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UNIVERSAL APPROACH TO THE TAKESAKI-TAKAI γ-DUALITY

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Abstract In this article, we generalize and simplify the proof of the Takesaki-Takai γ-duality theorem. Assume a morphism $\omega : G \to Aut\,(\mathrm{A})$ is a projective representation of the locally compact Abel group G in $Aut\,(\mathrm{A})$, mapping $\gamma : G \to G$ is continuous, and $(\mathrm{A},\ G,\ \omega)$ is a dynamic system then there exists isomorphism

$$\Upsilon \ : \ Env_{\hat{\omega}}{}^{\gamma}\left(L^1\left(\hat{G},\ Env_{\omega}{}^{\gamma}\left(L^1\left(G,\ \mathrm{A}\right)\right)\right)\right) \to \mathrm{A} \otimes LK\left(L^2\left(G\right)\right)$$

which is the equivariant for the double dual action

$$\hat{\hat{\omega}} \ : \ G \to Aut\left(Env_{\hat{\omega}}{}^{\gamma}\left(L^1\left(\hat{G},\ Env_{\omega}{}^{\gamma}\left(L^1\left(G,\ \mathrm{A}\right)\right)\right)\right)\right).$$

These results deepen our understanding of the representation theory and are especially interesting given their possible applications to problems of the quantum theory.

Keywords Takai duality, γ-duality, Wigner function, C^*-algebra, Pontryagin duality, induced representation, cross product.

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1. Introduction

Let G be a locally compact group, let $C_C\,(G)$ be a space of real-valued function with compact support.

Definition 1.1. A Radon measure on a locally compact group G is called a linear form μ on $C_C\,(G)$ such that for any compact set $K \subset G$ restriction of the linear form μ to subspace $C_C\,(K) \subset C_C\,(G)$ functions of $C_C\,(G)$ which support contains in K, is continuous in the topology of uniform convergence. The value $\mu\,(\psi)$ of the Radon measure μ on the continuous function $\psi \in C_C\,(G)$ with compact support is called a Radon integral of the function ψ.

As a consequence of the definition, we have that for any compact subset $K \subset G$ there exists a constant $\tilde{c}\,(K)$ dependent on K such that the equality

$$|\mu\,(\psi)| \le \tilde{c}\,\|\psi\|_{C_c(G)}$$

holds for all $\psi \in C_C\,(G)$.

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Let ${C_C}^{+}(G)$ be s set of all finite positive continuous functions with compact supports. We denote by $\wp_{+}(G)$ the set of all lower semicontinuous positive functions i.e., all functions ψ such that at every point g_0 of its domain satisfy the following condition

$$\lim_{g\to g_0} \inf_{g\in G} \psi(g) = \psi(g_0).$$

Definition 1.2. Let μ be positive Radon measure on G, then the upper integral $\mu^*(\psi)$ of a function $\psi \in \wp_{+}(G)$ is defined by

$$\mu^*(\psi) = \sup_{\varphi\in {C_C}^{+}(G), \quad \varphi\le\psi} \mu(\varphi).$$

The upper integral of an arbitrary positive function $\psi : G \to R^{+}$ is defined by

$$\mu^*(\psi) = \inf_{\varphi\in \wp_{+}(G), \quad \varphi\ge\psi} \mu^*(\varphi).$$

Definition 1.3. The outer measure $\mu^*(E)$ of an arbitrary subset $E \subset G$ is an upper integral $\mu^*(1_E)$ of the characteristic function 1_E of E.

The set $\mathrm{M}(G)$ of all Radon measures μ on the locally compact space G is the space of all linear forms on the vector space $C_C(G)$ and thus $\mathrm{M}(G)$ is a topological space with the $*$-weak or so-called wide topology of the weak convergence. If G is a compact group then the wide topology coincides with the classical weak topology.

Wide topology in $\mathrm{M}(G)$ can be defined by seminorms $\mu \mapsto \sup_{1\le i\le k} |\mu(\psi_i)|$, where $\{\psi_i\}_{1\le i\le k} \subset C_C(G)$ is an arbitrary finite sequence of functions of $C_C(G)$.

The dual group $\hat{G}$ consists of all homomorphisms (characters) from G to the circle group with natural measure $\hat{\mu}(\chi) = \int \overline{\chi(g)} d\mu(g)$, $\chi \in \hat{G}$.

The Fourier transform of a function $\psi \in L^1(G)$ is given by

$$\hat{\psi}(\chi) = \int_G \psi(g) \overline{\chi(g)} d\mu(g).$$

Let A be a C^*-algebra then we call a triplet (A, G, ω) a dynamical system where $\omega : G \to Aut(\mathrm{A})$ is a strongly continuous representation, and let H be a Hilbert space then a triplet (H, π, ρ) is called a covariant representation of (A, G, ω).

The Takai duality theory is a generalization of the Takesai duality theorem for the Neumann algebras, which are unital $*$-algebras of bounded operators on Hilbert spaces that are closed in the weak operator topology. The classical Takai duality theorem can be formulated as follows: let (A, ω) be an action of an Abelian group G then there exists an isomorphism Υ from the iterated product $(\mathrm{A} \times_\omega G) \times_{\hat{\omega}} \hat{G}$ to the maximal product $\mathrm{A} \otimes LK\left(L^2(G)\right)$.

Considerable interest in C^*-algebras is justified by many applications to the problems of quantum mechanics for instance so-called von Neumann algebras. Some applications of C^*-algebras to quantum physics are described in [9,15]. B. Abadie [1] considers the Cuntz-Krieger-Pimsner algebras that be a generalization of the crossed product by the set of integer numbers and Toeplitz and Cuntz-Krieger algebras. In [2,3], the Cuntz–Pimsner covariance condition is considered as a nondegeneracy condition for representations of cross algebras and a groupoid model for the Cuntz–Pimsner algebra is constructed; in [11], the author considers the C^*-envelope of a tensor algebra as the corresponding Cuntz- Krieger C^*-algebra.

We will consider the cross product of C^*-algebra $\mathrm{A}\times_\omega G$ as the universal enveloping C^*-algebra $Env_\omega\left(L^1\left(G,\ \mathrm{A}\right)\right)$ of the Banach algebra completed in the universal norm. The covariant representation $(H,\ \pi,\ \rho)$ can be unequivocally characterized by morphism $(\rho\propto\pi)\ :\ L^p\left(G,\ \mathrm{A}\right)E\to LB\left(H,\ H\right)$. This approach can be applied to generalize this theory to include the pseudo-differential operators for general quantization. Thus, we could define a binary operation as

$$\begin{aligned}&\left(\Psi_1\odot_\gamma\Psi_2\right)(g)\\&=\int_G\omega\left(\gamma\left(g\right)^{-1}\gamma\left(h\right)\right)\Psi_1\left(h\right)\omega\left(\gamma\left(g\right)^{-1}h\gamma\left(h^{-1}g\right),\ \Psi_2\left(h^{-1}g\right)\right)d\mu\left(h\right)\end{aligned}$$

and $\Psi_1{}^{\odot_\gamma}\left(g\right)=\omega\left(\gamma\left(g\right)^{-1}h\gamma\left(g^{-1}\right),\ \left(\Psi_1\left(g^{-1}\right)\right)^*\right)$ where $\gamma\ :\ G\to G$ is a continuous function. So, we could define a γ- quantization for $\gamma\ :\ G\to G$, and corresponding pseudo-differential operators, and recover the Weyl-Wigner theory; the next logical step in generalization is to consider p-Schatten classes. For further reading consider a list of references [1–7,9–12,15,16,18] and the most recent [8,13,14,17].

2. The C^*-algebra

Let A be a C^*-algebra. Let G be a locally compact group equipped with Haar measure μ. Let for each $g\in G$ we define a C^*-algebra isomorphism $\omega\left(g\right)\ :\ \mathrm{A}\to\mathrm{A}$, for each fixed $\psi\in\mathrm{A}$ the morphism $\omega\left(g,\ \psi\right)$ is a continuous mapping $\omega\left(\cdot,\ \psi\right)\ :\ G\to\mathrm{A}$ and satisfies the semigroup condition $\omega\left(g,\ \psi\right)\circ\omega\left(h,\ \psi\right)=\omega\left(gh,\ \psi\right)$ for all $g,\ h\in G$, a such defined morphism will be denoted $\omega\ :\ G\to Aut\left(\mathrm{A}\right)$. The triplet $(\mathrm{A},\ G,\ \omega)$ is called a dynamical system.

Definition 2.1. Let H be a separable Hilbert space, $\pi\ :\ G\to U\left(H\right)$ be a continuous unitary representation, $\rho\ :\ \mathrm{A}\to LB\left(H\right)$ be a $*$-representation, then the covariant representation is a set $(H,\ \pi,\ \rho)$ under the condition $\pi\left(g\right)\rho\left(\psi\right)\pi\left(g\right)^*=\rho\left(\omega\left(g,\ \psi\right)\right)$ for all $g\in G$ and $\psi\in\mathrm{A}$. Often, the triplet $(H,\ \pi,\ \rho)$ is abbreviated to duplet $(\pi,\ \rho)$.

Let $L^p\left(G,\ \mathrm{A}\right)$ be a Banach $*$-algebra of A- valued function on G, with the norm given by

$$\left\|\Psi\right\|_{L^p}{}^p=\int_G\left\|\Psi\left(g\right)\right\|_{\mathrm{A}}{}^pd\mu\left(g\right),$$

we assume $p=1$ and the multiplication operation $\odot\ :\ L^p\left(G,\ \mathrm{A}\right)\times L^p\left(G,\ \mathrm{A}\right)\to L^p\left(G,\ \mathrm{A}\right)$ is defined by

$$\left(\Psi_1\odot\Psi_2\right)(g)=\int_G\Psi_1\left(h\right)\omega\left(\Psi_2\left(h^{-1}g\right)\right)d\mu\left(h\right)$$

and

$$\Psi_1{}^{\odot}\left(g\right)=\omega\left(g,\ \left(\Psi_1\left(g^{-1}\right)\right)^*\right)$$

for any pair $\Psi_1,\ \Psi_2\in L^p\left(G,\ \mathrm{A}\right)$.

The universal enveloping C^*-algebra $Env\left(L^p\left(G,\ \mathrm{A}\right)\right)$ of the Banach $*$-algebra $L^p\left(G,\ \mathrm{A}\right)$ is constructed as follows. First, we construct the free tensor algebra

$$T\left(L^p\left(G,\ \mathrm{A}\right)\right)=G\oplus L^p\left(G,\ \mathrm{A}\right)\oplus\left(L^p\left(G,\ \mathrm{A}\right)\otimes L^p\left(G,\ \mathrm{A}\right)\right)\oplus$$

$$\oplus \left(L^p(G,\ \mathrm{A}) \otimes L^p(G,\ \mathrm{A}) \otimes L^p(G,\ \mathrm{A})\right) ...$$

where $\oplus$ is the direct sum and $\otimes$ is the tensor product. Second, the multiplication operation $\odot\ :\ L^p(G,\ \mathrm{A}) \times L^p(G,\ \mathrm{A}) \to L^p(G,\ \mathrm{A})$is bilinear and the tensor product is bilinear so the natural lift is accomplished in such a way as to preserve multiplication as a homomorphism. Third, the universal enveloping algebra $Env_\omega(L^p(G,\ \mathrm{A}))$ is a quotient space $Env_\omega(L^p(G,\ \mathrm{A})) = T(L^p(G,\ \mathrm{A}))/\sim$, where the equivalence relation is $\Psi_1 \otimes \Psi_2 - \Psi_2 \otimes \Psi_1 = \Psi_1 \odot \Psi_2$. The set I of all elements generated by elements given by $\Psi_1 \otimes \Psi_2 - \Psi_2 \otimes \Psi_1 - \Psi_1 \odot \Psi_2$ is a two-side ideal so I lies in the kernel of the quotient map, so we have the short **exact** sequence

$$0 \to I \to T(L^p(G,\ \mathrm{A})) \to T(L^p(G,\ \mathrm{A}))/I \to 0$$

since the sequence is exact, the kernel of the map coincides with the image of the mapping before. In this interpretation, the universal enveloping C^*-algebra $Env_\omega(L^p(G,\ \mathrm{A}))$ is defined as $Env_\omega(L^p(G,\ \mathrm{A})) = T(L^p(G,\ \mathrm{A}))/I$.

The universal norm is given as

$$\|\Psi\|_{Un} = \sup_{\Pi} \|\Pi(\Psi)\|_{LB(H)},$$

where mapping Π is a representation of $L^p(G,\ \mathrm{A})$ in $LB(H,\ H)$.

The integral transformation $(\rho \propto \pi)\ :\ L^p(G,\ \mathrm{A})\,E \to LB(H,\ H)$ defined by

$$(\rho \propto \pi)(\Psi) = \int_G \rho(\Psi(g))\,\pi(g)\,d\mu(g)$$

extends to mapping $(\rho \propto \pi)\ :\ Env_\omega(L^p(G,\ \mathrm{A})) \to LB(H,\ H)$ due to the university of enveloping C^*-algebra.

3. The Takesaki-Takai duality

Let morphism $\omega\ :\ G \to Aut(\mathrm{A})$ be a projective representation of the locally compact Abel group G in $Aut(\mathrm{A})$. We denote a C^*-algebra of compact operators on a separable Hilbert space H by $LK(H)$. The morphism $\omega\ :\ G \to Aut(\mathrm{A})$ is called an action of the group G. Let a triplet $(\mathrm{A}, G,\ \omega)$ be a dynamical system. We obtain the dual action as the homomorphism

$$\hat{\omega}\ :\ \hat{G} \to Aut\left(Env_\omega\left(L^1(G,\ \mathrm{A})\right)\right),$$

then the triplet $\left(Env_\omega\left(L^1(G,\ \mathrm{A})\right),\ \hat{G},\ \hat{\omega}\right)$ is called the dual dynamic system.

Theorem 3.1. *(Variant of the Takai duality). Let G be a locally compact Abelian group and let $(\mathrm{A},\ G,\ \omega)$ be the dynamic system. Then, $Env_{\hat{\omega}}\left(L^1\left(\hat{G},\ Env_\omega\left(L^1(G,\ \mathrm{A})\right)\right)\right)$ isomorphically equals $\mathrm{A} \otimes LK\left(L^2(G)\right)$, so there exists such isomorphism $\Upsilon\ :\ Env_{\hat{\omega}}\left(L^1\left(\hat{G},\ Env_\omega\left(L^1(G,\ \mathrm{A})\right)\right)\right) \to \mathrm{A} \otimes LK\left(L^2(G)\right)$ which is equivariant for the double dual action $\hat{\hat{\omega}}\ :\ G \to Aut\left(Env_{\hat{\omega}}\left(L^1\left(\hat{G},\ Env_\omega\left(L^1(G,\ \mathrm{A})\right)\right)\right)\right)$ and equivariant for $\omega \otimes Ad(\zeta)\ :\ G \to Aut\left(\mathrm{A} \otimes LK\left(L^2(G)\right)\right)$.*

Proof. The statement of the Takai duality theorem will be proven if we show that there is a sequence of the following isomorphisms:

$$Env_{\hat{\omega}}\left(L^1\left(\hat{G},\ Env_{\omega}\left(L^1\left(G,\ \mathrm{A}\right)\right)\right)\right)$$
$$\xrightarrow{\Upsilon_1} Env_{\hat{\lambda}^{-1}\otimes\omega}\left(L^1\left(G,\ Env_{Id}\left(L^1\left(\hat{G},\ \mathrm{A}\right)\right)\right)\right),$$
$$Env_{\hat{\lambda}^{-1}\otimes\omega}\left(L^1\left(G,\ Env_{Id}\left(L^1\left(\hat{G},\ \mathrm{A}\right)\right)\right)\right) \xrightarrow{\Upsilon_2} Env_{\lambda\otimes\omega}\left(L^1\left(G,\ C_0\left(G,\ \mathrm{A}\right)\right)\right),$$
$$Env_{\lambda\otimes\omega}\left(L^1\left(G,\ C_0\left(G,\ \mathrm{A}\right)\right)\right) \xrightarrow{\Upsilon_3} Env_{\lambda\otimes Id}\left(L^1\left(G,\ C_0\left(G,\ \mathrm{A}\right)\right)\right),$$
$$Env_{\lambda\otimes Id}\left(L^1\left(G,\ C_0\left(G,\ \mathrm{A}\right)\right)\right) \xrightarrow{\Upsilon_4} Env_{\lambda}\left(L^1\left(G\otimes \mathrm{A},\ C_0\left(G\right)\right)\right),$$
$$Env_{\lambda}\left(L^1\left(G\otimes \mathrm{A},\ C_0\left(G\right)\right)\right) \xrightarrow{\Upsilon_5} LK\left(L^2\left(G\right)\right)\otimes \mathrm{A},$$

so that the isomorphism in question can be presented as $\Upsilon = \Upsilon_5\circ\Upsilon_4\circ\Upsilon_3\circ\Upsilon_2\circ\Upsilon_1$, where λ is left translation.

Let K be compact, by construction, the set $C_C\left(K\times H,\ \mathrm{A}\right)$ is a dense subspace of $Env_{\beta}\left(L^1\left(H,\ Env_{\alpha}\left(L^1\left(K,\ \mathrm{A}\right)\right)\right)\right)$. Since the topology of $C_C\left(K,\ \mathrm{A}\right)$ is induced by the topology of L^1- norm, we presume $C_C\left(K,\ \mathrm{A}\right)\subset Env_{\alpha}\left(L^1\left(K,\ \mathrm{A}\right)\right)$ is invariant under homomorphism β and there is $f\left(h,\ g\right)\in C_C\left(H\times K,\ \mathrm{A}\right)$ such that $f\left(h,\ g\right)=\beta\left(\ell_f\left(h\right)\right)\left(g\right)$ where $\ell_f\in C_C\left(H,\ Env_{\alpha}\left(L^1\left(K,\ \mathrm{A}\right)\right)\right)$.

The proof will follow from the next statements.

Statement 1. The isomorphism

$$\Upsilon_1\ :\ Env_{\hat{\omega}}\left(L^1\left(\hat{G},\ Env_{\omega}\left(L^1\left(G,\ \mathrm{A}\right)\right)\right)\right)$$
$$\to Env_{\hat{\lambda}^{-1}\otimes\omega}\left(L^1\left(G,\ Env_{Id}\left(L^1\left(\hat{G},\ \mathrm{A}\right)\right)\right)\right)$$

maps dense subalgebras

$$\Upsilon_1\ :\ C_C\left(\hat{G}\times G,\ \mathrm{A}\right)\xrightarrow{onto} C_C\left(G\times\hat{G},\ \mathrm{A}\right)$$

so that $\Upsilon_1\left(f\right)\left(g,\ \chi\right)=\chi\left(g\right)f\left(\chi,\ g\right)$ for all $\left(g,\ \chi\right)\subset G\times\hat{G}$ and $f\in C_C\left(\hat{G}\times G,\ \mathrm{A}\right)$.

Statement 2. Let G be an Abelian group and let $C_C\left(\mathrm{A}, G,\ \omega\ \right)$ be a dynamical system then the mapping $\Upsilon_2\ :\ C_C\left(G\times\hat{G},\ \mathrm{A}\right)\to C_C\left(G,\ C_0\left(G,\ \mathrm{A}\right)\right)$ is given by $\Upsilon_2\left(f\right)\left(g,\ h\right)=\int_{\hat{G}} f\left(g,\ \chi\right)\overline{\chi\left(h\right)}\ d\hat{\mu}\left(\chi\right)$, the mapping

$$\Upsilon_2\ :\ Env_{\hat{\lambda}^{-1}\otimes\omega}\left(L^1\left(G,\ Env_{Id}\left(L^1\left(\hat{G},\ \mathrm{A}\right)\right)\right)\right)\to Env_{\lambda\otimes\omega}\left(L^1\left(G,\ C_0\left(G,\ \mathrm{A}\right)\right)\right)$$

is an isomorphism.

Statement 3. Let G be an Abelian group and let $\left(\mathrm{A}, G,\ \omega\ \right)$ be a dynamical system then there exists an isomorphism

$$\Upsilon_3\ :\ Env_{\lambda\otimes\omega}\left(L^1\left(G,\ C_0\left(G,\ \mathrm{A}\right)\right)\right)\to Env_{\lambda\otimes Id}\left(L^1\left(G,\ C_0\left(G,\ \mathrm{A}\right)\right)\right)$$

such that equality $\Upsilon_3\left(f\right)\left(g,\ h\right)=\omega^{-1}\left(h,\ f\left(g,\ h\right)\right)$ holds for all $f\in C_C\left(G,\ C_0\left(G,\ \mathrm{A}\right)\right)$.

Statement 4. Let G be a locally compact group and let $\left(\mathrm{A}, G,\ \omega\ \right)$ be a dynamical system then there exists an isomorphism $\tilde{\Upsilon}_5=\Upsilon_5\ \Upsilon_4$ such that

$$\tilde{\Upsilon}_5\ : Env_{\lambda\otimes\omega}\left(L^1\left(G,\ C_0\left(G,\ \mathrm{A}\right)\right)\right)\to LK\left(L^2\left(G\right)\right)\otimes \mathrm{A}.$$

Proof of Statement 1. In order to prove statement 1, we must show that $\|\Upsilon_1(f)\| = \|f\|$.

Let $f_1,\ f_2 \in C_C(H \times K,\ \mathrm{A})$ then

$$\left((h,\ g) \mapsto \ell_{f_1}(h) * \alpha\left(h,\ \ell_{f_1}\left(h^{-1}s\right)\right)(g)\right) \in C_C(H \times K,\ \mathrm{A}),$$

so that $\ell_{f_1} * \ell_{f_2} \in C_C(K,\ \mathrm{A}) \subset Env_\alpha\left(L^1(K,\ \mathrm{A})\right)$ and we have

$$\begin{aligned}&(\ell_{f_1} * \ell_{f_2})(s)(g)\\ &= \int_H \int_K \ell_{f_1}(h,t)\,\omega\left(t,\ \alpha\left(h, \left(\ell_{f_2}\left(h^{-1}s\right)\right),\ t^{-1}g\right)\right) d\mu_K(t)\, d\mu_H(h).\end{aligned}$$

Thus, we obtain those equalities

$$\begin{aligned}&(\ell_{f_1} * \ell_{f_2})(\chi)(g)\\ &= \int_{\hat{G}} \int_G \ell_{f_1}(\zeta,t)\,\overline{\zeta\left(t^{-1}g\right)}\,\omega\left(t,\ \alpha\left(\ell_{f_2}\left(\overline{\zeta}\chi\right)\right),\ t^{-1}g\right) d\mu(t)\, d\hat{\mu}(\zeta)\end{aligned}$$

and dual

$$\begin{aligned}&\left(\ell_{\tilde{f}_1} * \ell_{\tilde{f}_2}\right)(g)(\chi)\\ &= \int_G \int_{\hat{G}} \ell_{\tilde{f}_1}(t,\zeta)\,\overline{\zeta}(t)\,\chi(t)\,\omega\left(t,\ \alpha\left(\ell_{\tilde{f}_2}\left(t^{-1}g\right)\right),\ \overline{\zeta}\chi\right) d\hat{\mu}(\zeta)\, d\mu(t)\end{aligned}$$

hold for all $f_1,\ f_2 \in C_C\left(\hat{G} \times G,\ \mathrm{A}\right)$ and for all $\tilde{f}_1,\ \tilde{f}_2 \in C_C\left(G \times \hat{G},\ \mathrm{A}\right) \subset Env_{\hat{\lambda}\otimes\omega}\left(L^1\left(G,\ Env_{Id}\left(L^1(G,\ \mathrm{A})\right)\right)\right)$. Then, we have a homomorphism

$$\Upsilon_1\ :\ C_C\left(\hat{G} \times G,\ \mathrm{A}\right) \overset{\text{onto}}{\longrightarrow} C_C\left(G \times \hat{G},\ \mathrm{A}\right).$$

We write the equalities

$$\begin{aligned}\ell^*{}_{\Upsilon_1(f)}(g)(\chi) &= \left(\hat{\lambda}^{-1} \otimes \omega\right)(g)\left(\ell_{\Upsilon_1(f)}\left(g^{-1}\right)\right)(\chi)\\ &= \chi(g)\,\omega\left(g,\ \ell_{\Upsilon_1(f)}\left(g^{-1}\right)\right)^*(\chi)\\ &= \chi(g)\,\omega\left(g,\ \Upsilon_1(f)\left(g^{-1}, \overline{\chi}\right)\right)^*\\ &= \omega\left(g,\ f\left(\overline{\chi}, g^{-1}\right)\right)^*\\ &= \omega\left(g,\ \ell_f(\overline{\chi})\left(g^{-1}\right)\right)^*\\ &= \chi(g)\,\ell^*{}_f(\overline{\chi})(g)\\ &= \Upsilon_1\left(\ell^*{}_f\right)(\chi)(g).\end{aligned}$$

In general, every continuous in the inductive topology $*$ -homomorphism is bounded in the topology of the universal norm thus this $*$ -homomorphism extends to a representation on $Env_\omega\left(L^1(G,\ \mathrm{A})\right)$.

Let $U\ :\ G \to U(H)$ be a unitary representation and $(U,\ \rho)$ be a covariant representation of the dynamic system $\left(Env_{Id}\left(L^1\left(\hat{G},\ \mathrm{A}\right)\right),\ G,\ \hat{\lambda}^{-1} \otimes \omega\right)$, and $(V,\ \pi)$ be a covariant representation of $\left(\mathrm{A},\ \hat{G},\ Id\right)$, then we denote

$$\Lambda := (U,\ \rho)(f) = \int_G \rho(f(g))\, U(g)\, d\mu(g)$$

so that $\Lambda = (U,\ \rho)\ :\ Env_{\hat{\lambda}^{-1}\otimes\omega}\left(L^1\left(G,\ Env_{Id}\left(L^1\left(\hat{G},\ \mathrm{A}\right)\right)\right)\right) \to LB\,(H)$, and

$$\rho := (V,\ \pi)\,(f) = \int_{\hat{G}} \pi\,(f\,(\hat{g}))\,V\,(\hat{g})\,d\hat{\mu}\,(\hat{g})\,.$$

Next, we have $\hat{\lambda}^{-1}\,(g) \circ \phi\,(\overline{\chi}\zeta) = \chi\,(g)\,\phi\,(\overline{\chi}\zeta) \circ \hat{\lambda}^{-1}\,(g)$. Let us take $a \in \mathrm{A}$, $\psi \in C_C\,(G)$ and $\phi \in C_C\left(\hat{G}\right)$ so that all linear combinations $a \otimes \phi \otimes \psi$ constitute a dense subset of $C_C\left(\hat{G} \times G,\ \mathrm{A}\right)$, so that

$$\begin{aligned} U\,(g)\,V\,(\chi)\,\Lambda\,(a \otimes \phi \otimes \psi) &= U\,(g)\,V\,(\chi)\,\pi\,(a)\,V\,(\phi)\,U\,(\psi) \\ &= \pi\,(\omega\,(g, a))\,V\left(\hat{\lambda}^{-1}\,(g) \circ \phi\,(\overline{\chi}\zeta)\right)U\,(g)\,U\,(\psi) \\ &= \chi\,(g)\,V\,(\chi, \pi\,(\omega\,(g, a)))\,V\left(\hat{\lambda}^{-1}\,(g, \phi)\right)U\,(g)\,U\,(\psi) \\ &= \chi\,(g)\,V\,(\chi)\,U\,(g)\,\Lambda\,(a \otimes \phi \otimes \psi) \end{aligned}$$

and we obtain $U\,(g)\,V\,(\chi) = \chi\,(g)\,V\,(\chi)\,U\,(g)$. Next, we write

$$\begin{aligned} &U\,(g)\,\pi\,(b)\,\pi\,(a)\,V\,(\phi)\,U\,(\psi) \\ &= \pi\,(\omega\,(g,\ ba))\,V\left(\hat{\lambda}^{-1}\,(g,\ \phi)\right)U\,(g)\,U\,(\psi) \\ &= \pi\,(\omega\,(g,\ b))\,U\,(g)\,\pi\,(a)\,V\,(\phi)\,U\,(\psi)\,. \end{aligned}$$

We compute

$$\begin{aligned} \Lambda\,(\Upsilon_1\,(f)) &= \int_G \int_{\hat{G}} \pi\,(\Upsilon_1\,(f)\,(\chi,\ g))\,V\,(\chi)\,U\,(g)\,d\hat{\mu}\,(\chi)\,d\mu\,(g) \\ &= \int_G \int_{\hat{G}} \pi\,(f\,(\chi,\ g))\,\chi\,(g)\,V\,(\chi)\,U\,(g)\,d\hat{\mu}\,(\chi)\,d\mu\,(g) \\ &= \int_{\hat{G}} \int_G \pi\,(f\,(\chi,\ g))\,U\,(gV\,(\chi))\,d\mu\,(g)\,d\hat{\mu}\,(\chi) \end{aligned}$$

so, we obtain $\|\Upsilon_1\,(f)\| \leq\ \|f\|$, the similarly, we obtain $\|f\| \leq \|\Upsilon_1\,(f)\|$ and $\Upsilon_1\ :\ Env_{\hat{\omega}}\left(L^1\left(\hat{G},\ Env_\omega\left(L^1\,(G,\ \mathrm{A})\right)\right)\right) \to Env_{\hat{\lambda}^{-1}\otimes\omega}\left(L^1\left(G,\ Env_{Id}\left(L^1\left(\hat{G},\ \mathrm{A}\right)\right)\right)\right)$ is an isomorphism, statement 1 is proven. □

Proof of Statement 2. The isomorphism $Env_{Id}\left(L^1\left(\hat{G},\ \mathrm{A}\right)\right) \to C_0\,(G,\ \mathrm{A})$ given by $\langle\psi\overline{\chi}\rangle_{\hat{G}}$ can be constructed as an extension of the mapping $a \otimes \phi \mapsto a \otimes \hat{\phi}$ that is defined on the span of bases as $\mathrm{A} \otimes C^*\left(\hat{G}\right) \cong Env_{Id}\left(L^1\left(\hat{G},\ \mathrm{A}\right)\right) \to C_0\,(G,\ \mathrm{A}) \cong C_0\,(G) \otimes \mathrm{A}$. The mapping $\Upsilon_2 := \langle\psi\overline{\chi}\rangle_{\hat{G}} \otimes Id$ is equivariant isomorphism since

$$\begin{aligned} &(\lambda \otimes \omega)\,(g) \int_{\hat{G}} \psi\,(\chi)\,\overline{\chi\,(g)}\,d\hat{\mu}\,(\chi) \\ &= \int_{\hat{G}} \left(\lambda^{-1} \otimes \omega\right)(g)\,\psi\,(\chi)\,\overline{\chi\,(g)}\,d\hat{\mu}\,(\chi)\,, \end{aligned}$$

statement 2 is proven. □

Proof of Statement 3. Since $\omega^{-1}\,(h,\ \varphi\,(h))$ is an isomorphism $C_0\,(G,\ \mathrm{A}) \to C_0\,(G,\ \mathrm{A})$, statement 3 follows from

$$\omega^{-1}\,(g,\ (\lambda \otimes \omega)\,(g,\ \varphi))\,(h) = \omega^{-1}\left(h, \omega\ \left(g,\ \varphi\left(g^{-1}h\right)\right)\right)$$

$$=\omega^{-1}\left(g^{-1}h,\ \varphi\left(g^{-1}h\right)\right)$$
$$=\left(\lambda\otimes\omega\right)\left(g\right)\omega^{-1}\left(h,\ \varphi\left(h\right)\right),$$

so that $\Upsilon_3\left(f\right)\left(g,\ h\right)=\omega^{-1}\left(h,\ f\left(g,\ h\right)\right)$. □

Proof of Statement 4. Let Δ be a modular function on G, namely, $\Delta\ :\ G\to R_+$ is a continuous homomorphism and the equality

$$\Delta\left(g\right)\int_G\psi\left(hg\right)d\mu\left(h\right)=\int_G\psi\left(h\right)d\mu\left(h\right)$$

holds for all $\psi\in C_C\left(G\right)$. Next, we must show that $Env_\lambda\left(L^1\left(G,\ C_0\left(G\right)\right)\right)\cong LK\left(L^2\left(G\right)\right)$. The $Env_\lambda\left(L^1\left(G,\ C_0\left(G\right)\right)\right)$ is simple. We define a natural covariant representation $\left(M,\ l\right)$ of $\left(C_0\left(G\right),\ G,\ \lambda\right)$ as $M\left(\psi\right)\varphi\left(g\right)=\psi\left(g\right)\varphi\left(g\right)$ where $l\ :\ G\to U\left(L^2\left(G\right)\right)$ is the left-regular representation and M operator of pointwise multiplication. Let $k\in C_C\left(G\times G\right)$ then $\Delta\left(h^{-1}g\right)k\left(g,\ h^{-1}g\right)=\psi_k\left(h,g\right)$, $\psi_k\in C_C\left(G\times G\right)$ so that

$$\begin{aligned}&\int_G\left\langle M\left(\psi_k\left(g,\ \cdot\right)\right)l\left(g\right)\varphi_1,\ \varphi_2\right\rangle_{L^2}d\mu\left(g\right)\\=&\int_G\int_G\psi_k\left(g,\ h\right)\varphi_1\left(g^{-1}h\right)\overline{\varphi_2\left(h\right)}\,d\mu\left(g\right)d\mu\left(h\right)\\=&\int_G\int_G\Delta\left(g^{-1}\right)k\left(h,\ g^{-1}h\right)\varphi_1\left(g^{-1}h\right)\overline{\varphi_2\left(h\right)}\,d\mu\left(g\right)d\mu\left(h\right)\\=&\int_G\int_G k\left(h,\ g\right)\varphi_1\left(g\right)\overline{\varphi_2\left(h\right)}\,d\mu\left(g\right)d\mu\left(h\right).\end{aligned}$$

The kernel $k\in C_C\left(G\times G\right)\subset L^2\left(G\times G\right)$ defines a compact Hilbert-Schmidt operator. Since $C_C\left(G\right)$ is dense in $L^2\left(G\right)$ we have $LK\left(L^2\left(G\right)\right)$ belongs to the image of a compact Hilbert-Schmidt operator with kernel k mapping $Env_\lambda\left(L^1\left(G,\ C_0\left(G\right)\right)\right)$. Assume $\psi\in C_C\left(G\times G\right)$ we denote $k\left(h,\ g\right)=\Delta\left(g^{-1}\right)\psi\left(hg^{-1},\ h\right)$ so $\psi_k=\psi$, and $Env_\lambda\left(L^1\left(G,\ C_0\left(G\right)\right)\right)\cong LK\left(L^2\left(G\right)\right)$ follows from the density of $C_C\left(G\times G\right)$ in $Env_\lambda\left(L^1\left(G,\ C_0\left(G\right)\right)\right)$.

So, since

$$\begin{aligned}&\Delta\left(t\right)^{\frac{1}{2}}\int_G\psi\left(g,\ ht\right)\varphi\left(g^{-1}ht\right)\ d\mu\left(g\right)\\=&\int_G\left(\rho\otimes Id\right)\left(t,\ \psi\right)\left(g,h\right)\tau\left(t\right)\varphi\left(g^{-1}h\right)d\mu\left(g\right)\end{aligned}$$

the mapping given by integration $\int_G\psi\left(g,h\right)\left(g^{-1}h\right)d\mu\left(g\right)$ defines an equivariant isomorphism

$$\left(Env_\lambda\left(L^1\left(G,\ C_0\left(G\right)\right)\right),\ G,\ \rho\otimes Id\right)\to\left(LK\left(L^2\left(G\right)\right),\ G,\ Ad\left(\tau\right)\right),$$

where ρ is a right translation of the group G on itself.

Thus, we obtain the existence of the equivariant isomorphism

$$\tilde{\Upsilon}_5\ :\ Env_{\lambda\otimes Id}\left(L^1\left(G,\ C_0\left(G,\ \mathrm{A}\right)\right)\right)\to LK\left(L^2\left(G\right)\right)\otimes\mathrm{A},$$

statement 5 is proven so proof of the variant of the Takai duality theorem is completed. □

4. The general cross product C^*-algebra

Let $\gamma\ :\ G \to G$ be a continuous mapping, we define an enveloping C^*-algebra $Env_{\omega}{}^{\gamma}\left(L^p\left(G,\ \mathrm{A}\right)\right)$ as $T\left(L^p\left(G,\ \mathrm{A}\right)\right)/I$ where mapping I is the two-sided ideal generated by elements

$$\Psi_1 \otimes \Psi_2 - \Psi_2 \otimes \Psi_1 - \Psi_1 \odot_{\gamma} \Psi_2,$$

where a binary operation $\odot_{\gamma}$ is defined by

$$\begin{aligned}&\left(\Psi_1 \odot_{\gamma} \Psi_2\right)(g)\\ =&\int_G \omega\left(\gamma\left(g\right)^{-1}\gamma\left(h\right)\right)\Psi_1\left(h\right)\omega\left(\gamma\left(g\right)^{-1}h\gamma\left(h^{-1}g\right),\ \Psi_2\left(h^{-1}g\right)\right)d\mu\left(h\right).\end{aligned}$$

Thus, we generalized the **Takai duality theory** on γ-case as follows.

Theorem 4.1. (γ-variant of the Takai duality). *Let G be a locally compact Abelian group, let $\gamma\ :\ G \to G$ be a continuous mapping, and let $(\mathrm{A},\ G,\ \omega)$ be the dynamic system. Then, $Env_{\hat{\omega}}{}^{\gamma}\left(L^1\left(\hat{G},\ Env_{\omega}{}^{\gamma}\left(L^1\left(G,\ \mathrm{A}\right)\right)\right)\right)$ isomorphically equals $\mathrm{A} \otimes LK\left(L^2\left(G\right)\right)$, so there exists such isomorphism*

$$\Upsilon\ :\ Env_{\hat{\omega}}{}^{\gamma}\left(L^1\left(\hat{G},\ Env_{\omega}{}^{\gamma}\left(L^1\left(G,\ \mathrm{A}\right)\right)\right)\right) \to \mathrm{A} \otimes LK\left(L^2\left(G\right)\right),$$

which is equivariant for the double dual action

$$\hat{\hat{\omega}}\ :\ G \to Aut\left(Env_{\hat{\omega}}{}^{\gamma}\left(L^1\left(\hat{G},\ Env_{\omega}{}^{\gamma}\left(L^1\left(G,\ \mathrm{A}\right)\right)\right)\right)\right).$$

The proof is similar to the previous theorem.

5. Conclusions

This paper dedicated to dynamical systems and C^*-algebras. We establish that the enveloping C^*-algebra $Env_{\hat{\omega}}{}^{\gamma}\left(L^1\left(\hat{G},\ Env_{\omega}{}^{\gamma}\left(L^1\left(G,\ \mathrm{A}\right)\right)\right)\right)$ with a pointwise convergence topology is isomorphically identical to maximal product $\mathrm{A} \otimes LK\left(L^2\left(G\right)\right)$. In our future works, we will generalize this statement to include the classes of non-abelian groups G and wide class functions $\gamma\ :\ G \to G$, we also plan to develop a new approach to its application to symmetry in quantum mechanics.

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