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INITIAL VALUE PROBLEM FOR A CLASS OF SEMI-LINEAR FRACTIONAL ITERATIVE DIFFERENTIAL EQUATIONS

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Abstract An initial value problem of a class of semi-linear fractional order iterative differential equations is researched in this paper. The existence of solution is acquired in respect of Banach space $C(I, I)$ and $C_{K,q}(I, I)$ for fractional order iterative differential equations. Nevertheless, because the operator is Hölder continuous rather than Lipschitz continuous, uniqueness results can not be obtained. Additionally, a change of solution to $[k, \beta]$ for the $k \in I$ will arise from a small perturbation of the initial value. Our analysis is on the basis of the properties of Mittag-Leffler function and Schauder's fixed point theorem. Lastly, some examples are provided to demonstrate our results.

Keywords Fractional order iterative differential equations, Mittag-Leffler function, initial value problem, existence.

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1. Introduction

Iterative differential equations (IDE) are highly essential when people investigate the movement of charged particles with retarded interaction. On the other hand, because the use of IDE is one of the most practical approaches to research biological modeling of the bacteria reproduction, the usage of IDE has drawn the attention of numerous scholars.

Considering FDE with real variables can be employed in fields such as control theory [35], biology chemical physics [24], economics [27], electrical networks [16] that are naturally modeled by FDE, fractional derivatives are perceived as important tools to describe nonlinearity. The monographs [8, 17, 21, 30] are a great resource for fractional calculus theory and applications. There are many papers handling the existence or uniqueness of solutions to initial /boundary value problem (IVP or BVP) for some nonlinear FDE. For example, Barrett [2] proved the existence and uniqueness of solutions for a non-integer order IVP. In [5,37], the authors considered the IVP and BVP for FDE by applying the upper and lower solutions method. In [11–13], the authors presented the existence and uniqueness results for fractional nonlinear Volterra-Fredholm integro differential equations. In [15], the authors used Gagliardo-Nirenberg inequalities, fixed point theory and operator

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theory to consider the global/local well-posedness of the semi-linear time fractional Rayleigh-Stokes problem. In [14], the authors discussed the Hölder regularity result for ${}^{C}D^{q}_{\tau}x(\tau) + A(\tau)x(\tau) = f(\tau)$ concerning Caputo's fractional derivative.

IDE offer an effective method to explore the approximation solutions and people have researched them over the years because of their extensive applications. Some important results regarding the existence (uniqueness) of solutions for integer order differential equations or fractional iterative differential equation (FIDE) have been obtained. For example, the authors [1] provided sufficient conditions for the existence and uniqueness of solutions to the second order iterative dynamic BVP with mixed derivative operators. In [3, 26, 36], the authors proved the existence and uniqueness of solutions for first-order IDE. In [10], the authors investigated the existence, uniqueness, continuous dependence and Ulam stability theorems for iterative Caputo FDE. In [22, 31], the authors proved the existence and uniqueness of solutions for fractional iterative integro-differential equations. In [6, 20, 32, 34, 38], the authors proved the existence and uniqueness of solutions FIDE using some standard fixed point technology.

The author [4] discussed the existence, uniqueness and continuous dependence theorems for

$$x'(\tau) = f(\tau, x(x(\tau))), \quad x(\tau_0) = x_0, \quad \tau \in I = [\alpha, \beta],$$

The authors [5] considered the following nonlinear fractional relaxation differential equation

$$\begin{cases} {}^{C}D^{q}_{0^+}x(\tau) + \gamma x(\tau) = f(\tau, x(\tau)), & \tau \in (0,1], \\ x(0) = x_0 > 0, \end{cases}$$

where $q \in (0,1)$, $\gamma > 0$, $t_0,\ x_0 \in [0,1]$ and $f \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$. ${}^{C}D^{q}_{0^+}$ is standard Caputo fractional derivative.

The authors [7] discussed the existence of approximate solutions to FIDE

$$\begin{cases} {}^{C}D^{q}_{\alpha,\tau}x(\tau) = f(\tau, x(\tau), x(x^{v}(\tau))), & \tau \in I, \\ x(\tau_0) = x_0, \quad x'(\alpha) = 0, \end{cases}$$

where $v \in \mathbb{R}\backslash\{0\}$, $1 < q < 2$ and $f \in C(I^3, \mathbb{R})$.

The authors [18, 19] considered the following FIDE

$$D^{q}_{0^+}x(\tau) = f(\tau, x(\tau), x(x(\tau))), \quad x(0) = x_0,$$

where $q \in (0,1)$, $x_0 \in [0,T]$ and $f \in C([0,T]^3, \mathbb{R})$.

The authors [23] considered the following Caputo fractional quadratic IDE

$${}^{C}D^{q}_{0^+}x(\tau) = f(\tau, x(\tau), x(x(\tau))), \quad x(0) = x_0,$$

where $q \in (0,1)$, $x_0 \in [0,T]$ and $f \in C([0,T]^3, \mathbb{R})$.

Motivated by the papers [4, 5, 7, 18, 19, 23], we'll discuss the fractional iterative IVP with linear term

$${}^{C}D^{q}_{\alpha^+}x(\tau) + \gamma x(\tau) = f(\tau, x(\tau), x(x^{v}(\tau))), \quad \tau \in I, \quad 0 < q < 1 \tag{1.1}$$

subject to boundary condition

$$x(\alpha) = x_a, \tag{1.2}$$

where $x_a \in I$, $\gamma > 0$, $v \in \mathbb{R} \setminus \{0\}$, $0 \leq \alpha \leq \alpha^v$, $\beta^v \leq \beta$, ${}^C D^q_{\alpha^+}$ is starded Caputo fractional derivative and $f \in C(I^3, \mathbb{R})$.

As far as we are aware, no study on IVP (1.1)-(1.2) of fractional order iterative has been done, hence we hope to make some progress in this area with this paper. Compared to the paper [5], our nonlinear term f has two space variables, the latter of which is an iteration term. Compared to the papers [18, 19, 23], equation (1.1) involves a linear term $\gamma x(\tau)$, which can be considered as a perturbation to an equation. It can be handled quite differently without it. The existence results were obtained in Banach space $C(I, I)$ and $C_{K,q}(I, I)$ by employing the properties of Mittag-Leffler function and Schauder's fixed point theorem. However, the operator is just Hölder continuous rather than Lipschitz continuous, which prevents obtaining uniqueness results. This paper also discusses the continuous dependence of the solutions.

2. Some definitions and lemmas

Some fractional calculus theory concepts and other basic knowledge are presented in this section.

Let $C(I, \mathbb{R})$ be the set of all continuous functions from I into $\mathbb{R}$ with the norm $\|x\| = \sup\{\mid x(\tau) \mid; \tau \in I\}$.

Definition 2.1. (see [21, 28]) The Riemann-Liouville fractional integral of order $q > 0$ of a function $x \in C(I, R)$ is given by

$$I^q_{\alpha^+} x(\tau) = \int_\alpha^\tau \frac{(\tau - \varsigma)^{q-1}}{\Gamma(q)} x(\varsigma) d\varsigma$$

provided that the right-hand side is pointwise defined on $(0, \infty)$, and $\Gamma(q)$ is the Euler gamma function defined by $\Gamma(q) = \int_0^\infty \tau^{q-1} e^{-\tau} d\tau$.

Definition 2.2. (see [21, 28]) The Caputo's fractional order derivative of order $q > 0$, $n \in \mathbb{N}$ of a continuous function $h : (0, \infty) \to \mathbb{R}$ is given by

$${}^C D^q_{\alpha^+} h(\tau) = I^{n-q}_{\alpha^+} D^n h(\tau) = \frac{1}{\Gamma(n - q)} \int_\alpha^\tau \frac{h^{(n)}(\varsigma)}{(\tau - \varsigma)^{q+1-n}} d\varsigma,$$

where $n = [q] + 1$, and $[q]$ represents the integer part of the real number q. The Laplace transform is given as

$$L\{{}^C D^q_{\alpha^+} h(\tau)\} = \varsigma^q H(\varsigma) - \sum_{k=0}^{n-1} \varsigma^{q-k-1} h^{(k)}(\alpha),$$

where $H(\varsigma) = L\{h(\tau)\}$ denotes the Laplace transform of $h(\tau)$.

Let us review the Mittag-Leffler function

$$E_{q,p}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(qk + p)}, \quad q > 0,\ p \in \mathbb{R},\ z \in \mathbb{C},$$

and the Wright-type function

$$M_\theta(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(1-\theta(k+1))}, \quad \theta \in (0,1),\ z \in \mathbb{C}.$$

For additional information, read Kilbas [21] and Mainardi [25]. When limited to the real line, the function $E_{\alpha,\beta}$ is real analytic and represents a full function.

The Laplace transform for the Mittag-Leffler function with two parameters is as follows:

$$L\{\tau^{p-1}E_{q,p}(-\gamma\tau^q)\} = \frac{\varsigma^{q-p}}{\varsigma^q+\gamma}, \quad (\mathcal{R}(\varsigma) >\mid \gamma \mid^{\frac{1}{q}}), \tag{2.1}$$

where $\tau \geq 0$, $\mathcal{R}(\varsigma)$ denotes the real part of $\varsigma, \gamma \in \mathbb{R}$.

Lemma 2.1. (see [33, Lemma 2]) *Let $0 < q \leq 2$ and $p \in \mathbb{R}_+$. For all $z < 0$, $E_q(\cdot)$, $E_{q,p}(\cdot)$ and $E_{q,p}(\cdot)$ are non-negative.*

Moreover, $E_{q,q}(0) = \frac{1}{\Gamma(q)}$. For any $\tau_1,\ \tau_2 \geq 0$ and $\gamma > 0$,

$$E_{q,p}(-\gamma\tau_1^q) \to E_{q,p}(-\gamma\tau_2^q) \quad \text{as } \tau_1 \to \tau_2.$$

Lemma 2.2. (see [29]) *The following equality holds for $\gamma > 0$, $0 < q < 1$,*

$$\frac{d}{d\tau}E_q(-\gamma\tau^q) = -\gamma\tau^{q-1}E_{q,q}(-\gamma\tau^q), \quad \tau > 0,$$

where $E_q(\cdot) = E_{q,1}(\cdot)$.

Lemma 2.3. (see [29]) *The following equality holds for $\gamma > 0$, $q > 0$ and $m \in \mathbb{Z}$, then*

$$\frac{d^m}{d\tau^m}E_q(-\gamma\tau^q) = -\gamma\tau^{q-m}E_{q,q+1-m}(-\gamma\tau^q), \quad \tau > 0.$$

Lemma 2.4. (see [25]) *The Wright-type function has the following properties for each $\tau > 0$,*

$$\int_0^\infty \nu^\kappa M_\theta(\nu)d\nu = \frac{\Gamma(\kappa+1)}{\Gamma(\theta\kappa+1)}, \quad M_\theta(\tau) \geq 0, \quad \text{for} \quad -1 < \kappa < \infty. \tag{2.2}$$

Remark 2.1. (see [39]) The Mittag-Leffler function and Wright-type function have the following formula

$$E_q(-z) = \int_0^\infty M_q(\nu)e^{-z\nu}d\nu, \quad Z \in \mathbb{C}. \tag{2.3}$$

Lemma 2.5. (see [21, pp.140–141], [5]) *Let $h \in C(I,\mathbb{R})$, $\gamma > 0$, then the solution to FDE*

$$^C D^q_{\alpha^+}x(\tau) + \gamma x(\tau) = h(\tau), \quad \tau \in I, \quad 0 < q < 1 \tag{2.4}$$

is affected by the initial condition (1.2) which is the same as the integral equation

$$x(\tau) = x_a E_q(-\gamma(\tau-\alpha)^q) + \int_0^{\tau-\alpha} (\tau-\varsigma-\alpha)^{q-1}E_{q,q}(-\gamma(\tau-\varsigma-\alpha)^q)h(\varsigma)d\varsigma.$$

Sketch of Proof. By applying Laplace transform for both sides of (2.4), we obtain

$$\varsigma^q X(\varsigma) - \varsigma^{q-1} x_a + \gamma X(\varsigma) = H(\varsigma),$$

where $X(\varsigma)$ and $H(\varsigma)$ represent the Laplace transform of $x(\tau)$ and $h(\tau)$, respectively.
It then follows

$$X(\varsigma) = \frac{\varsigma^{q-1} x_a + H(\varsigma)}{\varsigma^q + \gamma}. \tag{2.5}$$

By applying inverse Laplace transform for both sides of (2.5), we have

$$x(\tau) = x_a E_q(-\gamma(\tau-\alpha)^q) + h(\tau) * [(\tau-\alpha)^{q-1} E_{q,q}(-\gamma(\tau-\alpha)^q)],$$

where $*$ is the convolution operator. Therefore, it follows

$$x(\tau) = x_a E_q(-\gamma(\tau-\alpha)^q) + \int_0^{\tau-\alpha} (\tau-\varsigma-\alpha)^{q-1} E_{q,q}(-\gamma(\tau-\varsigma-\alpha)^q) h(\varsigma) d\varsigma.$$

3. Existence result in $C(I, I)$

We investigate the existence of solutions to FDE (1.1)-(1.2) by Schauder's fixed point theorem [9].

The following conditions are provided to handle our problem,

(H1) The function $f \in C(I^3, \mathbb{R})$ is a Carathéodory function;

(H2) $\alpha,\ \beta \geq 0$ and satisfy $\alpha \leq \alpha^v$, $\beta^v \leq \beta$, $v \in \mathbb{R}\backslash\{0\}$.

Consider the operator $T : C(I, I) \to C(I, \mathbb{R})$ as follows:

$$\begin{aligned}(Tx)(\tau) =& x_a E_q(-\gamma(\tau-\alpha)^q) \\ &+ \int_0^{\tau-\alpha} (\tau-\varsigma-\alpha)^{q-1} E_{q,q}(-\gamma(\tau-\varsigma-\alpha)^q) f(\varsigma, x(\varsigma), x(x^v(\varsigma))) d\varsigma.\end{aligned} \tag{3.1}$$

Recall that we require $\alpha \leq x(\tau) \leq \beta$, for any $\alpha \leq \tau \leq \beta$, in a bid to make solutions to (1.1)-(1.2) be well-defined. Thus, if $x \in C(I, I)$ is a fixed point of the operator T, x is a solution to (1.1)-(1.2) so that $\alpha \leq (Tx)(\tau) \leq \beta$ for every $\tau \in I$. Then the next step is to investigate the operator equation

$$T(x) = x.$$

Theorem 3.1. *Let* (H1) and (H2) *hold. If*
(H3) *there exist constant constants f_m and f_M so that*

$$f_m \leq f(\tau, x, y) \leq f_M,\ (\tau, x, y) \in I^3, M = \max\{| f_m |, | f_M |\};$$

(H4) *one of the subsequent assumptions satisfies:*

(i) $M(1 - E_q(-\gamma(\beta-\alpha)^q)) \leq \gamma M_a$ *or*

(ii) $x_a = \alpha, \gamma(\beta - x_a) \geq f_M(1 - E_q(-\gamma(\beta-\alpha)^q))$,

$f_m(1 - E_q(-\gamma(\beta-\alpha)^q)) \geq \gamma x_a(1 - E_q(-\gamma(\beta-\alpha)^q)), f(t, x, y) \geq 0,$ *or*

(iii) $x_a = \beta,\ f_m \geq \gamma x_a(1 - E_q(-\gamma(\beta-\alpha)^q)),\ f(t, x, y) \leq 0,$

where $M_a = \max\{x_a E_q(-\gamma(\beta-\alpha)^q) - \alpha, \beta - x_a\}$.

Then there is a minimum of one solution for the fractional BVP (1.1)-(1.2) *in* $C(I, I)$.

Proof. We firstly prove that $T(C(I,I)) \subset C(I,I)$.

$E_q(-\gamma(\tau-\alpha)^q) \geq 0$ is known from Lemma 2.1, which means that for any $\tau - \alpha \geq \varsigma \geq 0$, $(\tau-\varsigma-\alpha)^{q-1}E_{q,q}(-\gamma(\tau-\varsigma-\alpha)^q) \geq 0$ is true. Furthermore, based on Lemma 2.3, when $m = 1$ and $E_q(0) = 1$, we know that

$$\int_0^{\tau-\alpha}(\tau-\varsigma-\alpha)^{q-1}E_{q,q}(-\gamma(\tau-\varsigma-\alpha)^q)d\varsigma = \frac{1}{\gamma}\big(1 - E_q(-\gamma(\tau-\alpha)^q)\big) \geq 0. \quad (3.2)$$

Lemma 2.2 tells us that $E_q(-\gamma(\tau-\alpha)^q)$ is monotonously decreasing concerning τ. Therefore, for $\tau \in I$, we have

$$0 < E_q(-\gamma(\beta-\alpha)^q) \leq E_q(-\gamma(\tau-\alpha)^q) \leq E_q(0) = 1. \quad (3.3)$$

For any $\tau \in I$, from (H3-H4), (3.2)-(3.3), we have

$$\begin{aligned} \mid (Tx)(\tau) \mid \leq & x_a E_q(-\gamma(\tau-\alpha)^q) + \mid \int_0^{\tau-\alpha}(\tau-\varsigma-\alpha)^{q-1}E_{q,q}(-\gamma(\tau-\varsigma-\alpha)^q) \\ & \times f(\varsigma, x(\varsigma), x(x^v(\varsigma)))d\varsigma \mid \\ \leq & x_a + \frac{1}{\gamma}M\big(1 - E_q(-\gamma(\tau-\alpha)^q)\big) \\ \leq & x_a + \frac{1}{\gamma}M\big(1 - E_q(-\gamma(\beta-\alpha)^q)\big) \\ \leq & \beta, \end{aligned}$$

and

$$\begin{aligned} \mid (Tx)(\tau) \mid \geq & x_a E_q(-\gamma(\tau-\alpha)^q) - \mid \int_0^{\tau-\alpha}(\tau-\alpha-\varsigma)^{q-1}E_{q,q}(-\gamma(\tau-\alpha-\varsigma)^q) \\ & \times f(\varsigma, x(\varsigma), x(x^v(\varsigma)))d\varsigma \mid \\ \geq & x_a E_q(-\gamma(\beta-\alpha)^q) - \frac{1}{\gamma}M\big(1 - E_q(-\gamma(\tau-\alpha)^q)\big) \\ \geq & x_a E_q(-\gamma(\beta-\alpha)^q) - \frac{1}{\gamma}M\big(1 - E_q(-\gamma(\beta-\alpha)^q)\big) \\ \geq & \alpha. \end{aligned}$$

It indicates that T is a self-mapping operator

$$T : C(I,I) \to C(I,I).$$

We handle the case (H4)(ii) and (H4)(iii) in a manner similar to that of (H4)(i). Next, we prove that T is an operator that is completely continuous.

Let $\{x_n\}$ be a sequence with $x_n \to x$ in $C(I,I)$. Then, from (3.2), for each $\tau \in I$, we get

$$\begin{aligned} & (\mid Tx_n)(\tau) - (Tx)(\tau) \mid \\ = & \mid \int_0^{\tau-\alpha}(\tau-\varsigma-\alpha)^{q-1}E_{q,q}(-\gamma(\tau-\varsigma-\alpha)^q)(f(\varsigma, x_n(\varsigma), x_n(x_n^v(\varsigma))) \end{aligned}$$

$$
\begin{aligned}
&\quad - f(\varsigma, x(\varsigma), x(x^v(\varsigma))))d\varsigma \mid \\
&\leq \sup_{\varsigma \in I} \mid f(\varsigma, x_n(\varsigma), x_n(x_n^v(\varsigma))) - f(\varsigma, x(\varsigma), x(x^v(\varsigma)))) \mid \frac{1}{\gamma}\big(1 - E_q(-\gamma(\tau-\alpha)^q)\big),
\end{aligned}
$$

which implies that

$$
\|Tx_n - Tx\| \leq \sup_{\varsigma \in I} \mid f(\varsigma, x_n(\varsigma), x_n^v(\varsigma)) - f(\varsigma, x(\varsigma), x^v(\varsigma))) \mid \frac{1}{\gamma}\big(1 - E_q(-\gamma(\tau-\alpha)^q)\big).
$$

We can sum up by using condition (H1) and Lebesgue's dominated convergence theorem that

$$
\|Tx_n - Tx\| \to 0 \text{ as } n \to \infty.
$$

T is hence continuous. For $\tau_1 = \alpha$ and $\alpha < \tau_2 \leq \beta$, from (3.2) and (3.3) and Lemma 2.1, we have

$$
\begin{aligned}
&\quad \mid (Tx)(\tau_2) - (Tx)(\tau_1) \mid \\
&= \mid \int_0^{\tau_2-\alpha} (\tau_2 - \varsigma - \alpha)^{q-1} E_{q,q}(-\gamma(\tau_2 - \varsigma - \alpha)^q) f(\varsigma, x(\varsigma), x(x^v(\varsigma)))d\varsigma \\
&\quad + x_a E_q(-\gamma(\tau_2-\alpha)^q) - x_a E_q(-\gamma(\tau_1-\alpha)^q) \mid \\
&\leq \frac{M}{\gamma}\big(1 - E_q(-\gamma(\tau_2-\alpha)^q)\big) + \mid x_a E_q(-\gamma(\tau_2-\alpha)^q) - x_a \mid \\
&= \left\{\frac{M}{\gamma} + x_a\right\}\big(1 - E_q(-\gamma(\tau_2-\alpha)^q)\big) \\
&\to\ 0 \ \text{ as } \ \tau_2 \to \alpha.
\end{aligned}
$$

For $\alpha < \tau_1 < \tau_2 \leq \beta$, from Lemma 2.1 and (3.2), we have

$$
\begin{aligned}
&\quad \mid (Tx)(\tau_2) - (Tx)(\tau_1) \mid \\
&= \mid x_a E_q(-\gamma(\tau_2-\alpha)^q) - x_a E_q(-\gamma(\tau_1-\alpha)^q) \\
&\quad + \int_0^{\tau_2-\alpha} (\tau_2 - \varsigma - \alpha)^{q-1} E_{q,q}(-\gamma(\tau_2 - \varsigma - \alpha)^q) f(\varsigma, x(\varsigma), x(x^v(\varsigma)))d\varsigma \\
&\quad - \int_0^{\tau_1-\alpha} (\tau_1 - \varsigma - \alpha)^{q-1} E_{q,q}(-\gamma(\tau_1 - \varsigma - \alpha)^q) f(\varsigma, x(\varsigma), x(x^v(\varsigma)))d\varsigma \mid \\
&\leq x_a \mid E_q(-\gamma(\tau_2-\alpha)^q) - E_q(-\gamma(\tau_1-\alpha)^q) \mid + M \mid \int_0^{\tau_2-\alpha} (\tau_2 - \varsigma - \alpha)^{q-1} \\
&\quad \times E_{q,q}(-\gamma(\tau_2 - \varsigma - \alpha)^q)d\varsigma - \int_0^{\tau_1-\alpha} (\tau_1 - \varsigma - \alpha)^{q-1} E_{q,q}(-\gamma(\tau_1 - \varsigma - \alpha)^q)d\varsigma \mid \\
&= x_a \mid E_q(-\gamma(\tau_2-\alpha)^q) - E_q(-\gamma(\tau_1-\alpha)^q) \mid \\
&\quad + \frac{M}{\gamma}(E_q(-\gamma(\tau_1-\alpha)^q) - E_q(-\gamma(\tau_2-\alpha)^q)) \\
&= \left\{\frac{M}{\gamma} + x_a\right\}(E_q(-\gamma(\tau_1-\alpha)^q) - E_q(-\gamma(\tau_2-\alpha)^q)).
\end{aligned}
\tag{3.4}
$$

From (2.5) and Remark 2.1, it follows from $1 - e^{-z} \leq y$ and $e^{-z} \leq 1$ for $z \geq 0$ that

$$\begin{aligned}
&E_q(-\gamma(\tau_1-\alpha)^q) - E_q(-\gamma(\tau_2-\alpha)^q)\\
\leq&\int_0^\infty M_\alpha(\nu) \mid e^{-\gamma(\tau_1-\alpha)^q\nu} - e^{-\gamma(\tau_2-\alpha)^q\nu} \mid d\nu\\
=&\int_0^\infty M_\alpha(\nu) e^{-\gamma(\tau_1-\alpha)^q\nu} \mid 1 - e^{-\gamma((\tau_2-\alpha)^q-(\tau_1-\alpha)^q)\nu} \mid d\nu\\
\leq&\gamma((\tau_2-\alpha)^q - (\tau_1-\alpha)^q)\int_0^\infty M_\alpha(\nu)\nu d\nu\\
=&\frac{\gamma\Gamma(2)}{\Gamma(1+q)}((\tau_2-\alpha)^q - (\tau_1-\alpha)^q)\\
\leq&\frac{\gamma\Gamma(2)}{\Gamma(1+q)}(\tau_2-\tau_1)^q,
\end{aligned} \tag{3.5}$$

where $\alpha < \tau_1 < \tau_2 \leq \beta$ and we use the inequality

$$(\tau_2-\alpha)^q - (\tau_1-\alpha)^q \leq \max\{1,q\}(\tau_2-\alpha)^{\max\{0,q-1\}}(\tau_2-\tau_1)^{\min\{q,1\}} \leq (\tau_2-\tau_1)^q.$$

Thus,

$$\begin{aligned}
\mid (Tx)(\tau_2) - (Tx)(\tau_1) \mid \leq& \left\{\frac{M}{\gamma} + x_a\right\}\frac{\gamma\Gamma(2)}{\Gamma(1+q)}(\tau_2-\tau_1)^q\\
\longrightarrow& 0 \quad \text{as} \quad \tau_2 \to \tau_1.
\end{aligned} \tag{3.6}$$

We assert that T is a completely continuous operator on the basis of Arzelá-Ascoli theorem. By Schauder's fixed point theorem, the problem (1.1)-(1.2) has at least one solution in $C(I, I)$. □

Corollary 3.1. *The fractional order iterative differential equation (1.1)-(1.2) has at least one continuous solution, if $\gamma \to 0$, $v \to 1$ and $\alpha \to 0^+$ as demonstrated by Ibrahim (2012) and (2013).*

4. Existence and an estimate of the result in $C_{K,q}(I, I)$

Assuming that K is a positive constant and $E \subset \mathbb{R}$ is a compact interval, we present the following set:

$$C_{K,q}(E,\mathbb{R}) = \{x \in C(E,\mathbb{R}) \mid\mid x(\tau_1) - x(\tau_2) \mid\leq K \mid \tau_1 - \tau_2 \mid^q\}$$

for any $\tau_1, \tau_2 \in \mathbb{R}$. Note that $C_{K,q}(E,\mathbb{R}) \subseteq C(E,\mathbb{R})$ is a complete metric space.

Theorem 4.1. *Let (H1)-(H4) hold. The solution to the IVP (1.1)-(1.2) can be acquired in $C_{K,q}(I, I)$, and $C_{K,q}(I, I)$ owns all its solutions for*

$$K = \left\{\frac{M}{\gamma} + x_a\right\}\frac{\gamma\Gamma(2)}{\Gamma(1+q)}.$$

Proof. Based on Theorem 3.1, we know that

$$T : C(I, I) \to C(I, I),$$

and T has a minimum of one solution in $C(I, I)$. Consider $\alpha < \tau_1 < \tau_2 \le \beta$. Then from (3.4), (3.5) and (3.6), we have

$$| (Tx)(\tau_2) - (Tx)(\tau_1) | \le K | \tau_1 - \tau_2 |^q .$$

Furthermore, for $\tau_1 = \alpha$ and $\alpha < \tau_2 \le \beta$, from Theorem 3.1, we have

$$| (Tx)(\tau_2) - (Tx)(\tau_1) | \le \left\{ \frac{M}{\gamma} + x_a \right\} | 1 - E_q(-\gamma(\tau_2 - \alpha)^q) | .$$

From Lemma 2.4 and Remark 2.1, we have

$$\begin{aligned} | 1 - E_q(-\gamma(\tau_2 - \alpha)^q) | &= | E_q(0) - E_q(-\gamma(\tau_2 - \alpha)^q) | \\ &= | \int_0^\infty M_\alpha(\nu) d\nu - \int_0^\infty M_\alpha(\nu) e^{-\gamma(\tau_2 - \alpha)^q \nu} d\nu | \\ &\le \int_0^\infty M_\alpha(\nu) | 1 - e^{-\gamma(\tau_2 - \alpha)^q \nu} | d\nu \\ &\le \gamma(\tau_2 - \alpha)^q \int_0^\infty M_\alpha(\nu) \nu d\nu \\ &= \frac{\gamma \Gamma(2)}{\Gamma(1+q)} (\tau_2 - \tau_1)^q, \end{aligned}$$

where we use the inequality

$$1 - e^{-z} \le z \quad \text{for} \quad z \ge 0.$$

In summary, for $\tau_1, \tau_2 \in I$, we have

$$| (Tx)(\tau_2) - (Tx)(\tau_1) | \le K | \tau_1 - \tau_2 |^q .$$

As a result, at least one solution for the IVP (1.1)-(1.2) exists in $C_{K,q}(I, I)$. □

Theorem 4.2. *Let* (H1-H4) *hold. There are constants* $K_1,\ K_2 > 0$ *such that*

(H5) $$| f(\tau, x_1, y_1) - f(\tau, x_2, y_2) | \le K_1 | x_1 - x_2 | + K_2 | y_1 - y_2 |$$

for $\tau \in I$, x_i, $y_i \in I (i = 1, 2)$. *Then two solutions* x_1 *and* x_2 *of problems (1.1)-(1.2) satisfy*

$$\| x_1 - x_2 \| \le K_T^{\frac{1}{1 - q \min\{1, v\}}} .$$

Where

$$\begin{aligned} K_T = &\frac{1}{\gamma} \big(1 - E_q(-\gamma(\beta - \alpha)^q)\big) [(K_1 + K_2) \beta^{1 - q \min\{1, v\}} \\ &+ \max\{1, v^q\} (\beta - \alpha)^{q \max\{v - 1, 0\}} K K_2]. \end{aligned}$$

If

$$\big(1 - E_q(-\gamma(\beta - \alpha)^q)\big)(K_1 + K_2) \le \gamma,$$

then

$$\| x_1 - x_2 \| \le \tilde{K}_T^{\frac{1}{1 - q \min\{1, v\}}},$$

where

$$\tilde{K}_T = \frac{\gamma - \big(1 - E_q(-\gamma(\beta-\alpha)^q)\big)(K_1+K_2)}{\big(1 - E_q(-\gamma(\beta-\alpha)^q)\big)\max\{1, v^q\}(\beta-\alpha)^{q\max\{v-1,0\}}KK_2}.$$

Proof. We can infer that there is at least one solution to problem (1.1)-(1.2) from Theorem 3.1. Let x_1 and x_2 be two solutions of (1.1)-(1.2). Then from Lemma 2.5, we have

$$\begin{aligned} x_1(\tau) =& x_a E_q(-\gamma(\tau-\alpha)^q) \\ &+ \int_0^{\tau-\alpha} (\tau-\varsigma-\alpha)^{q-1} E_{q,q}(-\gamma(\tau-\varsigma-\alpha)^q) f(\varsigma, x_1(\varsigma), x_1(x_1^v(s)))d\varsigma, \\ x_2(\tau) =& x_a E_q(-\gamma(\tau-\alpha)^q) \\ &+ \int_0^{\tau-\alpha} (\tau-\varsigma-\alpha)^{q-1} E_{q,q}(-\gamma(\tau-\varsigma-\alpha)^q) f(\varsigma, x_2(\varsigma), x_2(x_2^v(s)))d\varsigma. \end{aligned}$$

For $\tau \in I$, from (H5), we obtain

$$\begin{aligned} &\mid T(x_1)(\tau) - T(x_2)(\tau) \mid \\ =& \mid \int_0^{\tau-\alpha} (\tau-\varsigma-\alpha)^{q-1} E_{q,q}(-\gamma(\tau-\varsigma-a)^q) f(\varsigma, x_1(\varsigma), x_1(x_1^v(s)))d\varsigma \\ &- \int_0^{\tau-\alpha} (\tau-\varsigma-\alpha)^{q-1} E_{q,q}(-\gamma(\tau-\varsigma-\alpha)^q) f(\varsigma, x_2(\varsigma), x_2(x_2^v(s)))d\varsigma \mid \\ \leq& \int_0^{\tau-\alpha} (\tau-\varsigma-\alpha)^{q-1} E_{q,q}(-\gamma(\tau-\varsigma-\alpha)^q) \mid f(\varsigma, x_1(\varsigma), x_1(x_1^v(s))) \\ &- f(\varsigma, x_2(\varsigma), x_2(x_2^v(s))) \mid d\varsigma \\ \leq& \int_0^{\tau-\alpha} (\tau-\varsigma-\alpha)^{q-1} E_{q,q}(-\gamma(\tau-\varsigma-\alpha)^q)[K_1 \mid x_1(\varsigma) - x_2(\varsigma) \mid \\ &+ K_2 \mid x_1(x_1^v(s)) - x_2(x_2^v(s)) \mid] d\varsigma \\ \leq& \int_0^{\tau-\alpha} (\tau-\varsigma-\alpha)^{q-1} E_{q,q}(-\gamma(\tau-\varsigma-\alpha)^q)[K_1 \mid x_1(\varsigma) - x_2(\varsigma) \mid \\ &+ K_2 \mid x_1(x_1^v(s)) - x_1(x_2^v(s)) \mid] + K_2 \mid x_1(x_2^v(s)) - x_2(x_2^v(s)) \mid] d\varsigma \\ \leq& \int_0^{\tau-\alpha} (\tau-\varsigma-\alpha)^{q-1} E_{q,q}(-\gamma(\tau-\varsigma-\alpha)^q)[(K_1+K_2) \parallel x_1 - x_2 \parallel \\ &+ KK_2 \mid x_1^v(\varsigma) - x_2^v(\varsigma) \mid^q] d\varsigma \\ \leq& \int_0^{\tau-\alpha} (\tau-\varsigma-\alpha)^{q-1} E_{q,q}(-\gamma(\tau-\varsigma-\alpha)^q)[(K_1+K_2) \parallel x_1 - x_2 \parallel \\ &+ \max\{1, v^q\}(\beta-\alpha)^{q\max\{v-1,0\}} KK_2 \parallel x_1 - x_2 \parallel^{q\min\{1,q\}}] d\varsigma \\ \leq& \frac{1}{\gamma}\big(1 - E_q(-\gamma(\tau-\alpha)^q)\big)[(K_1+K_2) \parallel x_1 - x_2 \parallel \\ &+ \max\{1, v^q\}(\beta-\alpha)^{q\max\{v-1,0\}} KK_2 \parallel x_1 - x_2 \parallel^{q\min\{1,q\}}]. \end{aligned}$$

From $\| x_1 - x_2 \| \leq \beta$, we obtain

$$\begin{aligned}\| T(x_1) - T(x_2) \| \leq & \frac{1}{\gamma}\big(1 - E_q(-\gamma(\beta-\alpha)^q)\big)[(K_1+K_2)\beta^{1-q\min\{1,v\}} \\ & + \max\{1, v^q\}(\beta-\alpha)^{q\max\{v-1,0\}} K K_2] \| x_1 - x_2 \|^{q\min\{1,v\}} \\ = & K_T \| x_1 - x_2 \|^{q\min\{1,v\}} .\end{aligned}$$

We know that T is Hölder continuous rather than Lipschitz continuous since $\alpha \min \{1, v\} \leq \alpha < 1$. Assume that T has fixed points x_1 and x_2, then

$$\| x_1 - x_2 \| = \| T(x_1) - T(x_2) \| \leq K_T \| x_1 - x_2 \|^{q\min\{1,v\}} .$$

Thus, $\| u_1 - u_2 \| \leq K_T^{\frac{1}{1-q\min\{1,v\}}}$.

In general, we get

$$\begin{aligned}\| x_1 - x_2 \| \leq & \frac{1}{\gamma}\big(1 - E_q(-\gamma(\beta-\alpha)^q)\big)[(K_1+K_2) \| x_1 - x_2 \| \\ & + \max\{1, v^q\}(\beta-\alpha)^{q\max\{v-1,0\}} K K_2 \| x_1 - x_2 \|^{q\min\{1,v\}}] \\ \leq & \frac{1}{\gamma}\big(1 - E_q(-\gamma(\beta-\alpha)^q)\big)(K_1+K_2) \| x_1 - x_2 \| \\ & + \frac{1}{\gamma}\big(1 - E_q(-\gamma(\beta-\alpha)^q)\big) \max\{1, v^q\}(\beta-\alpha)^{q\max\{v-1,0\}} \\ & \times K K_2 \| x_1 - x_2 \|^{q\min\{1,v\}} .\end{aligned}$$

Thus,

$$\| x_1 - x_2 \| \leq \frac{\gamma - \big(1 - E_q(-\gamma(\beta-\alpha)^q)\big)(K_1+K_2) \| x_1 - x_2 \|^{q\min\{1,v\}}}{\big(1 - E_q(-\gamma(\beta-\alpha)^q)\big)\max\{1, v^q\}(\beta-\alpha)^{q\max\{v-1,0\}} K K_2},$$

which means that

$$\begin{aligned}& \| x_1 - x_2 \| \\ \leq & \left\{ \frac{\gamma - \big(1 - E_q(-\gamma(\beta-\alpha)^q)\big)(K_1+K_2)}{\big(1 - E_q(-\gamma(\beta-\alpha)^q)\big)\max\{1, v^q\}(\beta-\alpha)^{q\max\{v-1,0\}} K K_2} \right\}^{\frac{1}{1-q\min\{1,v\}}} .\end{aligned}$$

□

5. Continuous dependence of solutions

We address the continuous dependence of the solutions to (1.1) by utilizing integral inequalities as a useful tool. Let's give a slight modification to the initial value, that is,

$$x(\alpha) = x_a + \varepsilon, \tag{5.1}$$

where ε can be any kind of constant.

Theorem 5.1. *Assume that the Theorem 4.2's conditions are met. Suppose that IVP (1.1)-(1.2) has a solution $x(\tau)$ and $\omega(\tau)$ is the solution of*

$$\begin{cases} {}^C D^q_{\alpha,}\omega(\tau) + \gamma\omega(\tau) = f(\tau, \omega(\tau), \omega(\omega^v(\tau))), \\ \omega(\alpha) = x_a + \varepsilon, \quad \tau \in I. \end{cases} \tag{5.2}$$

Let γ_1 be a positive root of equation

$$\gamma_1 =\mid \varepsilon \mid +K_T\gamma_1^{q\min\{1,v\}}.$$

Then

$$\parallel x-\omega \parallel \leq \gamma_1. \tag{5.3}$$

If in addition

$$\big(1-E_q(-\gamma(\beta-\alpha)^q)\big)(K_1+K_2)\leq \gamma,$$

and let γ_2 be a positive root of equation

$$\gamma_2 =\mid \varepsilon \mid +\tilde{K}_T\gamma_2^{q\min\{1,v\}},$$

then

$$\parallel x-\omega \parallel \leq \gamma_2. \tag{5.4}$$

Proof. Recall that we just need to change x_a to $x_a+\varepsilon$ in (H4) in a bid to make the solution to (5.2) be well-defined.

Lemma 2.5 indicates that

$$\begin{aligned} x(\tau) =& x_aE_q(-\gamma(\tau-\alpha)^q)\\ &+\int_0^{\tau-\alpha}(\tau-\varsigma-\alpha)^{q-1}E_{q,q}(-\gamma(\tau-\varsigma-\alpha)^q)f(\varsigma,x(\varsigma),x(x^v(s)))d\varsigma, \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} \omega(\tau) =& (x_a+\varepsilon)E_q(-\gamma(\tau-\alpha)^q)\\ &+\int_0^{\tau-\alpha}(\tau-\varsigma-\alpha)^{q-1}E_{q,q}(-\gamma(\tau-\varsigma-\alpha)^q)f(\varsigma,\omega(\varsigma),\omega(\omega^v(s)))d\varsigma, \end{aligned} \tag{5.6}$$

which are the equivalent integral solution to (1.1)-(1.2) and (5.2), respectively.

(5.5) minus (5.6), given condition (H5), for any sufficiently small ε we get

$$\begin{aligned} \mid x(\tau)-\omega(\tau)\mid \leq& \mid \varepsilon \mid E_q(-\gamma(\tau-\alpha)^q)+\int_0^{\tau-\alpha}(\tau-\varsigma-\alpha)^{q-1}E_{q,q}(-\gamma(\tau-\varsigma-\alpha)^q)\\ &\times \mid f(\varsigma,x(\varsigma),x(x^v(s)))-f(\varsigma,\omega(\varsigma),\omega(\omega^v(s)))\mid d\varsigma\\ \leq& \mid \varepsilon \mid +\int_0^{\tau-\alpha}\mid E_{q,q}(-\gamma(\tau-\varsigma-\alpha)^q)[K_1\mid u(\varsigma)-\omega(\varsigma)\mid\\ &+K_2\mid x(x^v(s))-\omega(\omega^v(s))\mid]d\varsigma. \end{aligned}$$

Comparable to Theorem 4.2's proof, from $\parallel x-\omega\parallel\leq\beta$, we have

$$\parallel x-\omega \parallel \leq \mid \varepsilon \mid +K_T\parallel x-\omega \parallel^{q\min\{1,v\}},$$

which means (5.3). Generally speaking, we get

$$\parallel x-\omega \parallel \leq \mid \varepsilon \mid +\tilde{K}_T\parallel x-\omega \parallel^{q\min\{1,v\}},$$

which means (5.4). □

Remark 5.1. Theorem 5.1 tells us the solution to $[k,\beta]$ for k between α and β will vary if the condition (5.1) is slightly perturbed.

Remark 5.2. Our results can be used to the fractional IVP:

$$\begin{aligned} &{}^cD_{\alpha^+}^q x(\tau)+\gamma x(\tau)=f(\tau,x(\tau),x(x^{v_1}(\tau)),\cdots,x(x^{v_n}(\tau))),\quad n\in\mathbb{N}^+,\\ &x(\alpha)=x_a,\quad x_a\in I,\quad \tau\in I,\quad 0<q<1. \end{aligned}$$

6. Examples

Example 6.1. Consider the fractional IVP

$$\begin{aligned} &{}^cD_{0^+}^{0.5}x(\tau) + \gamma x(\tau) = \frac{1}{\sqrt{\tau^2+1}+99}\sin^2(x(x(\tau))), \\ &x(0) = 0, \quad \gamma > 0, \end{aligned} \tag{6.1}$$

where function f is defined by

$$f(\tau, x, x(x(\tau))) = \frac{1}{\sqrt{\tau^2+1}+99}\sin^2(x(x(\tau))),$$

$q = 0.5$, $v = 1$, $\alpha = 0$, $\beta = \pi$, $x_a = 0$. It is evident that (H1-H2) are valid. In addition, we obtain

$$0 \le f(\tau, x(\tau), x(x(\tau))(\tau)) \le \frac{1}{\sqrt{\tau^2+1}+99} \le \frac{1}{100}.$$

Taking $f_m = 0$, $f_M = \frac{1}{100}$, if γ satisfies the following inequality equation

$$\frac{1}{100\gamma}(1 - E_{0.5}(-\sqrt{\pi}\gamma)) \le \pi, \tag{6.2}$$

employing Theorem 3.1, we know that for every value of γ, the solution to (6.1) exists such that (6.2) holds.

Additionally, from the Mean Value Theorem we have

$$\frac{1}{\sqrt{\tau^2+1}+99} \mid \sin^2 v_1 - \sin^2 v_2 \mid = \frac{1}{\sqrt{\tau^2+1}+99} \mid \sin 2\theta \mid\mid v_1 - v_2 \mid \le \frac{1}{100} \mid v_1 - v_2 \mid,$$

where $\theta \in [0.\pi]$. Taking $K_1 = 0$ and $K_2 = \frac{1}{100}$, we get

$$\begin{aligned} &K = \left\{\frac{M}{\gamma} + x_a\right\}\frac{\gamma\Gamma(2)}{\Gamma(1+q)} = \frac{\Gamma(2)}{100\Gamma(1.5)}, \\ &1 - q\min\{1, v\} = \frac{1}{2}, \quad \frac{1}{1 - q\min\{1, v\}} = 2, \\ &\max\{1, v^q\}(\beta - \alpha)^{q\max\{v-1,0\}} = 1. \end{aligned}$$

By simple calculation, we have

$$K_T = \frac{1}{\gamma}(1 - E_{0.5}(-\sqrt{\pi}\gamma))\left[\frac{\sqrt{\pi}}{100} + \frac{\Gamma(2)}{10^4\Gamma(1.5)}\right],$$

where γ satisfies equation (6.2).

Thus, by Theorem 4.1, there is a solution for equation (6.1) in $C_{K,\frac{1}{2}}$, where $K = \frac{\Gamma(2)}{100\Gamma(1.5)}$. By Theorem 4.2, any two solutions x_1, $x_2 \in C_{K,\frac{1}{2}}(0,\pi)$ of (6.1) satisfy

$$\| x_1 - x_2 \| \le \frac{1}{\gamma^2}(1 - E_{0.5}(-\sqrt{\pi}\gamma))^2\left[\frac{\sqrt{\pi}}{100} + \frac{\Gamma(2)}{10^4\Gamma(1.5)}\right]^2.$$

Example 6.2. Consider the fractional IVP

$$
\begin{aligned}
&{}^{c}D_{0^+}^{0.5}x(\tau)+\gamma x(\tau)=\cos(\tau)\left(\frac{1}{\sqrt{1+x^2(\tau)}}+\sin(x(x(\tau)))\right),\\
&x(0)=0.1,\quad \gamma>0\ \ \tau\in[0,1],
\end{aligned}
\tag{6.3}
$$

where the function f is defined by

$$
f(\tau,x,x(x(\tau)))=\cos(\tau)\left(\frac{1}{\sqrt{1+x^2(\tau)}}+\sin(x(x(\tau)))\right),
$$

$q=0.5$, $v=1$, $\alpha=0$, $x_a=0.1$, $\beta=1$. It is evident that (H1-H2) are valid. In addition, we get

$$
-2\le f(\tau,x(\tau),x(x(\tau))(\tau))\le 2.
$$

Taking $f_m=-2$, $f_M=2$, we assume that γ satisfies the inequality equation that follows

$$
(1-E_{0.5}(-\gamma))\le 2\gamma M_a,\quad M_a=\max\{0.1E_{0.5}(-\gamma),0.9\}. \tag{6.4}
$$

Employing Theorem 3.1 we know that for every value of γ, the solution to (6.3) exists such that (6.4) holds. Furthermore, it is simple to infer that for any (x_1,y_1), $(x_2,y_2)\in\mathbb{R}\times\mathbb{R}$ and $\tau\in[0,1]$

$$
\begin{aligned}
|f(\tau,x_1,y_1)-f(\tau,x_2,y_2)|&\le|\cos(\tau)|\left[\left|\frac{1}{\sqrt{1+x_1^2}}-\frac{1}{\sqrt{1+x_2^2}}\right|+|\sin y_1^2-\sin y_2^2|\right]\\
&\le|x_1-x_2|+|y_1-y_2|.
\end{aligned}
$$

Taking $K_1=1$ and $K_2=1$, we get

$$
\begin{aligned}
&K=\left\{\frac{M}{\gamma}+x_a\right\}\frac{\gamma\Gamma(2)}{\Gamma(1+q)}=\frac{(20+\gamma)\Gamma(2)}{10\Gamma(1.5)},\\
&1-q\min\{1,v\}=\frac{1}{2},\quad \frac{1}{1-q\min\{1,v\}}=2,\\
&\max\{1,v^q\}(\beta-\alpha)^{q\max\{v-1,0\}}=1.
\end{aligned}
$$

By simple calculation, we have

$$
K_T=\frac{1}{\gamma}(1-E_{0.5}(-\gamma))(K+2),
$$

where γ satisfies equation (6.4).

Thus, according to Theorem 4.1, the solution to equation (6.3) can be found in $C_{K,\frac{1}{2}}$, where $K=\frac{(20+\gamma)\Gamma(2)}{10\Gamma(1.5)}$. By Theorem 4.2, any two solutions x_1, $x_2\in C_{K,\frac{1}{2}}(0,1)$ of (6.3) satisfy

$$
\| x_1-x_2\|\le\frac{1}{\gamma^2}(1-E_{0.5}(-\sqrt{\pi}\gamma))^2(K+2)^2.
$$

7. Conclusions

This paper addressed the existence, uniqueness and continuous dependence of solutions to the IVP of semi-linear FIDE. More specifically, we changed IVP into a fixed point problem employing the Laplace transform and some relations in fractional calculus. Schauder's fixed point theorem was utilized to explain the existence of solutions for FIDE. Unfortunately, because the operator is only Hölder continuous rather than Lipschitz continuous, uniqueness results can not be acquired. Moreover, continuous dependence of solutions offers a potential way to characterize the error estimates between approximate and explicit solutions to these kinds of problems. Lastly, some examples have been provided to demonstrate the findings.

Even if there have been a lot of works on FIDE thus far, there are still a lot of issues that need to be resolved. As far as we are aware, The FIDE coupled system has not been studied extensively. Hence, we intend to look into those more intriguing and challenging problems in our further research.

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