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ON GROUND STATE OF FRACTIONAL *P*-KIRCHHOFF EQUATION INVOLVING SUBCRITICAL AND CRITICAL EXPONENTIAL GROWTH*

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Abstract In this paper, we concern the existence of nontrivial ground state solutions of fractional p-Kirchhoff equation

$$\begin{cases} m\left(\|u\|^p\right)\left[(-\Delta)_p^s u+V(x)|u|^{p-2}u\right]=f(x,u) \quad \text{in}\,\mathbb{R}^N, \\ \|u\|=\left(\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy+\int_{\mathbb{R}^N}V(x)|u|^p dx\right)^{\frac{1}{p}}, \end{cases}$$

where $m:[0,+\infty)\to[0,+\infty)$ is a continuous function, $(-\Delta)_p^s$ is the fractional p-Laplacian operator with $0<s<1<p<\infty$, $V:\mathbb{R}^N\to[0,+\infty)$ is a continuous and 1 periodic function and $f\in C(\mathbb{R}^N\times\mathbb{R})$ is 1-periodic in $x_1,\cdots,x_N$. When the nonlinearity $f(x,u)$ has subcritical or critical exponential growth at ∞ without satisfying the Ambrosetti-Rabinowitz (AR) condition some existence results for nontrivial ground state solutions are obtained by using the minimax techniques, Nehari manifold methods combined with the fractional Moser-Trudinger inequality.

Keywords Fractional p-Kirchhoff equation, ground state, critical exponential growth, variational methods.

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1. Introduction

Let $s\in(0,1)$ and $N\geq 2$. In this paper we consider the following fractional Schrödinger-Kirchhoff type equation

$$m\left(\|u\|^p\right)\left[(-\Delta)_p^s u+V(x)|u|^{p-2}u\right]=f(x,u) \quad \text{in}\,\mathbb{R}^N, \tag{1.1}$$

where

$$\|u\|=\left([u]_{s,p}^p+\int_{\mathbb{R}^N}V(x)|u|^p dx\right)^{\frac{1}{p}},\ [u]_{s,p}=\left(\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy\right)^{\frac{1}{p}}, \tag{1.2}$$

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$m:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous function, $V:\mathbb{R}^N\to\mathbb{R}^+$ is a periodic potential, $1<p<+\infty$, $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is continuous and periodic in $x_1,\cdots,x_N$, and $(-\Delta)_p^s$ is the nonlinear nonlocal operator which is defined as

$$(-\Delta)_p^s u(x)=2\lim_{\epsilon\searrow 0}\int_{\mathbb{R}^N\setminus B_\epsilon(x)}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}dy,\quad x\in\mathbb{R}^N,$$

along functions $u\in C_0^\infty(\mathbb{R}^N)$, where $B_\epsilon(x)=\{z\in\mathbb{R}^N:|z-x|<\epsilon\}$ denotes the ball of $\mathbb{R}^N$ centered at $x\in\mathbb{R}^N$ and with radius $\epsilon>0$.

Recently, equations of the type (1.1) are receiving a lot of attentions, since they have wide applications in many fields of science, notably in continuum mechanics, phase transition phenomena, population dynamics, minimal surfaces, and anomalous diffusion, as they are typical outcomes of stochastic stabilization of Lévy processes, see [10,11] and the references therein. In the case of $ps<N$, in the paper [8], the authors first gave a detailed discussion on the physical meaning underling the fractional Kirchhoff model in bounded domain and their applications. In [20], Xiang et al. dealt with the following class of particular elliptic problem of Kirchhoff type involving convex-concave nonlinearities:

$$\begin{aligned}&\left[a+b\left(\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|}{|x-y|^{N+sp}}dxdy\right)^{\theta-1}\right](-\Delta)_p^s u\\&=\lambda\omega_1(x)|u|^{q-2}u+\omega_2(x)|u|^{r-2}u+h(x)\quad\text{in }\mathbb{R}^N,\end{aligned}\tag{1.3}$$

where $a,b\geq 0$, $\lambda>0$ is a real parameter, $0<s<1<p<\infty$ with $sp<N$, $1<q<p\leq\theta p<r<\frac{Np}{N-sp}$, ω_1,ω_2,h are functions which may change sign in $\mathbb{R}^N$. Under some suitable assumptions, they obtained the existence of two nontrivial solutions by using the Ekeland's variational principle and mountain pass theorem. A novelty of that paper is that a may be zero, which means that the above-mentioned problem is degenerate. In the case of non-degeneration, Pucci et al. in [16] did similar work for problem (1.3). In [17], Pucci et al. investigated the existence of entire solutions of the following stationary Kirchhoff type equations driven by the fractional p-Laplacian operator in $\mathbb{R}^N$:

$$m\left([u]_{s,p}^p\right)(-\Delta)_p^s u+V(x)|u|^{p-2}u=\lambda\omega(x)|u|^{q-2}u-h(x)|u|^{r-2}u,\tag{1.4}$$

where $ps<N$ with $0<s<1$, $m:\mathbb{R}^+\to\mathbb{R}^+$ is assumed to be continuous, $1<q<r<\infty$, the weight functions V has positive lower bound, h,ω is nonnegative and locally integrable in $\mathbb{R}^N$. They proved multiplicity results for problem (1.4) by using variational methods and topological degree theory and also obtained the existence of infinitely many pairs of entire solutions. A distinguished feature of that paper is still that $m(0)$ may be zero, which means that problem (1.4) is degenerate. On the study of critical fractional Kirchhoff type problem in the whole space, we refer to [3,15,22].

To the best of our knowledge, most of the works mentioned above on fractional Kirchhoff type problems involve the nonlinear terms satisfying polynomial growth, there are only a few paper (see [14,21]) dealing with the nonlinearity with exponential growth. Furthermore, we find that these papers mainly investigate the existence of nontrivial solutions and seldom consider the existence of ground state solutions.

Inspired by the above works and a very recent paper [23] devoted to the fractional Moser-Trudinger inequality, the purpose of this paper is to establish existence results of ground state solutions for problem (1.1) with $1 < p \leq \frac{N}{s}$ when the nonlinearity $f(x,.)$ has subcritical or critical exponential growth (improved subcritical polynomial growth) at ∞ and does not satisfy the classical (AR)-condition.

Let us now introduce our results: Suppose that the potential $V(x)$, the nonlinearity $f(x,u)$ and nonlocal function $m(t)$ respectively satisfy:

(V_1) $V \in C(\mathbb{R}^N)$ is 1-periodic in $x_1, \cdots, x_N$ and $V(x) > 0$.

(H_1) $f \in C(\mathbb{R}^N \times \mathbb{R})$ is 1-periodic in $x_1, \cdots, x_N$.

(H_2) $\lim\limits_{t\to\infty} \frac{f(x,t)}{|t|^{2p-2}t} = +\infty$ uniformly for $x \in \mathbb{R}^N$.

(H_3) For each $x \in \mathbb{R}^N$, $\frac{f(x,t)}{|t|^{2p-1}}$ is strict increasing for $t \in \mathbb{R}$ and $t \neq 0$.

(H_4) There exists $C_0 > 0$ such that for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$:

$$F(x,t) \leq C_0|t|^{2p} + C_0|f(x,t)|,$$

where $F(x,t) = \int_0^t f(x,s)ds$.

(M_1) there exists $m_0 > 0$ such that $m(t) \geq m_0$ for all $t \geq 0$ and $m(t)$ is increasing for $t \geq 0$.

(M_2) there exist constants $a_1, a_2 > 0$ and $S_0 > 0$ such that for some $\gamma \leq 1$:

$$m(t) \leq a_1 + a_2 t^\gamma, \quad \forall t \geq S_0.$$

(M_3) $\frac{m(t)}{t}$ is non-increasing for $t > 0$.

A typical example of a function m is given by

$$m(t) = a^* + b^* t,$$

where $a^*, b^* > 0$.

Our first main result will be to study problem (1.1) in the improved subcritical polynomial growth

$$(\text{SCPI}): \lim_{t\to\infty} \frac{f(x,t)}{|t|^{p^*-1}} = 0$$

uniformly for all $x \in \mathbb{R}^N$, where $p^* = \frac{Np}{N-sp}$. The condition was first introduced by Liu and Wang [13]. Note that in this case, Sobolev compact embedding theorem does not hold owing to the unboundedness of the domain. Our work is to study problem (1.1) where the nonlinearity f does not satisfy the (AR)-condition:

There exists $\theta > 2p$ such that

$$0 < \theta F(x,t) \leq t f(x,t), \text{ for any } x \in \mathbb{R}^N, \ t \neq 0.$$

By using the mountain pass theorem and its suitable version combined with Nehari manifold methods, we try to get the ground state solutions to problem (1.1) with $1 < p < \frac{N}{s}$.

Theorem 1.1. *Let $1 < p < \frac{N}{s}$ and assume that* (V_1), (H_1)-(H_3) *and* (M_1)-(M_3) *hold, if f has the improved subcritical polynomial growth on $\mathbb{R}^N$ (condition* (SCPI)*), then problem* (1.1) *has a ground state, i.e. a nontrivial solution u_* such that*

$$\mathcal{J}(u_*) = \inf\{\mathcal{J}(u) : u \neq 0 \text{ and } \langle \mathcal{J}'(u), u\rangle = 0\},$$

where definition of the functional $\mathcal{J}$ appears in the next section.

Remark 1.1. To our knowledge, problem (1.1) is rarely considered by other people when the nonlinearity f has a polynomial critical growth $f(x,t) \sim |t|^{p^*-1}$. Hence, our result is interesting since we considered the case where the nonlinearity f has slightly critical growth at infinity.

Now, we are interested in a borderline case of the Sobolev imbedding theorems, commonly known as Moser-Trudinger case, i.e., $ps = N$. For our purpose, we have to introduce a useful work as follows:

Proposition 1.1. (see [23]) *Let $s \in (0,1)$ and $ps = N$. Let $W^{s,p}(\mathbb{R}^N)$ be the space defined as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm*

$$u \mapsto \left(\|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{W^{s,p}(\mathbb{R}^N)}\right)^{\frac{1}{p}} = \|u\|_{W^{s,p}(\mathbb{R}^N)},$$

where

$$[u]_{W^{s,p}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy\right)^{\frac{1}{p}}.$$

Then there exists $\alpha_* > 0$ such that

$$\sup\left\{\int_{\mathbb{R}^N} \Phi_{N,s}\left(\alpha|u|^{\frac{N}{N-s}}\right) dx \mid u \in W^{s,p}(\mathbb{R}^N), \|u\|_{W^{s,p}(\mathbb{R}^N)} \leq 1\right\} < +\infty$$

for $\alpha \in [0, \alpha_*]$ and

$$\sup\left\{\int_{\mathbb{R}^N} \Phi_{N,s}\left(\alpha|u|^{\frac{N}{N-s}}\right) dx \mid u \in W^{s,p}(\mathbb{R}^N), \|u\|_{W^{s,p}(\mathbb{R}^N)} \leq 1\right\} = +\infty$$

for $\alpha \in (\alpha^*_{s,N}, \infty)$, where $\Phi_{N,s}(t) = e^t - \sum_{i=0}^{j_p-2} \frac{t^i}{i!}$, $j_p := \min\{j \in \mathbf{N} : j \geq p\}$ and

$$\alpha^*_{s,N} := N\left(\frac{2(N\omega_N)^2\Gamma(p+1)}{N!} \sum_{k=0}^{\infty} \frac{(N+k-1)!}{k!} \frac{1}{(N+2k)^p}\right)^{\frac{s}{N-s}}.$$

Here $\alpha^*_{s,N} \geq \alpha_*$. When $N = 1$, we also refer the reader to [18].

Next, we define the subcritical exponential growth and the critical exponential growth as follows:

(SCE): f has subcritical exponential growth on $\mathbb{R}^N$, i.e., $\lim_{t\to\infty} \frac{|f(x,t)|}{\exp\left(\alpha|t|^{\frac{N}{N-s}}\right)} = 0$ uniformly on $x \in \mathbb{R}^N$ for all $\alpha > 0$.

(CG): f has critical exponential growth on $\mathbb{R}^N$, i.e., there exists $\alpha_0 > 0$ such that

$$\lim_{t\to\infty} \frac{|f(x,t)|}{\exp\left(\alpha|t|^{\frac{N}{N-s}}\right)} = 0, \text{ uniformly on } x \in \mathbb{R}^N,\ \forall \alpha > \alpha_0$$

and

$$\lim_{t\to\infty}\frac{|f(x,t)|}{\exp\left(\alpha|t|^{\frac{N}{N-s}}\right)}=+\infty,\ \text{uniformly on}\, x\in\mathbb{R}^N,\ \forall\alpha<\alpha_0.$$

When $ps=N$ and f has the subcritical exponential growth (SCE), our work is still to study problem (1.1) where the nonlinearity f does not satisfy the (AR)-condition ($\theta>2p$) at infinity. To our knowledge, there are few works to study this problem for fractional p-Kirchhoff equation defined in the whole space. Hence, our result is new and our methods are more technical since we successfully used the above fractional Moser-Trudinger inequality established in whole space. Our result is as follows:

Theorem 1.2. *Let $ps=N$ and assume that* (V_1), (H_1)-(H_3) *and* (M_1)-(M_3) *hold, if f has the subcritical exponential growth on $\mathbb{R}^N$ (condition* (SCE)*), then problem* (1.1) *has a ground state solution.*

Remark 1.2. According to the condition (SCE), problem (1.1) is called subcritical exponential Kirchhoff-type problem defined in the whole space. It seems that there are few works to study the existence of ground state solutions of this problem at the present time. Hence, our result is new and interesting.

The study of problem (1.1) is more difficult than in the case of subcritical exponential growth when $ps=N$ and f has the critical exponential growth (CG) since our Euler-Lagrange functional does not satisfy the compactness condition at all level anymore. This point is completely similar to the case of the critical polynomial growth in $\mathbb{R}^N$ ($N\geq 3$) for the standard Laplacian studied by Brezis and Nirenberg in their pioneering work [2]. For the standard Laplacian problem, the authors [5,6] applied the extremal function sequences corresponded to Moser-Trudinger inequality to regain the compactness of Euler-Lagrange functional at some suitable level. The idea of choosing the testing functions firstly appeared in [2]. Here, by choosing particular testing functions for estimating some mountain pass level, we still study problem (1.1). Our result is as follows:

Theorem 1.3. *Let $sp=N$ and assume that* (V_1), (H_1)-(H_4) *and* (M_1)-(M_3) *hold. Furthermore, assume that*

$$(H_5)\ \lim_{u\to\infty} f(x,u)\exp(-\alpha_0\frac{\alpha^*_{s,N}}{\alpha_*}|u|^{\frac{N}{N-s}})u\geq\beta>m\left(\left[\frac{\alpha_*}{\alpha_0}\right]^{\frac{N-s}{s}}\right)\left[\frac{\alpha_*}{\alpha_0}\right]^{\frac{N-s}{s}}/(\omega_N\mathcal{M}),$$

uniformly in $(x,u)\in\mathbb{R}^N\times\mathbb{R}$, where ω_N denotes the volume of unit ball in $\mathbb{R}^N$ and

$$\mathcal{M}=\lim_{n\to\infty}N\ln n\int_0^1\exp\left(Nt^{\frac{N}{N-s}}\ln n-tN\ln n\right)dt.$$

If f has the critical exponential growth on $\mathbb{R}^N$ (condition (CG)*), then problem* (1.1) *has a ground state solution.*

Remark 1.3. Since $Nt^{\frac{N}{N-s}}-Nt\geq -Nt$ for $0\leq t\leq 1$, we get

$$\mathcal{M}\geq 1.$$

Remark 1.4. For fractional Kirchhoff problems with critical Trudinger-Moser nonlinearity defined in bounded domain, Xiang et al. [19] have recently obtained the existence of a ground state solution with positive energy by using minimax techniques combined with the fractional Moser-Trudinger inequality. However, for fractional p-

Kirchhoff problem (1.1) with periodic potential defined in the whole space involving critical exponential growth, there are few works to consider it. Hence our result is new and interesting.

The paper is arranged as follows. In Section 2, we introduce some necessary preliminary knowledge for working space. In Section 3, we prove some lemmas. In Section 4, we give the proofs of our main results.

2. Preliminaries

In this section, we introduce some preliminary knowledge which will be used in the sequel.

We first state the variation setting for problem (1.1). Let $1 \leq r \leq \infty$ and denote by $|\cdot|_r$ the norm of $L^r(\mathbb{R}^N)$. Let $0 < s < 1 < p < \infty$ be real numbers and define the fractional Sobolev space

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\},$$

equipped with the following norm

$$\|u\|_{s,p} = \left(|u|_p^p + [u]_{s,p}^p\right)^{\frac{1}{p}}.$$

The Gagliardo seminorm is defined for all measurable function $u : \mathbb{R}^N \to \mathbb{R}$ by

$$[u]_{s,p} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy\right)^{\frac{1}{p}}.$$

It is easy verify that $W^{s,p}(\mathbb{R}^N)$ is uniformly convex Banach space. The fractional Sobolev critical exponent is defined by

$$p^* = \begin{cases} \dfrac{Np}{N - sp}, & \text{if } sp < N, \\ \infty, & \text{if } sp \geq N. \end{cases}$$

Moreover, the fractional Sobolev space $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^\gamma(\mathbb{R}^N)$ is continuous for $p \leq \gamma \leq p^*$ if $sp < N$, and $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^\gamma(\mathbb{R}^N)$ is continuous for $p \leq \gamma < \infty$ if $sp = N$. For a detailed account on the properties of $W^{s,p}(\mathbb{R}^N)$ we refer the reader to [7].

For simplicity, from now on we denote $X = W^{s,p}(\mathbb{R}^N)$. Using condition ($V_1$), we can introduce a new norm $\|\cdot\|$ on $W^{s,p}(\mathbb{R}^N)$ as follow

$$\|u\| = \left(\int_{\mathbb{R}^N} V(x)|u|^p dx + [u]_{s,p}^p\right)^{\frac{1}{p}}, \ u \in X.$$

It is well known that the new norm $\|\cdot\|$ is equivalent to the standard $\|u\|_{s,p}$ in X. We rephrase variationally the fractional p-Laplacian as the nonlinear operator $A : X \to X^*$ defined for all $u, \varphi \in X$ by

$$\langle A(u), \varphi\rangle = m(\|u\|^p)\Big(\int_{\mathbb{R}^{2N}} \left[\frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}}\right] dxdy$$

$$+\int_{\mathbb{R}^N} V(x)|u(x)|^{p-2}u\varphi dx\big).$$

Thus, a weak solution of problem (1.1) is a function $u \in X$ such that

$$\langle A(u), \varphi\rangle = \int_{\mathbb{R}^N} f(x,u)\varphi dx \tag{2.1}$$

for all $\varphi \in X$. From condition (SCPI)(or (CG)), it is easy to know that (2.1) is the Euler-Lagrange equation of the functional

$$\mathcal{J}(u) = \frac{1}{p}M(\|u\|^p) - \int_{\mathbb{R}^N} F(x,u)dx,$$

where $M(t) = \int_0^t m(s)ds$. Next, we recall some definitions for compactness condition and a version of mountain pass theorem.

Definition 2.1. Let ($X, \|\cdot\|_X$) be a real Banach space with its dual space $(X^*, \|\cdot\|_{X^*})$ and $\mathcal{J} \in C^1(X,\mathbb{R})$. For $c \in \mathbb{R}$, we say that $\mathcal{J}$ satisfies the $(\mathrm{C})_{\mathrm{c}}$ condition if for any sequence $\{x_n\} \subset X$ with

$$\mathcal{J}(x_n) \to c, \quad \|\mathcal{J}'(x_n)\|_{X^*}(1+\|x_n\|_X) \to 0,$$

there is a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges strongly in X.

We have the following version of the mountain pass theorem (see [1, 4]):

Proposition 2.1. *Let X be a real Banach space and suppose that $\mathcal{J} \in C^1(X,\mathbb{R})$ satisfies the condition*

$$\max\{\mathcal{J}(0), \mathcal{J}(u_1)\} \le \alpha < \beta \le \inf_{\|u\|=\rho} \mathcal{J}(u),$$

for some $\alpha < \beta$, $\rho > 0$ and $u_1 \in X$ with $\|u_1\| > \rho$. Let $c \ge \beta$ be characterized by

$$c = \inf_{\gamma\in\Gamma}\max_{0\le t\le 1} \mathcal{J}(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1],X), \gamma(0) = 0, \gamma(1) = u_1\}$ is the set of continuous paths joining 0 and u_1. Then, there exists a $(\mathrm{C})_{\mathrm{c}}$ sequence $\{u_n\} \subset X$ such that

$$\mathcal{J}(u_n) \to c \ge \beta \ \text{ and } \ (1+\|u_n\|)\|\mathcal{J}'(u_n)\|_{X^*} \to 0 \ \text{ as } \ n \to \infty.$$

3. Some lemmas

Lemma 3.1. *Let $1 < p < \frac{N}{s}$ and assume that* (M_1), (M_2), (H_1)-(H_3) *and* (SCPI) *hold. Then:*

(i) *There exist $\rho, \alpha > 0$ such that $\mathcal{J}(u) \ge \alpha$ for all $u \in X$ with $\| u \|= \rho$,*
(ii) *There exists $\phi \in X$ such that $\mathcal{J}(t\phi) \to -\infty$ as $t \to +\infty$.*

Proof. By (SCPI) and (H_1)-(H_3), for any $\epsilon > 0$, there exist $A_1 = A_1(\epsilon)$, $B_1 = B_1(\epsilon)$, $p^* > p_1 > p$ and M large enough such that for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$,

$$F(x,t) \le \frac{\epsilon}{p_1}|t|^{p_1} + A_1|t|^{p^*}, \tag{3.1}$$

$$F(x,t) \geq \frac{M}{2p}|t|^{2p} - B_1. \tag{3.2}$$

Using the Sobolev inequality: $|u|_{p_1}^{p_1} \leq K_{p_1}\|u\|^{p_1}$, and $|u|_{p^*}^{p^*} \leq K_{p^*}\|u\|^{p^*}$, from (3.1), we obtain

$$\begin{aligned}\mathcal{J}(u) &\geq \frac{m_0}{p}\|u\|^p - \frac{\epsilon}{p_1}|u|_{p_1}^{p_1} - A_1|u|_{p^*}^{p^*}\\ &\geq \frac{1}{p}m_0\|u\|^p - \frac{K_{p_1}\epsilon}{p_1}\|u\|^{p_1} - A_1K_{p^*}\|u\|^{p^*}.\end{aligned}$$

So, part (i) holds if we choose $\epsilon > 0$ and $||u|| = \rho > 0$ small enough.

On the other hand, for all $t > S_0$, condition (M_2) means that

$$M(t) \leq \begin{cases} a_0 + a_1t + \dfrac{a_2}{\gamma+1}t^{\gamma+1}, & \text{if } \gamma \neq -1,\\ b_0 + a_1t + a_2\ln t, & \text{if } \gamma = -1, \end{cases} \tag{3.3}$$

where $a_0 = M(S_0) - (a_1S_0 + \frac{a_2}{\gamma+1}S_0^{\gamma+1})$ and $b_0 = M(S_0) - (a_1S_0 + a_2\ln(S_0))$ directly obtained by integration. Now, choosing nonzero $\phi \in C_0^\infty(\mathbb{R}^N)$ with $\|\phi\| = 1$, by (3.2) and (3.3), we get

$$\mathcal{J}(t\phi) \leq \begin{cases} \dfrac{a_0}{p} + \dfrac{a_1}{p}t^p + \dfrac{a_2}{p\gamma+p}t^{p\gamma+p} - \dfrac{M}{2p}t^{2p}|\phi|_{2p}^{2p} + B_1|\text{supp }\phi|, & \text{if } \gamma \neq -1,\\ \dfrac{b_0}{p} + \dfrac{a_1}{p}t^p + \dfrac{a_2\ln t}{p} - \dfrac{M}{2p}t^{2p}|\phi|_{2p}^{2p} + B_1|\text{supp }\phi|, & \text{if } \gamma = -1 \end{cases} \tag{3.4}$$

for all $t > S_0$, where $|\text{supp }\phi|$ denotes the volume of supp ϕ and M large enough. Provided $\gamma \leq 1$, then from (3.4), $\mathcal{J}(t\phi) \to -\infty$, as $t \to +\infty$. Thus part (ii) holds. □

Lemma 3.2. *Let $ps = N$ and assume that* (M_1), (M_2), (H_1)-(H_3) *and* (SCE) *(or* (CG)*) hold. Then:*

(i) *There exist $\rho, \alpha > 0$ such that $\mathcal{J}(u) \geq \alpha$ for all $u \in X$ with $\| u \| = \rho$,*
(ii) *There exists $\phi \in X$ such that $\mathcal{J}(t\phi) \to -\infty$ as $t \to +\infty$.*

Proof. By (SCE) (or (CG)) and (H_1)-(H_3), for any $\epsilon > 0$, there exist $A_1^* = A_1^*(\epsilon)$, M large enough, $B_1^* = B_1^*(\epsilon)$, $p_2 > \frac{N}{s}$, $\kappa > 0$ and $q_2 > p_2$ such that for all $(x,s) \in \mathbb{R}^N \times \mathbb{R}$,

$$F(x,s) \leq \frac{1}{p_2}\epsilon|s|^{p_2} + A_1^*\Phi_{N,s}\left(\kappa|s|^{\frac{N}{N-s}}\right)|s|^{q_2}, \tag{3.5}$$

$$F(x,s) \geq \frac{sM}{2N}|s|^{\frac{2N}{s}} - B_1^*. \tag{3.6}$$

From (3.5), the Hölder inequality, the fractional Moser-Trudinger embedding inequality (see proposition 1.1) and the Sobolev embedding inequalities we obtain

$$\begin{aligned}\mathcal{J}(u) &\geq \frac{m_0s}{N}\|u\|^{\frac{N}{s}} - \frac{\epsilon}{p_2}|u|_{p_2}^{p_2} - A_1\int_{\mathbb{R}^N}\Phi_{N,s}\left(\kappa|u|^{\frac{N}{N-s}}\right)|u|^{q_2}dx\\ &\geq \frac{sm_0}{N}\|u\|^{\frac{N}{s}} - \frac{K_{p_2}\epsilon}{p_2}\|u\|^{p_2}\end{aligned}$$

$$
\begin{aligned}
&-A_1\left(\int_{\mathbb{R}^N}\Phi_{N,s}\left(\kappa r\|u\|^{\frac{N}{N-s}}\left(\frac{|u|}{\|u\|}\right)^{\frac{N}{N-s}}\right)dx\right)^{\frac{1}{r}}\left(\int_{\mathbb{R}^N}|u|^{r'q_2}dx\right)^{\frac{1}{r'}}\\
&\geq\frac{sm_0}{N}\|u\|^{\frac{N}{s}}-\frac{K_{p_2}\epsilon}{p_2}\|u\|^{p_2}-C\|u\|^{q_2},
\end{aligned}
$$

where $r>1$ sufficiently close to 1, $\|u\|\leq\sigma$ and $\kappa r\sigma^{\frac{N}{N-s}}\leq\alpha_*$. So, part (i) holds if we choose $\epsilon>0$ and $\|u\|=\rho>0$ small enough.

On the other hand, by using (3.3) and (3.6), similar to the proof of (ii) of Lemma 3.1, we have

$$
\mathcal{J}(t\phi)\to-\infty,\quad \text{as } t\to+\infty,
$$

where $\phi\in C_0^\infty(\mathbb{R}^N)$ with $\|\phi\|=1$. Thus part (ii) holds. □

Lemma 3.3. *Let $1<p<\frac{N}{s}$ and assume that* (V_1), (M_1)-(M_3) *and* (H_1)-(H_3) *hold. If f has the improved subcritical polynomial growth on $\mathbb{R}^N$ (condition* (SCPI)*), then any* $(\mathrm{C})_{\mathrm{c}}$ *sequence of $\mathcal{J}$ is bounded.*

Proof. Let $\{u_n\}\subset X$ be a $(\mathrm{C})_c$ sequence such that

$$
\frac{M(\|u_n\|^p)}{p}-\int_{\mathbb{R}^N}F(x,u_n)dx\to c, \tag{3.7}
$$

$$
(1+\|u_n\|)\left(\langle A(u_n),\varphi\rangle-\int_{\mathbb{R}^N}f(x,u_n)\varphi dx\right)=o(1)\|\varphi\|,\quad \varphi\in X. \tag{3.8}
$$

By the contradiction, assume that $\|u_n\|\to\infty$ and set

$$
v_n=\frac{u_n}{\|u_n\|}.
$$

Then $\{v_n\}$ is bounded. Without loss of generality, we may assume that $\{v_n\}$ converges weakly to v in X, local converges strongly in $L^p(\mathbb{R}^N)$ and converges v a.e. $x\in\mathbb{R}^N$. Now, we will show that $v=0$. Dividing both sides of (3.7) by $\|u_n\|^{2p}$, from (M_2), we get

$$
\limsup_{n\to\infty}\int_{\mathbb{R}^N}\frac{F(x,u_n)}{\|u_n\|^{2p}}dx\leq\frac{a_2}{p(\gamma+1)}. \tag{3.9}
$$

Set

$$
\mathcal{A}=\{x\in\mathbb{R}^N: v(x)\neq 0\}.
$$

Using Hôpital's rule, it follows from (H_2) that

$$
\lim_{t\to\infty}\frac{F(x,t)}{|t|^{2p}}\to+\infty. \tag{3.10}
$$

This means

$$
\frac{F(x,u_n)}{|u_n|^{2p}}|v_n|^{2p}\to\infty,\quad x\in\mathcal{A}.
$$

If $|\mathcal{A}|$ is positive, then from (H_3), we have

$$
F(x,t)\geq 0,\quad \forall(x,t)\in\mathbb{R}^N\times\mathbb{R},
$$

and

$$
\int_{\mathbb{R}^N}\frac{F(x,u_n)}{\|u_n\|^{2p}}dx\geq\int_{\mathcal{A}}\frac{F(x,u_n)}{|u_n|^{2p}}|v_n|^{2p}dx\to\infty,
$$

which contradicts with (3.9).

By conditions (V_1), (H_1) and $v = 0$, we can prove that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_2(y)} |v_n|^p dx = 0. \tag{3.11}$$

Owing to (3.11), applying the Lions' Lemma [12, Lemma 1.1] we obtain

$$v_n \to 0 \quad \text{in}\, L^\gamma(\mathbb{R}^N), \quad \forall\, \gamma \in (p, p^*). \tag{3.12}$$

Let $t_n \in [0,1]$ such that

$$\mathcal{J}(t_n u_n) = \max_{t\in[0,1]} \mathcal{J}(tu_n).$$

For any given $R > 0$, by (H_1), (H_3) and (SCPI), there exist $C_1 > 0$ and $p^* > p_3 > p$ such that

$$F(x,t) \le C_1 |t|^{p_3} + \frac{1}{R^{p^*}} |t|^{p^*}, \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{3.13}$$

Also since $\|u_n\| \to \infty$, we get for n large enough:

$$\mathcal{J}(t_n u_n) \ge \mathcal{J}(\frac{R}{\|u_n\|} u_n) = \mathcal{J}(Rv_n). \tag{3.14}$$

Thus, from (3.13), and noting that $\|v_n\| = 1$, we have

$$\mathcal{J}(Rv_n) \ge m_0 R^p - C_1 R^{p_3} \int_{\mathbb{R}^N} |v_n(x)|^{p_3} dx - \frac{1}{R^{p^*}} \int_{\mathbb{R}^N} |Rv_n|^{p^*} dx. \tag{3.15}$$

Hence, from (3.12), for large enough R in above formula we get

$$\mathcal{J}(t_n u_n) \to \infty. \tag{3.16}$$

Noting that $\mathcal{J}(0) = 0$ and $\mathcal{J}(u_n) \to c$, we can suppose that $t_n \in (0,1)$. So from $\langle \mathcal{J}'(t_n u_n), t_n u_n \rangle = 0$, we have

$$m(t_n^p \|u_n\|^p) t_n^p \|u_n\|^p - \int_{\mathbb{R}^N} f(x, t_n u_n) t_n u_n dx = 0. \tag{3.17}$$

On the other hand, from (3.7) and (3.8), we imply that

$$\frac{1}{p} M(\|u_n\|^p) - \frac{1}{2p} m(\|u_n\|^p)\|u_n\|^p + \frac{1}{2p} \int_{\mathbb{R}^N} (f(x,u_n)u_n - 2pF(x,u_n))dx \to c. \tag{3.18}$$

Now, from (H_3) and (M_3), we get respectively

$$tf(x,t) - 2pF(x,t) \;\; \text{is strict increasing for}\, t > 0 \text{ and strict decreasing for}\, t < 0$$

and

$$\frac{1}{2} M(t) - \frac{1}{4} m(t)t \quad \text{is nondecreasing for } t \ge 0.$$

Thus, we have

$$\mathcal{J}(t_n u_n) = \frac{1}{p} M(\|t_n u_n\|^p) - \int_{\mathbb{R}^N} F(x, t_n u_n) dx$$

$$
\begin{aligned}
&= \frac{1}{p}M(\|t_nu_n\|^p) - \frac{1}{2p}m(\|t_nu_n\|^p) \\
&\quad + \frac{1}{2p}\int_{\mathbb{R}^N}[f(x,t_nu_n)t_nu_n - 2pF(x,t_nu_n)]dx \\
&\le \frac{1}{p}M(\|u_n\|^p) - \frac{1}{2p}m(\|u_n\|^p) + \frac{1}{2p}\int_{\mathbb{R}^N}[f(x,u_n)u_n - 2pF(x,u_n)]dx \\
&\le M,
\end{aligned}
$$

where M is a positive constant and the last inequality is followed by (3.18). This leads to a contradiction. □

Lemma 3.4. *Let $ps = N$ and assume that* (V_1), (M_1)-(M_3) *and* (H_1)-(H_3) *hold. If f has the subcritical exponential growth on $\mathbb{R}^N$ (condition* (SCE)*), then any* $(C)_c$ *sequence of $\mathcal{J}$ is bounded.*

Proof. Let $\{u_n\} \subset X$ be a $(C)_c$ sequence satisfying (3.7) and (3.8). Argue by the contradiction that $\|u_n\| \to \infty$ and set $v_n = \frac{u_n}{\|u_n\|}$. We can suppose that $v_n \rightharpoonup v$ in X. According to the previous section of proof of Lemma 3.3, we may similarly show that $v_n \rightharpoonup 0$.

Using conditions (V_1), (H_1) and $v = 0$, we can prove that

$$
\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_2(y)}|v_n|^p dx = 0. \tag{3.19}
$$

Owing to (3.19), applying the Lions' Lemma [12, Lemma 1.1] we obtain

$$
v_n \to 0 \quad \text{in}\, L^\gamma(\mathbb{R}^N), \quad \forall\, \gamma \in \left(\frac{N}{s}, +\infty\right). \tag{3.20}
$$

Again let $t_n \in [0,1]$ such that

$$
\mathcal{J}(t_nu_n) = \max_{t\in[0,1]}\mathcal{J}(tu_n).
$$

For any given $R > 0$, by (H_1), (H_3) and (SCE), there exists $C_2 > 0$ such that

$$
F(x,t) \le C_2|t|^{2p} + \Phi_{N,s}\left(\frac{\alpha_*}{2R^{\frac{N}{N-s}}}|t|^{\frac{N}{N-s}}\right), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.21}
$$

Also since $\|u_n\| \to \infty$, we get for n large enough:

$$
\mathcal{J}(t_nu_n) \ge \mathcal{J}(\frac{R}{\|u_n\|}u_n) = \mathcal{J}(Rv_n). \tag{3.22}
$$

Thus, from (3.21), and noting that $\|v_n\| = 1$, we have

$$
\mathcal{J}(Rv_n) \ge m_0R^p - C_2R^{2p}\int_{\mathbb{R}^N}|v_n(x)|^{2p}dx - \int_{\mathbb{R}^N}\Phi_{N,s}\left(\frac{\alpha_*}{2}|v_n|^{\frac{N}{N-s}}\right)dx. \tag{3.23}
$$

Hence, from (3.20), for large enough R in above formula we get

$$
\mathcal{J}(t_nu_n) \to \infty. \tag{3.24}
$$

Remained proof is completely similar to the last section of proof of Lemma 3.3. We omit it here. □

Lemma 3.5. *Assume that conditions* (M_1)-(M_2), (H_2), (H_3) *and* (H_5) *hold. If* f *has the critical exponential growth on* $\mathbb{R}^N$ *(condition (CG)), then*

$$c^* < M\left(\left[\frac{\alpha_*}{\alpha_0}\right]^{\frac{N-s}{s}}\right)\frac{s}{N},$$

where c^* *is defined by*

$$c^* := \inf_{\gamma\in\Gamma}\max_{0\le t\le 1}\mathcal{J}(\gamma(t)),$$

where $\mathbf{\Gamma} = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \quad \mathcal{J}(\gamma(1)) < 0\}$.

Proof. In order to get a more precise information about the minimax level c^*, let us denote by B the unit ball and consider the following sequence of nonnegative functions:

$$u_\epsilon(x) = \begin{cases} |\ln\epsilon|^{\frac{N-s}{N}}, & \text{if } |x|\le\epsilon,\\ \dfrac{|\ln|x||}{|\ln\epsilon|^{\frac{\epsilon}{N}}}, & \text{if } \epsilon<|x|<1,\\ 0, & \text{if } |x|\ge 1.\end{cases}$$

Set

$$\omega_n(x) = \frac{u_{\epsilon_n}}{\|u_{\epsilon_n}\|},$$

where $\epsilon_n = \frac{1}{n}$. Since $\|\omega_n\| = 1$, as in the proof of Lemma 3.2, we have that $\mathcal{J}(t\omega_n) \to -\infty$. Consequently,

$$c^* \le \max_{t>0}\mathcal{J}(t\omega_n), \ \forall n\in\mathbf{N}.$$

Thus it suffices to show that $\max_{t>0}\mathcal{J}(t\omega_n) < M\left(\left[\frac{\alpha_*}{\alpha_0}\right]^{\frac{N-s}{s}}\right)\frac{s}{N}$ for some $n\in\mathbf{N}$. Suppose by contradiction that this is not the case. So, for all n, this maximum is larger or equal to $M\left(\left[\frac{\alpha_*}{\alpha_0}\right]^{\frac{N-s}{s}}\right)\frac{s}{N}$. Let $t_n > 0$ such that

$$\mathcal{J}(t_n\omega_n) = \max\{\mathcal{J}(t\omega_n) : t\ge 0\} \ge M\left(\left[\frac{\alpha_*}{\alpha_0}\right]^{\frac{N-s}{s}}\right)\frac{s}{N}. \tag{3.25}$$

It follows from (H_3), (M_1) and (3.25) that

$$t_n^{\frac{N}{s}} \ge \left[\frac{\alpha_*}{\alpha_0}\right]^{\frac{N-s}{s}}. \tag{3.26}$$

Also at $t = t_n$, we have

$$m(t_n^{\frac{N}{s}})t_n^{\frac{N}{s}-1} - \int_{\mathbb{R}^N} f(x, t_n\omega_n)\omega_n dx = 0,$$

which implies that

$$m(t_n^{\frac{N}{s}})t_n^{\frac{N}{s}} = \int_{\mathbb{R}^N} f(x, t_n\omega_n)t_n\omega_n dx. \tag{3.27}$$

Moreover, it follows from (H_5) that given $\epsilon > 0$ there exists $R_\epsilon > 0$ such that

$$tf(x,t) \ge (\beta-\epsilon)\exp\left(\alpha_0\frac{\alpha^*_{s,N}}{\alpha_*}|t|^{\frac{N}{N-s}}\right), \ \forall t\ge R_\epsilon.$$

So from (3.27), we have

$$\begin{aligned} m(t_n^{\frac{N}{s}})t_n^{\frac{N}{s}} &= \int_{B_1(0)} f(x, t_n\omega_n)t_n\omega_n dx \\ &\geq \int_{B_{\frac{1}{n}}(0)} f(x, \frac{t_n}{\|u_{\frac{1}{n}}\|}(\ln n)^{\frac{N-s}{N}})\frac{t_n}{\|u_{\frac{1}{n}}\|}(\ln n)^{\frac{N-s}{N}} dx \\ &\geq (\beta-\epsilon)\int_{B_{\frac{1}{n}}(0)} \exp(\alpha_0\frac{\alpha_{s,N}^*}{\alpha_*}t_n^{\frac{N}{N-s}}\frac{\ln n}{\|u_{\frac{1}{n}}\|^{\frac{N}{N-s}}})dx \text{ (for large } n) \\ &= (\beta-\epsilon)\omega_N(\frac{1}{n})^N \exp(\alpha_0\frac{\alpha_{s,N}^*}{\alpha_*}t_n^{\frac{N}{N-s}}\frac{\ln n}{\|u_{\frac{1}{n}}\|^{\frac{N}{N-s}}}). \end{aligned}$$

Thus, we imply that, for large n

$$m(t_n^{\frac{N}{s}})t_n^{\frac{N}{s}} \geq (\beta-\epsilon)\omega_N \exp\left[\left(\frac{\alpha_0\frac{\alpha_{s,N}^*}{\alpha_*}t_n^{\frac{N}{N-s}}}{N\|u_{\frac{1}{n}}\|^{\frac{N}{N-s}}} - 1\right)N\ln n\right], \tag{3.28}$$

where ω_N denotes the volume of the unit ball and $\|u_{\frac{1}{n}}\| \to (\alpha_{s,N}^*/N)^{\frac{N-s}{N}}$ (see [23]). Thus, by (3.28), we know that $\{t_n\}$ is bounded. Using (3.26), we have

$$t_n^{\frac{N}{s}} \to \left[\frac{\alpha_*}{\alpha_0}\right]^{\frac{N-s}{s}}. \tag{3.29}$$

Let

$$A_n = \{x \in B : t_n\omega_n(x) \geq R_\epsilon\}, \quad B_n = B \setminus A_n,$$

and break the integral in (3.27) into a sum of integrals over A_n and B_n. Similar to the proof of (3.28), we have

$$m\left(\left[\frac{\alpha_*}{\alpha_0}\right]^{\frac{N-s}{s}}\right)\left[\frac{\alpha_*}{\alpha_0}\right]^{\frac{N-s}{s}} \geq (\beta-\epsilon)\lim_{n\to\infty}\int_B \exp\left[\alpha_{s,N}^*\omega_n^{\frac{N}{N-s}}\right]dx - (\beta-\epsilon)\omega_N. \tag{3.30}$$

The last integral in (3.30), denoted I_n is evaluated as follows:

$$I_n = \left\{\omega_N + N\omega_N \ln n\int_0^1 \exp\left(Nt^{\frac{N}{N-s}}\ln n - tN\ln n\right)dt\right\}.$$

So finally from (3.30) we have

$$m\left(\left[\frac{\alpha_*}{\alpha_0}\right]^{\frac{N-s}{s}}\right)\left[\frac{\alpha_*}{\alpha_0}\right]^{\frac{N-s}{s}} \geq (\beta-\epsilon)\omega_N\mathcal{M},$$

which implies $\beta \leq m\left(\left[\frac{\alpha_*}{\alpha_0}\right]^{\frac{N-s}{s}}\right)\left[\frac{\alpha_*}{\alpha_0}\right]^{\frac{N-s}{s}}/(\omega_N\mathcal{M})$. This happens a contradiction to (H_5). □

Lemma 3.6. *Let $ps = N$ and assume that* (V_1), (M_1)-(M_3) *and* (H_1)-(H_3) *hold. If f has the critical exponential growth on $\mathbb{R}^N$ (condition* (CG)*), then any* $(C)_c$ *sequence of $\mathcal{J}$ satisfying $c < M\left(\left[\frac{\alpha_*}{\alpha_0}\right]^{\frac{N-s}{s}}\right)\frac{s}{N}$ is bounded.*

Proof. Let $\{u_n\} \subset X$ be a $(C)_c$ sequence satisfying (3.7) and (3.8). Argue by the contradiction that $\|u_n\| \to \infty$ and set $v_n = \frac{u_n}{\|u_n\|}$. We can suppose that $v_n \rightharpoonup v$ in X. According to the previous section of proof of Lemma 3.3, we may similarly show that $v_n \rightharpoonup 0$.

From conditions (V_1), (H_1) and $v = 0$, we can prove that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_2(y)} |v_n|^p dx = 0. \tag{3.31}$$

Owing to (3.31), applying the Lions' Lemma [12, Lemma 1.1] we obtain

$$v_n \to 0 \quad \text{in } L^\gamma(\mathbb{R}^N), \quad \forall\, \gamma \in \left(\frac{N}{s}, +\infty\right). \tag{3.32}$$

Again let $t_n \in [0,1]$ such that

$$\mathcal{J}(t_n u_n) = \max_{t\in[0,1]} \mathcal{J}(tu_n).$$

Letting $R \in \left(0, [\frac{\alpha_*}{\alpha_0}]^{\frac{N-s}{N}}\right)$ and taking $\epsilon = \frac{\alpha_*}{R^{\frac{N}{N-s}}} - \alpha_0$, by (H_1), (H_3) and (CG), there exists $C_3 > 0$ such that

$$F(x,t) \le C_3|t|^{2p} + \epsilon\Phi_{N,s}\left((\alpha_0+\epsilon)|t|^{\frac{N}{N-s}}\right), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.33}$$

Also since $\|u_n\| \to \infty$, we get for n large enough:

$$\mathcal{J}(t_n u_n) \ge \mathcal{J}(\frac{R}{\|u_n\|} u_n) = \mathcal{J}(Rv_n). \tag{3.34}$$

Thus, from (3.33), and noting that $\|v_n\| = 1$, we have

$$\mathcal{J}(Rv_n) \ge \frac{s}{N} M(R^{\frac{N}{s}}) - C_3 R^{2p} \int_{\mathbb{R}^N} |v_n(x)|^{2p} dx - \epsilon \int_{\mathbb{R}^N} \Phi_{N,s}\left(\alpha_* |v_n|^{\frac{N}{N-s}}\right) dx. \tag{3.35}$$

Hence, by Proposition 1.1 and (3.32), letting $R \to [\frac{\alpha_*}{\alpha_0}]^{\frac{N-s}{N}}$ in (3.35) combining with (3.34), we get $\epsilon \to 0$ and

$$\liminf_{n\to\infty} \mathcal{J}(t_n u_n) \ge M\left(\left[\frac{\alpha_*}{\alpha_0}\right]^{\frac{N-s}{s}}\right) \frac{s}{N}. \tag{3.36}$$

Remained proof is completely similar to the last section of proof of Lemma 3.3. We omit it here. □

4. Proofs of the main results

Proof of Theorem 1.1. By Lemma 3.1 and Proposition 2.1, we know that there exists a $(C)_{c^*}$ sequence $\{u_n\}$ for the functional $\mathcal{J}$. Here c^* is defined by

$$c^* := \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} \mathcal{J}(\gamma(t)),$$

where $\mathbf{\Gamma} = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \quad \mathcal{J}(\gamma(1)) < 0\}$.

For the purpose of finding ground state solutions for problem (1.1), we first consider the Nehari manifold associated to the functional $\mathcal{J}$, namely,

$$\mathcal{N} := \{u \in X : \langle \mathcal{J}'(u), u\rangle = 0, u \neq 0\}$$

and the number $b := \inf\limits_{u\in\mathcal{N}} \mathcal{J}(u)$. We can easily get $c^* \leq b$. In fact, let $u \in \mathcal{N}$ and define $h : (0, +\infty) \to \mathbb{R}$ by $h(t) = \mathcal{J}(tu)$. Then $h(t)$ is differentiable and

$$h'(t) = \langle \mathcal{J}'(tu), u\rangle = m(t^p\|u\|^p)t^{p-1}\|u\|^p - \int_{\mathbb{R}^N} f(x, tu)udx, \ \forall t > 0.$$

Since $\langle \mathcal{J}'(u), u\rangle = 0$, that is, $m(\|u\|^p)\|u\|^p = \int_{\mathbb{R}^N} f(x, u)udx$, we have

$$h'(t)=t^{2p-1}\|u\|^{2p}\left[\frac{m(t^p\|u\|^p)}{t^p\|u\|^p} - \frac{m(\|u\|^p)}{\|u\|^p}\right]+t^{2p-1}\int_{\mathbb{R}^N}\left[\frac{f(x,u)}{u^{2p-1}} - \frac{f(x,tu)}{(tu)^{2p-1}}\right]u^{2p}dx.$$

Observing that $h'(1) = 0$, from (M$_3$) and (H$_3$), it follows that $h'(t) \geq 0$ for $0 < t < 1$ and $h'(t) \leq 0$ for $t > 1$. Hence,

$$\mathcal{J}(u) = \max_{t\geq 0} \mathcal{J}(tu).$$

Thus, we denote $g : [0, 1] \to X,\ g(t) = tt_0u$, where t_0 satisfying $\mathcal{J}(t_0u) < 0$, we have $g \in \mathbf{\Gamma}$ and therefore

$$c^* \leq \max_{t\geq 0} \mathcal{J}(g(t)) \leq \max_{t\geq 0} \mathcal{J}(tu).$$

Since $u \in \mathcal{N}$ is arbitrary, we have $c^* \leq b$.

Next, we by using Lemma 3.3 know that above the $(\mathrm{C})_{c^*}$ sequence $\{u_n\}$ is bounded in X. Set

$$\delta = \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_2(y)} |u_n|^p dx. \tag{4.1}$$

If $\delta = 0$, using the Lions' Lemma [12] again,

$$u_n \to 0 \text{ in } L^\gamma(\mathbb{R}^N),\ \forall \gamma \in (p, p^*).$$

Thus, from (H$_1$), (H$_3$) and (SCPI), we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(x, u_n)dx = 0, \tag{4.2}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(x, u_n)u_n dx = 0. \tag{4.3}$$

Hence, from (3.7) and (3.8), we have $c^* = 0$. This is a contradiction. Therefore $\delta > 0$. Using (4.1), we can choose $\{z_n\} \subset \mathbb{R}^N$ such that

$$\int_{B_2(z_n)} |u_n|^p dx \geq \frac{\delta}{2}.$$

It is easy to observe that the number of points in $\mathbb{Z}^N \cap B_2(z_n)$ is less than 4^N. So there exists $y_n \in \mathbb{Z}^N \cap B_2(z_n)$, such that

$$\int_{B_2(y_n)} |u_n|^p dx \geq m > 0. \tag{4.4}$$

Let $\tilde{u}_n = u_n(\cdot + y_n)$. Applying conditions ($V_1$) and ($H_1$), we get $\|\tilde{u}_n\| = \|u_n\|$ and

$$\int_{B_2(0)} |\tilde{u}_n|^p dx = \int_{B_2(y_n)} |u_n|^p dx \geq m > 0. \tag{4.5}$$

Going if necessary up to a subsequence, we obtain

$$\tilde{u}_n \rightharpoonup u^* \text{ in } X,\ \ \tilde{u}_n \to u^* \text{ in } L^\gamma_{loc}(\mathbb{R}^N),$$

and, from (4.5) we know that $u^* \neq 0$. Moreover, from the $\mathbb{Z}^N$ translation invariance of the problem, we know that $\{\tilde{u}_n\}$ is also $(C)_{c^*}$ sequence of $\mathcal{J}$. Without loss of generality, we can assume that $\|\tilde{u}_n\| \to \rho_0 > 0$. Thus for every $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} &m(\rho_0^p)\Big(\int_{\mathbb{R}^{2N}} \left[\frac{|u^*(x)-u^*(y)|^{p-2}(u^*(x)-u^*(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\right] dxdy \\ &\quad + \int_{\mathbb{R}^N} V(x)|u^*(x)|^{p-2}u^*\varphi dx\Big) \\ &= \int_{\mathbb{R}^N} f(x,u^*)\varphi dx. \end{aligned} \tag{4.6}$$

Now, we show that $\|u^*\| = \rho_0$. Since (M_1), argue by contradiction that

$$m(\|u^*\|^p)\|u^*\|^p < \int_{\mathbb{R}^N} f(x,u^*)u^* dx.$$

Using (M_1), (H_3) and Sobolev imbedding, we notice that $\langle \mathcal{J}'(tu^*), u^*\rangle > m_0 t^{p-1}\|u^*\|^p - lt^{2p-1}\|u^*\|^{2p} > 0$ for t small enough, where l is a positive constant. Then there exists $\sigma \in (0,1)$ such that $\langle \mathcal{J}'(\sigma u^*), \sigma u^*\rangle = 0$, i.e., $\sigma u^* \in \mathcal{N}$. Notice that $c^* < b$. From (H_3), we can follow that for each $x \in \mathbb{R}^N$,

$$tf(x,t) - 2pF(x,t) \text{ is strict increasing for } t > 0 \text{ and strict decreasing for } t < 0.$$

We also notice that m has similar above properties by (M_3). Thus, according to (M_3), (H_3), semicontinuity of norm and Fatou's lemma we get

$$\begin{aligned} c^* \leq b \leq \mathcal{J}(\sigma u^*) &= \mathcal{J}(\sigma u^*) - \frac{1}{2p}\langle \mathcal{J}'(\sigma u^*), \sigma u^*)\rangle \\ &= \frac{1}{p}M(\sigma u^*) - \frac{1}{2p}m(\|\sigma u^*\|^p)\|\sigma u^*\|^p \\ &\quad + \frac{1}{2p}\int_{\mathbb{R}^N} [f(x,\sigma u^*)\sigma u^* - 2pF(x,\sigma u^*)]dx \\ &< \frac{1}{p}M(u^*) - \frac{1}{2p}m(\|u^*\|^p)\|u^*\|^p \\ &\quad + \frac{1}{2p}\int_{\mathbb{R}^N} [f(x,u^*)u^* - 2pF(x,u^*)]dx \end{aligned}$$

$$
\begin{aligned}
&\leq \lim_{n\to\infty}[\mathcal{J}(u_n) - \frac{1}{2p}\langle \mathcal{J}'(u_n), u_n\rangle] \\
&= c^*,
\end{aligned}
$$

which leads to a contradiction. Hence, we have $\|u^*\| = \rho_0$ and u^* is a nontrivial ground state solution of problem (1.1). $\square$

Proof of Theorem 1.2. Using Lemma 3.2 and Proposition 2.1, we get that there exists a $(\mathrm{C})_{c^*}$ sequence $\{u_n\}$ for the functional $\mathcal{J}$. For the purpose of again finding ground state solutions for problem (1.1), we still consider the Nehari manifold corresponded to the functional $\mathcal{J}$, namely,

$$
\mathcal{N} := \{u \in X : \langle \mathcal{J}'(u), u\rangle = 0, u \neq 0\}
$$

and the number $b := \inf_{u\in\mathcal{N}} \mathcal{J}(u)$. Similar to the previous section of proof of Theorem 1.1, we can easily get $c^* \leq b$.

Now, by Lemma 3.4, we know that above the $(\mathrm{C})_{c^*}$ sequence $\{u_n\}$ is bounded in X. Set

$$
\delta = \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_2(y)} |u_n|^p dx. \tag{4.7}
$$

If $\delta = 0$, using the Lions' Lemma [12] again,

$$
u_n \to 0 \text{ in } L^\gamma(\mathbb{R}^N),\ \ \forall\gamma \in \left(\frac{N}{s}, \infty\right).
$$

Thus, by (H_1), (H_3) and (SCE), we obtain

$$
\lim_{n\to\infty} \int_{\mathbb{R}^N} F(x, u_n)dx = 0, \tag{4.8}
$$

and

$$
\lim_{n\to\infty} \int_{\mathbb{R}^N} f(x, u_n)u_n dx = 0. \tag{4.9}
$$

Hence, from (3.7) and (3.8), we get $c^* = 0$. This leads to a contradiction. Therefore $\delta > 0$. By (4.7), we can choose $\{z_n\} \subset \mathbb{R}^N$ such that

$$
\int_{B_2(z_n)} |u_n|^p dx \geq \frac{\delta}{2}.
$$

It is easy to know that the number of points in $\mathbb{Z}^N \cap B_2(z_n)$ is less than 4^N. So there exists $y_n \in \mathbb{Z}^N \cap B_2(z_n)$, such that

$$
\int_{B_2(y_n)} |u_n|^p dx \geq m_1 > 0. \tag{4.10}
$$

Let $\tilde{u}_n = u_n(\cdot + y_n)$. Applying conditions (V_1) and (H_1), we get $\|\tilde{u}_n\| = \|u_n\|$ and

$$
\int_{B_2(0)} |\tilde{u}_n|^p dx = \int_{B_2(y_n)} |u_n|^p dx \geq m_1 > 0. \tag{4.11}
$$

Going if necessary up to a subsequence, we obtain

$$
\tilde{u}_n \rightharpoonup u^* \text{ in } X,\ \ \tilde{u}_n \to u^* \text{ in } L^\gamma_{loc}(\mathbb{R}^N),
$$

and, from (4.11) we know that $u^* \neq 0$. Moreover, from the $\mathbb{Z}^N$ translation invariance of the problem, we know that $\{\tilde{u}_n\}$ is also $(\mathrm{C})_{c^*}$ sequence of $\mathcal{J}$. Without loss of generality, we can assume that $\|\tilde{u}_n\| \to \rho_0 > 0$. Thus for every $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} & m(\rho_0^p) \int_{\mathbb{R}^{2N}} \left[\frac{|u^*(x) - u^*(y)|^{p-2}(u^*(x) - u^*(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} \right] dxdy \\ & + \int_{\mathbb{R}^N} V(x)|u^*(x)|^{p-2} u^* \varphi dx \\ = & \int_{\mathbb{R}^N} f(x, u^*)\varphi. \end{aligned} \tag{4.12}$$

Remained proof is completely similar to the last section of proof of Theorem 1.1. We omit it here. □

Proof of Theorem 1.3. By Lemmas 3.2, 3.5 and 3.6, then there exists a bounded $(\mathrm{C})_{c^*}$ sequence $\{u_n\}$ at the level $0 < c^* < M\left(\left[\frac{\alpha_*}{\alpha_0}\right]^{\frac{N-s}{s}}\right)\frac{s}{N}$. Following the proof of Theorem 1.2, we only need to prove that $\delta > 0$ in (4.7). In fact by the contradiction that $\delta = 0$, we also obtain that

$$u_n \to 0 \text{ in } L^\gamma(\mathbb{R}^N), \quad \forall \gamma \in \left(\frac{N}{s}, \infty\right).$$

In addition, by using (H_4), similar to the proof of Proposition 5.2 in [9], we get

$$\int_{\mathbb{R}^N} F(x, u_n) dx \to 0. \tag{4.13}$$

So from (4.13) and (M_1), we have

$$\lim_{n\to\infty} \frac{s}{N} M(\|u_n\|^{\frac{N}{s}}) = c^* < M\left(\left[\frac{\alpha_*}{\alpha_0}\right]^{\frac{N-s}{s}}\right)\frac{s}{N}. \tag{4.14}$$

Since f has the critical exponential growth (CG) on $\mathbb{R}^N$, from (H_3), for any $\epsilon > 0$, we can find two constants $C_4 > 0$ and $\alpha > \alpha_0$ such that

$$|f(x,t)| \le \epsilon |t|^{\frac{N}{s}-1} + C_4 \Phi_{N,s}\left(\alpha |t|^{\frac{N}{N-s}}\right), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Thus, from the fractional case of the Moser-Trudinger inequality (see Theorem 1.3 in [23]),

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} f(x,u_n) u_n dx \right| \\ \le & C_4 \left(\int_{\mathbb{R}^N} \Phi_{N,s}\left(k\alpha |u_n|^{\frac{N}{N-s}}\right) dx \right)^{\frac{1}{k}} |u_n|_{k'} + \epsilon |u_n|_{\frac{N}{s}}^{\frac{N}{s}} \\ \le & C_4 \left(\int_{\mathbb{R}^N} \Phi_{N,s}\left(k\alpha \|u_n\|^{\frac{N}{N-s}} \left|\frac{u_n}{\|u_n\|}\right|^{\frac{N}{N-s}}\right) dx \right)^{\frac{1}{k}} |u_n|_{k'} + \epsilon |u_n|_{\frac{N}{s}}^{\frac{N}{s}} \\ \le & C_5 |u_n|_{k'} + \epsilon |u_n|_{\frac{N}{s}}^{\frac{N}{s}} \to 0, \end{aligned}$$

where $k > 1$ sufficiently close to 1 and k' is the dual number of k. Hence, from above formulas, we get

$$\|u_n\|^{\frac{N}{s}} \to 0.$$

This contradicts with (4.14). The proof is now completed. □

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