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LONG TIME BEHAVIOUR OF THE SOLUTIONS OF NONLINEAR WAVE EQUATION*

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Abstract In this paper, we consider the nonlinear wave equation

$$u_{tt} - \Delta u + mu + f(x,u) = 0,\ x \in \mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d,$$

where $m > 0$ and f is an analytic function of order at least two in u. The long time behaviour of its solutions is proved by Birkhoff normal form.

Keywords Long time behaviour, nonlinear wave equation, Birkhoff normal form.

MSC(2010) 37K45, 37K55.

1. Introduction and result

Consider nonlinear wave equation on d-dimensional torus

$$u_{tt} - \Delta u + mu + f(x,u) = 0, \quad x \in \mathbb{T}^d \tag{1.1}$$

with given initial data $u(0) \equiv u(0,x)$ and $\dot{u}(0) \equiv \partial_t u(0,x)$, where $m > 0$ and $f(x,u)$ is analytic function of order at least two with respect to u at the origin. For (1.1) with nonlinearity of the form $f(u)$, i.e., not containing the spatial variable x explicitly, the long time behaviour of the solutions has been proved by Bernier, Faou and Grébert in [5]. More precisely, they show that for almost all $m > 0$ and all $r \geq 2, s_0 > (d+1)/2$, there exists s_* depending on r, s_0 such that for any $s \geq s_*$, if the initial datum satisfies $\|(u(0),\dot{u}(0))\|_{H^s \times H^{s-1}} \leq \varepsilon$ for small enough $\varepsilon > 0$, then $\|u(t)_{\leq N_\varepsilon}\|_{H^{\frac{s}{2}}} \leq 2\varepsilon$ and $\|u(t)_{>N_\varepsilon}\|_{H^{s_0}} \leq \varepsilon^r$ for any $t \leq \varepsilon^{-\frac{r}{s_0+1}}$, where H^s is the Sobolev space on $\mathbb{T}^d$, $u(t)_{\leq N_\varepsilon}$ and $u(t)_{>N_\varepsilon}$ denote the low and high modes parts according to the threshold $N_\varepsilon = \varepsilon^{-\frac{2r}{s-2s_0}}$, respectively. In the following theorem, we study more general nonlinearity $f(x,u)$. For convenience, we keep fidelity with the notation and terminology from [5].

Theorem 1.1. *For almost all $m > 0$ and any given $r \geq 1$, there exists $\tau, \varepsilon_* > 0$ such that for any $s > 2s_0 > d+1$ with $s - 2s_0 \geq 2r(2r\tau + d)$, if the initial data $(u(0),\dot{u}(0)) \in H^s \times H^{s-1}$ satisfies $\varepsilon := \|(u(0),\dot{u}(0))\|_{H^s \times H^{s-1}} \leq \varepsilon_*$, then*

$$\|u(t)_{\leq N_\varepsilon}\|_{H^{\frac{s}{2}}} \leq c_0\varepsilon \quad \textit{and} \quad \|u(t)_{>N_\varepsilon}\|_{H^{s_0}} \leq \varepsilon^r \tag{1.2}$$

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for any $t \le \varepsilon^{-\frac{r}{2s_0}}$, *where* $N_\varepsilon = \varepsilon^{-\frac{2r}{s-2s_0}}$ *and the positive constant* c_0 *depends on* m, s.

Birkhoff normal form for long time behavior of solutions of Hamiltonian partial differential equations has been widely investigated by many authors. For nonlinear wave equations, see [1–5, 10, 17, 23] for example; for nonlinear Schrödinger equations, see [4, 9, 12, 14, 19, 20] for example; for equations with unbounded nonlinear vector field, see [8, 13, 15, 16, 18, 21, 22] for example; for equations without external parameters, see [6, 7, 11] for example.

For (1.1), the frequencies of linear equation are $\omega_a := \sqrt{|a|^2 + m}$, $a \in \mathbb{Z}^d$. If one wishes to get the usual long time stability, it is necessary to meet the following non-resonant conditions: any given integer $l \ge 3$, there exists $\gamma, \tau > 0$ such that for any $\boldsymbol{k} = (k_1, \cdots, k_p) \in (\mathbb{Z}^d)^p, \boldsymbol{h} = (h_1, \cdots, h_q) \in (\mathbb{Z}^d)^q$ with $p + q \le l$, one has

$$|\omega_{k_1} + \cdots + \omega_{k_p} - \omega_{h_1} - \cdots - \omega_{h_q}| \ge \frac{\gamma}{\mu_3(\boldsymbol{k}, \boldsymbol{h})^\tau}, \tag{1.3}$$

expect that $(|k_1|, \cdots, |k_p|)$ and $(|h_1|, \cdots, |h_q|)$ are equal up to a permutation, where $\mu_3(\boldsymbol{k}, \boldsymbol{h})$ denotes the third largest number among $\{|k_1|, \cdots, |k_p|, |h_1|, \cdots, |h_q|\}$. But when $d \ge 2$, (1.3) is not satisfied due to the combinations with two high frequencies in opposite signs, i.e., without loss of generality $|k_p|, |h_q| > N \ge \mu_3(\boldsymbol{k}, \boldsymbol{h})$ for some N large enough. In [5], the non-resonant conditions of this type are removed, and as a result, the corresponding monomials $z_{k_1} \cdots z_{k_p} \bar{z}_{h_1} \cdots \bar{z}_{h_q}$ are remained in the normal form. Of course, these terms essentially affect the long time stability. Novelly observing that these terms preserve the L^2 norm of high modes, the authors derive the result of long time behavior from the normal form. During the proof of energy estimate of high modes in higher Sobolev space H^s, the conservation of momentum, i.e.,

$$k_1 + \cdots + k_p - h_1 - \cdots - h_q = 0, \tag{1.4}$$

is crucially used to bound $|k_p|^{2s} - |h_q|^{2s}$.

However, in the present paper, the nonlinearity $f(x, u)$ contains the spatial variable x explicitly so that (1.4) is not true. Then we solve this problem by eliminating more terms than [5]. Precisely, we eliminate the monomials $z_{k_1} \cdots z_{k_p} \bar{z}_{h_1} \cdots \bar{z}_{h_q}$ with $|k_p|, |h_q| > N \ge \mu_3(\boldsymbol{k}, \boldsymbol{h})$ and $\big||k_p| - |h_q|\big| \ge C_0 N$ for some positive constants C_0. See Theorem 2.1 for the normal form and see (3.19) for the energy estimate of high modes in higher Sobolev space.

This paper is organized as follows: in Section 2, we give a normal form theorem, seeing Theorem 2.1, which is a modified version of Theorem 5 in [5]. In section 3, we apply Theorem 2.1 to the nonlinear wave equation (1.1) and thus prove Theorem 1.1. The main step is to control the high modes, and the key is to estimate the higher Sobolev norm with the help of L^2 norm, seeing (3.19). Besides, instead of Lemma 3 in [5], we estimate the high modes directly from the vector field in the same way as the low modes, seeing (3.21).

2. Normal form theorem

Define the Hilbert space $l^2_s(\mathbb{Z}^d, \mathbb{C})$ of the complex sequences $\xi = \{\xi_a\}_{a \in \mathbb{Z}^d}$ such that

$$\|\xi\|_s^2 := \sum_{a \in \mathbb{Z}^d} \langle a \rangle^{2s} |\xi_a|^2 < \infty \tag{2.1}$$

with $\langle a\rangle^2 := 1+|a|^2 = 1+a_1^2+\cdots+a_d^2$. Notice that for complex function $u(x) = \sum_{a\in\mathbb{Z}^d}\xi_a e^{\mathrm{i}a\cdot x}$ on $\mathbb{T}^d$ with $a\cdot x = a_1x_1+\cdots+a_dx_d$, the Sobolev norm $\|u\|_{H^s}$ is equivalent to the norm $\|\xi\|_s$. The scale of phase spaces

$$l_s^2\oplus l_s^2 \ni (\xi,\bar{\xi}) = (\{\xi_a\}_{a\in\mathbb{Z}^d},\{\bar{\xi}_a\}_{a\in\mathbb{Z}^d})$$

is endowed by the standard symplectic structure $-\mathrm{i}\sum_{a\in\mathbb{Z}^d} d\xi_a\wedge d\bar{\xi}_a$. For a Hamiltonian function $H(\xi,\bar{\xi})$, define its vector field

$$X_H(\xi,\bar{\xi}) = -\mathrm{i}\Big(\frac{\partial H}{\partial\bar{\xi}},-\frac{\partial H}{\partial\xi}\Big), \tag{2.2}$$

and for two Hamiltonian functions $H(\xi,\bar{\xi})$ and $F(\xi,\bar{\xi})$, define their Poisson bracket

$$\{H,F\} = -\mathrm{i}\sum_{a\in\mathbb{Z}^d}\Big(\frac{\partial H}{\partial\xi_a}\frac{\partial F}{\partial\bar{\xi}_a}-\frac{\partial H}{\partial\bar{\xi}_a}\frac{\partial F}{\partial\xi_a}\Big). \tag{2.3}$$

In a brief statement, we identify $\mathbb{C}^{\mathbb{Z}^d}\times\mathbb{C}^{\mathbb{Z}^d}\simeq\mathbb{C}^{\mathbb{U}_2\times\mathbb{Z}^d}$ with $\mathbb{U}_2=\{\pm1\}$ and use the convenient notation $z=(z_j)_{j=(\delta,a)\in\mathbb{U}_2\times\mathbb{Z}^d}$, where

$$z_j = \begin{cases}\xi_a, \text{ when } \delta=1,\\ \bar{\xi}_a, \text{ when } \delta=-1.\end{cases}$$

Set $\langle j\rangle = \langle a\rangle$ and define

$$\|z\|_s^2 := \sum_{j\in\mathbb{U}_2\times\mathbb{Z}^d}\langle j\rangle^{2s}|z_j|^2 = \|\xi\|_s^2+\|\bar{\xi}\|_s^2. \tag{2.4}$$

In particular, for any $j\in\mathbb{U}_2\times\mathbb{Z}^d$, decompose $z=z_{\le N}+z_{>N}$ with

$$(z_{\le N})_j = \begin{cases} z_j, \text{ for } \langle j\rangle\le N\\ 0, \text{ for } \langle j\rangle > N\end{cases} \quad\text{and}\quad (z_{>N})_j = \begin{cases} 0, \text{ for } \langle j\rangle\le N\\ z_j, \text{ for } \langle j\rangle > N\end{cases}. \tag{2.5}$$

For $\boldsymbol{j}=(j_1,\cdots,j_r)=(\delta_k,a_k)_{k=1}^r\in(\mathbb{U}_2\times\mathbb{Z}^d)^r$, denote the monomial $z_{\boldsymbol{j}} = z_{j_1}\cdots z_{j_r}$. For a homogeneous polynomial $P(z)$ of order r, namely

$$P(z) = \sum_{\boldsymbol{j}\in(\mathbb{U}_2\times\mathbb{Z}^d)^r} P_{\boldsymbol{j}} z_{\boldsymbol{j}},$$

define the μ-modulus

$$|P|_\mu = \sum_{a\in\mathbb{Z}^d} e^{\mu|a|}\sup_{\delta_1a_1+\cdots+\delta_ra_r=a}|P_{\boldsymbol{j}}|. \tag{2.6}$$

Similarly to the proof of Lemma 5.1 and Lemma 5.2 in [1], one has the following estimate of vector field

$$\|X_P(z)\|_s \le C|P|_\mu\|z\|_s^{r-1} \tag{2.7}$$

with constant C depending on r,s,μ. For two homogeneous polynomials P,Q of order r_1,r_2 respectively with finite μ-modulus, similarly to Lemma 5.9 of [1], one has the following estimate of Poisson bracket

$$|\{P,Q\}|_\mu \le r_1r_2|P|_\mu|Q|_\mu. \tag{2.8}$$

Theorem 2.1. *Fix a positive integer r. Consider the Hamiltonian function*

$$H = H_0 + P = \sum_{a\in\mathbb{Z}^d} \omega_a|\xi_a|^2 + P, \tag{2.9}$$

where the frequencies $\omega := \{\omega_a\}_{a\in\mathbb{Z}^d}$ and the higher order perturbation P satisfy the following two assumptions respectively:

(1) for any positive integer l, there exists $\gamma, \tau, C_0 > 0$ such that for any $N \geq 1$ and $\boldsymbol{j} = (\delta_k, a_k)_{k=1}^l \in (\mathbb{U}_2 \times \mathbb{Z}^d)^l$, $b_1, b_2 \in \mathbb{Z}^d$ with $\langle a_k\rangle \leq N$, $\langle b_1\rangle, \langle b_2\rangle > N$, we have

$$|\delta_1\omega_{a_1} + \cdots + \delta_l\omega_{a_l}| \geq \frac{\gamma}{N^\tau}, \quad \text{when } \boldsymbol{j} \notin \mathscr{A}_l, \tag{2.10}$$

$$|\delta_1\omega_{a_1} + \cdots + \delta_l\omega_{a_l} + \omega_{b_1}| \geq \frac{\gamma}{N^\tau}, \tag{2.11}$$

$$|\delta_1\omega_{a_1} + \cdots + \delta_l\omega_{a_l} + \omega_{b_1} + \omega_{b_2}| \geq \frac{\gamma}{N^\tau}, \tag{2.12}$$

$$|\delta_1\omega_{a_1} + \cdots + \delta_l\omega_{a_l} + \omega_{b_1} - \omega_{b_2}| \geq \frac{\gamma}{N^\tau}, \text{ when } \big||b_1| - |b_2|\big| \geq C_0N, \tag{2.13}$$

where $\mathscr{A}_l := \{\boldsymbol{j} = (\delta_k, a_k)_{k=1}^l \mid \exists$ permutation σ, s.t. $\forall k$, $\delta_k = -\delta_{\sigma_k}$, $|a_k| = |a_{\sigma_k}|\}$ is the set of resonant multi-indices;

(2) $P = \sum_{l\geq1} P_l$ with P_l homogenous of order $l+2$, and there exists $\mu, C_1, R_0 > 0$ such that

$$|P_l|_\mu \leq C_1 R_0^{-(l+2)}. \tag{2.14}$$

Then there exists a polynomial Hamiltonian $\chi = \sum_{l=1}^r \chi_l$ of order at most $r+2$ satisfying

$$|\chi_l|_\mu \leq C_2N^{r\tau} \tag{2.15}$$

with constant $C_2 > 0$ depending on r, C_1 and R_0 such that for any given $s > d/2$, the transformation of time one map Φ_χ^1 generated by χ, whose existence is guaranteed in a neighbourhood of the origin of $l_s^2 \oplus l_s^2$, puts H in normal form:

$$H \circ \Phi_\chi^1 = (H_0 + P) \circ \Phi_\chi^1 = H_0 + \mathscr{Z}^{(0)} + \mathscr{R}^{(ii)} + \mathscr{R}^{(iii)} + \mathscr{R}_{r+3}, \tag{2.16}$$

where

(i) the transformation fulfills the estimate

$$\|z - \Phi_\chi^1(z)\|_s \leq C_3N^{r\tau}\|z\|_s^2 \tag{2.17}$$

in a neighbourhood of the origin of $l_s^2 \oplus l_s^2$ with constant $C_3 > 0$ depending on s and C_2. Exactly, the same estimate is fulfilled by the inverse transformation;

(ii) $\mathscr{Z}^{(0)}$ is a polynomial of order at most $r+2$ and contains only resonant monomials, that is to say, for any $a \in \mathbb{Z}^d$,

$$\{J_a, \mathscr{Z}^{(0)}\} = 0, \tag{2.18}$$

where $J_a := \sum_{\langle b\rangle=\langle a\rangle} |z_b|^2$ is the super action;

(iii) $\mathscr{R}^{(ii)} = \sum_{l=1}^r \mathscr{R}_l^{(ii)}$ is a polynomial of order at most $r+2$ with two high modes in opposite signs and having small norm difference: they are (δ, b_1) and $(-\delta, b_2)$ with $\big||b_1| - |b_2|\big| < C_0N$. Moreover, the following estimate holds:

$$|\mathscr{R}_l^{(ii)}|_\mu \leq C_2N^{(r-1)\tau}; \tag{2.19}$$

(iv) $\mathscr{R}^{(iii)} = \sum_{l=1}^{r} \mathscr{R}_l^{(iii)}$ *is a polynomial of order at most* $r+2$ *with at least three high modes and*

$$|\mathscr{R}_l^{(iii)}|_\mu \le C_2 N^{(r-1)\tau}; \tag{2.20}$$

(v) $\mathscr{R}_{r+3}$ *is a polynomial of order at least* $r+3$ *and*

$$\|X_{\mathscr{R}_{r+3}}(z)\|_s \le C_3 N^{r\tau} \|z\|_s^{r+2}. \tag{2.21}$$

Proof. The proof is parallel to [5] except an essential difference: some terms with two high modes of opposite signs will be eliminated, while in [5], these terms are kept in $\mathscr{R}^{(ii)}$. For convenience, we introduce some notations.

For any $l = 1, \cdots, r$, decompose homogeneous polynomials P_l of order $l+2$ as follows:

$$P_l = P_l^{(0)} + P_l^{(i)} + P_l^{(ii)} + P_l^{(iii)},$$

where $P_l^{(0)}$ depends only on low modes, namely

$$P_l^{(0)}(z) = \sum_{\substack{\boldsymbol{j} \in (\mathbb{U}_2 \times \mathbb{Z}^d)^{l+2} \\ \mu_1(\boldsymbol{j}) \le N}} P_{l_{\boldsymbol{j}}}^{(0)} z_{\boldsymbol{j}},$$

$P_l^{(i)}$ contains only one high mode, namely

$$P_l^{(i)}(z) = \sum_{\substack{\boldsymbol{j} \in (\mathbb{U}_2 \times \mathbb{Z}^d)^{l+2} \\ \mu_2(\boldsymbol{j}) \le N < \mu_1(\boldsymbol{j})}} P_{l_{\boldsymbol{j}}}^{(i)} z_{\boldsymbol{j}},$$

$P_l^{(ii)}$ contains two high modes, namely

$$P_l^{(ii)}(z) = \sum_{\substack{\boldsymbol{j} \in (\mathbb{U}_2 \times \mathbb{Z}^d)^{l+2} \\ \mu_3(\boldsymbol{j}) \le N < \mu_2(\boldsymbol{j})}} P_{l_{\boldsymbol{j}}}^{(ii)} z_{\boldsymbol{j}},$$

and $P_l^{(iii)}$ contains at least three high modes, namely

$$P_l^{(iii)}(z) = \sum_{\substack{\boldsymbol{j} \in (\mathbb{U}_2 \times \mathbb{Z}^d)^{l+2} \\ \mu_3(\boldsymbol{j}) > N}} P_{l_{\boldsymbol{j}}}^{(iii)} z_{\boldsymbol{j}}$$

with $\mu_m(\boldsymbol{j})$ being the m-th largest number amongst the collection $\{\langle j_k \rangle\}_{k=1}^{l+2}$.

We will not only eliminate the non-resonant terms of $P_l^{(0)}$, all terms of $P_l^{(i)}$, and the terms of $P_l^{(ii)}$ with two high modes in same sign as in [5], but also eliminate the terms of $P_l^{(ii)}$ with two high modes in opposite signs and having large norm difference, i.e., (δ, b_1) and $(-\delta, b_2)$ with $\big||b_1| - |b_2|\big| \ge C_0 N$. For the latter, the corresponding homological equations are solved with the help of non-resonant condition (2.13).

Comparing with [5], although there is no condition of zero momentum, we still have the estimates of vector field and Poisson bracket, seeing (2.7) and (2.8). Besides, the remaining proof is a standard procedure of non-resonant Birkhoff normal form and thus we omit it. □

3. Proof the main theorem

Write the operator $\Lambda := (-\Delta + m)^{1/2}$ and let

$$z = \frac{1}{\sqrt{2}}(\Lambda^{\frac{1}{2}} u + \mathrm{i}\Lambda^{-\frac{1}{2}} \dot{u}), \tag{3.1}$$

and then (1.1) is equivalent to

$$\dot{z} = -\mathrm{i}\Lambda z - \frac{\mathrm{i}}{\sqrt{2}}\Lambda^{-\frac{1}{2}} f\big(x, \Lambda^{-\frac{1}{2}}(\frac{z+\bar{z}}{\sqrt{2}})\big). \tag{3.2}$$

Using the Fourier expansion $z(t,x) = \sum_{a\in\mathbb{Z}^d} \xi_a(t)\; e^{\mathrm{i}a\cdot x}$, for any $x \in \mathbb{T}^d$, rewrite (3.2) as

$$\dot{\xi}_a = -\mathrm{i}\frac{\partial H}{\partial \bar{\xi}_a} \tag{3.3}$$

with the Hamiltonian function

$$\begin{aligned} H &= H_0 + P \\ &= \sum_{a\in\mathbb{Z}^d} \omega_a |\xi_a|^2 + \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} F\big(x, \sum_{a\in\mathbb{Z}^d}(\frac{\xi_a e^{ia\cdot x} + \bar{\xi}_a e^{-ia\cdot x}}{\sqrt{2\omega_a}})\big)dx, \end{aligned} \tag{3.4}$$

where

$$\omega_a = \sqrt{|a|^2 + m} \tag{3.5}$$

and F is the primitive function of f with respect to the variable u, i.e., $f = \partial_u F$.

In the following, we identify the function z with its sequence of Fourier coefficients $\{\xi_a\}_{a\in\mathbb{Z}^d}$(or $\{z_j\}_{j\in\mathbb{U}_2\times\mathbb{Z}^d}$). In view of (3.1), there exists a constant $c \geq 1$ depending on m and s such that

$$\frac{1}{c}\|(u,\dot{u})\|_{H^s\times H^{s-1}} \leq \|z\|_{s-\frac{1}{2}} \leq c\|(u,\dot{u})\|_{H^s\times H^{s-1}}. \tag{3.6}$$

In the following, we will check that the Hamiltonian H in (3.4) meets two assumptions (1) and (2) in Theorem 2.1.

On the one hand, we show that for almost all $m > 0$, the family of frequencies $\{\omega_a\}_{a\in\mathbb{Z}^d}$ given in (3.5) is non-resonant, namely satisfies conditions (2.10)–(2.13). It is shown in [17] that for almost all $m > 0$ and any positive integer r, there exists $\gamma, \tau > 0$ such that (2.10)–(2.12) hold. Then we only need to check the condition (2.13). For any $N \geq 1$ and $\boldsymbol{j} = (\delta_k, a_k)_{k=1}^l \in (\mathbb{U}_2 \times \mathbb{Z}^d)^l$ with $l \leq r$ and $\langle a_k\rangle \leq N$, one has

$$\omega_{a_k} = \sqrt{|a_k|^2 + m} \leq \sqrt{N^2 - 1 + m} \leq \sqrt{1+m}\, N$$

and thus

$$|\delta_1\omega_{a_1} + \cdots + \delta_l\omega_{a_l}| \leq \sqrt{1+m}\, rN.$$

For any $\langle b_2\rangle > \langle b_1\rangle > N$ with $|b_2| - |b_1| \geq 2(1+m)rN$, one has

$$\begin{aligned} \omega_{b_2} - \omega_{b_1} &= \sqrt{|b_2|^2 + m} - \sqrt{|b_1|^2 + m} \\ &= \frac{|b_2|^2 - |b_1|^2}{\sqrt{|b_2|^2 + m} + \sqrt{|b_1|^2 + m}} \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{|b_2|^2-|b_1|^2}{\sqrt{1+m}(|b_2|+|b_1|)}\\
&=\frac{|b_2|-|b_1|}{\sqrt{1+m}}\\
&\geq 2\sqrt{1+m}\,rN.
\end{aligned}
$$

Hence, one has

$$
\begin{aligned}
|\delta_1\omega_{a_1}+\cdots+\delta_l\omega_{a_l}+\omega_{b_1}-\omega_{b_2}| &\geq \omega_{b_2}-\omega_{b_1}-|\delta_1\omega_{a_1}+\cdots+\delta_l\omega_{a_l}|\\
&\geq \sqrt{1+m}\,rN.
\end{aligned}
$$

Taking $C_0=2(1+m)r$, (2.13) holds when $\big||b_1|-|b_2|\big|\geq C_0N$.

On the other hand, since F is analytic with a zero of order at least two, then the assumption (2) in Theorem 2.1 holds.

Applying Theorem 2.1, there exists a normalizing transformation Φ_χ^1 such that $\Phi_\chi^{-1}(z)=z'$. Let $N=N_\varepsilon$ and then $N^{2r(2\tau r+d)}\leq N^{s-2s_0}=\varepsilon^{-2r}$. By (2.17) and (3.6), for any small enough ε, one has

$$
\|z'(0)\|_{s-\frac{1}{2}}\leq\|z(0)\|_{s-\frac{1}{2}}+\|z(0)-\Phi_\chi^{-1}(z(0))\|_{s-\frac{1}{2}}\leq 2c\varepsilon. \tag{3.7}
$$

Then recall the notation (2.5), we have

$$
\|z'(0)_{\leq N}\|_{\frac{s}{2}}\leq\|z'(0)\|_{s-\frac{1}{2}}\leq 2c\varepsilon, \tag{3.8}
$$

$$
\|z'(0)_{>N}\|_{s_0-\frac{1}{2}}\leq N^{s_0-s}\|z'(0)_{>N}\|_{s-\frac{1}{2}}<2c\varepsilon^{2r+1}. \tag{3.9}
$$

Define

$$
\tilde{t}=\inf\left\{t\geq 0 \mid \|z'(t)_{\leq N}\|_{\frac{s}{2}}=3c\varepsilon \text{ or } \|z'(t)_{>N}\|_{s_0-\frac{1}{2}}=\varepsilon^{r+1}\right\}.
$$

and we will prove that $\tilde{t}\geq\varepsilon^{-\frac{r}{2s_0}}$ in two parts.

(1) Control of the low modes $z'(t)_{\leq N}$.

Define $F_{\leq N}(z)=\|z_{\leq N}\|_{\frac{s}{2}}^2$ and then

$$
\begin{aligned}
|F_{\leq N}(z'(t))-F_{\leq N}(z'(0))| &= \left|\int_0^t\{H\circ\Phi_\chi^1,F_{\leq N}\}(z'(t))dt\right|\\
&\leq\int_0^t\left|\{\mathscr{R}^{(ii)}+\mathscr{R}^{(iii)}+\mathscr{R}_{r+3},F_{\leq N}\}(z'(t))\right|dt\\
&\leq |t|\|X_{\mathscr{R}^{(ii)}+\mathscr{R}^{(iii)}+\mathscr{R}_{r+3}}(z'(t))_{\leq N}\|_{\frac{s}{2}}\|z'(t)_{\leq N}\|_{\frac{s}{2}}.
\end{aligned} \tag{3.10}
$$

By (2.7), (2.19)–(2.21), there exists constant $C>0$ such that

$$
\begin{aligned}
\|X_{\mathscr{R}^{(ii)}+\mathscr{R}^{(iii)}}(z'(t))_{\leq N}\|_{\frac{s}{2}} &\leq N^{\frac{s-2s_0+1}{2}}\|X_{\mathscr{R}^{(ii)}+\mathscr{R}^{(iii)}}(z'(t))_{\leq N}\|_{s_0-\frac{1}{2}}\\
&\leq CN^{\frac{s-2s_0+1}{2}}N^{(r-1)\tau}\|z'(t)_{>N}\|_{s_0-\frac{1}{2}}^2\\
&\leq C\varepsilon^{r+\frac{3}{2}},
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
\|X_{\mathscr{R}_{r+3}}(z'(t))_{\leq N}\|_{\frac{s}{2}} &\leq CN^{r\tau}\big(N^{\frac{s-2s_0+1}{2}}\|z'(t)_{\leq N}\|_{s_0-\frac{1}{2}}^{r+1}\|z'(t)_{>N}\|_{s_0-\frac{1}{2}}\\
&\quad+\|z'(t)_{\leq N}\|_{\frac{s}{2}}^{r+2}\big)\\
&\leq C\varepsilon^{r+\frac{3}{2}}.
\end{aligned} \tag{3.12}
$$

When $t \le \varepsilon^{-\frac{r}{2s_0}}$, by (3.8), (3.10)–(3.12), one has

$$\begin{aligned}\|z'(t)_{\le N}\|^2_{\frac{s}{2}} &\le \|z'(0)_{\le N}\|^2_{\frac{s}{2}} + |F_{\le N}(z'(t)) - F_{\le N}(z'(0))| \\ &\le 4c^2\varepsilon^2 + C\varepsilon^{-\frac{r}{2s_0}}\varepsilon^{r+\frac{5}{2}} \\ &\le 9c^2\varepsilon^2. \end{aligned} \tag{3.13}$$

(2) Control of the high modes $z'(t)_{>N}$.

By Theorem 2.1, $\mathscr{R}^{(ii)}$ can be written as

$$\mathscr{R}^{(ii)} = \sum_{\substack{b_1,b_2\in\mathbb{Z}^d \\ \langle b_1\rangle,\langle b_2\rangle>N \\ ||b_1|-|b_2||<2(1+m)rN}} B_{b_1b_2}(z'_{\le N})z'_{b_1}\bar{z}'_{b_2} \tag{3.14}$$

with the estimate

$$|B_{b_1b_2}(z'_{\le N})| \le CN^{(r-1)\tau}\|z'_{\le N}\|_{\frac{s}{2}} \tag{3.15}$$

for some positive constants C. As the Hamiltonian is real, we have $\overline{B_{b_1b_2}(z'_{\le N})} = B_{b_2b_1}(z'_{\le N})$, i.e., the operator $\big(B_{b_1b_2}(z'_{\le N})\big)_{b_1,b_2}$ is Hermitian so that

$$\{\mathscr{R}^{(ii)}(z'(t)), \|z'(t)_{>N}\|_0^2\} = 0.$$

Thus one has

$$\begin{aligned}\Big|\|z'(t)_{>N}\|_0^2 - \|z'(0)_{>N}\|_0^2\Big| &= \left|\int_0^t \{H\circ\Phi_\chi^1(z'(t)), \|z'(t)_{>N}\|_0^2\}dt\right| \\ &\le \int_0^t \big|\{\mathscr{R}^{(iii)} + \mathscr{R}_{r+3}(z'(t)), \|z'(t)_{>N}\|_0^2\}\big|dt \\ &\le |t|\|X_{\mathscr{R}^{(iii)}+\mathscr{R}_{r+3}}(z'(t))_{>N}\|_0\|z'(t)_{>N}\|_0. \end{aligned} \tag{3.16}$$

Notice that by (2.7), (2.20) and (2.21), there exists constant $C>0$ such that

$$\|X_{\mathscr{R}^{(iii)}}(z'(t))_{>N}\|_{s_0-\frac{1}{2}} \le CN^{(r-1)\tau}\|z'(t)_{>N}\|^2_{s_0-\frac{1}{2}} \le C\varepsilon^{2r+\frac{3}{2}}, \tag{3.17}$$

$$\begin{aligned}\|X_{\mathscr{R}_{r+3}}(z'(t))_{>N}\|_{s_0-\frac{1}{2}} &\le CN^{r\tau}\big(N^{\frac{2s_0-s-1}{2}}\|z'(t)_{\le N}\|^{r+2}_{\frac{s}{2}} \\ &\quad + \|z'(t)_{\le N}\|^{r+1}_{s_0-\frac{1}{2}}\|z'(t)_{>N}\|_{s_0-\frac{1}{2}}\big) \\ &\le C\varepsilon^{2r+\frac{3}{2}}. \end{aligned} \tag{3.18}$$

By (3.9),(3.16)–(3.18) and the fact $s_0 > 1/2$, one has

$$\begin{aligned}\|z'(t)_{>N}\|_0^2 \le& \|z'(0)_{>N}\|_0^2 + |t|\|X_{\mathscr{R}^{(iii)}+\mathscr{R}_{r+3}}(z'(t))_{>N}\|_{s_0-\frac{1}{2}}\|z'(t)_{>N}\|_{s_0-\frac{1}{2}} \\ \le& C(1+t)\varepsilon^{3r+\frac{5}{2}}. \end{aligned} \tag{3.19}$$

Define $F_{>N}(z) = \|z_{>N}\|^2_{s_0-\frac{1}{2}}$ and then

$$\begin{aligned}\big|F_{>N}(z'(t)) - F_{>N}(z'(0))\big| &= \left|\int_0^t \{H\circ\Phi_\chi^1, F_{>N}\}(z'(t))dt\right| \\ &\le \int_0^t \big|\{\mathscr{R}^{(ii)} + \mathscr{R}^{(iii)} + \mathscr{R}_{r+3}, F_{>N}\}(z'(t))\big|dt. \end{aligned} \tag{3.20}$$

Using (3.14) and (3.15), one has

$$\begin{aligned}
&\left|\{\mathscr{R}^{(ii)}, F_{>N}\}(z'(t))\right| \\
=&\Big|\sum_{\substack{\langle b_1\rangle,\langle b_2\rangle>N\\ ||b_1|-|b_2||<2(1+m)rN}} (\langle b_1\rangle^{2s_0-1}-\langle b_2\rangle^{2s_0-1})B_{b_1b_2}(z'_{\le N})z'_{b_1}\bar z'_{b_2}\Big| \\
\le& \sum_{\substack{\langle b_1\rangle,\langle b_2\rangle>N\\ ||b_1|-|b_2||<2(1+m)rN}} 2s_0\big|\langle b_1\rangle-\langle b_2\rangle\big|(\langle b_1\rangle^{2s_0-2}+\langle b_2\rangle^{2s_0-2})\left|B_{b_1b_2}(z'_{\le N})\right| |z'_{b_1}\bar z'_{b_2}| \\
\le& CN^{(r-1)\tau+1}\|z'_{\le N}\|_{\frac{s}{2}} \sum_{\substack{|b_1|,|b_2|>N\\ ||b_1|-|b_2||<2(1+m)rN}} \langle b_1\rangle^{s_0-\frac32}\langle b_2\rangle^{s_0-\frac12}\left|z'_{b_1}\bar z'_{b_2}\right| \\
\le& CN^{(r-1)\tau+d+1}\|z'_{\le N}\|_{\frac{s}{2}}\|z'_{>N}\|_{s_0-\frac32}\|z'_{>N}\|_{s_0-\frac12} \\
\le& \|z'_{>N}\|_0^{\frac{2}{2s_0-1}}\|z'_{>N}\|_{s_0-\frac12}^{2-\frac{2}{2s_0-1}},
\end{aligned} \tag{3.21}$$

where the last inequality follows from the estimate $CN^{(r-1)\tau+d+1}\|z'_{\le N}\|_{\frac{s}{2}} \le 1$ and the Hölder inequality $\|z'_{>N}\|_{s_0-\frac32} \le \|z'_{>N}\|_0^{\frac{2}{2s_0-1}}\|z'_{>N}\|_{s_0-\frac12}^{1-\frac{2}{2s_0-1}}$. By (3.17) and ((3.18), there exists constant $C>0$ such that

$$\begin{aligned}
|\{\mathscr{R}^{(iii)}+\mathscr{R}_{r+3}, F_{>N}\}(z'(t))| \le& \|X_{\mathscr{R}^{(iii)}+\mathscr{R}_{r+3}}(z'(t))_{>N}\|_{s_0-\frac12}\|z'(t)_{>N}\|_{s_0-\frac12} \\
\le& C\varepsilon^{3r+\frac52}.
\end{aligned} \tag{3.22}$$

By (3.19)–(3.22), it is easy for any $t \le \varepsilon^{-\frac{r}{2s_0}}$ to get

$$\begin{aligned}
|F_{>N}(z'(t))-F_{>N}(z'(0))| \le& t(\|z'_{>N}\|_0^{\frac{2}{2s_0-1}}\|z'_{>N}\|_{s_0-\frac12}^{2-\frac{2}{2s_0-1}}+C\varepsilon^{3r+\frac52}) \\
\le& t\big(C(1+t)\varepsilon^{3r+\frac52}\big)^{\frac{1}{2s_0-1}}\varepsilon^{(r+1)(2-\frac{2}{2s_0-1})}+Ct\varepsilon^{3r+\frac52} \\
\le& \frac12\varepsilon^{2(r+1)}.
\end{aligned} \tag{3.23}$$

By (3.9) and (3.23), we obtain

$$\|z'(t)_{>N}\|^2_{s_0-\frac12} \le \|z'(0)_{>N}\|^2_{s_0-\frac12}+|F_{>N}(z'(t))-F_{>N}(z'(0))| \le \varepsilon^{2(r+1)}. \tag{3.24}$$

Combining (3.13) and (3.24), we can conclude that $\tilde t \ge \varepsilon^{-\frac{r}{2s_0}}$.

By (2.7) and (2.15), there exists constant $C>0$ such that

$$\begin{aligned}
\|X_\chi(z'(t))_{\le N}\|_{\frac{s}{2}} \le& CN^{r\tau}(\|z'(t)_{\le N}\|^2_{\frac{s}{2}}+N^{\frac{s-2s_0+1}{2}}\|z'(t)_{\le N}\|_{s_0-\frac12}\|z'(t)_{>N}\|_{s_0-\frac12}) \\
\le& C\varepsilon^{\frac32},
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
\|X_\chi(z'(t))_{>N}\|_{s_0-\frac12} \le& CN^{r\tau}(N^{\frac{2s_0-s-1}{2}}\|z'(t)_{\le N}\|^2_{\frac{s}{2}}+\|z'(t)_{\le N}\|_{s_0-\frac12}\|z'(t)_{>N}\|_{s_0-\frac12}) \\
\le& C\varepsilon^{r+\frac32}.
\end{aligned} \tag{3.26}$$

So by (3.13) and (3.24)–(3.26), for any $t \le \varepsilon^{-\frac{r}{2s_0}}$, we have

$$\|z(t)_{\le N}\|_{\frac{s}{2}} = \|\Phi^1_\chi(z'(t))_{\le N}\|_{\frac{s}{2}} \le \|z'(t)_{\le N}\|_{\frac{s}{2}}+\|X_\chi(z'(t))_{\le N}\|_{\frac{s}{2}} < 4c\varepsilon,$$

$$
\begin{aligned}
\|z(t)_{>N}\|_{s_0-\frac{1}{2}} &= \|\Phi_\chi^1(z'(t))_{>N}\|_{s_0-\frac{1}{2}} \le \|z'(t)_{>N}\|_{s_0-\frac{1}{2}} + \|X_\chi(z'(t))_{>N}\|_{s_0-\frac{1}{2}} \\
&\le 2\varepsilon^{r+1}.
\end{aligned}
$$

Going back to the original variables u, take $c_0 = 4c^2$ and by (3.6), we can obtain

$$
\|u(t)_{\le N}\|_{H^{\frac{s}{2}}} \le c_0\varepsilon \quad \text{and} \quad \|u(t)_{>N}\|_{H^{s_0}} \le \varepsilon^r.
$$

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