_4)}{a_4k_3^2\sqrt{k_3}}\left((k_3-k_4)N,-k_3\right) \text{ if } k_3 \neq k_1 \text{ and } k_3 = k_2, \\ \dfrac{i(k_3-k_4)\left(a_1a_2k_3+k_1^2\right)}{\sqrt{k_3}\left[k_3^2a_4\left(a_1a_2k_3+k_1^2\right)+k_0^2a_2(k_3-k_4)^2\right]}\left((k_3-k_4)N,-k_3\right) \\ \text{if } k_3 = k_1 = k_2. \end{cases} \tag{7.12}$$

On the other hand, from $(4.12)_2$, we have

$$\begin{aligned} \|\Phi_n\|_{\mathcal{H}}^2 &\geq k_3\left\|w_{n,x}+z_n\right\|^2 \qquad (7.13) \\ &= k_3|\alpha_{3,n}N-\alpha_{4,n}|^2\int_0^1 \sin^2(Nx)dx \\ &\geq \frac{k_3}{2}|\alpha_{3,n}N-\alpha_{4,n}|^2\int_0^1 \left[1-\cos\,(2Nx)\right]\,dx \\ &= \frac{k_3}{2}|\alpha_{3,n}N-\alpha_{4,n}|^2, \end{aligned}$$

then

$$\lambda_n^{-2}\left\|\Phi_n\right\|_{\mathcal{H}} \geq \sqrt{\frac{k_3}{2}}\lambda_n^{-2}|\alpha_{3,n}N-\alpha_{4,n}| = \frac{\sqrt{\frac{k_3}{2}}|\alpha_{3,n}N-\alpha_{4,n}|}{k_3N^2+k_0+\frac{k_3^2}{k_3-k_4}}, \tag{7.14}$$

hence (7.12) and (7.14) lead to

$$\lim_{n\to\infty}\lambda_n^{-2}\left\|\Phi_n\right\|_{\mathcal{H}}$$

$$\geq \begin{cases} \dfrac{(k_3-k_4)^2}{\sqrt{2}a_4k_3^3} & \text{if } k_3 \notin \{k_1,k_2\}, \\ \dfrac{a_1(k_3-k_4)^2}{\sqrt{2}\left[a_1a_4k_3^3+k_0^2(k_3-k_4)^2\right]} & \text{if } k_3=k_1 \text{ and } k_3\neq k_2, \\ \dfrac{(k_3-k_4)^2}{\sqrt{2}a_4k_3^3} & \text{if } k_3\neq k_1 \text{ and } k_3=k_2, \\ \dfrac{(k_3-k_4)^2\left(a_1a_2k_3+k_1^2\right)}{\sqrt{2}k_3\left[k_3^2a_4\left(a_1a_2k_3+k_1^2\right)+k_0^2a_2(k_3-k_4)^2\right]} & \text{if } k_3=k_1=k_2, \end{cases} \tag{7.15}$$

which implies $(7.3)_4$.

7.2. Case $(\tau_1, \tau_2, \tau_3, \tau_4) = (1, 1, 1, 0)$ and $k_3 \neq k_4$

We take

$$\beta_{2,n} = \beta_{4,n} = \beta_{6,n} = 0, \quad \beta_{8,n} = 1 \quad \text{and} \quad \lambda_n = \sqrt{k_4N^2 + \frac{k_3k_4}{k_4-k_3}}, \tag{7.16}$$

for $n \in \mathbb{N}$ such that $k_4N^2 + k_3 + \frac{k_3^2}{k_4-k_3} > 0$. We remark that $(7.3)_2$ and $(7.3)_3$ hold because, thanks to $(4.11)_2$, $(4.12)_3$, $(4.12)_4$ and (7.16), we have

$$\|F_n\|_{\mathcal{H}}^2 = \|f_{8,n}\|^2 = \int_0^1 \sin^2(Nx)dx \leq 1. \tag{7.17}$$

On the other hand, (7.5) admits the unique solution

$$\begin{cases} \alpha_{1,n} = \dfrac{k_0k_3J_2N}{\left(J_3J_4-k_3^2N^2\right)\left(J_1J_2-k_1^2N^2\right)-k_0^2J_2J_4}, \\ \alpha_{2,n} = \dfrac{-k_0k_1k_3N^2}{\left(J_3J_4-k_3^2N^2\right)\left(J_1J_2-k_1^2N^2\right)-k_0^2J_2J_4}, \\ \alpha_{3,n} = \dfrac{-k_3N\left(J_1J_2-k_1^2N^2\right)}{\left(J_3J_4-k_3^2N^2\right)\left(J_1J_2-k_1^2N^2\right)-k_0^2J_2J_4}, \\ \alpha_{4,n} = \dfrac{J_3\left(J_1J_2-k_1^2N^2\right)-k_0^2J_2}{\left(J_3J_4-k_3^2N^2\right)\left(J_1J_2-k_1^2N^2\right)-k_0^2J_2J_4}. \end{cases} \tag{7.18}$$

Similar computations to the ones done in subsection 7.1 show that

$$\left(J_3J_4-k_3^2N^2\right)\left(J_1J_2-k_1^2N^2\right)-k_0^2J_2J_4 \tag{7.19}$$

$$\sim \begin{cases} \dfrac{-ia_3k_3^2\sqrt{k_4}(k_4-k_1)(k_4-k_2)}{k_4-k_3}N^5 & \text{if } k_4 \notin \{k_1,k_2\}, \\ \dfrac{-a_1a_3k_3^2k_4(k_4-k_2)}{k_4-k_3}N^4 & \text{if } k_4=k_1 \text{ and } k_4\neq k_2, \\ \dfrac{-a_2a_3k_3^2k_4(k_4-k_1)}{k_4-k_3}N^4 & \text{if } k_4\neq k_1 \text{ and } k_4=k_2, \\ \dfrac{ia_3\sqrt{k_4}k_3^2\left(a_1a_2k_4+k_1^2\right)}{k_4-k_3}N^3 & \text{if } k_4=k_1=k_2 \end{cases}$$

and

$$J_3\left(J_1J_2-k_1^2N^2\right)-k_0^2J_2 \tag{7.20}$$
$$\sim\begin{cases}(k_4-k_1)(k_4-k_2)(k_4-k_3)N^6 & \text{if } k_4\notin\{k_1,k_2\},\\ -ia_1\sqrt{k_4}(k_4-k_2)(k_4-k_3)N^5 & \text{if } k_4=k_1 \text{ and } k_4\neq k_2,\\ -ia_2\sqrt{k_4}(k_4-k_1)(k_4-k_3)N^5 & \text{if } k_4\neq k_1 \text{ and } k_4=k_2,\\ -\left(a_1a_2k_4+k_1^2\right)(k_4-k_3)N^4 & \text{if } k_4=k_1=k_2,\end{cases}$$

then we deduce from $(7.18)_4$, (7.19) and (7.20) that

$$\alpha_{4,n}\sim\frac{i(k_4-k_3)^2}{a_3k_3^2\sqrt{k_4}}N. \tag{7.21}$$

On the other hand, from $(4.12)_2$, we have

$$\begin{aligned}\|\Phi_n\|_{\mathcal{H}}^2 &\geq k_4\left\|z_{n,x}\right\|^2 \\ &= k_4|\alpha_{4,n}|^2N^2\int_0^1\cos^2(Nx)dx \\ &\geq \frac{k_4}{2}|\alpha_{4,n}|^2N^2\int_0^1\left[1+\cos\,(2Nx)\right]\,dx \\ &= \frac{k_4}{2}|\alpha_{4,n}|^2N^2,\end{aligned} \tag{7.22}$$

then, according to (7.21) and the above inequality (7.22), we find $(7.3)_4$.

7.3. Case $(\tau_1,\tau_2,\tau_3,\tau_4)=(1,0,0,1)$ and $(k_1,k_3)\neq(k_2,k_4)$

Because $(k_1,k_3)\neq(k_2,k_4)$, we distinguish the two subcases $[k_1\neq k_2]$ and $[k_1=k_2$ and $k_3\neq k_4]$.

Subcase 1. $(\tau_1,\tau_2,\tau_3,\tau_4)=(1,0,0,1)$ **and** $k_1\neq k_2$. We choose

$$\beta_{2,n}=\beta_{6,n}=\beta_{8,n}=0,\quad \beta_{4,n}=1\quad\text{and}\quad \lambda_n=\sqrt{k_2N^2+\frac{k_1k_2}{k_2-k_1}}, \tag{7.23}$$

for $n\in\mathbb{N}$ such that $k_2N^2+k_1+\frac{k_1^2}{k_2-k_1}>0$. We observe that $(7.3)_3$ holds, and moreover, in virtue of $(4.11)_2$, $(4.12)_3$, $(4.12)_4$ and (7.23), we have

$$\|F_n\|_{\mathcal{H}}^2=\|f_{4,n}\|^2=\int_0^1\sin^2(Nx)dx\leq 1, \tag{7.24}$$

hence $(7.3)_2$ is satisfied. On the other hand, (7.5) has the unique solution

$$\begin{cases} \alpha_{1,n} = \dfrac{-k_1 N \left(J_3 J_4 - k_3^2 N^2\right)}{\left(J_3 J_4 - k_3^2 N^2\right)\left(J_1 J_2 - k_1^2 N^2\right) - k_0^2 J_2 J_4}, \\ \alpha_{2,n} = \dfrac{J_1 \left(J_3 J_4 - k_3^2 N^2\right) - k_0^2 J_4}{\left(J_3 J_4 - k_3^2 N^2\right)\left(J_1 J_2 - k_1^2 N^2\right) - k_0^2 J_2 J_4}, \\ \alpha_{3,n} = \dfrac{k_0 k_1 N J_4}{\left(J_3 J_4 - k_3^2 N^2\right)\left(J_1 J_2 - k_1^2 N^2\right) - k_0^2 J_2 J_4}, \\ \alpha_{4,n} = \dfrac{-k_0 k_1 k_3 N^2}{\left(J_3 J_4 - k_3^2 N^2\right)\left(J_1 J_2 - k_1^2 N^2\right) - k_0^2 J_2 J_4}. \end{cases} \tag{7.25}$$

As in subsections 7.1 and 7.2, direct computations lead to

$$\begin{aligned} &\left(J_3 J_4 - k_3^2 N^2\right)\left(J_1 J_2 - k_1^2 N^2\right) - k_0^2 J_2 J_4 \\ &\sim \begin{cases} \dfrac{-i a_1 k_1^2 \sqrt{k_2}(k_2 - k_3)(k_2 - k_4)}{k_2 - k_1} N^5 \text{ if } k_2 \notin \{k_3, k_4\}, \\ \dfrac{-i a_1 k_1^2 \sqrt{k_2}\left[(k_2 - k_4)\left(\dfrac{k_1 k_2}{k_2 - k_1} - k_0\right) - k_3^2\right]}{k_2 - k_1} N^3 \\ \text{if } k_0 \neq \dfrac{k_1 k_2}{k_2 - k_1} - \dfrac{k_3^2}{k_2 - k_4},\ k_2 = k_3 \text{ and } k_2 \neq k_4, \\ \dfrac{-\left[a_1 a_4 k_2 k_1^2 k_3^2 + k_0^2 k_1^2 (k_2 - k_4)^2\right]}{(k_2 - k_1)(k_2 - k_4)} N^2 \\ \text{if } k_0 = \dfrac{k_1 k_2}{k_2 - k_1} - \dfrac{k_3^2}{k_2 - k_4},\ k_2 = k_3 \text{ and } k_2 \neq k_4, \\ \dfrac{-a_1 a_4 k_1^2 k_2 (k_2 - k_3)}{k_2 - k_1} N^4 \text{ if } k_2 \neq k_3 \text{ and } k_2 = k_4, \\ \dfrac{i a_1 k_1^2 k_3^2 \sqrt{k_2}}{k_2 - k_1} N^3 \text{ if } k_2 = k_3 = k_4 \end{cases} \end{aligned} \tag{7.26}$$

and

$$\begin{aligned} &J_1\left(J_3 J_4 - k_3^2 N^2\right) - k_0^2 J_4 \\ &\sim \begin{cases} (k_2 - k_1)(k_2 - k_3)(k_2 - k_4) N^6 \text{ if } k_2 \notin \{k_3, k_4\}, \\ (k_2 - k_1)\left[(k_2 - k_4)\left(\dfrac{k_1 k_2}{k_2 - k_1} - k_0\right) - k_3^2\right] N^4 \\ \text{if } k_0 \neq \frac{k_1 k_2}{k_2 - k_1} - \dfrac{k_3^2}{k_2 - k_4},\ k_2 = k_3 \text{ and } k_2 \neq k_4, \\ -i a_4 \sqrt{k_2}(k_2 - k_1)\left(\dfrac{k_1 k_2}{k_2 - k_1} - k_0\right) N^3 \\ \text{if } k_0 = \dfrac{k_1 k_2}{k_2 - k_1} - \dfrac{k_3^2}{k_2 - k_4},\ k_2 = k_3 \text{ and } k_2 \neq k_4, \\ -i a_4 \sqrt{k_2}(k_2 - k_1)(k_2 - k_3) N^5 \text{ if } k_2 \neq k_3 \text{ and } k_2 = k_4, \\ -k_3^2 (k_2 - k_1) N^4 \text{ if } k_2 = k_3 = k_4, \end{cases} \end{aligned} \tag{7.27}$$

then, by combining $(7.25)_2$, (7.26) and (7.27), we get, for some $c > 0$,

$$|\alpha_{2,n}| \sim cN. \tag{7.28}$$

Moreover, from $(4.12)_1$, we see that

$$\begin{aligned} \|\Phi_n\|_{\mathcal{H}}^2 &\geq k_2 \|\psi_{n,x}\|^2 \\ &= k_2|\alpha_{2,n}|^2 N^2 \int_0^1 \cos^2(Nx)dx \\ &\geq \frac{k_2}{2}|\alpha_{2,n}|^2 N^2 \int_0^1 [1 + \cos(2Nx)]\, dx \\ &= \frac{k_2}{2}|\alpha_{2,n}|^2 N^2, \end{aligned} \tag{7.29}$$

then $(7.3)_4$ holds thanks to (7.28) and (7.29).

Subcase 2. $(\tau_1, \tau_2, \tau_3, \tau_4) = (1, 0, 0, 1)$, $k_1 = k_2$ **and** $k_3 \neq k_4$. The proof is similar to the one given in subsection 7.1 by considering the choices (7.7) to get (7.8), (7.9), $(7.10)_1$, $(7.11)_1$,

$$J_1 J_2 - k_1^2 N^2 \sim -k_1^2 N^2 \quad \text{if } k_1 = k_2 = k_3 \tag{7.30}$$

and

$$\left(J_3 J_4 - k_3^2 N^2\right)\left(J_1 J_2 - k_1^2 N^2\right) - k_0^2 J_2 J_4 \sim \frac{i a_4 k_1^2 k_3^2 \sqrt{k_3}}{k_3 - k_4} N^3 \quad \text{if } k_1 = k_2 = k_3 \tag{7.31}$$

(that is (7.30) and (7.31) correspond to $(7.10)_4$ and $(7.11)_4$, respectively, with $a_2 = 0$). Noticing that the two cases [$k_3 = k_1$ and $k_3 \neq k_2$] and [$k_3 \neq k_1$ and $k_3 = k_2$] considered in (7.10), (7.11) and (7.12) can not be considered here because $k_1 = k_2$. Then we deduce from $(7.9)_3$, $(7.9)_4$, $(7.10)_1$, $(7.11)_1$, (7.30) and (7.31) that, for some c_1, $c_2 > 0$,

$$|\alpha_{3,n}| \sim c_1 N \quad \text{and} \quad |\alpha_{4,n}| \sim c_2, \tag{7.32}$$

hence, by using (7.14) and (7.32), we arrive at $(7.3)_4$.

7.4. Case $(\tau_1, \tau_2, \tau_3, \tau_4) = (1, 0, 1, 0)$ and $(k_1, k_3) \neq (k_2, k_4)$

When $k_1 \neq k_2$, the proof is similar to the ones given in subsection 7.3 - subcase 1 by considering the same choices (7.23), so (7.24) and (7.25) hold, and therefore, by exploiting $(7.25)_2$, we get (7.28), and then $(7.3)_4$ holds according to (7.29). We omit the details here.

When $k_1 = k_2$ and $k_3 \neq k_4$, we follow the same arguments as in subsection 7.2 by considering the choices (7.16), we find (7.17), (7.18), $(7.19)_1$, $(7.19)_4$ with $a_2 = 0$, $(7.20)_1$ and $(7.20)_4$ with $a_2 = 0$ (the two cases [$k_4 = k_1$ and $k_4 \neq k_2$] and [$k_4 \neq k_1$ and $k_4 = k_2$] considered in (7.19) and (7.20) can not be considered here because $k_1 = k_2$), so (7.21) holds, and then, by combining $(7.18)_4$, (7.21) and (7.22), we deduce $(7.3)_4$.

7.5. Case $(\tau_1, \tau_2, \tau_3, \tau_4) = (0, 1, 0, 1)$ and $(k_1, k_3) \neq (k_2, k_4)$

We distinguish the three subcases [$k_1 \neq k_2$ and [$k_1 \neq k_3$ or $k_1 = k_4$]], [$k_3 \neq k_4$ and [$k_1 \neq k_3$ or $k_2 = k_3$]] and [$k_1 = k_3$ and $k_1 \notin \{k_2, k_4\}$]. We observe that these three subcases are equivalent to $(k_1, k_3) \neq (k_2, k_4)$.

Subcase 1. $(\tau_1, \tau_2, \tau_3, \tau_4) = (0, 1, 0, 1)$, $k_1 \neq k_2$ **and** $[k_1 \neq k_3$ **or** $k_1 = k_4]$. We choose

$$\beta_{4,n} = \beta_{6,n} = \beta_{8,n} = 0, \quad \beta_{2,n} = 1 \quad \text{and} \quad \lambda_n = \sqrt{k_1 N^2 + k_0 + \frac{k_1^2}{k_1 - k_2}}, \tag{7.33}$$

for $n \in \mathbb{N}$ such that $k_1 N^2 + k_0 + \frac{k_1^2}{k_1 - k_2} > 0$. We remark that $(4.11)_2$, $(4.12)_3$, $(4.12)_4$ and (7.33) lead to

$$\|F_n\|_{\mathcal{H}}^2 = \|f_{2,n}\|^2 = \int_0^1 \cos^2(Nx)dx \leq 1 \tag{7.34}$$

(which implies $(7.3)_2$) and (as for (7.13))

$$\begin{aligned} \|\Phi_n\|_{\mathcal{H}}^2 &\geq k_1 \|\varphi_{n,x} + \psi_n\|^2 \\ &= k_1 |\alpha_{1,n} N - \alpha_{2,n}|^2 \int_0^1 \sin^2(Nx)dx \\ &\geq \frac{k_1}{2} |\alpha_{1,n} N - \alpha_{2,n}|^2 \int_0^1 [1 - \cos\,(2Nx)]\;dx \\ &= \frac{k_1}{2} |\alpha_{1,n} N - \alpha_{2,n}|^2. \end{aligned} \tag{7.35}$$

On the other hand, according to (7.33), simple computations imply that the unique solution of (7.5) is

$$\begin{cases} \alpha_{1,n} = \dfrac{J_2\left(J_3 J_4 - k_3^2 N^2\right)}{\left(J_3 J_4 - k_3^2 N^2\right)\left(J_1 J_2 - k_1^2 N^2\right) - k_0^2 J_2 J_4}, \\ \alpha_{2,n} = \dfrac{-k_1 N\left(J_3 J_4 - k_3^2 N^2\right)}{\left(J_3 J_4 - k_3^2 N^2\right)\left(J_1 J_2 - k_1^2 N^2\right) - k_0^2 J_2 J_4}, \\ \alpha_{3,n} = \dfrac{-k_0 J_2 J_4}{\left(J_3 J_4 - k_3^2 N^2\right)\left(J_1 J_2 - k_1^2 N^2\right) - k_0^2 J_2 J_4}, \\ \alpha_{4,n} = \dfrac{k_0 k_3 N J_2}{\left(J_3 J_4 - k_3^2 N^2\right)\left(J_1 J_2 - k_1^2 N^2\right) - k_0^2 J_2 J_4}, \end{cases} \tag{7.36}$$

therefore

$$\begin{aligned} &\left(J_3 J_4 - k_3^2 N^2\right)\left(J_1 J_2 - k_1^2 N^2\right) - k_0^2 J_2 J_4 \\ &\sim \begin{cases} \dfrac{-i a_2 k_1^2 \sqrt{k_1}(k_1 - k_3)(k_1 - k_4)}{k_1 - k_2} N^5 & \text{if } k_1 \notin \{k_3, k_4\}, \\ -k_0^2 (k_1 - k_2)(k_1 - k_4) N^4 & \text{if } k_1 = k_3 \text{ and } k_1 \neq k_4, \\ \dfrac{-a_2 a_4 k_1^3 (k_1 - k_3)}{k_1 - k_2} N^4 & \text{if } k_1 \neq k_3 \text{ and } k_1 = k_4, \\ \dfrac{i\sqrt{k_1}\left[a_2 k_1^2 k_3^2 + a_4 k_0^2 (k_1 - k_2)^2\right]}{k_1 - k_2} N^3 & \text{if } k_1 = k_3 = k_4 \end{cases} \end{aligned} \tag{7.37}$$

and

$$
J_3J_4 - k_3^2N^2 \sim \begin{cases} (k_1-k_3)(k_1-k_4)N^4 & \text{if } k_1 \notin \{k_3,k_4\}, \\ \dfrac{k_1^2(k_2-k_4)}{k_1-k_2}N^2 & \text{if } k_1 = k_3,\ k_1 \neq k_4 \,\text{and}\, k_2 \neq k_4, \\ \dfrac{-ia_4k_1^2\sqrt{k_1}}{k_1-k_2}N & \text{if } k_1 = k_3,\ k_1 \neq k_4 \,\text{and}\, k_2 = k_4, \\ -ia_4\sqrt{k_1}(k_1-k_3)N^3 & \text{if } k_1 \neq k_3 \,\text{and}\, k_1 = k_4, \\ -k_3^2N^2 & \text{if } k_1 = k_3 = k_4, \end{cases} \tag{7.38}
$$

so, according to $(7.36)_1$, $(7.36)_2$, (7.37) and (7.38), $\alpha_{1,n}$ and $\alpha_{2,n}$ satisfy, for some $c_1,\ c_2 > 0$,

$$
(|\alpha_{1,n}|, |\alpha_{2,n}|) \sim \begin{cases} (c_1N, c_2) & \text{if } k_1 \notin \{k_3,k_4\}, \\ \left(c_1, \dfrac{c_2}{N}\right) & \text{if } k_1 = k_3,\ k_1 \neq k_4 \ \text{ and } \ k_2 \neq k_4, \\ \left(\dfrac{c_1}{N}, \dfrac{c_2}{N^2}\right) & \text{if } k_1 = k_3,\ k_1 \neq k_4 \ \text{ and } \ k_2 = k_4, \\ (c_1N, c_2) & \text{if } k_1 \neq k_3 \ \text{ and } \ k_1 = k_4, \\ (c_1N, c_2) & \text{if } k_1 = k_3 = k_4, \end{cases} \tag{7.39}
$$

we omit the details here. Because we are assuming in this subcase 1 that $[k_1 \neq k_3$ or $k_1 = k_4]$, then $(7.39)_2$ and $(7.39)_3$ can not be considered in this subcase 1, so the properties (7.35), $(7.39)_1$, $(7.39)_4$ and $(7.39)_5$ lead to $(7.3)_4$.

Subcase 2. $(\tau_1,\tau_2,\tau_3,\tau_4) = (0,1,0,1)$, $k_3 \neq k_4$ **and** $[k_1 \neq k_3$ **or** $k_2 = k_3]$. As in subsection 7.1, we consider the choices (7.7) and we get (7.8), (7.9) and (7.14). Moreover, we have

$$
\begin{aligned}
&\left(J_3J_4 - k_3^2N^2\right)\left(J_1J_2 - k_1^2N^2\right) - k_0^2J_2J_4 \qquad (7.40) \\
&\sim \begin{cases} \dfrac{-ia_4k_3^2\sqrt{k_3}(k_3-k_1)(k_3-k_2)}{k_3-k_4}N^5 & \text{if } k_3 \notin \{k_1,k_2\}, \\ -k_0^2(k_3-k_2)(k_3-k_4)N^4 & \text{if } k_1 = k_3 \ \text{ and } \ k_2 \neq k_3, \\ \dfrac{-a_2a_4k_3^3(k_3-k_1)}{k_3-k_4}N^4 & \text{if } k_1 \neq k_3 \ \text{ and } \ k_2 = k_3, \\ \dfrac{i\sqrt{k_3}\left[a_4k_3^2k_1^2 + a_2k_0^2(k_3-k_4)^2\right]}{k_3-k_4}N^3 & \text{if } k_1 = k_2 = k_3 \end{cases}
\end{aligned}
$$

and

$$J_1 J_2 - k_1^2 N^2 \sim \begin{cases} (k_3-k_1)(k_3-k_2)N^4 & \text{if } k_3 \notin \{k_1,k_2\}, \\ \dfrac{k_1^2(k_4-k_2)}{k_3-k_4}N^2 & \text{if } k_1=k_3,\ k_2\neq k_3 \text{ and } k_2\neq k_4, \\ \dfrac{-ia_2k_3^2\sqrt{k_3}}{k_3-k_4}N & \text{if } k_1=k_3,\ k_2\neq k_3 \text{ and } k_2=k_4, \\ -ia_2\sqrt{k_3}(k_3-k_1)N^3 & \text{if } k_1\neq k_3 \text{ and } k_2=k_3, \\ -k_1^2N^2 & \text{if } k_1=k_2=k_3, \end{cases} \tag{7.41}$$

so, as for (7.39), according to $(7.9)_3$, $(7.9)_4$, (7.40) and (7.41), $\alpha_{3,n}$ and $\alpha_{4,n}$ satisfy, for some c_1, $c_2 > 0$,

$$(|\alpha_{3,n}|,|\alpha_{4,n}|) \sim \begin{cases} (c_1N,c_2) & \text{if } k_3 \notin \{k_1,k_2\}, \\ \left(c_1,\dfrac{c_2}{N}\right) & \text{if } k_1=k_3,\ k_2\neq k_3 \text{ and } k_2\neq k_4, \\ \left(\dfrac{c_1}{N},\dfrac{c_2}{N^2}\right) & \text{if } k_1=k_3,\ k_2\neq k_3 \text{ and } k_2=k_4, \\ (c_1N,c_2) & \text{if } k_1\neq k_3 \text{ and } k_2=k_3, \\ (c_1N,c_2) & \text{if } k_1=k_2=k_3. \end{cases} \tag{7.42}$$

We remark that $(7.42)_2$ and $(7.42)_3$ can not be considered in this subcase 2 thanks to the assumption $[k_1 \neq k_3$ or $k_2 = k_3]$, then (7.14), $(7.42)_1$, $(7.42)_4$ and $(7.42)_5$ show that $(7.3)_4$ is satisfied.

Subcase 3. $(\tau_1,\tau_2,\tau_3,\tau_4)=(0,1,0,1)$, $k_1=k_3$ **and** $k_1 \notin \{k_2,k_4\}$. We take

$$\beta_{2,n}=\beta_{4,n}=\beta_{8,n}=0, \quad \beta_{6,n}=1 \quad \text{and} \quad \lambda_n=\sqrt{k_1N^2+k_0+b}, \tag{7.43}$$

for $n\in\mathbb{N}$ such that $k_1N^2+k_0+b>0$, where

$$b=\frac{k_1^2(2k_1-k_2-k_4)}{2(k_1-k_2)(k_1-k_4)}+\sqrt{k_0^2+\frac{k_1^4(k_2-k_4)^2}{4(k_1-k_2)^2(k_1-k_4)^2}}. \tag{7.44}$$

Then (7.8) and (7.9) hold. Moreover, we see that

$$J_2J_4 \sim (k_1-k_2)(k_1-k_4)N^4, \quad J_4N \sim (k_1-k_4)N^3 \tag{7.45}$$

and

$$\begin{aligned} &\left(J_3J_4-k_3^2N^2\right)\left(J_1J_2-k_1^2N^2\right)-k_0^2J_2J_4 \\ &=\left[b\left[(k_1-k_4)N^2-ia_4\lambda_n+b+k_0-k_1\right]-k_1^2N^2\right] \\ &\quad\times\left[b\left[(k_1-k_2)N^2-ia_2\lambda_n+b+k_0-k_1\right]-k_1^2N^2\right] \\ &\quad-k_0^2\left[(k_1-k_2)N^2-ia_2\lambda_n+b+k_0-k_1\right]\left[(k_1-k_4)N^2-ia_4\lambda_n+b+k_0-k_1\right], \end{aligned} \tag{7.46}$$

since $k_1 = k_3$. On the other hand, direct computations show that the coefficient of N^4 in the right hand side of (7.46) vanishes; that is

$$\left[(k_1 - k_4)b - k_1^2\right]\left[(k_1 - k_2)b - k_1^2\right] - k_0^2(k_1 - k_2)(k_1 - k_4) = 0,$$

therefore

$$\left(J_3J_4 - k_3^2N^2\right)\left(J_1J_2 - k_1^2N^2\right) - k_0^2J_2J_4 \sim \begin{cases} I_3N^3 & \text{if } I_3 \neq 0, \\ I_2N^2 & \text{if } I_3 = 0 \text{ and } I_2 \neq 0, \\ I_1N & \text{if } I_3 = I_2 = 0 \text{ and } I_1 \neq 0, \end{cases} \tag{7.47}$$

where

$$I_m = \begin{cases} i\sqrt{k_1}\left[a_2\left[k_0^2(k_1 - k_4) + k_1^2b - (k_1 - k_4)b^2\right]\right. \\ \left. +a_4\left[k_0^2(k_1 - k_2) + k_1^2b - (k_1 - k_2)b^2\right]\right] \text{ if } m = 3, \\ (b + k_0 - k_1)\left[\left(b - k_0^2\right)(2k_1 - k_2 - k_4) - 2k_1^2\right] - k_1a_2a_4\left(b^2 - k_0^2\right) \text{ if } m = 2, \\ -i\sqrt{k_1}(a_2 + a_4)(b + k_0 - k_1)\left(b^2 - k_0^2\right) \text{ if } m = 1. \end{cases}$$

Observing that $(I_1, I_2, I_3) \neq (0, 0, 0)$. Indeed, if $I_1 = 0$, then $b^2 = k_0^2$ or $b = k_1 - k_0$. If $b^2 = k_0^2$, then

$$I_3 = ik_1^2\sqrt{k_1}b(a_2 + a_4) \neq 0.$$

And if $b^2 \neq k_0^2$ and $b = k_1 - k_0$, then

$$I_2 = -k_1a_2a_4\left(b^2 - k_0^2\right) \neq 0.$$

Consequently, (7.47) implies that there exists $m \in \{1, 2, 3\}$ such that

$$\left(J_3J_4 - k_3^2N^2\right)\left(J_1J_2 - k_1^2N^2\right) - k_0^2J_2J_4 \sim I_mN^m. \tag{7.48}$$

Finally, we deduce from $(7.9)_1$, $(7.9)_2$, (7.45) and (7.48) the existence of c_1, $c_2 > 0$ such that

$$(|\alpha_{1,n}|, |\alpha_{2,n}|) \sim \left(c_1N^{4-m}, c_2N^{3-m}\right), \tag{7.49}$$

hence $(7.3)_4$ holds, since (7.35) and (7.49) lead to

$$\lambda_n^{-2}\|\Phi_n\|_{\mathcal{H}} \sim \frac{c_1}{\sqrt{2k_1}}N^{3-m}. \tag{7.50}$$

The proof of Theorem 7.1 is then ended. □

8. Lack of polynomial stability: Cases (3.1) and (3.2)

In the last cases (3.1) and (3.2) (where also the strong stability (2.17) holds but the exponential one (4.5) is not satisfied; see sections 3 and 4), we will prove that even the polynomial stability (6.6) does not hold in general.

Theorem 8.1. *For any $\delta > 0$, the polynomial decay* (6.6) *does not hold in the following two cases:*

$$(\tau_1, \tau_2, \tau_3, \tau_4) = (1,0,0,0), \quad (k_1, k_2) \in \{(k_3, k_3), (k_0, k_4)\} \quad \textit{and} \quad k_3 = \frac{k_0 k_4}{k_0 + k_4} \tag{8.1}$$

and

$$(\tau_1, \tau_2, \tau_3, \tau_4) = (0,0,1,0), \quad (k_3, k_4) \in \{(k_1, k_1), (k_0, k_2)\} \quad \textit{and} \quad k_1 = \frac{k_0 k_2}{k_0 + k_2}. \tag{8.2}$$

Proof. We need to treat only the case (8.1), since, by symmetry, the other case (8.2) can be treated in a similar way.

As in section 7, to prove Theorem 8.1, it is sufficient to show that, for any $m \in \mathbb{N}^*$,

$$\limsup_{\lambda \to \infty} \lambda^{-m} \left\| (i\lambda I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} > 0, \tag{8.3}$$

since (8.3) implies that (6.6) does not hold, for any $\delta > \frac{1}{m}$ (see [9,10]).

To get (8.3), it is sufficient to find sequences $(\lambda_n)_n \subset \mathbb{R}$, $(F_n)_n \subset \mathcal{H}$ and $(\Phi_n)_n \subset D(\mathcal{A})$, $n \in \mathbb{N}$, satisfying $(7.3)_1$, $(7.3)_2$, $(7.3)_3$ and (8.3). Let Φ_n, F_n, N and J_j, $j = 1,2,3,4$, as in section 7. Then $(7.3)_1$ is equivalent to (7.5). By considering (4.11) and (4.12), it is clear that $(F_n)_n \subset \mathcal{H}$ and $(\Phi_n)_n \subset D(\mathcal{A})$. Let $m \in \mathbb{N}^*$ and take

$$\beta_{2,n} = \beta_{4,n} = \beta_{6,n} = 0, \quad \beta_{8,n} = 1 \quad \text{and} \quad \lambda_n = \sqrt{k_2 N^2 + k_1 + N^{-m-1}}, \tag{8.4}$$

for $n \in \mathbb{N}$. It appears that $(7.3)_2$ and $(7.3)_3$ are satisfied (thanks to (7.17)) and the solution of (7.5) is given by (7.18). Moreover, we have $J_2 = N^{-m-1}$ and, according to the connections between k_j assumed in (8.1),

$$J_3 J_4 - k_3^2 N^2 = N^{-m-1} \left[(2k_2 - k_3 - k_4) N^2 + 2k_1 - k_0 - k_3 + N^{-m-1} \right],$$

therefore (noticing that $2k_2 - k_3 - k_4 \neq 0$ because of (8.1))

$$\left(J_3 J_4 - k_3^2 N^2 \right) \left(J_1 J_2 - k_1^2 N^2 \right) - k_0^2 J_2 J_4 \sim -k_1^2 (2k_2 - k_3 - k_4) N^{3-m}, \tag{8.5}$$

then $(7.18)_2$ and (8.5) imply that

$$|\alpha_{2,n}| \sim \frac{k_0 k_3}{k_1 |2k_2 - k_3 - k_4|} N^{m-1}, \tag{8.6}$$

hence, by using (7.29) and (8.6),

$$\begin{aligned} \lim_{n\to\infty} \lambda_n^{-m} \|\Phi_n\|_{\mathcal{H}} &\geq \lim_{n\to\infty} \frac{|\alpha_{2,n}|}{\sqrt{2} \left(\sqrt{k_2}\right)^{m-1} N^{m-1}} \\ &= \frac{k_0 k_3}{\sqrt{2} k_1 |2k_2 - k_3 - k_4| \left(\sqrt{k_2}\right)^{m-1}} \\ &> 0, \end{aligned} \tag{8.7}$$

which leads to (8.3). This ends the proof of Theorem 8.1. □

9. Comments and issues

We would like to point out in this section that there are several possible generalizations and various interesting open questions and promising research avenues.

1. Our results hold true for one of the following Dirichlet-Neumann boundary conditions:

$$\begin{cases} \varphi(0,t) = \psi_x(0,t) = w(0,t) = z_x(0,t) = 0 \text{ in } (0,\infty), \\ \varphi_x(1,t) = \psi(1,t) = w_x(1,t) = z(1,t) = 0 \text{ in } (0,\infty), \end{cases}$$

$$\begin{cases} \varphi_x(0,t) = \psi(0,t) = w_x(0,t) = z(0,t) = 0 \text{ in } (0,\infty), \\ \varphi_x(1,t) = \psi(1,t) = w_x(1,t) = z(1,t) = 0 \text{ in } (0,\infty), \end{cases} \tag{9.1}$$

and

$$\begin{cases} \varphi(0,t) = \psi_x(0,t) = w(0,t) = z_x(0,t) = 0 \text{ in } (0,\infty), \\ \varphi(1,t) = \psi_x(1,t) = w(1,t) = z_x(1,t) = 0 \text{ in } (0,\infty). \end{cases} \tag{9.2}$$

In cases (9.1) and (9.2), and without loss of generality (thanks to some change of variables as in Remark 2.1 of [17] for Bresse-type systems), one can, respectively, assume that

$$\int_0^1 \varphi(x,t)dx = \int_0^1 w(x,t)dx = 0$$

and

$$\int_0^1 \psi(x,t)dx = \int_0^1 z(x,t)dx = 0,$$

which allows to apply Poincaré's inequality to φ, ψ, w and z. The situation is more delicate when [φ and ψ] or [φ and z] or [ψ and w] or [w and z] have the same boundary condition at 0 or at 1, and also when [φ and w] or [ψ and z] have different boundary conditions at 0 or at 1.

2. Similar stability results to the ones proved in this paper can be obtained by replacing the coupling terms $-k_0(w-\varphi)$ and $k_0(w-\varphi)$ by $-k_0(z-\psi)$ and $k_0(z-\psi)$, respectively, and adding them to $(1.1)_2$ and $(1.1)_4$, respectively. Similarly, $-k_0(w-\varphi)$ and $k_0(w-\varphi)$ can be replaced by $-k_0(z-\varphi)$ and $k_0(z-\varphi)$, respectively, and added to $(1.1)_1$ and $(1.1)_4$, respectively, or they are replaced by $-k_0(w-\psi)$ and $k_0(w-\psi)$, respectively, and added to $(1.1)_2$ and $(1.1)_3$, respectively.

3. The frictional dampings $a_1\varphi_t$, $a_2\psi_t$, $a_3 w_t$ and $a_4 z_t$ (or some of them) can be replaced by other kinds of dissipation like, for example, memory, heat conduction and Kelvin-Voigt effects. Similar stability results to ours can be proved in these situations (see, for example, [1, 15, 16, 21] for other Timoshenko-type systems).

4. In section 7, we proved the optimality of the polynomial decay rate obtained in cases (6.3)-(6.5). However, in cases (6.1) and (6.2), we do not know if the polynomial deacy rates are optimal or not; perhaps, they can be improved.

5. The coupled two Timoshenko beams (1.1) studied in the present work is linaer. It would be very desirable to obtain analogous results in the presence of some

nonlinear terms, where such nonlinear models are more closer to the real world than the linear ones. In particular, when the frictional dampings (or some of them) are nonlinear; that is the linear frictional dampings $a_1\varphi_t$, $a_2\psi_t$, $a_3 w_t$ and $a_4 z_t$ are replaced, respectively, by the nonlinear ones $h_1(\varphi_t)$, $h_2(\psi_t)$, $h_3(w_t)$ and $h_4(z_t)$, where

$$h_j : s \in \mathbb{R} \mapsto h_j(s) \in \mathbb{R}, \quad j = 1, 2, 3, 4,$$

are fixed functions satisfying some smoothness and boundedness conditions. Another research avenue is to treat the local stability problem; that is the positive constants (or some of them) a_j, $j = 1, 2, 3, 4$, are replaced by nonnegative functions

$$a_j : x \in (0, 1) \mapsto a_j(x) \in \mathbb{R}_+, \quad j = 1, 2, 3, 4,$$

which can vanish on some parts of the interval $(0, 1)$. We aspire in future works to investigate these interesting questions.

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Conflict of interest

The author declares that he has no conflict of interest.

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