

# DEPENDENCE OF EIGENVALUES ON THE REGULAR FOURTH-ORDER STURM-LIOUVILLE PROBLEM\*

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**Abstract** In this paper, the eigenvalues of a regular fourth-order Sturm-Liouville (SL) problems are studied. The eigenvalues depend not only continuously but smoothly on the problem. An expression for the derivative of the eigenvalues with respect to a given parameter: an endpoint, a boundary condition, a coefficient, or the weight function, are found.

**Keywords** Fourth-order Sturm-Liouville problems, continuous dependence on parameters, eigenvalues.

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## 1. Introduction

In the early nineteenth century, Sturm and Liouville published a series of papers on second order linear ordinary differential equations including boundary value problems. The influence of their work was such that this subject became known as Sturm-Liouville theory. Sturm and Liouville were the first to see the need for finding properties of solutions directly from the equation even when no analytic expressions for solutions are available. A large amount of papers have been written since then. Among them, Pöschel and Trubowitz consider the eigenvalue of a regular second-order SL problem

$$-(py')' + qy = \lambda wy \quad (1.1)$$

with Dirichlet boundary conditions

$$y(a) = 0 = y(b)$$

in [22], they show that the  $n$ -th eigenvalue  $\lambda = \lambda_n(q)$  as a function of  $q$  is Frechet differentiable for  $q \in L^2(a, b)$ ,  $p = 1 = w$ , and give the expression of  $d\lambda_q$ . Dauge and Helffer in [5] show that the Neumann eigenvalues of a regular second order SL problem on an interval  $[a, b]$  are differentiable functions of the right endpoint

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b, they give the expression of  $\lambda'(b)$  and indicate that a similar equation holds for the eigenvalues of other separated BC. In [14] Kong and Zettl give a different proof of the Dauge-Helffer Theorem with substantially weaker hypotheses replace  $L^2[a, b]$  by  $L^1(a, b)$  and they obtained a similar result for coupled BC. In [15], Kong and Zettl further show that the eigenvalues of regular second-order SL problems are differentiable functions of all the data: the endpoints, the boundary conditions, as well as the coefficients and the weight functions and they give expressions for their derivatives. See Chapter 4 in [26] for an exposition of this theory. In [2], Battle proves continuity and differentiability of the eigenvalues for a more general second order problem. In [13], Kong et al. study that any isolated eigenvalue of a regular self-adjoint or non-self-adjoint ordinary linear  $n$ -th order BVP depends on the problem. In the classical case, these properties play an important role in the Bailey, Everitt, and Zettl code SLEIGN2 [1]. Yet, remarkably, this subject is an intensely active field of research today. Especially the research on dependence of the eigenvalues of boundary value problem has been extended in various aspects. Such as, the dependence of the eigenvalues of Sturm-Liouville problems with interface conditions, with transmission conditions, with eigenparameter-dependent boundary conditions, with distributional potentials [17, 18, 27–31], et al. At the same time, such research has also been extended to higher-order situations. For fourth-order case, Suo and Wang study the dependence of eigenvalues on the problem in [23] use the same method as [14, 15]. Ge, Wang and Suo study the dependence of eigenvalues on the boundary in [8]. Lv and Ao study the eigenvalues of fourth-order boundary value problems with self-adjoint canonical boundary conditions in [18]. Li and Ao et al. study the dependence of eigenvalues of fourth-order SL problems with discontinuous conditions in [17, 19, 28]. However, they considered the special fourth-order equation:

$$(p(x)y'')'' + q(x)y = \lambda w(x)y$$

with  $q = 1$  or  $q \neq 1$ .

In this paper, we consider a regular general fourth-order equation with separated conditions and coupled conditions. we show that the eigenvalues of the problems are differentiable functions of all the data: the endpoints, the boundary conditions, as well as the coefficients and the weight functions and we find formulas for their derivatives. It is necessary to study the dependence of eigenvalues of higher-order boundary value problems (BVPs), besides its theoretical importance, the continuous dependence of the eigenvalues and the eigenfunctions on the data is also fundamental from the numerical point of view. The major general purpose codes for the numerical computation of the eigenvalues and eigenfunctions of fourth-order Sturm-Liouville problems are SLEUTH (Sturm-Liouville eigenvalues using theta matrices) [9] and for more general problems see [3, 4, 10–12, 21].

## 2. Notation

Consider the fourth-order symmetric differential equation

$$(p_2(x)y'')'' - (p_1(x)y')' + q(x)y = \lambda w(x)y, x \in J' = (a', b'), -\infty \leq a' < b' \leq \infty, \lambda \in \mathbb{R}, \quad (2.1)$$

where

$$p_2, p_1, q, w : (a', b') \rightarrow \mathbb{R}, 1/p_2, p_1, q, w \in L_{loc}(a', b'), p_2 > 0, w > 0 \text{ a.e. on } (a', b'). \quad (2.2)$$

Let

$$J = [a, b], \quad a' < a < b < b' \quad (2.3)$$

and consider the BC

$$A \begin{pmatrix} y(a) \\ y'(a) \\ (p_2 y'')(a) \\ (p_2 y'')'(a) - (p_1 y')(a) \end{pmatrix} + B \begin{pmatrix} y(b) \\ y'(b) \\ (p_2 y'')(b) \\ (p_2 y'')'(b) - (p_1 y')(b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.4)$$

where the complex  $4 \times 4$  matrices A and B satisfy:

$$\text{rank}(A | B) = 4 \quad (2.5)$$

and

$$AEA^* = BEB^*, E = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (2.6)$$

A fourth-order SL boundary value problem (BVP) consists of Eq.(2.1) together with BC (2.4). With conditions (2.2), (2.3), (2.5) and (2.6) it is well known that this problem is a regular self-adjoint fourth-order SL problem. In this paper we fix all but one of the parameters that determine the SL problem, i.e., all but one of  $a$ ;  $b$ ;  $A$ ;  $B$ ;  $p_1$ ;  $1/p_2$ ;  $q$ ;  $w$  and study the dependence of the eigenvalues and eigenfunctions on that parameter.

From [24], we know that there are three basic types of self-adjoint boundary conditions (2.4), (2.5), (2.6): separated, coupled and mixed. In the separated case, there are many forms for the fourth-order problems. As other cases are similar, we will only study one form here, and we also study the coupled conditions:

1. Separated self-adjoint BC.

$$\cos \alpha y(a) - \sin \alpha y'(a) = 0, \quad (2.7)$$

$$\cos \alpha (p_2 y'')(a) - \sin \alpha [(p_2 y'')' - p_1 y'](a) = 0, \quad 0 \leq \alpha < \pi, \quad (2.8)$$

$$\cos \beta y(b) - \sin \beta y'(b) = 0, \quad (2.9)$$

$$\cos \beta (p_2 y'')(b) - \sin \beta [(p_2 y'')' - p_1 y'](b) = 0, \quad 0 < \beta \leq \pi. \quad (2.10)$$

2. Real coupled self-adjoint BC.

$$\begin{pmatrix} y(b) \\ y'(b) \\ (p_2 y'')(b) \\ (p_2 y'')'(b) - (p_1 y')(b) \end{pmatrix} = K \begin{pmatrix} y(a) \\ y'(a) \\ (p_2 y'')(a) \\ (p_2 y'')'(a) - (p_1 y')(a) \end{pmatrix}, \quad (2.11)$$

where  $K$  satisfies

$$K = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{pmatrix}, \quad k_{ij} \in R, \quad \det K = 1, \quad K E K^* = E. \quad (2.12)$$

### 3. Complex coupled self-adjoint BC

$$\begin{pmatrix} y(b) \\ y'(b) \\ (p_2 y'')(b) \\ (p_2 y'')(b) - (p_1 y')(b) \end{pmatrix} = e^{i\theta} K \begin{pmatrix} y(a) \\ y'(a) \\ (p_2 y'')(a) \\ (p_2 y'')(a) - (p_1 y')(a) \end{pmatrix}, \quad (2.13)$$

where  $K$  satisfies (2.12), and  $-\pi < \theta < 0$  or  $0 < \theta < \pi$ .

## 3. Continuity of eigenvalues and eigenfunctions

In this section, we establish the characterization of the eigenvalues as zeros of an entire function, and prove the continuity of the eigenvalues and eigenfunctions for the regular fourth-order SL problems. Let

$$\Omega = \{\omega = (a, b, A, B, p_1, 1/p_2, q, w)\} \quad (3.1)$$

such that (2.2), (2.3), (2.5), (2.6) hold. For the special case of separated BC (2.7)-(2.10) we also use the notation

$$\Omega_s = \{\omega = (a, b, \alpha, \beta, p_1, 1/p_2, q, w)\} \quad (3.2)$$

and for the coupled case (2.11)-(2.13) we let

$$\Omega_c = \{\omega = (a, b, \theta, K, p_1, 1/p_2, q, w)\} \quad (3.3)$$

when  $\theta = 0$  we shorten (3.3) to

$$\Omega_{rc} = \{\omega = (a, b, K, p_1, 1/p_2, q, w)\}. \quad (3.4)$$

We want to show that a small change of the problem results in only a small change of each eigenvalue and each eigenfunction. This means we have to compare the spectrum of different problems which may be defined on different intervals. Each  $\omega \in \Omega$  determines a unique SL problem:  $a, b$  the interval,  $A, B$  the boundary condition, and the restrictions of  $p_1, 1/p_2, q, w$  on  $[a, b]$  the equation. Observe that the values of  $p_1, 1/p_2, q, w$  outside the interval  $[a, b]$ , i.e. in  $(a', b') \setminus [a, b]$ , do not affect the spectrum of the problem determined by  $\omega$ . To account for this and to facilitate comparisons between eigenvalues of problems defined on different intervals we let

$$\tilde{\Omega} = \{\tilde{\omega} = (a, b, A, B, \tilde{p}_1, 1/\tilde{p}_2, \tilde{q}, \tilde{w})\} \quad (3.5)$$

where

$$\widetilde{p}_1 = \begin{cases} p_1, x \in [a, b] \\ 0, x \in (a', b') \setminus [a, b] \end{cases}, \quad \widetilde{1/p_2} = \begin{cases} 1/p_2, x \in [a, b] \\ 0, x \in (a', b') \setminus [a, b] \end{cases}$$

and  $\tilde{q}, \tilde{w}$  are defined similarly. Now we introduce the Banach space

$$X = \mathbf{R} \times \mathbf{R} \times M_{4 \times 4}(\mathbf{C}) \times M_{4 \times 4}(\mathbf{C}) \times L^1(a', b') \times L^1(a', b') \times L^1(a', b') \times L^1(a', b') \quad (3.6)$$

with its “natural” norm

$$\|\omega\| = \|\tilde{\omega}\| = |a| + |b| + \|A\| + \|B\| + \int_{a'}^{b'} (|\tilde{p}_1| + |\widetilde{1/p_2}| + |\tilde{q}| + |\tilde{w}|) \quad (3.7)$$

where  $\|A\|$  is any fixed matrix norm. We maintain that this space  $X$  is the “natural” setting for the study of regular SL problems. Note that, since  $p_1, 1/p_2, q, w$  are only assumed to be in  $L_{loc}(a', b')$ ,  $\omega$  is not a subset of  $X$  but  $\tilde{\omega}$  is since  $\tilde{p}_1, \widetilde{1/p_2}, \tilde{q}, \tilde{w}$  are in  $L^1(a', b')$ . Now we identify  $\Omega$  with  $\tilde{\Omega}$  as a subset of  $X$ . Then  $\Omega$  inherits the norm from  $X$ , and the convergence in  $\Omega$  is determined by this norm. It is easy to see that every point in  $\Omega$  is an accumulation point of  $\Omega$  with respect to the norm in  $X$ .

The isolated eigenvalues of a regular fourth-order SL problem depend continuously on the problem. More precisely we will give in Theorem 3.1. First we give a lemma.

**Lemma 3.1.** *A complex  $\lambda_0$  is an eigenvalue of the problem (2.1), (2.4) – (2.6) if and only if*

$$\Delta(\omega, \lambda_0) = \det[A + B\Phi(b, a, p_1, 1/p_2, q, w, \lambda_0)] = 0. \quad (3.8)$$

**Proof.** Where  $\Phi$  is defined in Theorem 3.1, the proof is omitted since it is routine.  $\square$

**Theorem 3.1.** *Let  $\omega_0 = (a_0, b_0, A_0, B_0, p_{10}, 1/p_{20}, q_0, w_0) \in \Omega$ . Assume that  $\mu = \lambda(\omega_0)$  is an isolated eigenvalue of the SL problem (2.1), (2.4) – (2.6) determined by  $\omega_0$  i.e.,  $\mu$  is an isolated eigenvalue of the SL problem*

$$(p_{20}(x)y'')'' - (p_{10}(x)y')' + q_0(x)y = \lambda w_0(x)y, \quad \text{on } [a_0, b_0], \quad A_0 Y_0(a_0) + B_0 Y_0(b_0) = 0.$$

*Then, given any  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for any  $\omega = (a, b, A, B, p_1, 1/p_2, q, w) \in \Omega$  satisfying the inequality*

$$\begin{aligned} \|\omega - \omega_0\| &= |a - a_0| + |b - b_0| + \|A - A_0\| + \|B - B_0\| \\ &\quad + \int_{a'}^{b'} (|\tilde{p}_1 - \tilde{p}_{10}| + |\widetilde{1/p_2} - \widetilde{1/p_{20}}| + |\tilde{q} - \tilde{q}_0| + |\tilde{w} - \tilde{w}_0|) \\ &< \delta, \end{aligned}$$

*the SL problem*

$$(p_2(x)y'')'' - (p_1(x)y')' + q(x)y = \lambda w(x)y, \quad \text{on } [a, b], \quad AY(a) + BY(b) = 0,$$

*has an isolated eigenvalue  $\lambda(\omega)$  satisfying the inequality*

$$|\lambda(\omega) - \lambda(\omega_0)| < \varepsilon.$$

**Proof.** For  $\omega \in \Omega$ , and  $\lambda \in R$ , let  $\Phi(b, a, p_1, 1/p_2, q, w, \lambda)$  be the matrix solution of the initial value problem

$$Y' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1/p_2 & 0 \\ 0 & p_1 & 0 & 1 \\ \lambda w - q & 0 & 0 & 0 \end{pmatrix} Y \quad \text{on } [a, b] \quad (3.9)$$

with  $\Phi(a) = I$ , where  $I$  is the identity matrix,  $Y = \begin{pmatrix} y \\ y' \\ p_2 y'' \\ (p_2 y'')' - (p_1 y') \end{pmatrix}$ . The

characteristic function  $\Delta$  of problem (2.1), (2.4) – (2.6) is defined as follows

$$\Delta(\omega, \lambda) = \det[A + B\Phi(b, a, p_1, 1/p_2, q, w, \lambda)], \quad \text{for } \omega \in \Omega, \lambda \in R \quad (3.10)$$

From Lemma 3.1,  $\lambda(\omega)$  is an eigenvalue of the problem (2.1), (2.4) – (2.6) if and only if  $\Delta(\omega, \lambda(\omega)) = 0$ . Furthermore, for any  $\omega \in \Omega$ ,  $\Delta(\omega, \lambda)$  is an entire function of  $\lambda$  and it is continuous in  $\omega$  see Theorems 2.7, 2.8 of [16], and  $\Delta(\omega_0, \mu) = 0$ . It is obvious that  $\Delta(\omega_0, \lambda)$  is not constant in  $\lambda$  since  $\mu$  is an isolated eigenvalue. Hence there exists  $p > 0$  such that  $\Delta(\omega_0, \lambda) \neq 0$  for  $\lambda \in S_p = \{\lambda \in C : |\lambda - \mu| = p\}$ . By the well known theorem on continuity of the roots of an equation as a function of parameters, see [25] or [6], the statement of Theorem 3.1 follows.  $\square$

**Remark 3.1.** Theorem 3.1 shows that for any fixed eigenvalue  $\mu$  associated with  $\omega = \omega_0$  there exists a continuous eigenvalue branch  $\lambda(\omega)$  satisfying  $\lambda(\omega_0) = \mu$ . However, this does not mean that for a fixed  $n$ , the  $n$ th eigenvalue  $\lambda_n(\omega)$  is always continuous in  $\omega$ , see Remark 3.7 in [15].

Below, each eigenvalue  $\lambda(\omega)$  of the BVP (2.1), (2.4) – (2.6) as a function of  $\omega$  for  $\omega \in \Omega$ , will always be assumed to be embedded in a continuous eigenvalue branch.

Next we state two lemmas needed in the later proofs which are also of independent interest. The first states that the unique solution of any initial value problem of Eq. (2.1) depends continuously on all parameters including the coefficients and the weight function in  $L^1$  norm.

**Lemma 3.2.** Let (2.2) hold, let  $c \in (a', b')$  and  $d, k, f, g \in \mathbf{C}$ . Consider the initial value problem

$$\begin{cases} (p_2(x)y'')'' - (p_1(x)y')' + q(x)y = \lambda w(x)y, \\ y(c) = d, y'(c) = k, (p_2 y'')(c) = f, (p_2 y'')'(c) - (p_1 y')(c) = g. \end{cases} \quad (3.11)$$

Then the unique solution  $y = y(\cdot, c, d, k, f, g, p_1, 1/p_2, q, w)$  is a continuous function of all its variables. More precisely, given  $\varepsilon > 0$  and any compact subinterval  $J$  of  $(a', b')$  there exists a  $\delta > 0$  such that if

$$|c - c_0| + |d - d_0| + |k - k_0| + |f - f_0| + |g - g_0|$$

$$+ \int_a^b (|p_1 - p_{10}| + |1/p_2 - 1/p_{20}| + |q - q_0| + |w - w_0|) < \delta \quad (3.12)$$

then

$$|y(x, c, d, k, f, g, p_1, 1/p_2, q, w) - y(x, c_0, d_0, k_0, f_0, g_0, p_{10}, 1/p_{20}, q_0, w_0)| < \varepsilon, \quad (3.13)$$

$$|y'(x, c, d, k, f, g, p_1, 1/p_2, q, w) - y'(x, c_0, d_0, k_0, f_0, g_0, p_{10}, 1/p_{20}, q_0, w_0)| < \varepsilon, \quad (3.14)$$

$$|p_2 y''(x, c, d, k, f, g, p_1, 1/p_2, q, w) - p_2 y''(x, c_0, d_0, k_0, f_0, g_0, p_{10}, 1/p_{20}, q_0, w_0)| < \varepsilon, \quad (3.15)$$

$$|[(p_2 y'')' - (p_1 y')](x, c, d, k, f, g, p_1, 1/p_2, q, w) - [(p_2 y'')' - (p_1 y')](x, c_0, d_0, k_0, f_0, g_0, p_{10}, 1/p_{20}, q_0, w_0)| < \varepsilon \quad (3.16)$$

for all  $x \in J$ .

**Proof.** This follows from Lemma 3.1 in [15].  $\square$

As a consequence of Theorem 3.1 and Lemma 3.2 we obtain.

**Lemma 3.3.** Let  $\omega_0 = (a_0, b_0, A_0, B_0, p_{10}, 1/p_{20}, q_0, w_0) \in \Omega$ . Let  $\lambda = \lambda(\omega)$  be an eigenvalue of SL problem (2.1), (2.4) – (2.6). If  $\lambda(\omega_0)$  is simple, then there exists a neighborhood  $M$  of  $\omega_0$  in  $\Omega$  such that  $\lambda(\omega)$  is simple for every  $\omega$  in  $M$ .

**Remark 3.2.** The conclusion of Lemma 3.3 holds if  $\omega_0$  is replaced by one of its components and  $\Omega$  by the corresponding subspace of  $\Omega$ .

**Proof.** See [17].  $\square$

**Definition 3.1.** A normalized eigenfunction  $u$  of an SL problem we mean an eigenfunction  $u$  that satisfies

$$\int_a^b |u|^2 w = 1. \quad (3.17)$$

**Theorem 3.2.** Let the notation and hypotheses of Theorem 3.1 hold.

- (i) Assume the eigenvalue  $\lambda(\omega_0)$  is simple for some  $\omega_0 \in \Omega$  and let  $u = u(\cdot, \omega_0)$  denote a normalized eigenfunction of  $\lambda(\omega_0)$ . Then there exist normalized eigenfunctions  $u = u(\cdot, \omega)$  of  $\lambda(\omega)$  for  $\omega \in \Omega$  such that when  $\omega \rightarrow \omega_0$  in  $\Omega$ , we have

$$\begin{aligned} u(\cdot, \omega) &\rightarrow u(\cdot, \omega_0), \\ u'(\cdot, \omega) &\rightarrow u'(\cdot, \omega_0), \\ p_2 u''(\cdot, \omega) &\rightarrow p_2 u''(\cdot, \omega_0), \\ [(p_2 u'')' - (p_1 u')](\cdot, \omega) &\rightarrow [(p_2 u'')' - (p_1 u')](\cdot, \omega_0), \end{aligned} \quad (3.18)$$

these uniformly on any compact subinterval  $J$  of  $(a', b')$ .

- (ii) Assume that  $\lambda(\omega)$  is a eigenvalue of multiplicity  $l$ , ( $l = 2, 3, 4$ ) for all  $\omega$  in some neighborhood  $M$  of  $\omega_0$  in  $\Omega$ . Then there exist  $l$  linearly independent normalized eigenfunctions  $u_k(\cdot, \omega)$  of  $\lambda(\omega)$  such that when  $\omega \rightarrow \omega_0$ , we have

$$\begin{aligned} u_k(\cdot, \omega) &\rightarrow u_k(\cdot, \omega_0), \\ u_k'(\cdot, \omega) &\rightarrow u_k'(\cdot, \omega_0), \end{aligned}$$

$$\begin{aligned} p_2 u_k''(\cdot, \omega) &\rightarrow p_2 u_k''(\cdot, \omega_0), \\ [(p_2 u_k'')' - (p_1 u_k')](\cdot, \omega) &\rightarrow [(p_2 u_k'')' - (p_1 u_k')](\cdot, \omega_0), \end{aligned} \quad (3.19)$$

these uniformly on any compact subinterval  $J$  of  $(a', b')$ . Note that in this case, given  $l$  linearly independent normalized eigenfunctions  $u_k$  of  $\lambda(\omega_0)$  there exist  $l$  linearly independent normalized eigenfunctions of  $\lambda(\omega)$  one of which converges to  $u_1$  and the other to  $u_2$  and so on as  $\omega \rightarrow \omega_0$  in  $\Omega$ .

**Proof.** See Theorem 3.5 of [13].  $\square$

## 4. Differentiability properties of eigenvalues

In this section we show that the isolated eigenvalues depend continuously on all the data, here we show that this dependence is in fact differentiable. Recall the definition of the Frechet derivative:

**Definition 4.1.** A map  $T$  from a Banach space  $X$  into a Banach space  $Y$  is Frechet differentiable at a point  $x \in X$ , if there exists a bounded linear operator  $dT_x : X \rightarrow Y$  such that for  $h \in X$

$$|T(x+h) - T(x) - dT_x(h)| = o(h) \quad (h \rightarrow 0).$$

**Lemma 4.1.** Assume  $u$  and  $v$  are solutions of (2.1) with  $\lambda = \mu$  and  $\lambda = \nu$  respectively. Then

$$\begin{aligned} &[u(p_2 \bar{v}'')' - \bar{v}(p_2 u'')' - u(p_1 \bar{v}') + \bar{v}(p_1 u') - u'(p_2 \bar{v}'') + \bar{v}'(p_2 u'')]_a^b \\ &:= [u(p_2 \bar{v}'')' - \bar{v}(p_2 u'')' - u(p_1 \bar{v}') + \bar{v}(p_1 u') - u'(p_2 \bar{v}'') + \bar{v}'(p_2 u'')](b) \\ &\quad - [u(p_2 \bar{v}'')' - \bar{v}(p_2 u'')' - u(p_1 \bar{v}') + \bar{v}(p_1 u') - u'(p_2 \bar{v}'') + \bar{v}'(p_2 u'')](a) \\ &= (\nu - \mu) \int_a^b u \bar{v} w. \end{aligned} \quad (4.1)$$

**Lemma 4.2.** Assume a real valued function  $f \in L_{loc}(a', b')$ . Then

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f = f(x) \quad \text{a.e. } (a', b'). \quad (4.2)$$

**Proof.** See Lemma 3.2 of [14].  $\square$

**Theorem 4.1.** (Eigenvalue-eigenfunction differential equation for special case of separated BVPs). Let (2.2) hold. Consider the BVP (2.1), (2.7) – (2.10) with  $0 \leq \alpha < \pi$  and  $\beta = \pi$ . Fix all the components of  $\omega$  except  $b$  and let  $\lambda = \lambda(b)$  and  $u = u(\cdot, b)$ . Then  $\lambda$  is differentiable a.e. and

$$\lambda'(b) = 2u'(b, b)(p_2 u'')'(b, b) - u'(b, b)(p_1 u')(b, b), \quad \text{a.e. in } [a, b']. \quad (4.3)$$

**Proof.** For small  $h$ , in (4.1) choose  $\mu = \lambda(b)$ ,  $\nu = \lambda(b+h)$ , and  $u = u(\cdot, b)$ ,  $v = u(\cdot, b+h)$ . Noting that

$$\begin{aligned} &[u(p_2 \bar{v}'')' - \bar{v}(p_2 u'')' - u(p_1 \bar{v}') + \bar{v}(p_1 u') - u'(p_2 \bar{v}'') + \bar{v}'(p_2 u'')](a) = 0, \\ &u(b, b) = 0, \quad (p_2 u'')(b, b) = 0, \end{aligned}$$



we have

$$\begin{aligned} & [\lambda(b+h) - \lambda(b)] \int_a^b u(s, b) u(s, b+h) w(s) ds \\ &= -u(b, b+h) (p_2 u'')'(b, b) - u'(b, b) (p_2 u'')(b, b+h) + u(b, b+h) (p_1 u')(b, b) \end{aligned} \quad (4.4)$$

and

$$\begin{aligned} & (p_2 u'')(b, b+h) \\ &= (p_2 u'')(b, b+h) - (p_2 u'')(b+h, b+h) \\ &= - \int_b^{b+h} (p_2 u'')'(s, b+h) ds \\ &= - \int_b^{b+h} (p_2 u'')'(s, b) ds + \int_b^{b+h} [(p_2 u'')'(s, b) - (p_2 u'')'(s, b+h)] ds \end{aligned}$$

$(p_2 u'')'(s, b) - (p_2 u'')'(s, b+h) \rightarrow 0$  as  $h \rightarrow 0$ . By lemma 4.2, we have

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{(p_2 u'')(b, b+h)}{h} &= -(p_2 u'')'(b, b), \\ \lim_{h \rightarrow 0} \frac{u(b, b+h)}{h} &= -u'(b, b). \end{aligned}$$

Dividing (4.4) by  $h$  and taking the limit as  $h \rightarrow 0$ , we get (4.3).  $\square$

**Theorem 4.2.** (*Eigenvalue-eigenfunction differential equation for special case of separated BVPs*). Let (2.2) hold. Consider the BVP (2.1), (2.7) – (2.10) with  $0 \leq \alpha < \pi$  and  $\beta = \frac{\pi}{2}$ . Fix all the components of  $\omega$  except  $b$  and let  $\lambda = \lambda(b)$  and  $u = u(\cdot, b)$ . Then  $\lambda$  is differentiable a.e. and

$$\lambda'(b) = -u^2(b, b) [\lambda(b)\omega(b) - q(b)] - u''(b, b)(p_2 u'')(b, b) \text{ a.e. in } [a, b']. \quad (4.5)$$

**Proof.** Since the proof is similar to that of Theorem 4.1. For small  $h$ , in (4.1) choose  $\mu = \lambda(b)$ ,  $\nu = \lambda(b+h)$  and  $u = u(\cdot, b)$ ,  $v = u(\cdot, b+h)$ . And from the boundary conditions, noting that

$$\begin{aligned} & [u(p_2 \bar{v}'')' - \bar{v}(p_2 u'')' - u(p_1 \bar{v}') + \bar{v}(p_1 u') - u'(p_2 \bar{v}'') + \bar{v}'(p_2 u'')](a) = 0, \\ & u'(b, b) = 0, \quad (p_2 u'')'(b, b) = 0 \end{aligned}$$

we have

$$\begin{aligned} & [\lambda(b+h) - \lambda(b)] \int_a^b u(s, b) u(s, b+h) w(s) ds \\ &= u(b, b)(p_2 u'')'(b, b+h) - u(b, b)(p_1 u')(b, b+h) + u'(b, b+h)(p_2 u'')(b, b) \end{aligned} \quad (4.6)$$

and

$$\begin{aligned} & (p_2 u'')'(b, b+h) \\ &= - \int_b^{b+h} (p_2 u'')''(s, b+h) ds \\ &= - \int_b^{b+h} [(p_1 u')' - qu + \lambda w u](s, b+h) ds \end{aligned}$$

$$\begin{aligned}
&= - \int_b^{b+h} (p_1 u')'(s, b+h) ds + \int_b^{b+h} q(s) u(s, b) ds \\
&\quad + q(s) \int_b^{b+h} [u(s, b+h) - u(s, b)] ds \\
&\quad - \lambda(b+h) \int_b^{b+h} u(s, b) w(s) ds - \lambda(b+h) \int_b^{b+h} [u(s, b+h) - u(s, b)] w(s) ds
\end{aligned}$$

as  $h \rightarrow 0$ ,  $u(s, b+h) - u(s, b) \rightarrow 0$ . By Lemma 4.2 and the continuity of  $\lambda$  at  $b$ , we have

$$\lim_{h \rightarrow 0} \frac{(p_2 u'')'(b, b+h)}{h} = -(p_2 u'')''(b, b) = -(p_1 u')'(b, b) - [\lambda(b)w(b) - q(b)]u(b, b).$$

Similarly

$$\lim_{h \rightarrow 0} \frac{(p_1 u')(b, b+h)}{h} = -(p_1 u')'(b, b)$$

and

$$\lim_{h \rightarrow 0} \frac{u'(b, b+h)}{h} = -u''(b, b).$$

Dividing (4.6) by  $h$  and taking the limit as  $h \rightarrow 0$ , we get (4.5).  $\square$

**Theorem 4.3.** (Eigenvalue-eigenfunction differential equation for separated BVPs). Let (2.2) hold. Consider the BVP (2.1), (2.7) – (2.10) with  $0 \leq \alpha < \pi$  and  $0 < \beta \leq \pi$ .

- (1) Fix all the components of  $\omega$  except  $a$  and let  $\lambda = \lambda(a)$ ,  $u = u(\cdot, a)$ . Then  $\lambda$  is differentiable a.e. and

$$\begin{aligned}
\lambda'(a) &= u^2(a, a)[\lambda(a)w(a) - q(a)] - 2u'(a, a)(p_2 u'')'(a, a) + u'(a, a)(p_1 u')(a, a) \\
&\quad + \frac{1}{p_2(a)}(p_2 u'')^2(a, a) \quad \text{a.e. in } (a', b].
\end{aligned} \tag{4.7}$$

- (2) Fix all the components of  $\omega$  except  $b$  and let  $\lambda = \lambda(b)$ ,  $u = u(\cdot, b)$ . Then  $\lambda$  is differentiable a.e. and

$$\begin{aligned}
\lambda'(b) &= 2u'(b, b)(p_2 u'')'(b, b) - u^2(b, b)[\lambda(b)w(b) - q(b)] - u'(b, b)(p_1 u')(b, b) \\
&\quad - \frac{1}{p_2(b)}(p_2 u'')^2(b, b) \quad \text{a.e. in } [a, b').
\end{aligned} \tag{4.8}$$

**Proof.** Since the proof of (4.7), (4.8) are similar we just prove (4.8). The proof is similar to that of Theorem 4.1 and 4.2. For small  $h$ , in (4.1) choose  $\mu = \lambda(b)$ ,  $\nu = \lambda(b+h)$  and  $u = u(\cdot, b)$ ,  $v = u(\cdot, b+h)$ . And from the boundary conditions, noting that

$$[u(p_2 \bar{v}'')' - \bar{v}(p_2 u'')' - u(p_1 \bar{v}') + \bar{v}(p_1 u') - u'(p_2 \bar{v}'') + \bar{v}'(p_2 u'')](a) = 0$$

we have

$$\begin{aligned}
&[\lambda(b+h) - \lambda(b)] \int_a^b u(s, b) u(s, b+h) w(s) ds \\
&= u(b, b)(p_2 u'')'(b, b+h) - u(b, b+h)(p_2 u'')'(b, b) - u(b, b)(p_1 u')(b, b+h) \\
&\quad + u(b, b+h)(p_1 u')(b, b) - u'(b, b)(p_2 u'')(b, b+h) + u'(b, b+h)(p_2 u'')(b, b)
\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}
& (p_2 u'')'(b, b+h) \\
&= - \int_b^{b+h} (p_2 u'')''(s, b+h) ds \\
&= - \int_b^{b+h} [(p_1 u')' - qu + \lambda w u](s, b+h) ds \\
&= - \int_b^{b+h} (p_1 u')'(s, b+h) ds + \int_b^{b+h} q(s) u(s, b) ds \\
&\quad + \int_b^{b+h} q(s) [u(s, b+h) - u(s, b)] ds \\
&\quad - \lambda(b+h) \int_b^{b+h} u(s, b) w(s) ds - \lambda(b+h) \int_b^{b+h} [u(s, b+h) - u(s, b)] w(s) ds
\end{aligned}$$

as  $h \rightarrow 0$ ,  $u(s, b+h) - u(s, b) \rightarrow 0$ . by Lemma 4.2, we have

$$\begin{aligned}
\lim_{h \rightarrow 0} \frac{(p_2 u'')'(b, b+h)}{h} &= -(p_2 u'')''(b, b) = -(p_1 u')'(b, b) - [\lambda(b)w(b) - q(b)]u(b, b), \\
\lim_{h \rightarrow 0} \frac{(p_1 u')(b, b+h)}{h} &= -(p_1 u')'(b, b), \\
\lim_{h \rightarrow 0} \frac{(p_2 u'')(b, b+h)}{h} &= -(p_2 u'')'(b, b), \\
\lim_{h \rightarrow 0} \frac{u'(b, b+h)}{h} &= -u''(b, b), \\
\lim_{h \rightarrow 0} \frac{u(b, b+h)}{h} &= -u'(b, b).
\end{aligned}$$

Dividing (4.9) by  $h$  and taking the limit as  $h \rightarrow 0$ , we get (4.8).  $\square$

**Theorem 4.4.** (*Eigenvalue-eigenfunction differential equation for coupled BVPs*). Let (2.2) hold. Consider the coupled BVP (2.1) with (2.13), (2.12) where  $-\pi < \theta \leq \pi$ .

- (1) Fix all the components of  $\omega$  except  $a$  and let  $\lambda = \lambda(a)$  and  $u = u(\cdot, a)$ . Then  $\lambda$  is differentiable a.e. and

$$\begin{aligned}
\lambda'(a) &= -2\operatorname{Re}[u'(a)(p_2 \bar{u}'')'(a)] + p_1(a) |(u')(a)|^2 + \frac{1}{p_2(a)} |(p_2 u'')(a)|^2 \\
&\quad - |u(a)|^2 [q(a) - \lambda(a)w(a)].
\end{aligned} \tag{4.10}$$

- (2) Fix all the components of  $\omega$  except  $b$  and let  $\lambda = \lambda(b)$  and  $u = u(\cdot, b)$ . Then  $\lambda$  is differentiable a.e. and

$$\begin{aligned}
\lambda'(b) &= 2\operatorname{Re}[u'(b)(p_2 \bar{u}'')'(b)] - p_1(b) |(u')(b)|^2 - \frac{1}{p_2(b)} |(p_2 u'')(b)|^2 \\
&\quad + |u(b)|^2 [q(b) - \lambda(b)w(b)].
\end{aligned} \tag{4.11}$$

**Proof.** Since the proof of (4.10) and (4.11) are similar we just prove (4.11). The proof is similar to that of Theorem 4.3, we have

$$[\lambda(b+h) - \lambda(b)] \int_a^b u \bar{v} w$$

$$\begin{aligned}
&= [u(p_2 \bar{v}'')' - \bar{v}(p_2 u'')' - u(p_1 \bar{v}') + \bar{v}(p_1 u') - u'(p_2 \bar{v}'') + \bar{v}'(p_2 u'')]_a^b \\
&= ((p_2 \bar{v}'')' - (p_1 \bar{v}'), -(p_2 \bar{v}''), \bar{v}', -\bar{v})(b) \begin{pmatrix} u \\ u' \\ p_2 u'' \\ (p_2 u'')' - p_1 u' \end{pmatrix} (b) \\
&\quad - ((p_2 \bar{v}'')' - (p_1 \bar{v}'), -(p_2 \bar{v}''), \bar{v}', -\bar{v})(a) \begin{pmatrix} u \\ u' \\ p_2 u'' \\ (p_2 u'')' - p_1 u' \end{pmatrix} (a) \\
&= ((p_2 \bar{v}'')' - (p_1 \bar{v}'), -(p_2 \bar{v}''), \bar{v}', -\bar{v})(b) \begin{pmatrix} u \\ u' \\ p_2 u'' \\ (p_2 u'')' - p_1 u' \end{pmatrix} (b) \\
&\quad - ((p_2 \bar{v}'')' - (p_1 \bar{v}'), -(p_2 \bar{v}''), \bar{v}', -\bar{v})(b+h) \begin{pmatrix} u \\ u' \\ p_2 u'' \\ (p_2 u'')' - p_1 u' \end{pmatrix} (b) \\
&= - [((p_2 \bar{v}'')' - (p_1 \bar{v}'), -(p_2 \bar{v}''), \bar{v}', -\bar{v})(b+h) \\
&\quad - ((p_2 \bar{v}'')' - (p_1 \bar{v}'), -(p_2 \bar{v}''), \bar{v}', -\bar{v})(b)] \begin{pmatrix} u \\ u' \\ p_2 u'' \\ (p_2 u'')' - p_1 u' \end{pmatrix} (b), \\
&\quad \lim_{h \rightarrow 0} \frac{[(p_2 \bar{v}'')' - (p_1 \bar{v}')(b+h) - [(p_2 \bar{v}'')' - (p_1 \bar{v}')](b)]}{h} \\
&= [(p_2 \bar{u}'')' - (p_1 \bar{u}')](b) \\
&= \bar{u}(b)[\lambda(b)\omega(b) - q(b)], \\
&\quad \lim_{h \rightarrow 0} \frac{-(p_2 \bar{v}'')(b+h) + (p_2 \bar{v}'')(b)}{h} = -(p_2 \bar{u}'')(b), \\
&\quad \lim_{h \rightarrow 0} \frac{\bar{v}'(b+h) - \bar{v}'(b)}{h} = \bar{u}''(b), \\
&\quad \lim_{h \rightarrow 0} \frac{\bar{v}(b) - \bar{v}(b+h)}{h} = -\bar{u}'(b).
\end{aligned}$$

Then we get (4.11).  $\square$

**Theorem 4.5.** Let  $\omega = (a, b, A, B, p_1, 1/p_2, q, w) \in \Omega$ . Let  $\lambda = \lambda(b)$  and let  $u = u(\cdot, b)$  be a normalized eigenfunction of  $\lambda$  for the BVP (2.1), (2.4) – (2.6). Assume that  $\lambda(\omega)$  is an eigenvalue of multiplicity  $l$  ( $l = 1, 2, 3, 4$ ) in some neighborhood  $M \subset \Omega$  of  $\omega$ . Then  $\lambda$  is continuously differentiable with respect to each variable  $\alpha, \beta$  for the separated BC (2.7) – (2.10); continuously differentiable with respect to each variable  $\theta, k$  for the coupled BC (2.11) – (2.13); continuously differentiable with respect to each variable  $p_1, 1/p_2, q, w$  for the general BC (2.4) – (2.6) in the appropriate sense. Then derivatives are given by:

1. Fix all components of  $\omega$  except  $\alpha$  and let  $\lambda = \lambda(\alpha)$  and  $u = u(\cdot, \alpha)$ . Then  $\lambda$  is differentiable and

$$\lambda'(\alpha) = 2 \{u(a)(p_2 u'')(a) + u'(a)[(p_2 u'')'(a) - (p_1 u')(a)]\}. \quad (4.12)$$

2. Fix all components of  $\omega$  except  $\beta$  and let  $\lambda = \lambda(\beta)$  and  $u = u(\cdot, \beta)$ . Then  $\lambda$  is differentiable and

$$\lambda'(\beta) = -2 \{u(b)(p_2 u'')(b) + u'(b)[(p_2 u'')'(b) - (p_1 u')(b)]\}. \quad (4.13)$$

3. Fix all components of  $\omega$  except  $\theta$  and let  $\lambda = \lambda(\theta)$  and  $u = u(\cdot, \theta)$ . Then  $\lambda$  is differentiable at  $\theta$  for any  $\theta$  satisfying  $-\pi < \theta < 0$  or  $0 < \theta < \pi$  and

$$\lambda'(\theta) = 2\text{Im}[u(b)(p_2 \bar{u}'')(b) + \bar{u}(b)(p_1 u')(b) + \bar{u}'(b)(p_2 \bar{u}'')(b)]. \quad (4.14)$$

4. Fix all components of  $\omega$  except  $K$  and let  $\lambda = \lambda(K)$  and  $u = u(\cdot, K)$ . Assume  $K$  satisfies (2.12). Then  $\lambda$  is differentiable and its Frechet derivative is given by:

$$d\lambda_K(H) = -((p_2 \bar{u}'')' - (p_1 \bar{u}'), -p_2 \bar{u}'', \bar{u}', -\bar{u})(b) H K^{-1} \begin{pmatrix} u \\ u' \\ p_2 u'' \\ (p_2 u'')' - p_1 u' \end{pmatrix} (b). \quad (4.15)$$

5. Fix all components of  $\omega$  except  $1/p_2$  and consider  $\lambda$  as a function of  $1/p_2 \in L^1(a, b)$ . Then  $\lambda$  is Frechet differentiable and its Frechet derivative is given by:

$$d\lambda_{1/p_2}(h) = - \int_a^b |p_2 u''|^2 h, h \in L^1(a, b). \quad (4.16)$$

6. Fix all components of  $\omega$  except  $p_1$  and consider  $\lambda$  as a function of  $p_1 \in L^1(a, b)$ . Then  $\lambda$  is Frechet differentiable and its Frechet derivative is given by:

$$d\lambda_{p_1}(h) = \int_a^b |u'|^2 h, h \in L^1(a, b). \quad (4.17)$$

7. Fix all components of  $\omega$  except  $q$  and consider  $\lambda$  as a function of  $q \in L^1(a, b)$ . Then  $\lambda$  is Frechet differentiable and its Frechet derivative is given by:

$$d\lambda_q(h) = \int_a^b |u|^2 h, h \in L^1(a, b). \quad (4.18)$$

8. Fix all components of  $\omega$  except  $w$  and consider  $\lambda$  as a function of  $w \in L^1(a, b)$ . Then  $\lambda$  is Frechet differentiable and its Frechet derivative is given by:

$$d\lambda_w(h) = -\lambda \int_a^b |u|^2 h, h \in L^1(a, b). \quad (4.19)$$

**Proof.** Since the proofs of (4.12), (4.13) are similar we just prove (4.13). From the BVP we have

$$[u(p_2 v'')' - v(p_2 u'')' - u(p_1 v') + v(p_1 u') - u'(p_2 v'') + v'(p_2 u'')](a) = 0.$$

Hence

$$\begin{aligned} & [\lambda(\beta + h) - \lambda(\beta)] \int_a^b uvw \\ &= [u(p_2 v'')' - v(p_2 u'')' - u(p_1 v') + v(p_1 u') - u'(p_2 v'') + v'(p_2 u'')]'_a^b \\ &= [u(p_2 v'')' - v(p_2 u'')' - u(p_1 v') + v(p_1 u') - u'(p_2 v'') + v'(p_2 u'')](b) \\ &= \tan \beta u'(b)[(p_2 v'')'(b) - (p_1 v')(b)] - \tan(\beta + h) u'(b)[(p_2 v'')'(b) - (p_1 v')(b)] \\ &\quad - \tan(\beta + h) v'(b)[(p_2 u'')'(b) - (p_1 u')(b)] + \tan \beta v'(b)[(p_2 u'')'(b) - (p_1 u')(b)] \\ &= -[\tan(\beta + h) - \tan \beta] u'(b)[(p_2 v'')'(b) - (p_1 v')(b)] \\ &\quad - [\tan(\beta + h) - \tan \beta] v'(b)[(p_2 u'')'(b) - (p_1 u')(b)], \\ &\lambda'(\beta) \\ &= -\sec^2 \beta u'(b)[(p_2 u'')'(b) - (p_1 u')(b)] - \sec^2 \beta v'(b)[(p_2 u'')'(b) - (p_1 u')(b)] \\ &= -\tan^2 \beta u'(b)[(p_2 u'')'(b) - (p_1 u')(b)] - u'(b)[(p_2 u'')'(b) - (p_1 u')(b)] \\ &\quad - \tan^2 \beta v'(b)[(p_2 u'')'(b) - (p_1 u')(b)] - v'(b)[(p_2 u'')'(b) - (p_1 u')(b)] \\ &= -2 \{u(b)(p_2 u'')(b) + u'(b)[(p_2 u'')'(b) - (p_1 u')(b)]\}. \end{aligned}$$

To proof (4.14),

$$\begin{aligned} & [\lambda(\theta + h) - \lambda(\theta)] \int_a^b u \bar{v} w \\ &= [u(p_2 \bar{v}'')' - \bar{v}(p_2 u'')' - u(p_1 \bar{v}') + \bar{v}(p_1 u') - u'(p_2 \bar{v}'') + \bar{v}'(p_2 u'')]'_a^b \\ &= ((p_2 \bar{v}'')' - (p_1 \bar{v}'), -p_2 \bar{v}'', \bar{v}', -\bar{v})(b) \begin{pmatrix} u \\ u' \\ p_2 u'' \\ (p_2 u'')' - p_1 u' \end{pmatrix} (b) \\ &\quad - ((p_2 \bar{v}'')' - (p_1 \bar{v}'), -p_2 \bar{v}'', \bar{v}', -\bar{v})(a) \begin{pmatrix} u \\ u' \\ p_2 u'' \\ (p_2 u'')' - p_1 u' \end{pmatrix} (a) \end{aligned}$$

$$\begin{aligned}
&= e^{i\theta}((p_2 \bar{v}'')' - (p_1 \bar{v}'), -p_2 \bar{v}'', \bar{v}', -\bar{v})(b) K \begin{pmatrix} u \\ u' \\ p_2 u'' \\ (p_2 u'')' - p_1 u' \end{pmatrix} (a) \\
&\quad - e^{i(\theta+h)}((p_2 \bar{v}'')' - (p_1 \bar{v}'), -p_2 \bar{v}'', \bar{v}', -\bar{v})(b) K \begin{pmatrix} u \\ u' \\ p_2 u'' \\ (p_2 u'')' - p_1 u' \end{pmatrix} (a) \\
&= -e^{i\theta}((p_2 \bar{v}'')' - (p_1 \bar{v}'), -p_2 \bar{v}'', \bar{v}', -\bar{v})(b) K \begin{pmatrix} u \\ u' \\ p_2 u'' \\ (p_2 u'')' - p_1 u' \end{pmatrix} (a) (e^{ih} - 1).
\end{aligned}$$

Dividing both sides of above equation by  $h$  and taking the limit as  $h \rightarrow 0$  we obtain

$$\begin{aligned}
&\lambda'(\theta) \\
&= -ie^{i\theta}((p_2 \bar{u}'')' - (p_1 \bar{u}'), -p_2 \bar{u}'', \bar{u}', -\bar{u})(b) K \begin{pmatrix} u \\ u' \\ p_2 u'' \\ (p_2 u'')' - p_1 u' \end{pmatrix} (a) \\
&= -i((p_2 \bar{u}'')' - (p_1 \bar{u}'), -p_2 \bar{u}'', \bar{u}', -\bar{u})(b) \begin{pmatrix} u \\ u' \\ p_2 u'' \\ (p_2 u'')' - p_1 u' \end{pmatrix} (b) \\
&= -i[u(p_2 \bar{u}'')' - \bar{u}(p_2 u'')' - u(p_1 \bar{u}') + \bar{u}(p_1 u') - u'(p_2 \bar{u}'') + \bar{u}'(p_2 u'')](b) \\
&= 2\text{Im}[u(p_2 \bar{u}'')' + \bar{u}(p_1 u') + \bar{u}'(p_2 u'')](b).
\end{aligned}$$

To proof (4.15), let  $u = u(\cdot, K)$ ,  $v = u(\cdot, K + H)$ ,

$$\begin{aligned}
&[\lambda(K + H) - \lambda(K)] \int_a^b u \bar{v} w \\
&= [u(p_2 \bar{v}'')' - \bar{v}(p_2 u'')' - u(p_1 \bar{v}') + \bar{v}(p_1 u') - u'(p_2 \bar{v}'') + \bar{v}'(p_2 u'')]_a^b
\end{aligned}$$

$$\begin{aligned}
&= e^{i\theta}((p_2 \bar{v}'')' - (p_1 \bar{v}'), -p_2 \bar{v}'', \bar{v}', -\bar{v})(b)K \begin{pmatrix} u \\ u' \\ p_2 u'' \\ (p_2 u'')' - p_1 u' \end{pmatrix} (a) \\
&\quad - e^{i\theta}((p_2 \bar{v}'')' - (p_1 \bar{v}'), -p_2 \bar{v}'', \bar{v}', -\bar{v})(b)(K + H) \begin{pmatrix} u \\ u' \\ p_2 u'' \\ (p_2 u'')' - p_1 u' \end{pmatrix} (a) \\
&= -e^{i\theta}((p_2 \bar{v}'')' - (p_1 \bar{v}'), -p_2 \bar{v}'', \bar{v}', -\bar{v})(b)H \begin{pmatrix} u \\ u' \\ p_2 u'' \\ (p_2 u'')' - p_1 u' \end{pmatrix} (a) \\
&= -((p_2 \bar{u}'')' - (p_1 \bar{u}'), -(p_2 \bar{u}'', \bar{u}', -\bar{u})(b)HK^{-1} \begin{pmatrix} u \\ u' \\ p_2 u'' \\ (p_2 u'')' - p_1 u' \end{pmatrix} (b) + o(H).
\end{aligned}$$

Hence

$$\begin{aligned}
&\lambda(K + H) - \lambda(K) \\
&= -((p_2 \bar{u}'')' - (p_1 \bar{u}'), -(p_2 \bar{u}'', \bar{u}', -\bar{u})(b)HK^{-1} \begin{pmatrix} u \\ u' \\ p_2 u'' \\ (p_2 u'')' - p_1 u' \end{pmatrix} (b) + o(H).
\end{aligned}$$

Defined by Frechet differential, we obtain (4.15).

To proof (4.16)

$$\begin{aligned}
&[\lambda(1/p_{2h}) - \lambda(1/p_2)] \int_a^b u \bar{v} w \\
&= \int_a^b \lambda(1/p_{2h}) u \bar{v} w - \int_a^b \lambda(1/p_2) u \bar{v} w \\
&= \int_a^b [(p_{2h} \bar{v}'')'' - (p_1 \bar{v}')' + q \bar{v}] u - \int_a^b [(p_2 u'')'' - (p_1 u')' + q u] \bar{v} \\
&= \int_a^b u d(p_{2h} \bar{v}'')' - \int_a^b u d(p_1 \bar{v}') + \int_a^b q \bar{v} u - \int_a^b \bar{v} d(p_2 u'')' + \int_a^b \bar{v} d(p_1 u') - \int_a^b q u \bar{v}
\end{aligned}$$



$$\begin{aligned}
&= [u(p_{2h}\bar{v}'')' - \bar{v}(p_2u'')' - u(p_1\bar{v}') + \bar{v}(p_1u')]_a^b - \int_a^b u' d(p_{2h}\bar{v}'') + \int_a^b \bar{v}' d(p_2u'') \\
&= [u(p_{2h}\bar{v}'')' - \bar{v}(p_2u'')' - u(p_1\bar{v}') + \bar{v}(p_1u') - u'(p_{2h}\bar{v}'') + \bar{v}'(p_2u'')]_a^b \\
&\quad + \int_a^b (p_{2h} - p_2)\bar{v}''u'',
\end{aligned}$$

where

$$[u(p_{2h}\bar{v}'')' - \bar{v}(p_2u'')' - u(p_1\bar{v}') + \bar{v}(p_1u') - u'(p_{2h}\bar{v}'') + \bar{v}'(p_2u'')]_a^b = 0.$$

Then

$$\begin{aligned}
&[\lambda(1/p_{2h}) - \lambda(1/p_2)](1 + o(1)) \\
&= \int_a^b (p_{2h} - p_2)u''\bar{v}'' + o(h) \\
&= - \int_a^b p_2p_{2h}hu''\bar{v}'' + o(h) \\
&= - \int_a^b (p_2u'')(p_{2h}\bar{v}'')h + o(h) \\
&= - \int_a^b |p_2u''|^2h + o(h), \\
&\lambda(1/p_{2h}) - \lambda(1/p_2) \\
&= [- \int_a^b |p_2u''|^2h + o(h)](1 + o(1))^{-1} \\
&= - \int_a^b |p_2u''|^2h + o(h).
\end{aligned}$$

Hence

$$d\lambda_{1/p_2}(h) = - \int_a^b |p_2u''|^2h, h \in L^1(a, b).$$

To proof (4.17)

$$\begin{aligned}
&[\lambda(p_1 + h) - \lambda(p_1)] \int_a^b u\bar{v}w \\
&= \int_a^b \lambda(p_1 + h)u\bar{v}w - \int_a^b \lambda(p_1)u\bar{v}w \\
&= \int_a^b [(p_2\bar{v}'')'' - ((p_1 + h)\bar{v}')' + q\bar{v}]u - \int_a^b [(p_2u'')'' - (p_1u')' + qu]\bar{v} \\
&= \int_a^b [(p_2\bar{v}'')'' - (p_1\bar{v}')' + q\bar{v}]u - \int_a^b [(p_2u'')'' - (p_1u')' + qu]\bar{v} - \int_a^b (h\bar{v}')'u \\
&= [u(p_2\bar{v}'')' - \bar{v}(p_2u'')' - u(p_1\bar{v}') + \bar{v}(p_1u') - u'(p_2\bar{v}'') + \bar{v}'(p_2u'')]_a^b - \int_a^b (h\bar{v}')'u
\end{aligned}$$

where

$$[u(p_2\bar{v}'')' - \bar{v}(p_2u'')' - u(p_1\bar{v}') + \bar{v}(p_1u') - u'(p_2\bar{v}'') + \bar{v}'(p_2u'')]_a^b = 0.$$

Then

$$[\lambda(p_1 + h) - \lambda(p_1)](1 + o(1)) = \int_a^b |u'|^2 h + o(h), \quad (4.20)$$

$$\lambda(p_1 + h) - \lambda(p_1) = \left[ \int_a^b |u'|^2 h + o(h) \right] (1 + o(1))^{-1} = \int_a^b |u'|^2 h + o(h). \quad (4.21)$$

Hence

$$d\lambda_{p_1}(h) = \int_a^b |u'|^2 h, h \in L^1(a, b).$$

To proof (4.18)

$$\begin{aligned} & [\lambda(q + h) - \lambda(q)] \int_a^b u \bar{v} w \\ &= \int_a^b \lambda(q + h) u \bar{v} w - \int_a^b \lambda(q) u \bar{v} w \\ &= \int_a^b [(p_2 \bar{v}'')'' - (p_1 \bar{v}')' + (q + h) \bar{v}] u - \int_a^b [(p_2 u'')'' - (p_1 u')' + q u] \bar{v} \\ &= \int_a^b [(p_2 \bar{v}'')'' - (p_1 \bar{v}')' + q \bar{v}] u - \int_a^b [(p_2 u'')'' - (p_1 u')' + q u] \bar{v} + \int_a^b h \bar{v} u \\ &= [u(p_2 \bar{v}'')' - \bar{v}(p_2 u'')' - u(p_1 \bar{v}') + \bar{v}(p_1 u') - u'(p_2 \bar{v}'') + \bar{v}'(p_2 u'')]_a^b + \int_a^b h \bar{v} u \end{aligned}$$

where

$$[u(p_2 \bar{v}'')' - \bar{v}(p_2 u'')' - u(p_1 \bar{v}') + \bar{v}(p_1 u') - u'(p_2 \bar{v}'') + \bar{v}'(p_2 u'')]_a^b = 0.$$

Then

$$[\lambda(q + h) - \lambda(q)](1 + o(1)) = \int_a^b |u|^2 h + o(h), \quad (4.22)$$

$$\lambda(q + h) - \lambda(q) = \left[ \int_a^b |u|^2 h + o(h) \right] (1 + o(1))^{-1} = \int_a^b |u|^2 h + o(h). \quad (4.23)$$

Hence

$$d\lambda_q(h) = \int_a^b |u|^2 h + o(h), h \in L^1(a, b).$$

The proof of (4.19) is similar to that of (4.18) and hence omitted.  $\square$

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