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MONOTONE ITERATIVE TECHNIQUE FOR FRACTIONAL MEASURE DIFFERENTIAL EQUATIONS IN ORDERED BANACH SPACE*

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Abstract This article is based on the monotonic iterative method in the presence of upper and lower solutions, and investigates the existence of S-asymptotic ω-periodic mild solutions for a class of fractional measure differential equations with nonlocal conditions in an ordered Banach spaces. Firstly, in the case of upper and lower solutions, a monotonic iterative method is constructed to obtain the maximal and minimal S-asymptotically ω-periodic mild solution to our concern problem. Secondly, we establish an existence result of S-asymptotically ω-periodic mild solutions for the mentioned without assuming the existence of upper and lower S-asymptotically ω-periodic mild solutions under generalized monotonic conditions and non compactness measure conditions of nonlinear terms. Finally, as an application of abstract results, an example is provided to illustrate our main findings.

Keywords Regulated functions, Henstock-Lebesgue-Stieltjes integral, measure differential equations, monotone iterative technique.

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1. Introduction

Fractional calculus has been widely applied in the study of linear and nonlinear fractions differential equations (FDE) have emerged as challenges in the real world. Many researchers in certain regions use FDEs extensively, making some problems easier to approach, such as modeling nonlinear phenomena, optimal control of complex systems, and other scientific research (e.g., see [42,43]). In addition, fractional differential systems can describe nonlinear phenomena in physics, mathematics, and engineering. These types of equations have attracted widespread attention in recent years, as shown in [56,64,66] and their references.

The theory of measure differential equations (MDEs) encompasses some well-known situations. When absolute continuous functions, step functions, or the sum of absolute continuous functions and step functions are given, these systems cor-

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respond to typical ordinary differential equations, difference equations, or impulse differential equations, respectively. Another advantage of considering MDE is that we can model Zeno trajectories, as gas as a bounded function of change may exhibit infinite discontinuity within a finite interval. This type of system appears in many fields of applied mathematics, such as non smooth mechanics, game theory, etc. see [5, 53, 67]. The early investigation of MDEs was conducted by [19, 55, 62, 65]. For a complete introduction to measure differential systems, reference can be made to [7–10, 12]. Recently, the MDEs theory in the $\mathbb{R}^n$ space has been developed to some extent [27, 28, 52, 68].

On the other hand, as is well known, the periodic law of development or motion of things, It is a common phenomenon in nature and human activities. However, in real life, many the phenomenon does not have strict periodicity. In order to better describe these mathematics, many scholars have introduced other definitions of generalized periodicity, such as almost periodicity, asymptotic periodicity, and asymptotic almost periodicity, pseudo almost periodicity and S-asymptotic periodicity, see [1, 20–22]. Due to the S-asymptotically periodic functions first studied functions in Banach space by Henríquez et al. [38], including some literature on the S- asymptotic periodic solutions of fractional evolution equations, we can refer to [14, 18, 46, 47, 58, 60]. It is worth noting that S-asymptotically period function is first proposed and established by Henríquez et al. [38] is a more general approximate periodic function between asymptotic periodic functions and asymptotic almost periodic functions.

The properties of periodic solutions to functional differential equations, integral equations and partial differential equations have been extensively studied. Specially, because fractional derivative has genetic or memory properties, the solutions of periodic boundary value problems for fractional differential equations can not be extended periodically to time t in $\mathbb{R}^+$. Therefore, many scholars began to study various extended solutions of periodic solutions of fractional evolution equations(such as almost periodic solutions, asymptotically almost periodic solutions, pseudo almost periodic solutions, asymptotically periodic solutions, S-asymptotically periodic solutions and so on). For the related research on the S-asymptotically periodic solutions of fractional evolution equations, one can refer to [6,13,13–18,46,47,60]. In [63], Shu et al. discussed the existence and uniqueness of positive S-asymptotically ω-periodic mild solutions for a class of semilinear neutral fractional evolution equations with delay by using the contraction mapping principle on positive cones. In [48], Li et al. discussed the positive S-asymptotically ω-periodic mild solutions for the abstract fractional evolution equation on infinite interval.

Due to the structures of such equations, investigating their solutions is challenging. To the best of the authors' knowledge, the existence of S-asymptotically ω-periodic mild solutions for abstract damped elastic systems with delay is a subject that has not been treated in the literature. This fact and the interesting relationship between S-asymptotically ω-periodic mild solutions and S-asymptotically ω-periodic functions are the main motivations of this work.

Based on previous work ideas and methods [24, 25, 27, 30, 32, 39], in this work, we investigate the existence of S-asymptotically ω-periodic mild solution to fractional measure differential equations with nonlocal conditions and delay

$$\begin{cases} {}^cD_t^{1+\beta}u(t) + \sum\limits_{k=1}^{n} \alpha_k {}^cD_t^{\gamma_k}u(t) = Au(t) + F(t, u(t), u_t)dg(t), \quad t \geq 0, \\ u(t) = Q(\sigma(u), u)(t) + \varphi(t), \quad t \in [-r, 0], \\ u'(0) = Q_0(u) + \psi, \end{cases} \tag{1.1}$$

where $u(\cdot)$ take values in a Banach space E; ${}^cD_t^{\eta}$ stand for the Caputo fractional derivative of order η, $\alpha_k > 0$ and all γ_k, $k = 1, 2, \cdots, n, n \in \mathbb{N}$, are positive real numbers such that $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$. We assume that $A : \mathcal{D}(A) \subset E \to E$ is a κ-sectorial operator, and A generates a strongly continuous family $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$ of bounded and linear operators on E, $f : \mathbb{R}^+ \times E \times \mathcal{B} \to E$ is a suitable nonlinear function and $g : \mathbb{R}^+ \to \mathbb{R}$ is nondeacreasing and continuous from the left, dg denote the distributional derivative of g (see [68]), the functions $Q : \mathbb{R}^+ \times G(\mathbb{R}^+, E) \to E, Q_0 : G(\mathbb{R}^+, E) \to E$, $\sigma : G(\mathbb{R}^+, E) \to E$ will be specified later, where $G(\mathbb{R}^+, E)$ denotes the space of regulated functions on $\mathbb{R}^+$, $\mathcal{B} := G([-r, 0], E)$. For $t \geq 0$, $u_t \in \mathcal{B}$ is the history state defined by $u_t(s) = u(t + s)$ for $s \in [-r, 0]$, $\varphi \in \mathcal{B}$ and $\varphi(0) \in \mathcal{D}(A)$, $\psi \in E$, $r > 0$ is a constant.

The highlights and advantages of this paper are presented as follows:

(1) This paper is to construct the general principle for lower and upper solutions coupled with the monotone iterative technique for the delay evolution equation involving nonlocal in ordered Banach space, and obtain the existence of maximal and minimal S-asymptotically ω-periodic mild solutions, which will fill the research gap in this area.

(2) The main method used in this paper is the monotone iterative technique in the presence of the lower and upper solutions, which is an effective and widely used method to study the nonlinear differential equations as an application of the ordered fixed point theorem. This method can not only study the solvability of the equations, but also obtain the iterative sequence of the solutions, which provides a reasonable and effective theoretical basis for solving the approximate solutions by computer.

(3) The existence results of S-asymptotic ω-periodic mild solutions were derived using monotonic iteration technique, filling the research gap in this field by using regulated functions, Henstock Lebesgue Stieltjes integral is set to measure driven equation involving multi-term time fractional derivatives.

(4) Some authors choose topological methods to study the existence of S-asymptotic ω- periodic solutions, which is known as fixed point theory, which has become a very powerful and important tool for studying nonlinear phenomena. Specifically, the author utilized the contraction mapping principle, Leray-Schauder alternative theorem, Schauder theorem, and Krasnoselkii's theorem. However, the monotonic iterative method with upper and lower solutions is the first to be used to study related problems in ordered Banach spaces. Therefore, our results are novel and meaningful.

The organizational structure of this article is as follows. The second part of the paper presents preliminary details. The third part uses monotonic iteration method to compare the upper and lower solutions with (β, γ_k)-resolvent family, it is proved that S- asymptotic ω-periodic mild solution. Finally, an example was provided to illustrate the application of the obtained results. The conclusion section concludes this article.

2. Preliminaries

Throughout this paper, let $(E, \|\cdot\|)$ be an ordered Banach space with partial order "$\leq$" induced by the positive cone $K = \{u \in E | u \geq \theta\}$ (θ is the zero element of E), K is normal with normal constant N. Let $r > 0$ be constants, we denote by $C_b(\mathbb{R}^+, E)$ the Banach space of all bounded and continuous functions from $\mathbb{R}^+$ to E equipped with the norm

$$\|u\|_\infty = \sup_{t\in\mathbb{R}^+} \|u(t)\|$$

and $G(\mathbb{R}^+, E)$ denotes the Banach space of regulated functions on $\mathbb{R}^+$ equipped with a norm $\|u\|_\infty = \sup\limits_{t\in\mathbb{R}^+} \|u(t)\|$, $\mathcal{B} := G([-r, 0], E)$ the Banach space of regulated functions from $[-r, 0]$ to E with the norm

$$\|\phi\|_{\mathcal{B}} = \sup_{s\in[-r,0]} \|\phi(s)\|.$$

Let $SAP_\omega(E)$ represent the subspace of $C_b(\mathbb{R}^+, E)$ consisting all the E-value S-asymptotically ω-periodic functions endowed with the uniform convergence norm denoted by $\|\cdot\|$. Then $SAP_\omega(E)$ is a Banach space (see [38], Proposition 3.5]). If $u \in SAP_\omega(E)$, then it is not difficult to test and verify that the function $t \to u_t$ belongs to $SAP_\omega(\mathcal{B})$ (see [46, 47]).

For the rest of this paper, we define by

$$\Omega := \{u \in G([-r, \infty), E) \cap C_b([-r, \infty), E)|\ u|_{[-r,0]} \in \mathcal{B} \text{ and } u|_{\mathbb{R}^+} \in SAP_\omega(E)\},$$

that Ω is a Banach space equipped with the norm

$$\|u\|_\Omega = \sup_{t\in[-r,\infty)} \|u(t)\|.$$

Define a positive cone K_Ω by

$$K_\Omega = \{u \in \Omega | u(t) \in K, \quad t \in [-r, \infty)\},$$

with the normal constant N. Then Ω is an ordered Banach space with the partial order relation "$\leq$" induced by the cone K_Ω. Similarly, $\mathcal{B}$ is also an order Banach space whose partial ordering "$\leq$" induced by a positive cone

$$K_{\mathcal{B}} = \{\phi \in \mathcal{B} | \phi(s) \in K, s \in [-r, 0]\}$$

with the normal constant N.

A partition of $[a, b]$ is a finite collection of pairs $\{([t_{i-1}, t_i], e_i), i = 1, 2, \cdots, n\}$, where $[t_{i-1}, t_i]$ are nonoverlapping subintervals of $[a, b]$, $e_i \in [t_{i-1}, t_i]$, $i = 1, \cdots, n$ and $\bigcup_{i=1}^n [t_{i-1}, t_i] = [a, b]$. A gauge δ on $[a, b]$ is a positive function on $[a, b]$. For a given guage δ we say that a partition is δ-fine if $[t_{i-1}, t_i] \subset (e_i - \delta(e_i), e_i + \delta(e_i))$, $i \in \{1, \cdots, n\}$. Let $u(t^-)$ and $u(t^+)$ denote the left limit and right limit of the function u at the point t, respectively.

Definition 2.1. [61] A function $u : [a, b] \to E$ is said to be regulated on $[a, b]$, if the limits

$$\lim_{s\to t^-} u(s) = u(t^-),\ t \in (a, b] \ \text{ and } \ \lim_{s\to t^+} u(s) = u(t^+),\ t \in [a, b)$$

exist and are finite.

Denote by $G([a, b], E)$ the space of all regulated function from $[a, b]$ into E. Obviously, the space $G([a, b], E)$ is a Banach space endowed with the supremum norm.

Definition 2.2. [61] A set $B \subset G([a, b], E)$ is called equiregulated, if for every $\epsilon > 0$ and $\tau \in [a, b]$, there exists $\delta > 0$ such that

(i) If $u \in B, t \in [a, b]$ and $t \in (\tau - \delta, \tau)$, then $\|u(\tau^-) - u(t)\|_E < \epsilon$.

(ii) If $u \in B, t \in [a, b]$ and $t \in (\tau, \tau + \delta)$, then $\|u(t) - u(\tau^+)\|_E < \epsilon$.

Lemma 2.1. [61] *Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of functions from $[a, b]$ to E. If u_n converge pointwisely to u_0 as $n \to \infty$ and the sequence $\{u_n\}_{n=1}^{\infty}$ is equiregulated, then u_n converges uniformly to u_0.*

Lemma 2.2. [9, 61] *Let $B \subset G([a, b], E)$. If B is bounded and equiregulated, then the set $\overline{co}(B)$ is also bounded and equiregulated, where $\overline{co}(B)$ define the closed convex hull of B.*

Lemma 2.3. [51] *Assume that $B \subset G([a, b], E)$ is equiregulated and, for every $t \in [a, b]$ the set $\{u(t) : u \in B\}$ is relatively compact in E. Then the set B is relatively compact in $G([a, b], E)$.*

Next, we will review the definition of Henstock-Lebesgue-Stieljes integral.

Definition 2.3. [61] A function $\psi : [0, b] \to E$ is said to be Henstock-Lebesgue-Stieltjes integrable w.r.t. $g : [0, b] \to \mathbb{R}$, if there exists a function denoted by $(HLS)\int_a^{\cdot} : [0, b] \to E$ such that, for every $\epsilon > 0$, there is a gauge δ_ϵ on $[0, b]$ with

$$\Big\|\sum_{i=1}^{n}\psi(e_i)(g(t_i)-g(t_{i-1}))-\Big((HLS)\int_0^{t_i}\psi(s)dg(s)-(HLS)\int_0^{t_{i-1}}\psi(s)dg(s)\Big)\Big\| < \epsilon,$$

for every δ_ϵ-fine partition $\{(e_i, [t_{i-1}, t_i]) : i = 1, 2, \dots, n\}$ of $[0, b]$.

Denote by $\mathbb{HLS}_g^p([a, b], \mathbb{R})(p > 1)$ the space of all p-ordered Henstock-Lebesgue-Stieltjes integral regulated from $[a, b]$ to $\mathbb{R}$ with respect to g, which norm $\|\cdot\|_{\mathbb{HLS}_g^p}$ defined by

$$\|\psi\|_{\mathbb{HLS}_g^p} = \Big((HLS)\int_a^b \|\psi(s)\|^p dg(s)\Big)^{\frac{1}{p}}.$$

Lemma 2.4. [32] *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $\Psi \in \mathbb{HLS}_g^p([a, b], \mathbb{R}^+)$ and $g : [a, b] \to \mathbb{R}$ be regulated. Then the function $H(t) = \int_0^t (t - s)^\beta \Psi(s)dg(s)$ is regulated and*

$$H(t) - H(t^-) \le \Big(\int_{t^-}^{t}(t - s)^{q\beta}dg(s)\Big)^{\frac{1}{q}}\Psi(t)(\Delta^- g(t))^{\frac{1}{p}}, \quad t \in (a, b],$$

$$H(t^+) - H(t) \le \Big(\int_{t^+}^{t}(t^+ - s)^{q\beta}dg(s)\Big)^{\frac{1}{q}}\Psi(t)(\Delta^+ g(t))^{\frac{1}{p}}, \quad t \in [a, b),$$

where $\Delta^+ g(t) = g(t^+) - g(t)$ and $\Delta^- g(t) = g(t) - g(t^-)$.

Lemma 2.5. [23] *Let for $t \in [a,b]$, $Z(t)$ be weakly relatively compact in E. Suppose that $B \subset L^1_\mu([a,b],E)$ is a bounded set and there is a function $N(\cdot) \in L^1_\mu([a,b],\mathbb{R}^+)$ such that $\|b(t)\|_E \le N(t)$ μ-a.e. $t \in [a,b]$ for all $b \in B$. If for every $b \in B, b(t) \in Z(t)$ for μ-a.e. $t \in [a,b]$, then B is weakly relatively compact in $L^1_\mu([a,b],E)$, where $L^1_\mu([a,b],E)$ be the set of all μ-integrable functions, μ is a measure.*

Definition 2.4. [57] An (β,γ_k)-resolvent family $\{S_{\beta,\gamma_k}(t)\}_{t\ge 0}$ on E is said to be positive, if $S_{\beta,\gamma_k}(t)x \ge \theta$ for each $x \ge \theta, x \in E$, and $t \ge 0$.

Definition 2.5. [57] An (β,γ_k)-resolvent family $\{S_{\beta,\gamma_k}(t)\}_{t\ge 0}$ on E is said to be equicontinuous, if the function $t \to S_{\beta,\gamma_k}(t)$ is continuous from $(0,\infty) \to \mathcal{L}(E)$ on the operator norm $\|\cdot\|_{\mathcal{L}(E)}$.

Definition 2.6. The Riemann-Liouville fractional integral of a function $f \in L^1_{loc}([0,\infty),E)$ of order $\eta > 0$ with lower limit zero is defined as follows

$$\mathbb{I}^\eta f(t) = \int_0^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)} f(s)ds, \quad t > 0$$

and $\mathbb{I}^0 f(t) = f(t)$, provided that side integral is point-wise defined in $[0,\infty)$.

Definition 2.7. Let $\eta > 0$ be given and denote $m = [\eta]$. The Caputo fractional derivative of order $\eta > 0$ of a function $f \in C^m([0,\infty),E)$ with lower limit zero is given by

$${}^cD^\eta f(t) = \mathbb{I}^{m-\eta} D^m f(t) = \int_0^t \frac{(t-s)^{m-\eta-1}}{\Gamma(m-\eta)} D^m f(s)ds,$$

and ${}^cD^0 f(t) = f(t)$, where $D^m = d^m/dt^m$ and $[\cdot]$ is ceiling function.

Let A be a closed linear operator on the Banach space E with domain $\mathcal{D}(A)$ and denote by $\rho(A)$ the resolvent set of A.

Definition 2.8. [42]. Let E be a Banach space and let $\beta > 0, \gamma_k, \alpha_k, k = 1,2,\ldots n$ be real positive numbers. Then A is called the generator of (β,γ_k)-resolvent family if there exists $\kappa \ge 0$ and a strongly continuous function $S_{\beta,\gamma_k} : \mathbb{R}^+ \to \mathcal{L}(E)$ such that

$$\Big\{\lambda^{\beta+1} + \sum_{k=1}^n \alpha_k \lambda^{\gamma_k} : Re(\lambda) > \kappa\Big\} \subset \rho(A)$$

and

$$\lambda^\beta \Big(\lambda^{\beta+1} + \sum_{k=1}^n \alpha_k \lambda^{\gamma_k} - A\Big)^{-1} u = \int_0^\infty S_{\beta,\gamma_k}(t)udt,$$

where $Re(\lambda) > \kappa$ and $u \in E$.

An operator A is said to be κ-sectorial of angle θ if there exist $\theta \in [0, \frac{\pi}{2})$ and $\kappa \in \mathbb{R}$ such that its resolvent is in the sector

$$\kappa + S_\theta := \Big\{\kappa + \lambda : \lambda \in \mathbb{C}, |arg(\lambda)| < \frac{\pi}{2} + \theta\Big\} \backslash \{\omega\},$$

and

$$\|(\lambda - A)^{-1}\| \le \frac{M}{|\lambda - \omega|}, \quad \lambda \in \omega + S_\theta.$$

Lemma 2.6. [42] *Let $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$ and $\alpha_k \geq 0, k = 1, 2, \dots n$ be given, $\mu > 0$ and $\kappa < 0$. Assume that A is a κ-sectorial operator of angle $\frac{\gamma_k \pi}{2}$. Then A generates a (β, γ_k)-resolvent family $S_{\beta,\gamma_k}(t)$ satisfying the estimate*

$$\|S_{\beta,\gamma_k}(t)\| \leq \frac{C}{1 + |\kappa|(t^{\beta+1} + \sum_{k=1}^{n} \alpha_k t^{\gamma_k})}, \quad t \geq 0, \tag{2.1}$$

for some constant $C > 0$ depending only on β, γ_k.

Definition 2.9. [38] A function $u \in C_b(\mathbb{R}^+, E)$ is called S-asymptotically ω-periodic if there exists ω such that

$$\lim_{t\to\infty} \|u(t + \omega) - u(t)\| = 0, \quad \forall t \geq 0.$$

In this case, we say that ω is an asymptotic of u. It is clear that if ω is an asymptotic period for u, then every $k\omega, k = 1, 2, \cdots$, is also an asymptotic period of u.

In view of Lemma 2.14 in paper [29], we give the definition of a mild solution for the problem (1.1) below.

Definition 2.10. Let $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$ and $\alpha_k \geq 0, k = 1, 2, \dots n$ be given and A be a generator of a bounded (β, γ_k)-resolvent family $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$. Then a regulated function $u(\cdot) : \mathbb{R}^+ \to E$ is said to be mild solution of problem (1.1) if $u(t) = Q(\sigma(u), u)(t) + \varphi(t), u'(0) = Q_0(u) + \psi$ and satifies the following integral equation

$$\begin{aligned} u(t) =& S_{\beta,\gamma_k}(t)[Q(\sigma(u), u)(0) + \varphi(0)] + (\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(u)] \\ &+ \sum_{k=1}^{n} \alpha_k (\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(\sigma(u), u)(0) + \varphi(0)] \\ &+ \int_0^t T_{\beta,\gamma_k}(t - s)F(s, u(s), u_s)dg(s), \quad t \geq 0, \end{aligned} \tag{2.2}$$

where $T_{\beta,\gamma_k}(t) = (\varphi_\beta * S_{\beta,\gamma_k})(t)$. Moreover, if u is S-asymptotically ω-periodic, then it is called S-asymptotically ω-periodic mild solution of problem (1.1).

Moreover, we noted that by the estimate (2.1) and (2.3) and (2.4) in paper [29], hence there exists a constant $C > 0$ such that, we have

$$\|T_{\beta,\gamma_k}(t)\|_{\mathcal{L}(E)} \leq Ct^{\beta-\gamma_k}. \tag{2.3}$$

Denote $M := \sup_{t\geq 0} \|S_{\beta,\gamma_k}(t)\| < +\infty, M > 0.$, in view of (2.1) and (2.3) and (2.4) in paper [29], we have

$$\begin{aligned} \|\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k}(t)\| &\leq C \int_0^t \varphi_{1+\beta-\gamma_k}(t - \tau)\varphi_{\gamma_k-\beta-\varepsilon}(\tau)d\tau \\ &= C\varphi_{1-\varepsilon}(t) \\ &= Ct^{-\varepsilon}. \end{aligned} \tag{2.4}$$

And then, we note that

$$\int_0^\infty \frac{1}{1 + |\kappa|t^{\beta+1}} dt = \frac{|\kappa|^{-\frac{1}{\beta+1}} \pi}{(\beta + 1)\sin(\frac{\pi}{\beta+1})}$$

for $1 < \beta + 1 < 2$ and therefore $S_{\beta,\gamma_k}(t)$ is integrable. Hence, we have

$$\begin{aligned}\|(\varphi_1 * S_{\beta,\gamma_k})(t)\| &= \Big\| \int_0^t S_{\beta,\gamma_k}(s)ds \Big\| \\ &\leq \int_0^t \|S_{\beta,\gamma_k}(s)\| ds \\ &\leq \int_0^t \frac{C}{1 + |\kappa|(s^{\beta+1} + \sum_{k=1}^n \alpha_k s^{\gamma_k})} ds \\ &< C \int_0^\infty \frac{1}{1 + |\kappa| s^{\beta+1}} ds \\ &= \frac{C|\kappa|^{-\frac{1}{\beta+1}} \pi}{(\beta+1)\sin(\frac{\pi}{\beta+1})}.\end{aligned}$$

Moreover, we denote

$$\widetilde{M} := \sup_{t \geq 0} \|(\varphi_1 * S_{\beta,\gamma_k})(t)\| = \frac{C|\kappa|^{-\frac{1}{\beta+1}} \pi}{(\beta+1)\sin(\frac{\pi}{\beta+1})}. \tag{2.5}$$

In addition, we present the definitions of lower and upper solutions for the nonlocal problem (1.1).

Definition 2.11. If a function $v \in \Omega$ with $v|_{\mathbb{R}^+} \in C(\mathbb{R}^+, E) \cap C^{1+\beta}(\mathbb{R}^+, E)$ satisfies $Av(0) \leq A[Q(u)(0) + \varphi(0)]$ and

$$\begin{cases} {}^cD_t^{1+\beta} v(t) + \sum_{k=1}^n \alpha_k {}^cD_t^{\gamma_k} v(t) \leq Av(t) + F(t, v(t), v_t)dg(t), & t \geq 0, \\ v(t) \leq Q(\sigma(v), v)(t) + \varphi(t), & t \in [-r, 0], \\ v'(0) \leq Q_0(v) + \psi, \end{cases}$$

then $v(t)$ is named a lower solution of nonlocal problem (1.1). And if the inequalities in above are all reversed, then $v(t)$ is named an upper solution of nonlocal problem (1.1).

We give the definition of upper and lower mild solution to problem (1.1).

Definition 2.12. If a function $v \in \Omega$ satisfies

$$\begin{aligned} v(t) \leq & S_{\beta,\gamma_k}(t)[Q(\sigma(v), v)(0) + \varphi(0)] + (\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(v)] \\ & + \sum_{k=1}^n \alpha_k (\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(\sigma(v), v)(0) + \varphi(0)] \\ & + \int_0^t T_{\beta,\gamma_k}(t-s) F(s, v(s), v_s) dg(s), \end{aligned}$$

then $v(t)$ is named a lower mild solution of nonlocal problem (1.1). And if the inequality in above is reversed, then $v(t)$ is named an upper mild solution of nonlocal problem (1.1).

The Hausdorff measure of noncompactness of a bounded subset S of E is defined to be the infimum of the set of all real numbers $\epsilon > 0$ such that S can be covered by a finite number of balls radius smaller than ϵ, that is,

$$\alpha(S) = \inf\{\epsilon > 0 : S \subset \cup_{i=1}^{n} B(\xi_i, r_i), \xi_i \in E, r_i < \epsilon (i = 1, \dots, n), n \in \mathbb{N}\},$$

where $B(\xi_i, r_i)$ denotes the open ball centered at ξ_i and of radius r_i.

Lemma 2.7. [4,41] *Let S, T be bounded subsets of E and $\lambda \in \mathbb{R}$. Then*

(1) $\alpha(S) = 0$ if and only if S is relatively compact;

(2) $S \subseteq T$ implies $\alpha(S) \leq \alpha(T)$;

(3) $\alpha(\overline{S}) = \alpha(S)$;

(4) $\alpha(S \cup T) = \max\{\alpha(S), \alpha(T)\}$;

(5) $\alpha(\lambda S) = |\lambda|\alpha(S)$, where $\lambda S = \{x = \lambda z : z \in S\}$;

(6) $\alpha(S + T) \leq \alpha(S) + \alpha(T)$, where $S + T = \{x = y + z : y \in S, z \in Z\}$;

(7) $\alpha(co(S)) = \alpha(S)$.

Let W be a subset of $G([a, b], E)$. For each fixed $t \in [a, b]$, we denote $W(t) = \{x(t) : x \in W\}$.

Lemma 2.8. [9] *Let $W \subset G([a, b], E)$ be bounded and equiregulated on $[a, b]$. Then $\alpha(W(t))$ is regulated on $[a, b]$.*

Lemma 2.9. [9] *Let $W \subset G([a, b], E)$ be bounded and equiregulated on $[a, b]$. Then $\alpha(W) = \sup\{\alpha(W(t)) : t \in [a, b]\}$.*

Lemma 2.10. [11] *Let E be a Banach space and $B \subset E$ be bounded. Then there exists a countable subset $B_0 \subset B$, such that $\alpha(B) \leq \alpha(B_0)$.*

Denote by $\mathcal{LS}_g([a, b], E)$ the space of all functions $f : [a, b] \to E$ that are Lebesgue-Stieltjes integrable with respect to g. Let μ_g be the Lebesgue-stieltjes measure on $[a, b]$ induced by g.

Lemma 2.11. [36] *Let $W_0 \subset \mathcal{LS}_g([a, b], E)$ be a countable set. Assume that there exists a positive function $k \in \mathcal{LS}_g([a, b], \mathbb{R}^+)$ such that $\|w(t)\| \leq k(t)$ μ_g-a.e. holds for all $w \in W_0$. Then we have*

$$\alpha\Big(\int_a^b W_0(t)dg(t)\Big) \leq 2\int_a^b \alpha(W_0(t))dg(t).$$

Lemma 2.12. [44] *Let $T > 0$. Assume that $a, m \in G([0, T], \mathbb{R}^+)$. If the function $y \in G([0, T], \mathbb{R}^+)$ satisfies the inequality*

$$y(t) \leq m(t) + \int_0^t a(s)y(s)dg(s)$$

for every $t \in [0, T]$, then

$$y(t) \leq m(t) + \int_0^t a(s)m(s)e^{\int_s^t a(\tau)dg(\tau)}dg(s).$$

3. Main result

For $v, w \in \Omega$ with $v \leq w$, we denote the order interval $\{u|v \leq u \leq w\} \subset \Omega$ by $[v, w]$. Furthermore, we denote $\{u(t)|v(t) \leq u(t) \leq w(t), t \in [-r, \infty)\}$ in E by $[v(t), w(t)]$ and $\{u_t|v_t \leq u_t \leq w_t, t \in [0, \infty)\}$ in $\mathcal{B}$ by $[v_t, w_t]$, respectively.

Theorem 3.1. *Let E be an ordered Banach space, whose positive cone $K \subset E$ is normal. Let $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$ and $\alpha_k \geq 0, k = 1, 2, \ldots n$ be given. A is an κ-sectorial operator of angle $\frac{\gamma_k \pi}{2}, k = 1, 2, \cdots, n$ with $\kappa < 0$, and A generates a positive and compact (β, γ_k)-resolvent family $\{S_{\beta,\gamma_k}(t)\}_{t \geq 0}$ on E. Assume that $\omega > 0$ is a constant and the nonlocal problem (1.1) has a lower mild solution $v^{(0)}$ and an upper mild solution $w^{(0)}$ with $v^{(0)} \leq w^{(0)}$. If $\varphi \in K_{\mathcal{B}}$, $Q(\sigma(u), u)(0) + \varphi(0) \in K \cap \mathcal{D}(A)$ and $\psi \in K$, $F : \mathbb{R}^+ \times E \times \mathcal{B} \to E$ is continuous as well as the following conditions are established:*

$(H1)$ *For each constant $R > 0$, there exists $P(\cdot) \in \mathbb{HLS}^p_g(\mathbb{R}^+, \mathbb{R}^+)$ for some $p > 1$ such that*

$$\sup_{\|u\| \leq R} \|F(t, u(t), u_t)\| \leq P(t)W(R),\ t \geq 0,$$

where $W : [0, +\infty) \to \mathbb{R}^+$ is a continuous nondecreasing function and

$$\lim_{R \to +\infty} \inf \frac{W(R)}{R} = w_0 < +\infty.$$

$(H2)$ *(1) There exists $\omega > 0$ such that for all $x \in E, \phi \in \mathcal{B}$*

$$\lim_{t \to \infty} \|F(t + \omega, x, \phi) - F(t, x, \phi)\| = 0.$$

(2) $F(t, u, u_t)$ is measurable for all $u \in G(\mathbb{R}^+, E)$.

$(H3)$ *For any $t \in \mathbb{R}^+$, $x_1, x_2 \in E$ and $\phi_1, \phi_2 \in \mathcal{B}$ with $v^{(0)}(t) \leq x_1 \leq x_2 \leq w^{(0)}(t)$ and $v^{(0)}_t \leq \phi_1 \leq \phi_2 \leq w^{(0)}_t$,*

$$F(t, x_2, \phi_2) - F(t, x_1, \phi_1) \geq \theta.$$

$(H4)$ *(1) The nonlocal functions $Q(\sigma(u), u), Q_0(u)$ is increasing in order interval $[v^{(0)}, w^{(0)}]$;*

(2) $Q : \mathbb{R}^+ \times G([-r, +\infty), E) \to E, Q_0 : G([-r, +\infty), E) \to E$ are continuous and compact mapping, $\sigma : G([-r, +\infty), E) \to E$ is continuous and there are two positive constants c_0, c_1, d_0, d_1 such that

$$\|Q_0(u)\| \leq c_0\|u\| + d_0, \quad \|Q(\sigma(u), u)\| \leq c_1\|u\| + d_1, \quad u \in G([-r, +\infty), E).$$

Then the problem (1.1) has minimal and maximal S-asymptotically ω-periodic mild solutions $\underline{u}, \overline{u} \in [v^{(0)}, w^{(0)}]$, which can be obtained by the monotone iterative procedures starting from $v^{(0)}$ and $w^{(0)}$, respectively.

Proof. For each $u \in [v^{(0)}, w^{(0)}]$, we have $u_t \in [v^{(0)}_t, w^{(0)}_t] = [v^{(0)}(t + s), w^{(0)}(t + s)] \subset SAP_\omega(\mathcal{B})$ for $t \in \mathbb{R}^+, s \in [-r, 0]$. Define an operator $\mathcal{Q} : [v^{(0)}, w^{(0)}] \to$

$G([-r,+\infty),E)$ by

$$(\mathcal{Q}u)(t)=\begin{cases} S_{\beta,\gamma_k}(t)[Q(\sigma(u),u)(0)+\varphi(0)]+(\varphi_1 * S_{\beta,\gamma_k})(t)[\psi+Q_0(u)] \\ +\sum_{k=1}^{n}\alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(\sigma(u),u)(0)+\varphi(0)] \\ +\int_0^t T_{\beta,\gamma_k}(t-s)F(s,u(s),u_s)dg(s), \qquad t\geq 0, \\ Q(\sigma(u),u)(t)+\varphi(t), \qquad t\in[-r,0]. \end{cases} \tag{3.1}$$

By (H2)(2), the integral $\int_0^t T_{\beta,\gamma_k}(t-s)F(s,u(s),u_s)dg(s)$ is well defined. Clearly, if $\mathcal{Q}$ admit a fixed point in $G([-r,+\infty),E)$, then the system (1.1) admits a mild solution.

Now, we complete the proof by six steps.

Step I. The set $\{\mathcal{Q}u : u(\cdot)\in\Omega\}$ is equiregulated.

For any $b>0$, $t_0\in[-r,b)$, we have

$$\begin{aligned} &\|(\mathcal{Q}u)(t)-(\mathcal{Q}u)(t_0^+)\| \\ \leq&\|(S_{\beta,\gamma_k}(t)-S_{\beta,\gamma_k}(t_0^+))[Q(\sigma(u),u)(0)+\varphi(0)]\| \\ &+\|[(\varphi_1 * S_{\beta,\gamma_k})(t)-(\varphi_1 * S_{\beta,\gamma_k})(t_0^+)][\psi+Q_0(u)]\| \\ &+\sum_{k=1}^{n}\frac{\alpha_k M}{\Gamma(1+\beta-\gamma_k)}\Big|\int_0^t (t-s)^{\beta-\gamma_k}ds \\ &-\int_0^{t_0^+}(t_0^+-s)^{\beta-\gamma_k}ds\Big|\|[Q(\sigma(u),u)(0)+\varphi(0)]\| \\ &+\int_0^{t_0^+}\|[T_{\beta,\gamma_k}(t-s)-T_{\beta,\gamma_k}(t_0^+-s)]F(s,u(s),u_s)\|dg(s) \\ &+\int_{t_0^+}^{t}\|T_{\beta,\gamma_k}(t-s)F(s,u(s),u_s)\|dg(s) \\ \leq&\|S_{\beta,\gamma_k}(t)-S_{\beta,\gamma_k}(t_0^+)\|_{L(\mathbb{E})}\cdot\|\varphi(0)\|+M|t-t_0^+|\cdot\|[\psi+Q_0(u)]\| \\ &+\sum_{k=1}^{n}\alpha_k M\Big|\frac{t^{1+\beta-\gamma_k}-(t_0^+)^{1+\beta-\gamma_k}}{\Gamma(2+\beta-\gamma_k)}\Big|\|Q(\sigma(u),u)(0)+\varphi(0)\| \\ &+W(R)\int_0^{t_0^+}\|T_{\beta,\gamma_k}(t-s)-T_{\beta,\gamma_k}(t_0^+-s)\|_{\mathcal{L}(E)}P(s)dg(s) \\ &+CW(R)\int_{t_0^+}^{t}(t-s)^{\beta-\gamma_k}P(s)dg(s) \\ =&J_1(t)+J_2(t)+J_3(t)+J_4(t)+J_5(t), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} &J_1(t)=\|S_{\beta,\gamma_k}(t)-S_{\beta,\gamma_k}(t_0^+)\|_{L(E)}\cdot\|Q(\sigma(u),u)(0)+\varphi(0)\|, \\ &J_2(t)=M|t-t_0^+|\cdot\|\psi+Q_0(u)\|, \\ &J_3(t)=\sum_{k=1}^{n}\alpha_k M\Big|\frac{t^{1+\beta-\gamma_k}-(t_0^+)^{1+\beta-\gamma_k}}{\Gamma(2+\beta-\gamma_k)}\Big|\cdot\|Q(\sigma(u),u)(0)+\varphi(0)\|, \end{aligned}$$

$$J_4(t) = W(r)\int_0^{t_0^+} \|T_{\beta,\gamma_k}(t-s) - T_{\beta,\gamma_k}(t_0^+ - s)\|_{L(E)} P(s)dg(s),$$
$$J_5(t) = CW(r)\int_{t_0^+}^{t} (t-s)^{\beta-\gamma_k} P(s)dg(s).$$

By $J_2(t)$ and $J_3(t)$, we get that $J_2(t) \to 0$ and $J_3(t) \to 0$ as $t \to t_0^+$ independently of $u \in \Omega$. Since the compactness of $S_{\beta,\gamma_k}(t)$ and $T_{\beta,\gamma_k}(t)$ for $t > 0$, we have $J_1(t) \to 0$ and applying dominated convergence theorem, we get that $J_4(t) \to 0$ as $t \to t_0^+$ independently of $u \in \Omega$. Let $H(t) = \int_0^t (t-s)^{\beta-\gamma_k} P(s)dg(s)$. In view of Lemma 2.4, we get $H(t)$ is a regulated function on $\mathbb{R}^+$. Therefore,

$$\begin{aligned} J_5(t) =& CW(r)\int_{t_0^+}^{t} (t-s)^{\beta-\gamma_k} P(s)dg(s) \\ \le& CW(r)\Big(\|H(t) - H(t_0^+)\| + \int_0^{t_0^+} \|((t-s)^{\beta-\gamma_k} \\ & - (t_0^+ - s)^{\beta-\gamma_k})P(s)\|dg(s)\Big) \\ \to& 0 \quad \text{as} \quad t \to t_0^+, \end{aligned}$$

we have $\|(\mathcal{Q}u)(t) - (\mathcal{Q}u)(t_0^+)\|_\Omega \to 0$ as $t \to t_0^+$ independently of $u \in \Omega$.

Similarly, one can demonstrate that for any $t_0 \in (-r, b]$, $\|(\mathcal{Q}u)(t) - (\mathcal{Q}u)(t_0^+)\|_\Omega \to 0$ as $t \to t_0^+$. According to the arbitrariness of b, one can find that $u(t)$ is defined on $[-r, +\infty)$. On the other hand, it is easy to see $\lim_{t\to\infty} \|u(t+\omega) - u(t)\| = 0$. Hence, assert that $\{\mathcal{Q}u : u(\cdot) \in \Omega\}$ is equiregulated.

Step II. We verify that $\mathcal{Q} : \Omega \to \Omega$ is continuous operator.

Let $\{u^{(n)}\} \subset \Omega$ be a sequence such that $u^{(n)} \to u(t)$ in Ω as $n \to \infty$, then, $u^{(n)}(t) \to u(t)$ in E and $u_t^{(n)} \to u_t$ in $\mathcal{B}$ for every $t \ge 0$ as $n \to \infty$. For $t \in \mathbb{R}^+$, by the continuity of F and Q, Q_0, when $n \to \infty$, we have

$$\begin{aligned} &F(t, u^{(n)}(t), u_t^{(n)}) \to F(t, u(t), u_t), \quad Q_0(u^{(n)}) \to Q_0(u), \\ &Q(\sigma(u^{(n)}, u^{(n)}) \to Q(\sigma(u), u) \end{aligned}$$

and

$$\|F(t, u^{(n)}(t), u_t^{(n)}) - F(t, u(t), u_t)\| \le 2P(t)W(r). \tag{3.3}$$

Moreover, for each $t \ge 0$, we have

$$\begin{aligned} &\|\mathcal{Q}(u^n)(t) - \mathcal{Q}(u)(t)\| \\ \le& S_{\beta,\gamma_k}(t)\|Q(\sigma(u^{(n)}), u^{(n)}) - Q(\sigma(u), u)\| \\ &+ (\varphi_1 * S_{\beta,\gamma_k})(t)\|Q_0(u^{(n)}) - Q_0(u)\| \\ &+ \sum_{k=1}^{n} \alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)\|Q(\sigma(u^{(n)}), u^{(n)}) - Q(\sigma(u), u)\| \\ &+ C\int_0^t (t-s)^{\beta-\gamma_k}\|F(s, u^{(n)}(s), u_s^{(n)}) - F(s, u(s), u_s)\|dg(s). \end{aligned} \tag{3.4}$$

By (3.3)-(3.4) and the dominated convergence theorem for the Henstock-Lebesgue-Stieltjes integral, for each $t \ge 0$, we get that $\|\mathcal{Q}(u^{(n)})(t) - \mathcal{Q}(u)(t)\|_\Omega \to 0$ as $n \to \infty$.

Moreover, by Step I, it can shown that $\{\mathcal{Q}(u^{(n)})\}_{n=1}^{\infty}$ is equiregulated. Therefore, by Lemma 2.1, we get that $\mathcal{Q}(u^{(n)})$ converge uniformly to $\mathcal{Q}(u)$. Thus, $\mathcal{Q}$ is a continuous operator.

Step III. We show that $\mathcal{Q}([v^{(0)}, w^{(0)}]) \subset \Omega$.

For any $u \in [v^{(0)}, w^{(0)}]$, the operator $(\mathcal{Q}u)$ is defined on $[-r, \infty)$, and since $\varphi \in \mathcal{B}$, we have $\mathcal{Q}u|_{[-r,0]} \in \mathcal{B}$. Thus, we show that the function

$$\begin{aligned} f : t \to & S_{\beta,\gamma_k}(t)[Q(\sigma(u), u)(0) + \varphi(0)] + (\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(u)] \\ & + \sum_{k=1}^{n} \alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(\sigma(u), u)(0) + \varphi(0)] \\ & + \int_0^t T_{\beta,\gamma_k}(t-s)F(s, u(s), u_s)dg(s) \in SAP_\omega(E), \quad t \geq 0. \end{aligned} \tag{3.5}$$

Since $u|_{\mathbb{R}^+} \in SAP_\omega(E)$ and $u_t \in SAP_\omega(\mathcal{B})$ for all $t \geq 0$, $\|u(t+\omega) - u(t)\| \leq \epsilon$ and $\|u_{t+\omega} - u_t\|_{\mathcal{B}} \leq \epsilon$ become arbitrarily small by choosing t large enough. Hence, by the continuity of F, there exists a constant $t_{\epsilon,1} > 0$ such that, for every $t \geq t_{\epsilon,1}$, we have

$$\|F(t, u(t+\omega), u_{t+\omega}) - F(t, u(t), u_t)\| \leq \frac{\epsilon}{2} \tag{3.6}$$

and we can find a positive constant $t_{\epsilon,2}$ sufficiently large such that for $t \geq t_{\epsilon,2}$, by (H1), we have

$$\|F(t+\omega, u(t+\omega), u_{t+\omega}) - F(t, u(t+\omega), u_{t+\omega})\| \leq \frac{\epsilon}{2}. \tag{3.7}$$

Then for $t > t_\epsilon := \max\{t_{\epsilon,1}, t_{\epsilon,2}\}$, by (3.5), we get

$$\begin{aligned} & f(t+\omega) - f(t) \\ = & S_{\beta,\gamma_k}(t+\omega)[Q(\sigma(u), u)(0) + \varphi(0)] \\ & + (\varphi_1 * S_{\beta,\gamma_k})(t+\omega)[\psi + Q_0(u)] \\ & + \sum_{k=1}^{n} \alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t+\omega)[Q(\sigma(u), u)(0) + \varphi(0)] \\ & + \int_0^t T_{\beta,\gamma_k}(t+\omega-s)F(s, u(s), u_s)dg(s) \\ & - S_{\beta,\gamma_k}(t)\varphi(0) - (\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(u)] \\ & - \sum_{k=1}^{n} \alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(\sigma(u), u)(0) + \varphi(0)] \\ & - \int_0^t T_{\beta,\gamma_k}(t-s)F(s, u(s), u_s)dg(s) \\ = & S_{\beta,\gamma_k}(t+\omega)\varphi(0) - S_{\beta,\gamma_k}(t)[Q(\sigma(u), u)(0) + \varphi(0)] \\ & + \Big((\varphi_1 * S_{\beta,\gamma_k})(t+\omega) - (\varphi_1 * S_{\beta,\gamma_k})(t)\Big)[\psi + Q_0(u)] \\ & + \sum_{k=1}^{n} \alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t+\omega)[Q(\sigma(u), u)(0) + \varphi(0)] \\ & - \sum_{k=1}^{n} \alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(\sigma(u), u)(0) + \varphi(0)] \end{aligned}$$

$$
\begin{aligned}
&+\int_0^{\omega} T_{\beta,\gamma_k}(t+\omega-s)F(s,u(s),u_s)dg(s)\\
&+\int_0^{t} T_{\beta,\gamma_k}(t-s)(F(s+\omega,u(s+\omega),u_{s+\omega})-F(s,u(s),u_s))dg(s)\\
&+\int_0^{t} T_{\beta,\gamma_k}(t-s)(F(s,u(s+\omega),u_{s+\omega})-F(s,u(s),u_s))dg(s)\\
:=&J_1(t)+J_2(t)+J_3(t)+J_4(t)+J_5(t).
\end{aligned}
\tag{3.8}
$$

Then

$$
\|f(t+\omega)-f(t)\|\leq\|J_1(t)\|+\|J_2(t)\|+\|J_3(t)\|+\|J_4(t)\|+\|J_5(t)\|. \tag{3.9}
$$

Hence, we have

$$
\begin{aligned}
&\|J_1(t)\|\\
\leq&\|S_{\beta,\gamma_k}(t+\omega)[Q(\sigma(u),u)(0)+\varphi(0)]\|+\|S_{\beta,\gamma_k}(t)[Q(\sigma(u),u)(0)+\varphi(0)]\|\\
\leq&(\|S_{\beta,\gamma_k}(t+\omega)\|+\|S_{\beta,\gamma_k}(t)\|)\cdot\|Q(\sigma(u),u)(0)+\varphi(0)\|\\
\leq&\frac{2C\|Q(\sigma(u),u)(0)+\varphi(0)\|}{1+|\kappa|(t^{\beta+1}+\sum_{k=1}^n\alpha_k t^{\gamma_k})},
\end{aligned}
$$

it is implies that $\|J_1(t)\|$ tend to 0 as $t\to\infty$.

On the other hand, by (2.1) we have $\sup_{t>\tau}\|tS_{\beta,\gamma_k}(t)\|<\infty$, for each $\tau>0$. Since A is an ω-sectorial of angle $\gamma_k\frac{\pi}{2}$ then $\|\mathcal{L}[S_{\beta,\gamma_k}](\lambda)\|\to 0$ as $\lambda\to 0$. Thus, by the vector-valued Hardy-Littlewood theorem (see [2], Theorem 4.2.9), we obtain

$$
\|(\varphi_1 * S_{\beta,\gamma_k})(t)\|\to 0 \quad\text{as}\quad t\to\infty. \tag{3.10}
$$

By (2.4), it is implies that

$$
\|\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k}(t)\|\to 0 \text{ as } \ t\to\infty. \tag{3.11}
$$

Hence, we have

$$
\begin{aligned}
\|J_2(t)\|&\leq\|(\varphi_1 * S_{\beta,\gamma_k})(t+\omega)[\psi+Q_0(u)]-(\varphi_1 * S_{\beta,\gamma_k})(t)[\psi+Q_0(u)]\|\\
&\leq\|(\varphi_1 * S_{\beta,\gamma_k})(t+\omega)-(\varphi_1 * S_{\beta,\gamma_k})(t)\|\cdot\|\psi+Q_0(u)\|.
\end{aligned}
$$

By (3.10), we deduce that $\|J_2(t)\|$ tend to 0 as $t\to\infty$.

$$
\begin{aligned}
\|J_3(t)\|\leq&\|\sum_{k=1}^n\alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t+\omega)[Q(\sigma(u),u)(0)+\varphi(0)]\\
&-\sum_{k=1}^n\alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(\sigma(u),u)(0)+\varphi(0)]\|\\
\leq&\sum_{k=1}^n\alpha_k\|[(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t+\omega)\\
&-(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)]\|\cdot\|Q(\sigma(u),u)(0)+\varphi(0)\|.
\end{aligned}
$$

By (3.11), we have $\|J_3(t)\|$ tend to 0 as $t\to\infty$.

By (H1), we have

$$\|J_4\| \leq \int_0^\omega \|T_{\beta,\gamma_k}(t+\omega-s)\| \cdot \|F(s,u(s),u_s)\| dg(s)$$
$$\leq CW(r)\int_0^\omega (t+\omega-s)^{\beta-\gamma_k}P(s)dg(s).$$

Thus, $\|J_4(t)\|$ tend to 0 as $t \to \infty$. By (3.6), (3.7) and (H1), we have

$$\begin{aligned}
&\|J_5\| \\
\leq &\int_0^t \|T_{\beta,\gamma_k}(t-s)\| \cdot \|F(s+\omega,u(s+\omega),u_{s+\omega}) \\
&- F(s,u(s+\omega),u_{s+\omega})\| dg(s) \\
&+ \int_0^t \|T_{\beta,\gamma_k}(t-s)\| \cdot \|F(s,u(s+\omega),u_{s+\omega}) - F(s,u(s),u_s)\| dg(s) \\
\leq &4CW(r)\Big(\int_0^{t_\epsilon} (t-s)^{q(\beta-\gamma_k)}dg(s)\Big)^{\frac{1}{q}}\|P\|_{\mathbb{HLS}_g^p} + \epsilon\int_{t_\epsilon}^t \|T_{\beta,\gamma_k}(t-s)\|dg(s) \\
&+ 4CW(r)\Big(\int_0^{t_\epsilon} (t-s)^{q(\beta-\gamma_k)}dg(s)\Big)^{\frac{1}{q}}\|P\|_{\mathbb{HLS}_g^p} + \epsilon\int_{t_\epsilon}^t \|T_{\beta,\gamma_k}(t-s)\|dg(s) \\
\leq &8CW(r)\Big(\int_0^{t_\epsilon} (t-s)^{q(\beta-\gamma_k)}dg(s)\Big)^{\frac{1}{q}}\|P\|_{\mathbb{HLS}_g^p} + 2\epsilon\int_0^t \|T_{\beta,\gamma_k}(t-s)\|dg(s),
\end{aligned}$$

which implies that $\|J_5(t)\|$ tends to 0 ad $t \to \infty$. Thus, from the above results, we have

$$\lim_{t\to\infty} \|f(t+\omega) - f(t)\| = 0.$$

Combining this with the definition $\mathcal{Q}$, we have $\mathcal{Q}(SPA_\omega(E)) \subset SPA_\omega(E)$, and combining this fact with Step II, we obtain $(\mathcal{Q}u) \in \Omega$ for any $u \in [v^{(0)}, w^{(0)}]$, $\mathcal{Q}([v^{(0)}, w^{(0)}]) \subset \Omega$.

Step IV. We check that $\mathcal{Q} : [v^{(0)}, w^{(0)}] \to [v^{(0)}, w^{(0)}]$ is a monotonically increasing operator. Since $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$ is positive, thus $(\varphi_1 * S_{\beta,\gamma_k})(t)$, $(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)$ and $T_{\beta,\gamma_k}(t) = (\varphi_\beta * S_{\beta,\gamma_k})(t)$ are also positive. On the one hand, in view of Definition 2.10, Definition 2.11, and the positivity of operators $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$, $(\varphi_1 * S_{\beta,\gamma_k})(t)$, $(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)$ and $T_{\beta,\gamma_k}(t) = (\varphi_\beta * S_{\beta,\gamma_k})(t)$, we can deduce that for any $t \in [0,\infty)$, $v^{(0)}(t) \leq (\mathcal{Q}v^{(0)})(t)$, together with $v^{(0)}(t) = \varphi(t) = (\mathcal{Q}v^{(0)})(t)$ for $t \in [-r, 0]$, we get $v^{(0)} \leq \mathcal{Q}v^{(0)}$. Similarly, $\mathcal{Q}w^{(0)} \leq w^{(0)}$ is available.

On the other hand, let $u^{(1)}, u^{(2)} \in [v^{(0)}, w^{(0)}]$ with $u^{(1)} \leq u^{(2)}$, we can see

$$v^{(0)}(t) \leq u^{(1)}(t) \leq u^{(2)}(t) \leq w^{(0)}(t), \quad t \in [-r, a],$$

$$v_t^{(0)} \leq u_t^{(1)} \leq u_t^{(2)} \leq w_t^{(0)}, \quad t \in [0,\infty).$$

Thus, by (H1), (H2) and the positivity of $S_{\beta,\gamma_k}(t)(t \geq 0), T_{\beta,\gamma_k}(t)(t \geq 0)$, we can get

$$\mathcal{Q}u^{(1)} \leq \mathcal{Q}u^{(2)}.$$

Consequently, $\mathcal{Q} : [v^{(0)}, w^{(0)}] \to [v^{(0)}, w^{(0)}]$ is a monotonically increasing operator.

Now, we establish two iterative sequences $\{v^{(n)}\}$ and $\{w^{(n)}\}$ in $[v^{(0)}, w^{(0)}]$ by

$$v^{(n)} = \mathcal{Q}v^{(n-1)}, \quad w^{(n)} = \mathcal{Q}w^{(n-1)}, \qquad n = 1, 2, \cdots. \tag{3.12}$$

Using the monotonicity of $\mathcal{Q}$, we can easily confirm that $\{v^{(n)}\}$ and $\{w^{(n)}\}$ satisfy:

$$v^{(0)} \leq v^{(1)} \leq v^{(2)} \leq \cdots \leq v^{(n)} \leq \cdots \leq w^{(n)} \leq \cdots \leq w^{(2)} \leq w^{(1)} \leq w^{(0)}. \quad (3.13)$$

Step V. We prove that $\{v^{(n)}\}$ and $\{w^{(n)}\}$ are convergent in Ω.

For any $a > 0$, let $V = \{v^{(n)} | n \in \mathbb{N}\}$ and $V_0 = \{v^{(n-1)} | n \in \mathbb{N}\}$. Then $V(t) = (\mathcal{Q}V_0)(t)$ for $t \in [-r, a]$. In fact, $v^{(n)}(t) = \varphi(t)$ for $t \in [-r, 0]$, thus, $\{v^{(n)}(t)\}$ is relatively compact on E for $t \in [-r, 0]$. For $\forall \epsilon \in (0, t)$, we define a set $\{(\mathcal{Q}^\epsilon V_0)(t)\}$ by

$$\mathcal{Q}^\epsilon V_0(t) := \{Q^\epsilon v^{(n)}(t) | v^{(n)} \in V_0, t \in [0, a]\},$$

where

$$\begin{aligned}
&\mathcal{Q}^\epsilon v^{(n)}(t)\\
=&S_{\beta,\gamma_k}(t)[Q(\sigma(v^{(n-1)}), v^{(n-1)})(0) + \varphi(0)]\\
&+ (\varphi_1 * S_{\beta,\gamma_k})(t-\epsilon)[\psi + Q_0(v^{(n-1)})]\\
&+ \sum_{k=1}^{n} \alpha_k \int_0^{t-\epsilon} \frac{(t-s)^{\beta-\gamma_k}}{\Gamma(1+\beta-\gamma_k)} S_{\beta,\gamma_k}(t)[Q(\sigma(v^{(n-1)}), v^{(n-1)})(0) + \varphi(0)]ds\\
&+ \int_0^{t-\epsilon} T_{\beta,\gamma_k}(t-s) F(s, v^{(n-1)}(s), v_s^{(n-1)}) dg(s), \quad t \geq 0.
\end{aligned}$$

And by the compactness of $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$, we obtain that the set $\mathcal{Q}^\epsilon V_0(t)$ is relatively compact in E for all $\epsilon \in (0, t)$. Moreover, for every $v^{(n)} \in V_0$ and $t \in [0, a]$, from the following inequality

$$\begin{aligned}
&\|\mathcal{Q}v^{(n)}(t) - \mathcal{Q}^\epsilon v^{(n)}(t)\|\\
\leq&\|((\varphi_1 * S_{\beta,\gamma_k}(t) - \varphi_1 * S_{\beta,\gamma_k}(t-\epsilon))[\psi + Q_0(v^{(n-1)})]\|\\
&+ \|\sum_{k=1}^{n} \alpha_k \int_{t-\epsilon}^{t} \frac{(t-s)^{\beta-\gamma_k}}{\Gamma(1+\beta-\gamma_k)} S_{\beta,\gamma_k}(s)[Q(\sigma(v^{(n-1)}), v^{(n-1)})(0) + \varphi(0)]ds\|\\
&+ \|\int_{t-\epsilon}^{t} T_{\beta,\gamma_k}(t-s) F(s, v^{(n-1)}(s), v_s^{(n-1)}) dg(s)\|\\
\leq&M\epsilon(\|\psi\| + c_0\|v^{(n)}\|_\infty + d_0) + \sum_{k=1}^{n} \frac{\alpha_k M \epsilon^{1+\beta-\gamma_k}}{\Gamma(2+\beta-\gamma_k)} (\|\varphi(0)\| + c_1\|v^{(n)}\|_\infty + d_1)\\
&+ CW(r) \int_{t-\epsilon}^{t} (t-s)^{\beta-\gamma_k} P(s) dg(s)\\
\leq&M\epsilon(\|\psi\| + c_0\|v^{(n)}\|_\infty + d_0) + \sum_{k=1}^{n} \frac{\alpha_k M \epsilon^{1+\beta-\gamma_k}}{\Gamma(2+\beta-\gamma_k)} (\|\varphi(0)\| + c_1\|v^{(n)}\|_\infty + d_1)\\
&+ CW(r)\Big(\|H(t) - H(t-\epsilon)\|\\
&+ \int_0^{t-\epsilon} |(t-s)^{\beta-\gamma_k} - (t-\epsilon-s)^{\beta-\gamma_k}| P(s) dg(s)\Big)\\
\to&0 \quad \text{as} \quad \epsilon \to 0,
\end{aligned}$$

thus, the set $\{(\mathcal{Q}V_0)(t)\}$ is relatively compact, which implies that $\{v^{(n)}(t)\}$ is relatively compact on E for $t \in [0, a]$. Thus, we have proved that $\{v^{(n)}(t)\}$ is relatively compact on E for $t \in [-r, a]$.

Therefore, $\{v^{(n)}\}$ is relatively compact in $G([-r,a],E)$ by the Arzelà-Ascoli Theorem, which implies that there is convergent subsequence in $v^{(n)}$. Combing with the monotonicity and the normality of the cone, we have $\{v^{(n)}\}$ and $\{w^{(n)}\}$ themselves are convergent, i.e., there exist $\underline{u}, \overline{u} \in G([-r,a],E)$, such that

$$\underline{u}(t) = \lim_{n\to\infty} v^{(n)}(t), \quad \overline{u}(t) = \lim_{n\to\infty} w^{(n)}(t), \qquad t \in [-r,a].$$

By the arbitrariness of a, we have $\underline{u}$ and $\overline{u}$ are defined on $[-r,\infty)$. On the other hand, it is easy to see $\lim_{t\to\infty} \|\underline{u}(t+\omega)-\underline{u}(t)\| = 0$ and $\lim_{t\to\infty} \|\overline{u}(t+\omega)-\overline{u}(t)\| = 0$. Hence, we can deduce that there exist $\underline{u}, \overline{u} \in \Omega$, such that

$$\underline{u}(t) = \lim_{n\to\infty} v^{(n)}(t), \quad \overline{u}(t) = \lim_{n\to\infty} w^{(n)}(t), \qquad t \in [-r,\infty). \tag{3.14}$$

Taking limit in (3.12), we have

$$\underline{u} = \mathcal{Q}\underline{u}, \qquad \overline{u} = \mathcal{Q}\overline{u}.$$

Therefore $\underline{u}, \overline{u} \in \Omega$ are fixed points of $\mathcal{Q}$ and they are the S-asymptotically ω-periodic mild solution of the problem (1.1).

Step VI. We claim that $\underline{u}$ and $\overline{u}$ are the minimal and maximal S-asymptotically ω-periodic mild solutions of the nonlocal problem (1.1), respectively.

Taking limit of both ends of (3.12), we can deduce from (3.14) that

$$\underline{u} = \mathcal{Q}\underline{u}, \quad \overline{u} = \mathcal{Q}\overline{u}. \tag{3.15}$$

Applying (3.13), we can get $\underline{u}, \overline{u} \in [v^{(0)}, w^{(0)}] \subset \Omega$ that are fixed points of $\mathcal{Q}$ and $\underline{u} \leq \overline{u}$. In fact, let $u \in [v^{(0)}, w^{(0)}]$ is an arbitrary fixed point of $\mathcal{Q}$, then for every $t \in [-r,\infty)$, we have $v^{(0)}(t) \leq u(t) \leq w^{(0)}(t)$, and

$$v^{(1)}(t) = (\mathcal{Q}v^{(0)})(t) \leq (\mathcal{Q}u)(t) = u(t) \leq (\mathcal{Q}w^{(0)})(t) = w^{(1)}(t),$$

namely,

$$v^{(1)} \leq u \leq w^{(1)}.$$

Repeat this process, we get

$$v^{(n)} \leq u \leq w^{(n)}, \qquad n = 1, 2, \cdots.$$

Let $n \to \infty$, we can see $\underline{u} \leq u \leq \overline{u}$. Therefore $\underline{u}$ and $\overline{u}$, respectively, are the minimal and maximal S-asymptotically ω-periodic mild solutions of nonlocal problem (1.1) in $[v^{(0)}, w^{(0)}]$, and $\underline{u}, \overline{u}$ can be obtained by the iterative sequences (3.12) starting from $v^{(0)}$ and $w^{(0)}$, respectively. $\square$

Theorem 3.2. *Let E be an ordered Banach space, whose positive cone $K \subset E$ is normal. Let $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$ and $\alpha_k \geq 0, k = 1, 2, \ldots n$ be given. A is an κ-sectorial operator of angle $\frac{\gamma_k \pi}{2}, k = 1, 2, \cdots, n$ with $\kappa < 0$, and A generates a positive and equicontinuous (β, γ_k)-resolvent family $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$ on E. Assume that $\omega > 0$ is a constant and the nonlocal problem (1.1) has a lower mild solution $v^{(0)}$ and an upper mild solution $w^{(0)}$ with $v^{(0)} \leq w^{(0)}$. If $\varphi \in K_{\mathcal{B}}$, $Q(\sigma(u), u)(0) + \varphi(0) \in K \cap \mathcal{D}(A)$ and $\psi \in K$, $F : \mathbb{R}^+ \times E \times \mathcal{B} \to E$ is continuous and satisfies the conditions (H1)-(H4) and the following conditions*

(*H*5) *For each* $t \in \mathbb{R}^+$, *and monotone sequence* $\{u^{(n)}\} \subset [v^{(0)}, w^{(0)}]$, *there exist constants* $L_f \geq 0, 0 < L_h < \frac{\Gamma(2+\beta-\gamma_k)(1-2\widetilde{M}L_g)}{2M(\Gamma(2+\beta-\gamma_k)+\sum_{k=1}^{n}|\alpha_k|a^{\beta-\gamma_k+1})}$ *such that*

$$\alpha(\{F(t,u^{(n)}(t),u_t^{(n)})\}) \leq L_f\Big(\alpha(\{u^{(n)}(t)\}) + \sup_{s\in[-r,0]}\alpha(\{u_t^{(n)}(s)\})\Big),$$

$$\alpha(\{Q_0(u^{(n)}(t))\}) \leq L_g\alpha(\{u^{(n)}(t)\}),\ \alpha(\{Q(\sigma(u^{(n)}(t)),u^{(n)}(t))\}) \leq L_h\alpha(\{u^{(n)}(t)\}).$$

Then the nonlocal problem (1.1) has minimal and maximal S-asymptotically ω-periodic mild solutions $\underline{u}, \overline{u} \in [v^{(0)}, w^{(0)}]$, *which can be obtained by the monotone iterative procedures starting from* $v^{(0)}$ *and* $w^{(0)}$, *respectively.*

Proof. Let $\mathcal{Q}$ be defined by (3.1). From the proof of Theorem 3.1, we know that $\mathcal{Q} : [v^{(0)}, w^{(0)}] \to [v^{(0)}, w^{(0)}]$ is a continuous increasing operator and $v^{(0)} \leq \mathcal{Q}v^{(0)}$, $\mathcal{Q}w^{(0)} \leq w^{(0)}$. Hence, the iterative sequences $v^{(n)}$ and $w^{(n)}$ defined by (3.12) satisfy (3.13). By $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$ is an equicontinuous resolvent family, by the Step. I of proof of Theorem 3.1, we get $\{v^{(n)}\}$ and $\{w^{(n)}\}$ are bounded and equiregulated in $t \in [-r, +\infty)$.

Next, we prove that $\{v^{(n)}\}$ and $\{w^{(n)}\}$ are convergent in Ω.

For $\forall a > 0$, restrict $\{v^{(n)}\}$ to interval $[-r, a]$. Let $V = \{v^{(n)}|n \in \mathbb{N}\}$ and $V_0 = \{v^{(n-1)}|n \in \mathbb{N}\}$. Then $V = (\mathcal{Q}V_0)$. From $V_0 = V \cup \{v^{(0)}\}$ it follow that $\alpha(V_0(t)) = \alpha(V(t))$ for $t \in [-r, a]$.

For $t \in [-r, 0]$, in view of the fact that $v^{(n)}(t) = \mathcal{Q}v^{(n-1)}(t) = \varphi(t)$, we can see

$$\alpha(V(t)) = 0, \quad t \in [-r, 0]. \tag{3.16}$$

For $t \in [0, a]$, we have

$$\sup_{s\in[-r,0]}\alpha(\{v_t^{(n)}(s)\}) = \sup_{s\in[-r,0]}\alpha(\{v^{(n)}(t+s)\}) \leq \alpha(\{v^{(n)}(t)\}). \tag{3.17}$$

By Lemma 2.2, we have

$$\begin{aligned}
\alpha(V(t)) =& \alpha(\{\mathcal{Q}V_0(t)\}) \\
=& \alpha\Big(\Big\{S_{\beta,\gamma_k}(t)[Q(\sigma(v^{(n-1)}), v^{(n-1)})(0) + \varphi(0)] \\
& + (\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(v^{(n-1)})] \\
& + \sum_{k=1}^{n}\alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(\sigma(v^{(n-1)}), v^{(n-1)})(0) + \varphi(0)] \\
& + \int_0^t T_{\beta,\gamma_k}(t-s)F(s, v^{(n-1)}(s), v_s^{(n-1)})dg(s)\Big\}\Big) \\
\leq& \alpha(\{S_{\beta,\gamma_k}(t)[Q(\sigma(v^{(n-1)}), v^{(n-1)})(0) + \varphi(0)]\}) \\
& + \alpha\Big(\Big\{\int_0^t S_{\beta,\gamma_k}(t)[\psi + Q_0(v^{(n-1)})]ds\Big\}\Big) \\
& + \alpha\Big(\Big\{\sum_{k=1}^{n}\alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(\sigma(v^{(n-1)}), v^{(n-1)})(0) + \varphi(0)]\Big\}\Big) \\
& + \alpha\Big(\Big\{\int_0^t T_{\beta,\gamma_k}(t-s)F(s, v^{(n-1)}(s), v_s^{(n-1)})dg(s)\Big\}\Big)
\end{aligned}$$

$$=\alpha_0+\alpha_1+\alpha_2+\alpha_3.$$

First of all, we have

$$\begin{aligned}\alpha_0 :=&\alpha(S_{\beta,\gamma_k}(t)[Q(\sigma(v^{(n-1)}),v^{(n-1)})(0)+\varphi(0)]\})\\ \le&2M\alpha(\{Q(\sigma(v^{(n-1)}),v^{(n-1)})(0)\})\\ =&2ML_h\alpha(V(t)).\end{aligned}$$

By Lemma 2.12, (H5), (2.5) and (3.17), we have

$$\begin{aligned}\alpha_1 :=&\ \alpha\Big(\Big\{(\varphi_1 * S_{\beta,\gamma_k})(t)[\psi+Q_0(v^{(n-1)})]\Big\}\Big)\\ \le&\ 2\widetilde{M}\alpha(\{(\psi+Q_0(v^{(n-1)})(t))\})\\ \le&\ 2\widetilde{M}\alpha(\{Q_0(v^{(n-1)})(t)\})\\ =&\ 2\widetilde{M}L_g\alpha(V(t)),\end{aligned}$$

$$\begin{aligned}\alpha_2 :=&\ \alpha\Big(\Big\{\sum_{k=1}^{n}\alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(\sigma(v^{(n-1)}),v^{(n-1)})(0)+\varphi(0)]\Big\}\Big)\\ \le&\ 2\sum_{k=1}^{n}|\alpha_k|\int_0^t\frac{(t-s)^{\beta-\gamma_k}}{\Gamma(1+\beta-\gamma_k)}\|S_{\beta,\gamma_k}(t-s)\|_{\mathcal{L}(E)}\\ &\times\alpha(\{Q(\sigma(v^{(n-1)}),v^{(n-1)})(0)\})ds\\ \le&\ \frac{2ML_h\sum_{k=1}^{n}|\alpha_k|a^{\beta-\gamma_k+1}}{\Gamma(2+\beta-\gamma_k)}\alpha(V(t)),\end{aligned}$$

$$\begin{aligned}\alpha_3 :=&\ \alpha\Big(\Big\{\int_0^t T_{\beta,\gamma_k}(t-s)F(s,v^{(n-1)}(s),v_s^{(n-1)})dg(s)\Big\}\Big)\\ \le&\ 2\int_0^t\|T_{\beta,\gamma_k}(t-s)\|_{\mathcal{L}(E)}\cdot\alpha(\{F(s,v^{(n-1)}(s),v_s^{(n-1)})\})ds\\ \le&\ 2C\int_0^t(t-s)^{\beta-\gamma_k}\alpha(\{F(s,v^{(n-1)}(s),v_s^{(n-1)})\})dg(s)\\ \le&\ \frac{2Ca^{\beta-\gamma_k+1}}{\beta-\gamma_k+1}L_f\int_0^t\alpha(V(s))dg(s).\end{aligned}$$

Consequently, for $t\in[0,a]$, we have

$$\begin{aligned}\alpha(V(t))\le&2ML_h\alpha(V(t))+\frac{2ML_h\sum_{k=1}^{n}|\alpha_k|a^{\beta-\gamma_k+1}}{\Gamma(2+\beta-\gamma_k)}\alpha(V(t))\\ &+2\widetilde{M}L_g\alpha(V(t))+\frac{2Ca^{\beta-\gamma_k+1}}{\beta-\gamma_k+1}L_f\int_0^t\alpha(V(s))dg(s).\end{aligned}$$

Since $\Gamma(2+\beta-\gamma_k)[1-2ML_h-2\widetilde{M}L_g]>2ML_h\sum_{k=1}^{n}|\alpha_k|a^{\beta-\gamma_k+1}$, it gives that

$$\alpha(V(t))\le\frac{2Ca^{\beta-\gamma_k+1}L_f\Gamma(1+\beta-\gamma_k)\int_0^t\alpha(V(s))dg(s)}{\Gamma(2+\beta-\gamma_k)[1-2ML_h-2\widetilde{M}L_g]-2ML_h\sum_{k=1}^{n}|\alpha_k|a^{\beta-\gamma_k+1}}.$$

Hence, by Bellman inequality, $\alpha(V(t)) \equiv 0$ in $[0,a]$. Combining with (3.16), we have $\alpha(V(t)) \equiv 0$ in $[-r,a]$, which shows that $\{v^{(n)}(t)\}$ is precompact on E for any $t \in [-r,a]$. We can similarly show that $\{w^{(n)}(t)\}$ is also precompact on E for $t \in [-r,a]$. Hence, there are convergent subsequences in $\{v^{(n)}\}$ and $\{w^{(n)}\}$. Combining with the monotonicity and the normality of the cone, it is clear that $\{v^{(n)}\}$ and $\{w^{(n)}\}$ themselves are convergent, i.e., there exist $\underline{u}, \overline{u} \in C([-r,a],E)$, such that

$$\underline{u}(t) = \lim_{n\to\infty} v^{(n)}(t), \quad \overline{u}(t) = \lim_{n\to\infty} w^{(n)}(t), \qquad t \in [-r,a].$$

According to the arbitrariness of a, one can find that $\underline{u}$ and $\overline{u}$ are defined on $[-r,\infty)$. On the other hand, it is easy to see $\lim_{t\to\infty} \|\underline{u}(t+\omega) - \underline{u}(t)\| = 0$ and $\lim_{t\to\infty} \|\overline{u}(t+\omega) - \overline{u}(t)\| = 0$. Hence, we can deduce that there exist $\underline{u}, \overline{u} \in \Omega$, such that

$$\underline{u}(t) = \lim_{n\to\infty} v^{(n)}(t), \quad \overline{u}(t) = \lim_{n\to\infty} w^{(n)}(t), \qquad t \in [-r,\infty). \tag{3.18}$$

And by the Step.VI of proof of Theorem 3.1, $\underline{u}, \overline{u}$ are the minimal and maximal S-asymptotically ω-periodic mild solutions of the problem (1.1), which can be obtained by monotone iterative sequences starting from $v^{(0)}$ and $w^{(0)}$. $\square$

Theorem 3.3. *Let E be an ordered Banach space, whose positive cone $K \subset E$ is normal, Let $0 < \beta \le \gamma_n \le \cdots \le \gamma_1 \le 1$ and $\alpha_k \ge 0, k = 1,2,\ldots n$ be given. A is an κ-sectorial operator of angle $\frac{\gamma_k \pi}{2}, k = 1,2,\cdots,n$ with $\kappa < 0$, and A generates positive and equicontinuous (β,γ_k)-resolvent family $\{S_{\beta,\gamma_k}(t)\}_{t\ge 0}$ on E. Assume that $\omega > 0$ is a constant, $\varphi \in K_{\mathcal{B}}$, $Q(u)(0) + \varphi(0) \in K \cap \mathcal{D}(A)$ and $\psi \in K$, $F : \mathbb{R}^+ \times K \times K_{\mathcal{B}} \to E$, $Q : \mathbb{R}^+ \times G([-r,+\infty),E) \to E, Q_0 : G([-r,+\infty),E) \to E$ are continuous, $\sigma : G([-r,+\infty),E) \to E$ is continuous and $F(t,\theta,\theta) \ge \theta$ for $t \ge 0$. If the condition (H2) and the following conditions are established:*

(H6) For any $R > 0$, $t \ge 0$, $x_1, x_2 \in K$ with $\theta \le x_1 \le x_2$, $\|x_i\| \le R$ and $\phi_1, \phi_2 \in K_{\mathcal{B}}$ with $\theta \le \phi_1 \le \phi_2$, $\|\phi_i\|_{\mathcal{B}} \le R$,

$$F(t,x_2,\phi_2) \ge F(t,x_1,\phi_1) \ge \theta.$$

(H7) For any $t \ge 0$, $x \in E$ and $\phi \in \mathcal{B}$, there exist functions $p_i(\cdot) \in \mathbb{HLS}^p_g(\mathbb{R}^+,\mathbb{R}^+)$ for some $p > 1$ and nondecreasing functions $\mathcal{F}_i \in C(\mathbb{R}^+,\mathbb{R}^+)(i = 1,2)$ as well as a positive constant $\overline{K}$ such that

$$\|F(t,x,\phi)\| \le p_1(t)\mathcal{F}_1(\|x\|) + p_2(t)\mathcal{F}_2(\|\phi\|_{\mathcal{B}}) + \overline{K},$$

where $\mathcal{F}_i$ and p_i satisfy

$$\liminf_{l\to+\infty} \frac{\mathcal{F}_i(l)}{l} := \zeta_i < +\infty, \ i = 1,2.$$

(H8) The nonlocal functions $Q(\sigma(u),u), Q_0(u)$ are bounded and increasing for $u \in G([-r,+\infty),K)$ with $\|u\|_C \le R$ such that

$$\liminf_{R\to\infty} \frac{Q_0(R)}{R} := \eta < +\infty, \ \liminf_{R\to\infty} \frac{Q(\sigma(R),R)}{R} := \eta_1 < +\infty.$$

$(H9)$ *For any* $R > 0$, $t \geq 0$, *and the monotone increasing sequence* $\{u^{(n)}\} \subset \bar{B}(\theta, R)$, *there exist constants* $L_f, L_g, L_h \geq 0$ *such that*

$$\alpha(\{F(t, u^{(n)}(t), u^{(n)}_t)\}) \leq L_f\Big(\alpha(\{u^{(n)}(t)\}) + \sup_{s\in[-r,0]} \alpha(\{u^{(n)}_t(s)\})\Big),$$

$$\alpha(\{Q_0(u^{(n)}(t))\}) \leq L_g\alpha(\{u^{(n)}(t)\}),\ \alpha(\{Q(\sigma(u^{(n)}(t)), u^{(n)}(t))\}) \leq L_h\alpha(\{u^{(n)}(t)\}).$$

$(H10)$ *The function* $s \mapsto \int_0^{\cdot}(\cdot - s)^{\beta-\gamma_k}dg(s)$ *belongs to* $\mathbb{HLS}^q_g(\mathbb{R}^+, \mathbb{R}^+)$.

Then the nonlocal problem (1.1) has at least a S*-asymptotically* ω*-periodic positive mild solution* $u \in G([-r, \infty), K)$ *provided that*

$$\begin{aligned}&(M + \sum_{k=1}^{n} \alpha_k M_1)\eta_1 + \widetilde{M}\eta + C \sup_{t\geq 0}\Big(\int_0^t (t-s)^{q(\beta-\gamma_k)}dg(s)\Big)^{\frac{1}{q}}\\ &\times (\zeta_1\|p_1\|_{\mathbb{HLS}^p_g} + \zeta_2\|p_2\|_{\mathbb{HLS}^p_g}) < 1, \qquad (3.19)\end{aligned}$$

and $\frac{1}{q} + \frac{1}{p} = 1$.

Proof. Let a be any positive constant. For given $\varphi \in K_{\mathcal{B}}$, $\|\varphi\|_{\mathcal{B}} \leq R$. Define

$$\Omega_R = \big\{u \in C([-r, \infty), K)\big|\|u(t)\| \leq R, t \in \mathbb{R}^+; u|_{[-r,0]} \in \mathcal{B}, u(t) = \varphi(t), t \in [-r, 0]\big\},$$

and the operator $\mathcal{Q} : \Omega_R \to K$ by

$$(\mathcal{Q}u)(t) = \begin{cases} S_{\beta,\gamma_k}(t)[Q(\sigma(u), u)(0) + \varphi(0)] + (\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(u)] \\ + \sum_{k=1}^{n} \alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(\sigma(u), u)(0) + \varphi(0)] \\ + \int_0^t T_{\beta,\gamma_k}(t-s)F(s, u(s), u_s)dg(s), \qquad t \in [0, a], \\ Q(u)(t) + \varphi(t), \qquad t \in [-r, 0]. \end{cases} \qquad (3.20)$$

From the hypothesis (H6)-(H8), the positivity of $S_{\beta,\gamma_k}(t)(t \geq 0)$ and the definition of Ω_R, it follows that the positive mild solution of nonlocal problem (1.1) in $\mathbb{R}^+$ is equivalent to the fixed point of $\mathcal{Q}$.

Step I. We check that there is a constant $R_0 > 0$ such that $\mathcal{Q}(\Omega_{R_0}) \subset \Omega_{R_0}$.

In view of (2.4), we observe that as $M_1 := \sup_{t\geq 0} \|\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k}(t)\| < +\infty$.

Indeed, if this were not so, it would follows that for any $R > 0$, there exists $u \in \Omega_R$ such that $\|\mathcal{Q}u\| > R$. In view of (2.5) and (3.20), for any $t \geq 0$, we have

$$\begin{aligned}&\|(\mathcal{Q}u)(t)\|\\ \leq&\|S_{\beta,\gamma_k}(t)[Q(\sigma(u), u)(0) + \varphi(0)] + (\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(u)]\\ &+ \sum_{k=1}^{n} \alpha_k \int_0^t \frac{(t-s)^{\beta-\gamma_k}}{\Gamma(1+\beta-\gamma_k)} S_{\beta,\gamma_k}(s)[Q(\sigma(u), u)(0) + \varphi(0)]ds\\ &+ \int_0^t T_{\beta,\gamma_k}(t-s)F(s, u(s), u_s)dg(s)\|\\ \leq&\|S_{\beta,\gamma_k}(t)[Q(\sigma(u), u)(0) + \varphi(0)]\| + \|(\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(u)]\|\end{aligned}$$

$$
\begin{aligned}
&+\Big\|\sum_{k=1}^{n}\alpha_k\int_0^t\frac{(t-s)^{\beta-\gamma_k}}{\Gamma(1+\beta-\gamma_k)}S_{\beta,\gamma_k}(s)[Q(\sigma(u),u)(0)+\varphi(0)]ds\Big\|\\
&+\Big\|\int_0^t T_{\beta,\gamma_k}(t-s)F(s,u(s),u_s)dg(s)\Big\|\\
\le&\|S_{\beta,\gamma_k}(t)[Q(\sigma(u),u)(0)+\varphi(0)]\|+\|(\varphi_1*S_{\beta,\gamma_k})(t)[\psi+Q_0(u)]\|\\
&+\|\sum_{k=1}^{n}\alpha_k\int_0^t\frac{(t-s)^{\beta-\gamma_k}}{\Gamma(1+\beta-\gamma_k)}S_{\beta,\gamma_k}(s)[Q(\sigma(u),u)(0)+\varphi(0)]ds\|\\
&+\int_0^t\|T_{\beta,\gamma_k}(t-s)\|\cdot\|F(s,u(s),u_s)\|dg(s)\\
\le&M(\|\varphi\|_{\mathcal{B}}+Q(\sigma(R),R)+\widetilde{M}(\|\psi\|+Q_0(R))\\
&+\Big(\sum_{k=1}^{n}\alpha_kM_1\Big)(\|\varphi\|_{\mathcal{B}}+Q(\sigma(R),R)\\
&+C\Big(\mathcal{F}_1(R)\Big(\int_0^t[p_1(s)]^pdg(s)\Big)^{\frac{1}{p}}+\mathcal{F}_2(R)\Big(\int_0^t[p_2(s)]^pdg(s)\Big)^{\frac{1}{p}}+\overline{K}\Big)\\
&\times\Big(\int_0^t(t-s)^{q(\beta-\gamma_k)}dg(s)\Big)^{\frac{1}{q}}\\
\le&(M+\sum_{k=1}^{n}\alpha_kM_1)(\|\varphi\|_{\mathcal{B}}+Q(\sigma(R),R)+\widetilde{M}(\|\psi\|+Q_0(R))\\
&+C\sup_{t\ge0}\Big(\int_0^t(t-s)^{q(\beta-\gamma_k)}dg(s)\Big)^{\frac{1}{q}}(\mathcal{F}_1(R)\|p_1\|_{\mathbb{HLS}_g^p}\\
&+\mathcal{F}_2(R)\|p_2\|_{\mathbb{HLS}_g^p}+\overline{K}).
\end{aligned}\tag{3.21}
$$

Hence, according to the above calculation, we can see

$$
\begin{aligned}
R<&(M+\sum_{k=1}^{n}\alpha_kM_1)(\|\varphi\|_{\mathcal{B}}+Q(\sigma(R),R))+\widetilde{M}(\|\psi\|+Q_0(R))\\
&+C\sup_{t\ge0}\Big(\int_0^t(t-s)^{q(\beta-\gamma_k)}dg(s)\Big)^{\frac{1}{q}}\\
&\times(\mathcal{F}_1(R)\|p_1\|_{\mathbb{HLS}_g^p}+\mathcal{F}_2(R)\|p_2\|_{\mathbb{HLS}_g^p}+\overline{K}).
\end{aligned}
$$

Dividing both sides by R and taking the lower limit as $R\to\infty$, we can get

$$
\begin{aligned}
&(M+\sum_{k=1}^{n}\alpha_kM_1)\eta_1+\widetilde{M}\eta+C\sup_{t\ge0}\Big(\int_0^t(t-s)^{q(\beta-\gamma_k)}dg(s)\Big)^{\frac{1}{q}}\\
&\times(\zeta_1\|p_1\|_{\mathbb{HLS}_g^p}+\zeta_2\|p_2\|_{\mathbb{HLS}_g^p})\ge1,
\end{aligned}
$$

which is a contradiction (3.19). Thus, there is a constant $R_0>0$ such that $\mathcal{Q}(\Omega_{R_0})\subset\Omega_{R_0}$.

Step II. The set $\{\mathcal{Q}u:u(\cdot)\in\Omega_R\}$ is equiregulated. For any $b>0$, any $t_0\in[-r,b)$, we have

$$\|(\mathcal{Q}u)(t)-(\mathcal{Q}u)(t_0^+)\|$$

$$
\begin{aligned}
\leq & \|(S_{\beta,\gamma_k}(t)-S_{\beta,\gamma_k}(t_0^+))[Q(\sigma(u),u)(0)+\varphi(0)]\| \\
& +\|[(\varphi_1 * S_{\beta,\gamma_k})(t)-(\varphi_1 * S_{\beta,\gamma_k})(t_0^+)][\psi+Q_0(u)]\| \\
& +\sum_{k=1}^{n}\frac{\alpha_k M}{\Gamma(1+\beta-\gamma_k)}\Big|\int_0^t (t-s)^{\beta-\gamma_k}ds \\
& -\int_0^{t_0^+}(t_0^+-s)^{\beta-\gamma_k}ds\Big|\|Q(\sigma(u),u)(0)+\varphi(0)\| \\
& +\int_0^{t_0^+}\|[T_{\beta,\gamma_k}(t-s)-T_{\beta,\gamma_k}(t_0^+-s)]F(s,u(s),u_s)\|dg(s) \\
& +\int_{t_0^+}^{t}\|T_{\beta,\gamma_k(t-s)}(t-s)F(s,u(s),u_s)\|dg(s) \\
\leq & \|S_{\beta,\gamma_k}(t)-S_{\beta,\gamma_k}(t_0^+)\|_{L(\mathbb{E})}\cdot\|Q(\sigma(u),u)(0)+\varphi(0)\| \\
& +\widetilde{M}|t-t_0^+|\cdot\|\psi+Q_0(u)\| \\
& +\sum_{k=1}^{n}\alpha_k M\Big|\frac{t^{1+\beta-\gamma_k}-(t_0^+)^{1+\beta-\gamma_k}}{\Gamma(2+\beta-\gamma_k)}\Big|\|Q(\sigma(u),u)(0)+\varphi(0)\| \\
& +(\mathcal{F}_1(\|u\|))\int_0^{t_0^+}\|T_{\beta,\gamma_k}(t-s)-T_{\beta,\gamma_k}(t_0^+-s)\|_{L(E)}p_1(s)dg(s) \\
& +\mathcal{F}_2(\|u_s\|_{\mathcal{B}})\int_0^{t_0^+}\|T_{\beta,\gamma_k}(t-s)-T_{\beta,\gamma_k}(t_0^+-s)\|_{L(E)}p_2(s)dg(s) \\
& +\overline{K}\int_0^{t_0^+}\|T_{\beta,\gamma_k}(t-s)-T_{\beta,\gamma_k}(t_0^+-s)\|_{L(E)}dg(s) \\
& +C\mathcal{F}_1(\|u\|)\int_{t_0^+}^{t}(t-s)^{\beta-\gamma_k}p_1(s)dg(s) \\
& +C\mathcal{F}_2(\|u_s\|_{\mathcal{B}})\int_{t_0^+}^{t}(t-s)^{\beta-\gamma_k}p_2(s)dg(s) \\
& +C\overline{K}\int_{t_0^+}^{t}(t-s)^{\beta-\gamma_k}dg(s) \\
= & I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t)+I_6(t)+I_7(t)+I_8(t)+I_9(t),
\end{aligned}
\tag{3.21}
$$

where

$$
\begin{aligned}
& I_1(t)=\|S_{\beta,\gamma_k}(t)-S_{\beta,\gamma_k}(t_0^+)\|_{L(E)}\cdot\|Q(\sigma(u),u)(0)+\varphi(0)\|, \\
& I_2(t)=\widetilde{M}|t-t_0^+|\cdot\|\varphi+Q_0(u)\|, \\
& I_3(t)=\sum_{k=1}^{n}\alpha_k M\Big|\frac{t^{1+\beta-\gamma_k}-(t_0^+)^{1+\beta-\gamma_k}}{\Gamma(2+\beta-\gamma_k)}\Big|\cdot\|Q(\sigma(u),u)(0)+\varphi(0)\|, \\
& I_4(t)=(\mathcal{F}_1(\|u\|))\int_0^{t_0^+}\|T_{\beta,\gamma_k}(t-s)-T_{\beta,\gamma_k}(t_0^+-s)\|_{L(E)}p_1(s)dg(s),
\end{aligned}
$$

$$I_5(t) = \mathcal{F}_2(\|u_s\|_{\mathcal{B}}) \int_0^{t_0^+} \|T_{\beta,\gamma_k}(t-s) - T_{\beta,\gamma_k}(t_0^+ - s)\|_{L(E)} p_2(s) dg(s),$$

$$I_6(t) = \overline{K} \int_0^{t_0^+} \|T_{\beta,\gamma_k}(t-s) - T_{\beta,\gamma_k}(t_0^+ - s)\|_{L(E)} dg(s),$$

$$I_7(t) = C\mathcal{F}_1(\|u\|) \int_{t_0^+}^t (t-s)^{\beta-\gamma_k} p_1(s) dg(s),$$

$$I_8(t) = C\mathcal{F}_2(\|u_s\|_{\mathcal{B}}) \int_{t_0^+}^t (t-s)^{\beta-\gamma_k} p_2(s) dg(s),$$

$$I_9(t) = C\overline{K} \int_{t_0^+}^t (t-s)^{\beta-\gamma_k} dg(s).$$

By the expression of $I_2(t)$ and $I_3(t)$, we derive that $I_2(t) \to 0$ and $I_3(t) \to 0$ as $t \to t_0^+$ independently of $u \in \Omega$. Since the compactness of $S_{\beta,\gamma_k}(t)$ and $T_{\beta,\gamma_k}(t)$ for $t > 0$ yields the continuity in the sense of uniform operator topology. We get that $I_1(t) \to 0$ and applying dominated convergence theorem on $I_4(t), I_5(t), I_6(t)$ and $I_9(t)$, we can derive that $I_4(t), I_5(t), I_6(t), I_9(t) \to 0$ as $t \to t_0^+$ independently of $u \in \Omega$. Let $H_1(t) = \int_0^t (t-s)^{\beta-\gamma_k} p_1(s) dg(s), H_2(t) = \int_0^t (t-s)^{\beta-\gamma_k} p_2(s) dg(s)$. By Lemma 2.4, we known that $H(t)$ is a regulated function on $\mathbb{R}^+$. Therefore, we have

$$\begin{aligned}
I_7(t) =& C\mathcal{F}_1(R) \int_{t_0^+}^t (t-s)^{\beta-\gamma_k} p_1(s) dg(s) \\
\leq& C\mathcal{F}_1(R)\Big(\|H_1(t) - H_1(t_0^+)\| + \int_0^{t_0^+} \|((t-s)^{\beta-\gamma_k} \\
&- (t_0^+ - s)^{\beta-\gamma_k}) p_1(s)\| dg(s)\Big) \\
\to& 0 \ \text{ as } \ t \to t_0^+ \ \text{ independently of } \ u,
\end{aligned}$$

and

$$\begin{aligned}
I_8(t) =& C\mathcal{F}_2(R) \int_{t_0^+}^t (t-s)^{\beta-\gamma_k} p_2(s) dg(s) \\
\leq& C\mathcal{F}_2(R)\Big(\|H_2(t) - H_2(t_0^+)\| + \int_0^{t_0^+} \|((t-s)^{\beta-\gamma_k} \\
&- (t_0^+ - s)^{\beta-\gamma_k}) p_2(s)\| dg(s)\Big) \\
\to& 0 \ \text{ as } \ t \to t_0^+ \ \text{ independently of } \ u.
\end{aligned}$$

Therefore, $\|(\mathcal{Q}u)(t) - (\mathcal{Q}u)(t_0^+)\|_\Omega \to 0$ as $t \to t_0^+$. independently of $u \in \Omega_R$.

Similarly, one can demonstrate that for any $t_0 \in (-r, b]$, $\|(\mathcal{Q}u)(t) - (\mathcal{Q}u)(t_0^+)\|_\Omega \to 0$ as $t \to t_0^+$. According to the arbitrariness of b, one can find that $u(t)$ is defined on $[0, \infty)$. On the other hand, it is easy to see $\lim_{t\to\infty} \|u(t+\omega) - u(t)\| = 0$. Hence, assert that $\{\mathcal{Q}u : u(\cdot) \in \Omega_R\}$ is equiregulated.

Step III. We finally show that the operator $\mathcal{Q}$ has a positive fixed point on Ω_{R_0}.

We know that $\mathcal{Q} : \Omega_{R_0} \to \Omega_{R_0}$ is a monotonic increasing operator based on

(H6)-(H8) and the proof Theorem 3.1.

Let $v^0 = \theta \in K$ and establish the iterative sequence $\{v^{(n)}\}$ by

$$v^{(n)} = \mathcal{Q}v^{(n-1)}, \quad n = 1, 2, \cdots . \tag{3.22}$$

Then according to the monotonicity of $\mathcal{Q}$, one can find $\{v^{(n)}\} \subset K$ and

$$\theta = v^{(0)} \leq v^{(1)} \leq \cdots \leq v^{(n)} \leq \cdots . \tag{3.23}$$

Similar to the proof of Theorem 3.2, we can get $\alpha(\{v^{(n)}(t)\}) \equiv 0$ in $[-r, a]$, that is, $\{v^{(n)}(t)\}$ is precompact, hence, it has a convergent subsequence $v^{(n_k)} \to u \in \Omega_1$, combined with its monotonicity (3.23) and the normality of cone K, it is easy to know that

$$v^{(n)} \to u \in G([-r, a], K), \qquad n \to \infty.$$

Taking limit of both ends of (3.22), and by the continuity of $\mathcal{Q}$, we can get $u = \mathcal{Q}u$, which shows that $u \in G([-r, a], K)$ is a positive mild solution of the nonlocal problem (1.1). By the arbitrariness of a, we get that $u(t)$ is defined on $[-r, \infty)$. On the other hand, by the method of Step.III of Theorem 3.1, it is easy to see $\lim_{t\to\infty} \|u(t + \omega) - u(t)\| = 0$, which implies that $u(t)$ is a S-asymptotically ω-periodic mild solution for $t \geq 0$. Hence, we know that the nonlocal problem (1.1) has at least a S-asymptotically ω-periodic positive mild solution u in $G([-r, \infty), K)$. □

4. Applications

In this section, we give an example to illustrate our main results. Let

$\beta, \gamma_k > 0 (k = 1, 2, \ldots, n)$ be such that $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$. Consider the following measure driven differential equation:

$$\begin{cases} {}^cD_{0+}^{1+\beta} u(t, x) + \sum\limits_{k=1}^{n} \alpha_k {}^cD^{\gamma_k} u(t, x) = \Delta u(t, x) + \tau u(t, x) \\ +\dfrac{\sin(u(t + s))}{1 + e^{2t}} dg(t), \quad (t, x) \in \mathbb{R}^+ \times [0, \pi],\ s \in [-r, 0], \\ u(t, 0) = u(t, \pi) = 0, \quad t \in \mathbb{R}^+, \\ u(t, x) = \int_0^a \theta(t, s) \log(1 + |u(s, x)|) ds + \varphi(t, x), \quad (t, x) \in [-r, 0] \times [0, \pi], \\ \dfrac{\partial u(t, x)}{\partial t}|_{t=0} = \dfrac{|u(t, x)|}{6 + |u(t, x)|} + \psi(x), \quad x \in [0, \pi], \end{cases} \tag{4.1}$$

where Δ is Laplace operator, $a > 0, \tau < 0$ are constant, $g : [0, \pi] \to \mathbb{R}$ is a non-deacresing, left continuous function, $\theta(t, s)$ is a continuous function from $[0, \infty) \times [-r, 0]$ to $\mathbb{R}^+$. Furthermore, define the operator $A : \mathcal{D}(A) \subset E \to E$ by $Au = \Delta u + \tau u$ and

$$\mathcal{D}(A) = \{u \in E : u, u' \quad \text{are absolutely continuous}, u'' \in E, u(0) = u(\pi) = 0\}.$$

Then it is well known that the operator A is κ-sectorial with $\kappa = \tau < 0$ and angle $\frac{\pi}{2}$ (and hence of angle $\frac{\gamma_k \pi}{2}$) for all $\gamma_k \leq 1, k = 1, 2, \cdots, n$). Since $\beta, \gamma_k > 0, k = 1, 2, \cdots, m$ be such that $0 < \beta \leq \gamma_m \leq \cdots \leq \gamma_1 \leq 1$, by Lemma 2.7 , we deduce that A generates a bounded (β, γ_k)-resolvent family $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$.

We choose the workspace $E = L^2([0,\pi],\mathbb{R})$, which is an ordered Banach space with L^2-norm $\|\cdot\|_2$ and partial-order " $\leq$ ", $K = \{u \in L^2([0,\pi],\mathbb{R}) : u(x) \geq 0,\ a.e.\ x \in [0,\pi]\}$ is a normal cone. Note $\mathcal{B} := G([-r,0]\times[0,\pi], E)$ with the normal cone $K_{\mathcal{B}} = \{u \in \mathcal{B} : u(t,x) \in K, t \in [-r,0],\ a.e.\ x \in [0,\pi]\}$. We define

$$
\begin{aligned}
&f(t,x,u(t,x),u(t+s,x)) = \frac{\sin(u(t+s))}{1+e^{2t}},\ \ t \in \mathbb{R}^+,\ s \in [-r,0],\\
&Q_0(u(t,x)) = \frac{|u(t,x)|}{6+|u(t,x)|},\ Q(u(t,x)) = \int_0^a \theta(t,s)\log(1+|u(s,x)|)ds.
\end{aligned}
$$

For $u \in [0,\pi]$, we set $\varphi(t) = \varphi(t,\cdot)$, $\psi = \psi(\cdot)$, $u(t) = u(t,\cdot)$, $u_t(s) = u(t+s,\cdot)$ and

$$
\begin{aligned}
&F(t,u(t),u_t) = f(t,\cdot,u(t,\cdot),u(t+s,\cdot)), \quad Q_0(u) = \frac{|u|}{6+|u|},\\
&Q(u) = \int_0^a \theta(t,s)\log(1+|u|)ds.
\end{aligned}
$$

Then, equation (4.1) can be transformed into the form of abstract nonlocal problem (1.1) in $L^2([0,\pi],\mathbb{R})$.

Further, from the definition of functions f and Q_0, we have

$$
\|F(t,u(t),u_t)\| \leq \frac{1}{2}\|u\|, \quad \|Q_0(u(t,x))\| \leq \frac{1}{6}\|u\|, \|Q(u(t,x))\| \leq \int_0^a \theta(t,s)ds\|u\|.
$$

We deduce that condition (H4) is satisfied with $c_0 = \frac{1}{6}$ and $d_0 = 0$, $c_1 = \int_0^a \theta(t,s)ds$ and $d_1 = 0$. Additionally, (H1) is satisfied with $P(t) = \frac{1}{2}$ and $W(r) = r$.

Theorem 4.1. *Assume that $\omega > 0$, $f : \mathbb{R}^+ \times [0,\pi] \times K \times K_{\mathcal{B}} \to E$ be continuous and the conditions (H2) is satisfied. If the following conditions*
(A1) $f(t,x,0,0) \geq 0$ for $(t,x) \in \mathbb{R}^+ \times [0,\pi]$, and there is a function $0 \leq w = w(t,x) \in G([-r,\infty)\times[0,\pi])$ satisfying $\lim_{t\to\infty} w(t+\omega,\cdot) - w(t,\cdot) = 0$, such that

$$
\begin{cases}
\quad {}^cD_{0+}^{1+\beta} w(t,x) + \displaystyle\sum_{k=1}^{n} \alpha_k {}^cD^{\gamma_k} w(t,x)\\
\geq \Delta u(t,x) + \tau u(t,x) + f(t,x,w(t,x),w(t+s,x))dg(t),\\
\quad (t,x) \in \mathbb{R}^+ \times [0,\pi], s \in [-r,0],\\
w(t,0) = w(t,\pi) = 0, \qquad t \in \mathbb{R}^+,\\
w(t,x) \geq \displaystyle\int_0^a \theta(t,s)\log(1+|w(s,x)|)ds + \varphi(t,x), \qquad (t,x) \in \mathbb{R}^+ \times [0,\pi],\\
\dfrac{\partial w(x,0)}{\partial t} \geq Q_0(w(t,x)) + \psi(x), \qquad x \in [0,\pi].
\end{cases}
$$

(A2) there exists a constant $l > 0$ such that for any $x \in [0,\pi], t \in \mathbb{R}^+$ and $0 \leq x_1 \leq x_2 \leq w(\cdot,t), 0 \leq \phi_1 \leq \phi_2 \leq w_t$,

$$
f(t,x_2,\phi_2) - f(t,x_1,\phi_1) \geq \theta
$$

hold, then all the conditions in Theorem 3.1 are satisfied, our results can be applied to system (4.1). Also, the problem (4.1) has minimal and maximal S-asymptotically ω-periodic solutions $\underline{u}, \overline{u} \in G([-r,\infty), L^2([0,\pi],\mathbb{R}) \cap SAP_\omega(L^2([0,\pi],\mathbb{R}))$ between 0 *and w, which can be obtained by monotone iterative sequences starting from* 0 *and ω.*

Proof. From the condition (A1), it follows that $v_0 \equiv 0$ and $w_0 = w(x,t) \geq 0$ are lower and upper S-asymptotically ω-periodic mild solutions of the problem (4.1), respectively. Thus, by the condition (A2), one can find that the condition (H2) holds. Therefore, from Theorem 3.1, we can obtain that the problem (4.1) has minimal and maximal time S-asymptotically ω-periodic mild solutions $\underline{u}, \overline{u} \in G([-r,\infty), E) \cap SAP_\omega(E)$, which can be obtained by monotone iterative sequences starting from 0 and w, respectively. □

5. Conclusions

This article establishes some results on the existence of maximal and minimal S-asymptotic ω- periodic mild solutions of fractional measure differential equations in order Banach space by using the method of upper and lower solutions. In further work, we investigate the existence of S-asymptotic ω-periodic mild solutions for a class of Hilfer fractional measure differential equations.

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References

[1] D. Araya and C. Lizama, *Almost automorphic mild solutions to fractional differential equations*, Nonlinear Analysis, 2008, 69(11), 3692–3705.

[2] W. Arendt, C. Batty, M. Hieber and F. Neubrander, *Vector-valued Laplace Transforms and Cauchy-Problems*, Monographs in Mathematics, 2001, 96. Birkhäuser, Basel.

[3] J. Banas and K. Goebel, *Measure of Noncompactness in Banach Spaces*, Lect. Notes Pure Appl. Math., New York: Marcel Dekker, 1980.

[4] J. Banaś and M. Mursaleen, *Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations*, Springer, 2014.

[5] B. Brogliato, *Nonsmooth Mechanics: Models, Dynamics, and Control*, Springer, 1996.

[6] A. Caicedo and C. Cuevas, *S-asymptotically ω-periodic solutions of abstract partial neutral integro-differential equations*, Functional Differential Equations, 2010, 17(1–2).

[7] Y. Cao and J. Sun, *Existence of solutions for semilinear measure driven equations*, Journal of Mathematical Analysis and Applications, 2015, 425(2), 621–631.

[8] Y. Cao and J. Sun, *Measures of noncompactness in spaces of regulated functionwith application to semilinear measure driven equations*, Boundary Value Problems, 2016, 2016(38), 1–17.

[9] Y. Cao and J. Sun, *On existence of nonlinear measure driven equations involving non-absolutely convergent integrals*, Nonlinear Anal. Hybrid Syst., 2016, 20, 72–81.

[10] Y. Cao and J. Sun, *Approximate controllability of semilinear measure driven systems*, Mathematische Nachrichten, 2018, 291(13), 1979–1988.

[11] X. Chen and L. Cheng, *On countable determination of the Kuratowski measure of noncompactness*, J. Math. Anal. Appl., 2021, 504, 125370.

[12] M. Cichoń and B. R. Satco, *Easure differential inclusions-between continuous and discrete*, Adv. Difference Equ., 2014, 56.

[13] C. Cuevas and J. C. De Souza, *S-asymptotically ω-periodic solutions of semilinear fractional integro-differential equations*, Applied Mathematics Letters, 2009, 22(6), 865–870.

[14] C. Cuevas, H. R. Henriquez and H. Soto, *Asymptotically periodic solutions of fractional differential equations*, Appl. Math. Comput., 2014, 236, 524–545.

[15] C. Cuevas and C. Lizama, *Almost automorphic solutions to a class of semilinear fractional differential equations*, Applied Mathematics Letters, 2008, 21, 1315–1319.

[16] C. Cuevas and C. Lizama, *Almost automorphic solutions to integral equations on the line*, Semigroup Forum., 2009, 79, 461–472.

[17] C. Cuevas and M. Pinto, *Existence and uniqueness of pseudo almost periodic solutions of semilinear Cauchy problems with non dense domain*, Nonlinear Analysis, 2001, 45, 73–83.

[18] C. Cuevas and J. Souza, *Existence of S-asymptotically ω-periodic solutions for fractional order functional integro-differential equations with infinite delay*, Nonlinear Anal., 2010, 72, 1683–1689.

[19] P. C. Das and R. R. Sharma, *Existence and stability of measure differential equations*, Czechoslovak Math. J., 1972, 22(97), 145–158.

[20] B. de Andrade and C. Cuevas, *Almost automorphic and pseudo almost automorphic solutions to semilinear evolution equations with non dense domain*, Journal of Inequalities and Applications, 2009, 2009(8), Article ID 298207.

[21] B. de Andrade and C. Cuevas, *Compact almost automorphic solutions to semilinear Cauchy problems with nondense domain*, Applied Mathematics and Computation, 2009, 215, 2843–2849.

[22] B. de Andrade and C. Cuevas, S-asymptotically ω-periodic and asymptotically ω-periodic solutions to semilinear Cauchy problems with non dense domain, Nonlinear Analysis Series A: Theory, Methods and Applications, 2010, 72, 3190–3208.

[23] J. Diestel, W. M. Ruess and W. Schachermayer, *Weak compactness in $l^1(\mu, x)$*, Proc. Amer. Math. Soc., 1993, 118, 447–453.

[24] A. Diop, *On approximate controllability of multi-term time fractional measure differential equations with nonlocal conditions*, Fractional Calculus and Applied Analysis, 2022, 25, 2090–2112.

[25] A. Diop, *Existence of mild solutions for multi-term time fractional measure differential equations*, The Journal of Analysis, 2022, 30, 1609–1623.

[26] A. Diop, M. A. Diop, K. Ezzinbi and Paul dit A. Guindo, *Optimal controls problems for some impulsive stochastic integro-differential equations with state-dependent delay*, Stochastics, 2022, 94(5/8), 1186–1220.

[27] M. Federson, J. G. Mesquita and A. Slavìk, *Measure functional differential equations and functional dynamic equations on time scales*, J. Differential Equations, 2012, 252, 3816–3847.

[28] M. Federson, J. G. Mesquita and A. Slavík, *Basic results for functional differential and dynamic equations involving impulses*, Math. Nachr., 2013, 286(2–3), 181–204.

[29] H. Gou, *On the S-asymptotically ω-periodic mild solutions for multi-time fractional measure differential equations*, Topological Methods in Nonlinear Analysis, 2023, 62(2), 569–590.

[30] H. Gou and Y. Li, *Existence and Approximate Controllability of Semilinear Measure Driven Systems with Nonlocal Conditions*, Bulletin of the Iranian Mathematical Society, 2022, 48, 769–789.

[31] R. C. Grimmer, *Asymptotically almost periodic solutions of differential equations*, SIAM Journal on Applied Mathematics, 1969, 17, 109–115.

[32] H. Gu and Y. Sun, *Nonlocal controllability of fractional measure evolution equation*, Journal of Inequalities and Applications, 2020, 1(2020), 1–18.

[33] D. Guo, *Nonlinear Functional Analysis*, Shandong Science and Technology, Jinan, (Chinese), 1985.

[34] D. Guo and J. Sun, *Ordinary Differential Equations in Abstract Spaces*, Shandong Science and Technology, Jinan, (Chinese), 1989.

[35] G. Haiyin, W. Ke, W. Feng and D. Xiao, *Massera-type theorem and asymptotically periodic logistic equations*, Nonlinear Analysis: Real World Applications, 2006, 7, 1268–1283.

[36] H. P. Heinz, *On the behaviour of measures of noncompactness with respect to differential and integration of vector-valued functions*, Nonlinear Anal., 1983, 7, 1351–1371.

[37] H. Henríquez and C. Lizama, *Compact almost automorphic solutions to integral equations with infinite delay*, Nonlinear Analysis, 2009, 71, 6029–6037.

[38] H. R. Henríquez, M. Pierri and P. Táboas, *On S-asymptotically ω-periodic functions on Banach spaces and applications*, J. Math. Anal. Appl., 2008, 343, 1119–1130.

[39] E. Hernandez and D. O'Regan, *On a new class of abstract impulsive differential equations*, Proc. Am. Math. Soc., 2013, 141, 1641–1649.

[40] Y. Hino, T. Naito, N. V. Minh and J. Shin, *Almost Periodic Solutions of Differential Equations in Banach Spaces*, Taylor & Francis, London, New York, 2002.

[41] M. I. Kamenskii, V. V. Obukhovskll and P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, Walter de Gruyter, 2001.

[42] V. Keyantuo, C. Lizama and M. Warma, *Asymptotic behavior of fractional order semilinear evolution equations*, Differential and Integral Equations, 2013, 26 (7–8), 757–780.

[43] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 2006, 204. Amsterdam: Elsevier Science B. V.

[44] Y. J. Kim, *Stieltjes derivatives and its applications to integral inequalities of Stieltjes type*, J. Korean Soc. Math. Educ. Ser. B Pure App. Math., 2011, 18(1), 63–78.

[45] R. I. Leine and T. F. Heimsch, *Global uniform symptotic attractive stability of the non-autonomous bouncing ball system*, Phys. D., 2012, 241, 2029–2041.

[46] F. Li, J. Liang and H. Wang, *S-asymptotically ω-periodic solution for fractional differential equations of order $q \in (0, 1)$ with finite delay*, Adv. Difference Equ., 2017, 83, 14 pp.

[47] F. Li and H. Wang, *S-asymptotically ω-periodic mild solutions of neutral fractional differential equations with finite delay in Banach space*, Mediterr. J. Math., 2017, 14, 57.

[48] Q. Li, G. Wang and M. Wei, *Monotone iterative technique for time-space fractional diffusion equations involving delay*, Nonlinear Anal: Model., 2021, 26, 241–258.

[49] Y. Li, *The positive solutions of abstract semilinear evolution equations and their applications*, Acta Math. Sin. (Chinese), 1996, 39, 666–672.

[50] S. M. Manou-Abi and W. Dimbour, *On the p-th mean S-asymptotically omega periodic solution for some stochastic evolution equation driven by Q-brownian motion*, 2018. https://doi/arXiv:1711.03767v1.

[51] J. G. Mesquita, *Measure Functional Differential Equations and Impulsive Functional Dynamic Equations on Time Scales*, Universidade de Sao Paulo, Brazil, Ph.D. thesis, 2012.

[52] B. M. Miller and E. Y. Rubinovich, *Impulsive Control in Continuous and Discrete Continuous Systems*, Kluwer Academic Publishers, NewYork, Boston, Dordrecht, London, Moscow, 2003.

[53] J. J. Moreau, *Unilateral contact and dry friction in finite freedom dynamics*, in: Nonsmooth Mechanics and Applications, Springer-Verlag, NewYork, 1988, 1–82.

[54] G. M. N'Guérékata, *Existence and uniqueness of almost automorphic mild solutions of some semilinear abstract differential equations*, Semigroup Forum, 2004, 69, 80–86.

[55] S. G. Pandit and S. G. Deo, *Differential Systems Involving Impulses*, Springer, 1982.

[56] E. A. Pardo and C. Lizama, *Mild solutions for multi-term time-fractional differential equations with nonlocal initial conditions*, Electronic Journal of Differential Equations, 2020, 39, 1–10.

[57] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[58] M. Pierri, *On S-asymptotically ω-periodic functions and applications*, Nonlinear Anal, 2012, 75, 651–661.

[59] I. Podlubny, *Fractional differential equations*, New York, NY: Academic Press, 1999.

[60] L. Ren, J. Wang and M. Fečkan, *Asymptotically periodic solutions for Caputo type fractional evolution equations*, Fract. Calc. Appl. Anal., 2018, 21, 1294–1312.

[61] B. Satco, *Regulated solutions for nonlinear measure driven equations*, Nonlinear Anal. Hybrid Syst., 2014, 13, 22–31.

[62] R. R. Sharma, *An abstract measure differential equation*, Proc. Amer. Math. Soc., 1972, 32, 503–510.

[63] X. Shu, F. Xu and Y. Shi, *S-asymptotically ω-positive periodic solutions for a class of neutral fractional differential equations*, Appl. Math. Comput., 2015, 270, 768–776.

[64] V. Singh and D. N. Pandey, *Controllability of multi-term time-fractional differential systems*, Journal of Control and Decision, 2020, 7(2), 109–125.

[65] K. Surendra and P. A. Ravi, *Existence of solution non-autonomous semilinear measure driven equations*, Differential Equation & Application, 2020, 12(3), 313–322.

[66] L. V. Trong, *Decay mild solutions for two-term time fractional differential equations in Banach spaces*, Journal Fixed Point Theory and Applications, 2016, 18, 417–432.

[67] N. V. Wouw and R. I. Leine, *Tracking control for a class of measure differential inclusions*, in: Proceedings of the 47th IEEE Conference on Decision and Control, 2008.

[68] S. T. Zavalishchin and A. N. Sesekin, *Dynamic Impulse Systems: Theory and Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997.