Journal of Applied Analysis and Computation
Volume 14, Number 5, October 2024, 2845–2861
Website:http://www.jaac-online.com
DOI:10.11948/20230454

UNICITY OF MEROMORPHIC FUNCTIONS CONCERNING DERIVATIVES-DIFFERENCE AND SMALL FUNCTIONS*

Ge Wang[1], Zhiying He[1] and Mingliang Fang[1,†]

Abstract In this paper, we study unicity of meromorphic functions concerning derivatives-differences and small functions and improve the results due to Chen and Zhang [Ann. Math. Ser.A 42 (2021)] and Liu and Chen [J. Korean Soc. Math. Educ. Ser. B: Pure Apple. Math. 30 (2023)]. Meanwhile, we give negative answer to the problems posed by Chen and Xu [Comput. Methods Funct. Theory 22 (2022)], Banerjee and Maity[Bull. Korean Math. Soc. 58 (2021)].

Keywords Meromorphic functions, unicity, difference, small functions.

MSC(2010) 30D35.

1. Introduction

In this paper, we assume that the reader is familiar with the basic notions of Nevanlinna's value distribution theory, see [10, 23, 24]. In the following, meromorphic always means meromorphic in the whole complex plane.

By $S(r, f)$, we denote any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ possible outside of an exceptional set E with finite logarithmic measure $\int_E dr/r < \infty$. A meromorphic function a is said to be a small function of f if it satisfies $T(r, a) = S(r, f)$.

Let f be a nonconstant meromorphic function. The order $\rho(f)$ and the hyper-order $\rho_2(f)$ of f are defined by

$$\rho(f) = \varlimsup_{r\to\infty} \frac{\log^+ T(r, f)}{\log r}, \quad \rho_2(f) = \varlimsup_{r\to\infty} \frac{\log^+ \log^+ T(r, f)}{\log r}.$$

If $\rho(f) < \infty$, then the function f is called meromorphic function of finite order.

Let η be a nonzero complex number. The difference operator is defined as

$$\Delta_\eta f = f(z + \eta) - f(z) \quad \text{and} \quad \Delta^n_\eta f = \Delta^{n-1}_\eta(\Delta_\eta f),$$

where $n(\geq 2)$ is a positive integer.

†The corresponding author.
1Department of Mathematics, Hangzhou Dianzi University, Hangzhou 310018, China
*The authors were supported by the National Natural Science Foundation of China (Nos. 12171127, 12371074).
Email: gwang@hdu.edu.cn(G. Wang), zhiyinggood@163.com(Z. He), mlfang@hdu.edu.cn(M. Fang)

Let f be a transcendental meromorphic function, and let a be a small function of f. The deficiency of a small function a with respect to f is defined by

$$\delta(a,f)=\varliminf_{r\to\infty}\frac{m\left(r,\frac{1}{f-a}\right)}{T(r,f)}=1-\varlimsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-a}\right)}{T(r,f)}.$$

It is easy to see $0\le\delta(a,f)\le 1$. If $\delta(a,f)>0$, then a is called a deficient function of f, and if a is a constant, then a is called a deficient value. And we define

$$\lambda(f-a)=\varlimsup_{r\to\infty}\frac{\log^+N\left(r,\frac{1}{f-a}\right)}{\log r}.$$

If $\lambda(f-a)<\rho(f)$ for $\rho(f)>0$ and $N\left(r,\frac{1}{f-a}\right)=O(\log r)$ for $\rho(f)=0$, then a is called a Borel exceptional small function of f. If a is a constant, then a is called a Borel exceptional value of f.

Let f and g be two meromorphic functions, and let a either be a small function of both f and g or be a constant. We say that f and g share a CM(IM) if $f-a$ and $g-a$ have the same zeros counting multiplicities(ignoring multiplicities). $N(r,a)$ is a counting function of zeros of both $f-a$ and $g-a$ with the same multiplicity and the multiplicity is counted. If

$$N\left(r,\frac{1}{f-a}\right)+N\left(r,\frac{1}{g-a}\right)-2N(r,a)\le S(r,f)+S(r,g),$$

then we call that f and g share a CM almost. Set $E(a,f)=\{z|f-a=0\}$, where a zero with multiplicity m is counted m times in the set.

Let k be a positive integer, we denote by $N_{k)}\left(r,\frac{1}{f-1}\right)$ the counting function for 1-points of f with multiplicity $\le k$, where multiplicity is counted, and by $\overline{N}_{k)}\left(r,\frac{1}{f-1}\right)$ the corresponding one for which multiplicity is not counted. Let $N_{(k}\left(r,\frac{1}{f-1}\right)$ be the counting function for 1-points of f with multiplicity $\ge k$, where multiplicity is counted, and by $\overline{N}_{(k}\left(r,\frac{1}{f-1}\right)$ the corresponding one for which multiplicity is not counted. Let $E_{k)}(1,f)$ denotes the set of those 1-points of f with multiplicity $\le k$, where a 1-point with multiplicity $m(\le k)$ is counted m times in the set.

Recently, many papers studied the uniqueness of transcendental entire function and their higer order difference operators sharing small function, and obtained many interesting results, see [14, 17, 18, 20, 21].

In 1926, Nevanlinna [24] proved the following famous five-value theorem.

Theorem 1.1. *Let f and g be two nonconstant meromorphic functions, and let a_i $(i=1,2,3,4,5)$ be five distinct values in the extended complex plane. If f and g share a_i $(i=1,2,3,4,5)$ IM, then $f\equiv g$.*

In 2000, Li and Qiao [15] improved Theorem 1.1 as follows.

Theorem 1.2. *Let f and g be two nonconstant meromorphic functions, and let a_i $(i=1,2,3,4,5)$ be five distinct small functions of both f and g. If f and g share a_i $(i=1,2,3,4,5)$ IM, then $f\equiv g$.*

In 2014, Chen and Li [2] proved.

Theorem 1.3. *Let f be a nonconstant entire function of finite order, let η be a positive integer, and let a be a periodic entire small function of f whose period is η. If $f, \Delta_\eta f, \Delta_\eta^n f$ $(n \geq 2)$ share a CM, then $\Delta_\eta^n f \equiv \Delta_\eta f$.*

In 2021, Chen and Zhang [4] proved.

Theorem 1.4. *Let f be a transcendental entire function of finite order with a Borel exceptional entire small function a satisfying $\rho(a) < 1$, and let η be a nonzero complex number such that $\Delta_\eta^2 f \not\equiv 0$. If $\Delta_\eta^2 f$ and $\Delta_\eta f$ share $\Delta_\eta a$ CM, where $\Delta_\eta a$ is a small function of $\Delta_\eta^2 f$, then*

$$f(z) = a(z) + Be^{Az},$$

where A and B are two nonzero constants and $a(z)$ reduces to a constant.

In 2023, Liu and Chen [16] extended Theorem 1.4 as follows.

Theorem 1.5. *Let f be a transcendental entire function of finite order with a Borel exceptional entire small function a satisfying $\rho(a) < 1$, let n be a positive integer, and let η be a nonzero complex number such that $\Delta_\eta^{n+1} f \not\equiv 0$. If $\Delta_\eta^{n+1} f$ and $\Delta_\eta^n f$ share $\Delta_\eta^n a$ CM, where $\Delta_\eta^n a$ is a small function of $\Delta_\eta^{n+1} f$, then*

$$f(z) = a(z) + Be^{Az},$$

where A and B are two nonzero constants and $a(z)$ reduces to a constant.

By Theorems 1.1–1.5, we natural pose the following problem.

Problem 1.1. *Whether "$\rho(a) < 1$" can be deleted or not in Theorems 1.4 and 1.5?*

In this paper, we give a positive answer to Problem 1.1 and prove the following result.

Theorem 1.6. *Let f be a transcendental entire function of finite order with a Borel exceptional entire small function a, let n be a positive integer, and let η be a nonzero finite complex number such that $\Delta_\eta^{n+1} f \not\equiv 0$. If $\Delta_\eta^{n+1} f$ and $\Delta_\eta^n f$ share b CM, where b is a small function of f, then*

$$f(z) = a(z) + Be^{Az},$$

where A and B are two nonzero constants and $a(z)$ is a polynomial with $\deg a \leq n - 1$.

Remark 1.1. In Theorem 1.5 and Theorem 1.6, "$a(z)$ reduces to a constant" is not valid.

Example 1.1. Let $f = a(z) + Be^{Az}$, where $a(z) = z^{n-1}$ and A, B are nonzero finite complex numbers satisfying $e^{A\eta} = 2$, and let $b = 0$. Obviously, $\Delta_\eta^{n+1} f(z) = B(e^{A\eta} - 1)^{n+1} e^{Az} = B(e^{A\eta} - 1)^n e^{Az} = \Delta_\eta^n f(z)$. Hence $\Delta_\eta^n f(z)$ and $\Delta_\eta^{n+1} f(z)$ share b CM, but $a(z)$ is not a constant.

In 2011, Heittokangas et al. [12] started to consider the uniqueness of meromorphic function with its shifts and proved.

Theorem 1.7. *Let f be a nonconstant entire function of finite order, and let η be a nonzero finite complex number. If $f(z)$ and $f(z+\eta)$ share two distinct finite values a, b IM, then $f(z) \equiv f(z+\eta)$.*

In 2020, Qi et al. [19] proved.

Theorem 1.8. *Let f be a nonconstant meromorphic function of finite order, and let a, η be two nonzero finite complex numbers. If $f'(z)$ and $f(z+\eta)$ share a CM, and $E(0, f(z+\eta)) \subset E(0, f'(z))$, $E(\infty, f'(z)) \subset E(\infty, f(z+\eta))$, then $f'(z) \equiv f(z+\eta)$.*

In 2022, Chen and Xu [5] proved.

Theorem 1.9. *Let f be a nonconstant meromorphic function with $\rho_2(f) < 1$, let η be a nonzero finite complex number, and let k be a positive integer. If $f^{(k)}(z)$ and $f(z+\eta)$ share $0, \infty$ CM and 1 IM, then $f^{(k)}(z) \equiv f(z+\eta)$.*

Chen and Xu [5] posed the following problem.

Problem 1.2. *Let f be a nonconstant meromorphic function with $\rho_2(f) < 1$, and let η be a nonzero finite complex number. If $f^{(k)}$ and $f(z+\eta)$ share $0, \infty$ CM and $E_{1)}\left(1, f^{(k)}(z)\right) = E_{1)}\left(1, f(z+\eta)\right)$, then $f^{(k)}(z) \equiv f(z+\eta)$?*

In this paper, we give a negative answer to Problem 1.2.

Example 1.2. Let $f(z) = \sin z$, $\eta = \pi$, $k = 4$. Obviously $\rho(f) = 1$. By a simple calculation, we know that $f^{(4)}(z) = \sin z$ and $f(z+\eta) = -\sin z$. In this case, we have $f^{(4)}(z)$ and $f(z+\eta)$ share $0, \infty$ CM, and $E_{1)}(1, f^{(4)}(z)) = E_{1)}(1, f(z+\eta)) = \varnothing$, but $f^{(4)}(z) \not\equiv f(z+\eta)$.

In addition, we further studied this problem and have proved.

Theorem 1.10. *Let f be a nonconstant meromorphic function with $\rho_2(f) < 1$, let η be a nonzero finite complex number, and let k be a positive integer. If $E(0, f(z+\eta)) \subset E\left(0, f^{(k)}(z)\right)$, $E\left(\infty, f^{(k)}(z)\right) \subset E(\infty, f(z+\eta))$, $E_{2)}(1, f^{(k)}(z)) = E_{2)}\left(1, f(z+\eta)\right)$, then $f^{(k)}(z) \equiv f(z+\eta)$.*

In the following,

$$L_\eta f(z) = \sum_{j=0}^{k} b_j f(z+j\eta), \quad L_\eta^b f(z) = \sum_{j=0}^{k} b_j f(z+j\eta),$$

where $b_j \in \mathbb{C}$, $b_k \neq 0$ and $b = \sum_{j=0}^{k} b_j$.

In 2021, Banerjee and Maity [1] proved the following results.

Theorem 1.11. *Let f be a nonconstant meromorphic function with $\rho_2(f) < 1$, let η be a nonzero complex number, and let a be a small periodic function of f whose period is η. If $L_\eta^0 f \not\equiv 0$, and $E(0, f) \subset E(0, L_\eta^0 f)$, $E(a, f) \subset E(a, L_\eta^0 f)$, $E(\infty, L_\eta^0 f) \subset E(\infty, f)$, then $L_\eta^0 f \equiv f$.*

Theorem 1.12. *Let f be a nonconstant meromorphic function of finite order, and let η, b_0, a_1, a_2 be nonzero complex numbers with $a_1 \neq a_2$. If $L_\eta^0 f \not\equiv 0$, and $L_\eta^0 f, f$ share a_1, a_2, ∞ CM, then $L_\eta^0 f \equiv f$.*

Theorem 1.13. *Let f be a nonconstant meromorphic function with $\rho_2(f) < 1$, let η be a nonzero complex number, and let a_1, a_2 be two distinct periodic small functions* *of f whose period are η. If $L_\eta^1 f \not\equiv 0$, and $E(a_1,f)\subset E(a_1,L_\eta^1 f)$, $E(a_2,f)\subset E(a_2,L_\eta^1 f)$, $E(\infty,L_\eta^1 f)\subset E(\infty,f)$, then $L_\eta^1 f\equiv f$.*

Banerjee and Maity [1] posed the following problem.

Problem 1.3. *Are Theorems 1.11–1.13 valid or not for $L_\eta^b f$ where $b\neq 0,1$ or $L_\eta f$?*

In this paper, we give a negative answer to Problem 1.3.

Example 1.3. Let $f(z)=\frac{e^{2z}+1}{e^{2z}-1}$, and let $L_\eta f(z)=f(z)+f(z+\eta)-f(z+2\eta)-f(z+3\eta)-f(z+4\eta)=-\frac{e^{2z}+1}{e^{2z}-1}$, where $\eta=\pi i$. Obviously, $f(z)\neq\pm1$, $L_\eta f(z)\neq\pm1$. Hence, $f(z)$ and $L_\eta f(z)$ share $1,-1,\infty$ CM, but $f(z)\not\equiv L_\eta f(z)$.

2. Some Lemmas

In order to prove our results, we need the following lemmas.

Lemma 2.1. ([13]) *Let f be a nonconstant entire function of finite order. If a is a Borel exceptional entire function of f, then $\delta(a,f)=1$.*

Lemma 2.2. ([10]) *Let f be a nonconstant meromorphic function, and let k be a positive integer. Then*

$$m\left(r,\frac{f^{(k)}}{f}\right)=S(r,f).$$

Lemma 2.3. ([6,9]) *Let f be a nonconstant meromorphic function with $\rho_2(f)<1$, and let η be a nonzero finite complex number. Then*

$$m\left(r,\frac{f(z+\eta)}{f(z)}\right)=S(r,f),\quad m\left(r,\frac{f(z)}{f(z+\eta)}\right)=S(r,f).$$

Especially, if $\rho(f)<+\infty$, then for any $\varepsilon>0$, we have

$$m\left(r,\frac{f(z+\eta)}{f(z)}\right)=O\left(r^{\rho(f)-1+\varepsilon}\right).$$

Lemma 2.4. ([6,9]) *Let f be a nonconstant meromorphic function with $\rho_2(f)<1$, and let η be a nonzero finite complex number. Then*

$$N\left(r,f(z+\eta)\right)=N(r,f(z))+S(r,f),$$
$$\overline{N}\left(r,\frac{1}{f(z+\eta)}\right)=\overline{N}\left(r,\frac{1}{f(z)}\right)+S(r,f).$$

Lemma 2.5. ([11]) *Let η be a nonzero finite complex number, let n be a positive integer, and let f be a transcendental meromorphic function of finite order satisfying $\delta(a,f)=1,\delta(\infty,f)=1$, where a is a small function of f. If $\Delta_\eta^n f\not\equiv 0$, then*
(1) $T(r,\Delta_\eta^n f)=T(r,f)+S(r,f)$;
(2) $\delta\left(\Delta_\eta^n a,\Delta_\eta^n f\right)=\delta\left(\infty,\Delta_\eta^n f\right)=1$.

Lemma 2.6. ([10]) *Let f be a nonconstant meromorphic function, and let a,b be two distinct small functions of f. Then*

$$T(r,f)\le\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{f-b}\right)+S(r,f).$$

Lemma 2.7. ([23]) *Let f be a meromorphic function. If $f \neq 0, \infty$, then there exists an entire function α such that $f(z) = e^{\alpha(z)}$.*

Lemma 2.8. ([3]) *Let a be a finite complex number, let f be a transcendental meromorphic function of finite order with two Borel exceptional values a, ∞, and let η be a nonzero finite complex number such that $\Delta_\eta f \not\equiv 0$. If f and $\Delta_\eta f$ share a, ∞ CM, then $a = 0, f(z) = e^{Az+B}$, where $A(\neq 0), B$ are two finite constants.*

Lemma 2.9. ([22, 23]) *Let $n \geq 3$ be a positive integer, let $f_j(j = 1, \cdots, n)$ be meromorphic functions which are not constants except for f_n, and let $\sum_{j=1}^{n} f_j \equiv 1$. If $f_n \not\equiv 0$, and*

$$\sum_{j=1}^{n} N\left(r, \frac{1}{f_j}\right) + (n-1)\sum_{j=1}^{n} \overline{N}(r, f_j)\, (\lambda + o(1))\, T(r, f_k),$$

where I is a set of $r \in (0, \infty)$ with infinite linear measure, $r \in I, k = 1, 2, \cdots, n - 1, \lambda < 1$, then $f_n \equiv 1$.

Lemma 2.10. ([8, 23]) *Let f and g be two nonconstant meromorphic functions satisfying*

$$\delta(0, f) = \delta(\infty, f) = 1, \quad \delta(0, g) = \delta(\infty, g) = 1.$$

If f and g share 1 CM almost, then either $f \equiv g$ or $fg \equiv 1$.

Lemma 2.11. ([11]) *Let f be a meromorphic function of finite order, and let η, c, d be three nonzero finite complex numbers. If $f(z + \eta) = cf(z)$, then either $T(r, f) \geq dr$ for sufficiently large r or f is a constant.*

Lemma 2.12. ([7]) *Let f be a meromorphic function with $\rho(f) < 1$, and let η be a nonzero finite complex number. Then for each given $\varepsilon > 0$, and a positive integer n, there exists a set $E \subset (1, \infty)$ that depends on f, and it has finite logarithmic measure, such that for all z satisfying $|z| = r \notin E \bigcup [0, 1]$, we have*

$$\left|\frac{\Delta_\eta^n f(z)}{f(z)}\right| \leq |z|^{n(\rho(f)-1)+\varepsilon}.$$

Lemma 2.13. *Let α be an entire function with $\rho(\alpha) \leq 1$, let n be a positive integer, and let η, d be two nonzero finite complex numbers. If $\Delta_\eta^n \alpha \equiv 0$, then either $T(r, \alpha) \geq dr$ for sufficiently large r or α is a polynomial with $\deg \alpha \leq n - 1$.*

Proof. We prove the lemma by mathematical induction. In the following, d denote a positive number, not necessarily the same at each occurrence. For $n = 1$ we have

$$\alpha(z + \eta) = \alpha(z). \tag{2.1}$$

Then by Lemma 2.11 and (2.1) we know that Lemma 2.13 is valid for $n = 1$.

Suppose that for $n = k - 1$ the lemma is valid. Next we consider the case $n = k$. From $\Delta_\eta^k \alpha \equiv 0$ and above discussion we deduce that either $T\left(r, \Delta_\eta^{k-1}\alpha\right) \geq dr$ for sufficiently large r or $\Delta_\eta^{k-1}\alpha$ is a constant.

If $T\left(r, \Delta_\eta^{k-1}\alpha\right) \geq dr$ for sufficiently large r, then by $\rho(\alpha) \leq 1$, Lemma 2.3 (setting $\varepsilon = \frac{1}{2}$) and for sufficiently large r, we obtain

$$T\left(r, \Delta_\eta^{k-1}\alpha\right) = m\left(r, \Delta_\eta^{k-1}\alpha\right)$$

$$\leq m(r,\alpha)+m\left(r,\frac{\Delta_\eta^{k-1}\alpha}{\alpha}\right)+O(1)$$
$$\leq T(r,\alpha)+Mr^{\frac{1}{2}}$$
$$\leq T(r,\alpha)+\frac{1}{2}dr, \tag{2.2}$$

where M is a positive number. Since $T\left(r,\Delta_\eta^{k-1}\alpha\right)\geq dr$, then by (2.2) we have $T(r,\alpha)\geq d_0 r$, where $d_0=\frac{d}{2}$.

If $\Delta_\eta^{k-1}\alpha\equiv C$, where C is a constant, then $p(z)=\frac{C}{(k-1)!\eta}z^{k-1}$ is a solution of $\Delta_\eta^{k-1}\alpha\equiv C$. Let $\beta(z)$ be any solution of $\Delta_\eta^{k-1}\alpha\equiv 0$. Then we know that either $T(r,\beta)\geq dr$ for sufficiently large r or β is a polynomial with $\deg\beta\leq k-2$. From above argument we have either $T(r,\beta+p)\geq T(r,\beta)-T(r,p)\geq\frac{d}{2}r$ or $\beta+p$ is a polynomial with $\deg(\beta+p)\leq k-1$. It follows that either $T(r,\alpha)\geq dr$ for sufficiently large r or α is a polynomial with $\deg\alpha\leq k-1$.

Thus Lemma 2.13 is proved. □

Lemma 2.14. ([7]) *Let f be a meromorphic function of finite order, and let η be a nonzero finite complex number. Then for each positive integer k, $\rho\left(\Delta_\eta^k f\right)\leq\rho(f)$.*

Lemma 2.15. ([24]) *Let f be a meromorphic function. Then $\rho(f)=\rho(f')$.*

3. Proof of Theorem 1.6

First, we claim $\rho(f)>0$. Suppose on the contrary that $\rho(f)=0$. Set $F(z)=f(z)-a(z)$. Since a is a Borel exceptional entire small function of f, we obtain

$$N\left(r,\frac{1}{F}\right)=N\left(r,\frac{1}{f-a}\right)=O(\log r).$$

Hence F has finitely many zeros. We assume that $a_1,a_2,\cdots,a_n$ are all zeros of F, where n is a positive integer.

From $\rho(f)=0$, we deduce $\frac{F}{(z-a_1)(z-a_2)\cdots(z-a_n)}=e^h$, where h is a constant. Then we have $F(z)=c(z-a_1)(z-a_2)\cdots(z-a_n)$, where $c=e^h$. It follows that

$$T(r,F)=n\log r+O(1). \tag{3.1}$$

By (3.1) we deduce that f is a nonzero polynomial. Since b is a small function of f, then we know that b is a constant, which contradicts with $\Delta_\eta^{n+1}f$ and $\Delta_\eta^n f$ share b CM. Hence $\rho(f)>0$.

Since a is a Borel exceptional entire small function of f, then by Lemma 2.1, we obtain $\delta(a,f)=1$. Obviously, $\delta(\infty,f)=1$. It follows from Lemma 2.5 that

$$\delta(\Delta_\eta^n a,\Delta_\eta^n f)=1,\quad \delta(\Delta_\eta^{n+1}a,\Delta_\eta^{n+1}f)=1, \tag{3.2}$$
$$\delta(\infty,\Delta_\eta^n f)=1,\quad \delta(\infty,\Delta_\eta^{n+1}f)=1. \tag{3.3}$$

Now, we consider three cases

Case 1. $b\equiv\Delta_\eta^{n+1}a$.

Case 1.1. $\Delta_\eta^{n+1}a\not\equiv\Delta_\eta^n a$.

Since $\Delta_\eta^n f$ and $\Delta_\eta^{n+1} f$ share b CM, then by (3.2), (3.3), Lemma 2.5 and Lemma 2.6 we have

$$\begin{aligned} T(r,f) &= T(r,\Delta_\eta^n f) + S(r,f) \\ &\le \overline{N}(r,\Delta_\eta^n f) + \overline{N}\left(r,\frac{1}{\Delta_\eta^n f - \Delta_\eta^n a}\right) + \overline{N}\left(r,\frac{1}{\Delta_\eta^n f - b}\right) + S(r,f) \\ &\le \overline{N}\left(r,\frac{1}{\Delta_\eta^{n+1} f - b}\right) + S(r,f) \\ &\le S(r,f), \end{aligned}$$

a contradiction.

Case 1.2. $\Delta_\eta^{n+1} a \equiv \Delta_\eta^n a$.

Set

$$G = \Delta_\eta^n f - \Delta_\eta^n a. \tag{3.4}$$

Then we have

$$\Delta_\eta G = \Delta_\eta^{n+1} f - \Delta_\eta^n a.$$

Since $\Delta_\eta^n f$ and $\Delta_\eta^{n+1} f$ share $b(\equiv \Delta_\eta^n a)$ CM, we obtain that G and $\Delta_\eta G$ share $0, \infty$ CM.

It follows from (3.2) and (3.3) that

$$\delta(0,G) = 1, \quad \delta(0,\Delta_\eta G) = 1, \tag{3.5}$$

$$\delta(\infty,G) = 1, \quad \delta(\infty,\Delta_\eta G) = 1. \tag{3.6}$$

By $\delta(a,f) = 1$, $\delta(\infty,f) = 1$ and Lemma 2.5, we obtain

$$T(r,G) = T(r,f) + S(r,f). \tag{3.7}$$

Since a is a Borel exceptional function of f, then by $\rho(f) > 0$ we have

$$\overline{\lim_{r\to\infty}} \frac{\log^+ N\left(r,\frac{1}{f-a}\right)}{\log r} < \rho(f). \tag{3.8}$$

By Lemma 2.3 and Nevanlinna's first fundamental theorem we have

$$m\left(r,\frac{1}{f-a}\right) \le m\left(r,\frac{1}{\Delta_\eta^n (f-a)}\right) + m\left(r,\frac{\Delta_\eta^n (f-a)}{f-a}\right) + S(r,f),$$

$$T(r,f-a) - N\left(r,\frac{1}{f-a}\right) \le T\left(r,\Delta_\eta^n (f-a)\right) - N\left(r,\frac{1}{\Delta_\eta^n (f-a)}\right) + S(r,f).$$

Hence, by Lemma 2.5 we have

$$N\left(r,\frac{1}{\Delta_\eta^n (f-a)}\right) \le N\left(r,\frac{1}{f-a}\right) + S(r,f). \tag{3.9}$$

By Lemma 2.3 (setting $\varepsilon = \frac{1}{2}$), we obtain

$$S(r,f) \le M r^{\rho(f)-\frac{1}{2}}, \tag{3.10}$$

where M is a positive number.

It follows from (3.8) that

$$N\left(r,\frac{1}{f-a}\right)\leq r^{\frac{\rho(f)+\lambda(f-a)}{2}}. \tag{3.11}$$

By (3.10) and (3.11) we have

$$N\left(r,\frac{1}{f-a}\right)+S(r,f)\leq (1+M)r^{M_1}, \tag{3.12}$$

where $M_1=\max\left\{\rho(f)-\frac{1}{2},\frac{\rho(f)+\lambda(f-a)}{2}\right\}$.

It follows from (3.8), (3.9) and (3.12) that

$$\frac{\log^+ N\left(r,\frac{1}{\Delta_\eta^n(f-a)}\right)}{\log r}\leq\frac{\log(1+M)r^{M_1}}{\log r}\leq M_1+\frac{\log(1+M)}{\log r}.$$

Then we have

$$\overline{\lim_{r\to\infty}}\frac{\log^+ N\left(r,\frac{1}{G}\right)}{\log r}=\overline{\lim_{r\to\infty}}\frac{\log^+ N\left(r,\frac{1}{\Delta_\eta^n(f-a)}\right)}{\log r}\leq M_1<\rho(f). \tag{3.13}$$

By (3.7) and (3.13) we deduce that 0 is a Borel exceptional value of G. It follows from Lemma 2.8 that $G=e^{A_1z+B_1}$, where $A_1(\neq 0), B_1$ are two constants.

From (3.4) we get

$$\Delta_\eta^n\left(f(z)-a(z)\right)=e^{A_1z+B_1}. \tag{3.14}$$

By Hadamard's factorization theorem, we obtain

$$f(z)-a(z)=\beta(z)e^{p(z)}, \tag{3.15}$$

where $\beta(z)$ is an entire function such that $\rho(\beta)=\lambda(\beta)<\rho(f)$, and $p(z)$ is a nonconstant polynomial with $\deg p=\rho(f)$. Hence we have

$$T(r,\beta)=S(r,e^p). \tag{3.16}$$

It follows from (3.14) and (3.15) that $\Delta_\eta^n\left(\beta(z)e^{p(z)}\right)=e^{A_1z+B_1}$. That is

$$\sum_{i=0}^{n}(-1)^iC_n^i\beta\left(z+(n-i)\eta\right)e^{p(z+(n-i)\eta)}=e^{A_1z+B_1}. \tag{3.17}$$

Next, we consider two subcases.

Case 1.2.1. $\deg p\geq 2$.

By (3.17) we have

$$\sum_{i=0}^{n}(-1)^iC_n^i\frac{\beta\left(z+(n-i)\eta\right)}{e^{A_1z+B_1}}e^{p(z+(n-i)\eta)}\equiv 1. \tag{3.18}$$

If $n=1$, then by (3.18) we have

$$\frac{\beta\left(z+\eta\right)}{e^{A_1z+B_1}}e^{p(z+\eta)}-\frac{\beta(z)}{e^{A_1z+B_1}}e^{p(z)}\equiv 1. \tag{3.19}$$

Obviously, $T\left(r, e^{A_1 z+B_1}\right) = S(r, e^p)$. Then by (3.16), (3.19) and Nevanlinna's second fundamental theorem we have

$$\begin{aligned}
T(r,e^p) \leq & T\left(r, \frac{\beta(z)}{e^{A_1 z+B_1}}e^p\right) + S(r,e^p) \\
\leq & \overline{N}\left(r, \frac{\beta(z)}{e^{A_1 z+B_1}}e^p\right) + \overline{N}\left(r, \frac{1}{\frac{\beta(z)}{e^{A_1 z+B_1}}e^p}\right) + \overline{N}\left(r, \frac{1}{\frac{\beta(z)}{e^{A_1 z+B_1}}e^p+1}\right) \\
& + S\left(r, \frac{\beta(z)}{e^{A_1 z+B_1}}e^p\right) \\
\leq & S(r,e^p),
\end{aligned}$$

a contradiction.

If $n \geq 2$, then by (3.18) and Lemma 2.9 we get a contradiction.

Case 1.2.2. $\deg p = 1$.

Set $p(z) = kz + t$, where $k(\neq 0), t$ are two finite complex numbers. Next we consider two subcases.

Case 1.2.2.1. $A_1 \neq k$.

Then by (3.17) we have

$$\sum_{i=0}^{n}(-1)^i C_n^i d_i \beta\left(z+(n-i)\eta\right) e^{(k-A_1)z} \equiv 1, \tag{3.20}$$

where $d_i = e^{(n-i)k\eta+t-B_1}$.

By (3.20) and $A_1 \neq k$ we have $\sum_{i=0}^{n}(-1)^i C_n^i d_i \beta(z+(n-i)\eta) \neq 0, \infty$. From Lemma 2.7 and $\rho(\beta) < \rho(f) = 1$ we know that there exists a polynomial $\gamma(z)$ such that $\sum_{i=0}^{n}(-1)^i C_n^i d_i \beta(z+(n-i)\eta) = e^{\gamma(z)}$. Since $\rho(\beta) < \rho(f) = 1$, we know that $\gamma(z)$ is a constant. Combining with (3.20) we deduce that $e^{(k-A_1)z}$ is a constant, a contradiction.

Case 1.2.2.2. $A_1 = k$.

Thus by (3.17) we have

$$\sum_{i=0}^{n}(-1)^i C_n^i \beta\left(z+(n-i)\eta\right) e^{k\eta(n-i)} \equiv e^{B_1-t}. \tag{3.21}$$

If $\beta' \equiv 0$, we know that β is a constant. It follows from (3.15) that $f(z) = a(z) + Be^{Az}$ where A, B are two nonzero constants.

Since $b = \Delta_\eta^n a$, then by $\Delta_\eta^{n+1} a = \Delta_\eta^n a$ we have

$$\Delta_\eta b = b.$$

It follows that $b(z+\eta) = 2b(z)$. By Lemma 2.11 we know that either $T(r,b) > dr$ for sufficiently large r or b is a constant, then by b is a small function of f, we know that b is a constant. Obviously $\Delta_\eta^n a(z) = b = 0$.

From a is a Borel exceptional entire small function of f, we have $\rho(a) \leq 1$. It follows from Lemma 2.13 that a is a polynomial with $\deg a \leq n-1$. Therefore, $f(z) = a(z) + Be^{Az}$, where A, B are two nonzero constants and $a(z)$ is a polynomial with $\deg a \leq n-1$.

If $\beta' \not\equiv 0$, then by (3.21) we have

$$\sum_{i=0}^{n}(-1)^i C_n^i \frac{\beta'\left(z+(n-i)\eta\right)}{\beta'(z)} e^{k\eta(n-i)} \equiv 0. \tag{3.22}$$

We now rewrite equation (3.22) in the form

$$(e^{k\eta})^n \frac{\Delta_\eta^n \beta'(z)}{\beta'(z)} + D_{n-1}\frac{\Delta_\eta^{n-1}\beta'(z)}{\beta'(z)} + \cdots + D_1 \frac{\Delta_\eta \beta'(z)}{\beta'(z)} = D_0, \tag{3.23}$$

where $D_{n-1}, \cdots, D_1, D_0$ are constants.

By Lemma 2.15 we know that $\rho(\beta') = \rho(\beta) < \rho(f) = 1$. Now we choose ε such that $0 < \varepsilon < 1 - \rho(\beta')$. Then by Lemma 2.12 we know that there exists a set $E \subset (1, \infty)$ with finite logarithmic measure, such that for all z satisfying $|z| = r \notin E \bigcup [0,1]$, and for $1 \leq j \leq n$, we have

$$\frac{\Delta_\eta^j \beta'(z)}{\beta'(z)} = o(1). \tag{3.24}$$

Let $|z| = r \notin E \bigcup [0,1]$ and $|z| \to \infty$. By (3.23) and (3.24) we have $D_0 = 0$. Thus we have

$$\left(e^{k\eta}\right)^n \Delta_\eta^n \beta'(z) + D_{n-1}\Delta_\eta^{n-1}\beta'(z) + \cdots + D_1 \Delta_\eta \beta'(z) = 0. \tag{3.25}$$

Case a. $\Delta_\eta \beta' \equiv 0$.

By Lemma 2.13 we deduce that either $T(r, \beta') > dr$ for sufficiently large r or β' is a constant, then by $\beta \not\equiv 0$ and $\rho(\beta') = \rho(\beta) < 1$ we know that β' is a nonzero constant.

By (3.22) we have

$$\sum_{i=0}^{n}(-1)^i C_n^i e^{k\eta(n-i)} = 0.$$

Hence $\left(e^{k\eta} - 1\right)^n = 0$, which yields $e^{k\eta} = 1$.

Set $\beta(z) = c_0 z + c_1$ where $c_0 (\neq 0), c_1$ are two constants. By (3.15) and $A_1 = k$ we have $f(z) = a(z) + (c_0 z + c_1)\, e^{kz+B_1}$. Thus,

$$\Delta_\eta^n f(z) = \Delta_\eta^n a(z) + \Delta_\eta^n \left((c_0 z + c_1) e^{kz+B_1}\right). \tag{3.26}$$

If $n = 1$, then by (3.26), $e^{k\eta} = 1$ and $b = \Delta_\eta^{n+1} a = \Delta_\eta^n a$ we have

$$\begin{aligned}\Delta_\eta f(z) &= \Delta_\eta a(z) + (c_0 z + c_0\eta + c_1) e^{k(z+\eta)+B_1} - (c_0 z + c_1) e^{kz+B_1} \\ &= \Delta_\eta a(z) + c_0 \eta e^{kz+B_1} \\ &= b + c_0 \eta e^{kz+B_1},\end{aligned}$$

and

$$\begin{aligned}\Delta_\eta^2 f(z) &= \Delta_\eta \left(\Delta_\eta a(z) + c_0\eta e^{kz+B_1}\right) \\ &= \Delta_\eta^2 a(z) + c_0 \eta e^{k(z+\eta)+B_1} - c_0 \eta e^{kz+B_1}\end{aligned}$$

$$
\begin{aligned}
&= \Delta_\eta^2 a(z) \\
&= b. \qquad (3.27)
\end{aligned}
$$

Hence by $\Delta_\eta f(z)$ and $\Delta_\eta^2 f(z)$ share b CM, we get a contradiction.

If $n \geq 2$, then by a is a polynomial with $\deg a \leq n-1$ and (3.27) we have $\Delta_\eta^{n+1} f(z) = \Delta_\eta^{n+1} a(z) \equiv 0$, a contradiction.

Case b. $\Delta_\eta \beta'(z) \not\equiv 0$.

It follows from Lemmas 2.14, 2.15 that $\rho\left(\Delta_\eta \beta'\right) \leq \rho\left(\beta'\right) = \rho(\beta) < 1$. Therefore by (3.25) and Lemma 2.12 we have $D_1 = 0$. Now we suppose that $D_l \neq 0$, where $2 \leq l \leq n$, and $D_{l-1} = \cdots = D_1 = 0$. Then by (3.25) we have

$$\left(e^{k\eta}\right)^n \Delta_\eta^n \beta'(z) + D_{n-1} \Delta_\eta^{n-1} \beta'(z) + \cdots + D_l \Delta_\eta^l \beta'(z) = 0.$$

We claim $\Delta_\eta^l \beta'(z) \equiv 0$. Otherwise, we have

$$\left(e^{k\eta}\right)^n \frac{\Delta_\eta^n \beta'(z)}{\Delta_\eta^l \beta'(z)} + D_{n-1} \frac{\Delta_\eta^{n-1} \beta'(z)}{\Delta_\eta^l \beta'(z)} + \cdots + D_{l+1} \frac{\Delta_\eta^{l+1} \beta'(z)}{\Delta_\eta^l \beta'(z)} = -D_l. \qquad (3.28)$$

By (3.28) and Lemma 2.12 we have $D_l = 0$, a contradiction. Hence $\Delta_\eta^l \beta'(z) \equiv 0$.

It follows from Lemma 2.13 that either $T(r, \beta') > dr$ for sufficiently large r or β' is a polynomial with $\deg \beta' \leq l-1$, then by $\rho(\beta') = \rho(\beta) < 1$ we know that β' is a polynomial with $\deg \beta' \leq l-1$. From (3.22) we have $\sum_{i=0}^n (-1)^i C_n^i e^{k\eta(n-i)} = 0$, which yields $e^{k\eta} = 1$.

By (3.21) we deduce that $\sum_{i=0}^n (-1)^i C_n^i \beta\left(z + (n-i)\eta\right) \equiv e^{B_1 - t}$. That is $\Delta_\eta^n \beta \equiv C_1$, where $C_1 = e^{B_1 - t}$. By (3.15) we have $f(z) = a(z) + \beta(z) e^{kz + B_1}$. Thus, by $e^{k\eta} = 1$ and $b = \Delta_\eta^{n+1} a = \Delta_\eta^n a$ we have

$$
\begin{aligned}
\Delta_\eta^n f(z) &= \Delta_\eta^n a(z) + \Delta_\eta^n \left(\beta(z) e^{kz + B_1}\right) \\
&= \Delta_\eta^n a(z) + \sum_{i=0}^n (-1)^i C_n^i \beta(z + (n-i)\eta) e^{k(z + (n-i)\eta) + B_1} \\
&= \Delta_\eta^n a(z) + \sum_{i=0}^n (-1)^i C_n^i \beta(z + (n-i)\eta) e^{kz + B_1} \\
&= \Delta_\eta^n a(z) + \Delta_\eta^n \beta(z) e^{kz + B_1} \\
&= \Delta_\eta^n a(z) + C_1 e^{kz + B_1} \\
&= b + C_1 e^{kz + B_1},
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta_\eta^{n+1} f(z) &= \Delta_\eta \left(\Delta_\eta^n a(z) + C_1 e^{kz + B_1}\right) \\
&= \Delta_\eta^{n+1} a(z) + C_1 e^{k(z+\eta) + B_1} - C_1 e^{kz + B_1} \\
&= \Delta_\eta^{n+1} a(z) \\
&= b.
\end{aligned}
$$

Hence by $\Delta_\eta^n f(z)$ and $\Delta_\eta^{n+1} f(z)$ share b CM, we get a contradiction.

Case 2. $b \equiv \Delta_\eta^n a$.

Case 2.1. $\Delta_\eta^{n+1} a \not\equiv \Delta_\eta^n a$.

Since $\Delta_\eta^n f$ and $\Delta_\eta^{n+1} f$ share b CM, then by (3.2), (3.3), Lemma 2.5 and Lemma 2.6 we have

$$\begin{aligned} T(r,f) &= T(r,\Delta_\eta^{n+1} f) + S(r,f) \\ &\leq \overline{N}(r,\Delta_\eta^{n+1} f) + \overline{N}\left(r, \frac{1}{\Delta_\eta^{n+1} f - \Delta_\eta^{n+1} a}\right) + \overline{N}\left(r, \frac{1}{\Delta_\eta^{n+1} f - b}\right) + S(r,f) \\ &\leq \overline{N}\left(r, \frac{1}{\Delta_\eta^{n} f - b}\right) + S(r,f) \\ &\leq S(r,f), \end{aligned}$$

a contradiction.

Case 2.2. $\Delta_\eta^{n+1} a \equiv \Delta_\eta^n a$.

Using the same argument as used in Case 1.2, we get $f(z) = a(z) + Be^{Az}$, where A, B are two nonzero constants and $a(z)$ is a polynomial with $\deg a \leq n-1$.

Case 3. $b \not\equiv \Delta_\eta^n a$ and $b \not\equiv \Delta_\eta^{n+1} a$.

Set

$$F_1 = \frac{\Delta_\eta^n f - \Delta_\eta^n a}{b - \Delta_\eta^n a}, \quad F_2 = \frac{\Delta_\eta^{n+1} f - \Delta_\eta^{n+1} a}{b - \Delta_\eta^{n+1} a}. \tag{3.29}$$

It follows from (3.2), (3.3) and (3.29) that

$$\delta(0,F_1) = \delta(\infty,F_1) = 1, \tag{3.30}$$

$$\delta(0,F_2) = \delta(\infty,F_2) = 1. \tag{3.31}$$

From $\Delta_\eta^n f$ and $\Delta_\eta^{n+1} f$ share b CM, we know that F_1 and F_2 share 1 CM almost. It follows from (3.30), (3.31) and Lemma 2.10 that either $F_1F_2 \equiv 1$ or $F_1 \equiv F_2$.

If $F_1F_2 \equiv 1$, from (3.29) we obtain

$$\left(\Delta_\eta^n f - \Delta_\eta^n a\right)\left(\Delta_\eta^{n+1} f - \Delta_\eta^{n+1} a\right) = \left(b - \Delta_\eta^n a\right)\left(b - \Delta_\eta^{n+1} a\right). \tag{3.32}$$

By (3.32), $\delta(a,f) = 1$, Lemma 2.3 and Nevanlinna's first fundamental theorem we have

$$\begin{aligned} 2T(r,f) &\leq T\left(r, \frac{1}{(f-a)^2}\right) + S(r,f) \\ &\leq m\left(r, \frac{1}{(f-a)^2}\right) + S(r,f) \\ &\leq m\left(r, \frac{\Delta_\eta^n f - \Delta_\eta^n a}{f-a}\right) + m\left(r, \frac{\Delta_\eta^{n+1} f - \Delta_\eta^{n+1} a}{f-a}\right) \\ &\quad + m\left(r, \frac{1}{\left(b - \Delta_\eta^n a\right)\left(b - \Delta_\eta^{n+1} a\right)}\right) + S(r,f) \\ &\leq S(r,f), \end{aligned}$$

a contradiction. Therefore $F_1 \equiv F_2$.

It follows that

$$\frac{\Delta_\eta^n f - \Delta_\eta^n a}{b - \Delta_\eta^n a} \equiv \frac{\Delta_\eta^{n+1} f - \Delta_\eta^{n+1} a}{b - \Delta_\eta^{n+1} a}. \tag{3.33}$$

By (3.33) we have

$$\frac{\Delta_\eta^{n+1} f - b}{\Delta_\eta^n f - b} \equiv \frac{b - \Delta_\eta^{n+1} a}{b - \Delta_\eta^n a}. \tag{3.34}$$

Since f is a transcendental entire function of finite order and $\Delta_\eta^n f$ and $\Delta_\eta^{n+1} f$ share b CM, then by Lemma 2.7 we know that there exists a polynomial $\mu(z)$ satisfying $\deg \mu \leq \rho(f)$ such that

$$\frac{\Delta_\eta^{n+1} f - b}{\Delta_\eta^n f - b} \equiv e^{\mu(z)}. \tag{3.35}$$

It follows from (3.33)-(3.35) that

$$\frac{\Delta_\eta^{n+1} f - \Delta_\eta^{n+1} a}{\Delta_\eta^n f - \Delta_\eta^n a} = e^{\mu(z)}. \tag{3.36}$$

By $G = \Delta_\eta^n f - \Delta_\eta^n a$ and (3.36) we have

$$\Delta_\eta G = e^{\mu(z)} G.$$

Using the same argument as used in Case 1.2, we get $f(z) = a(z) + Be^{Az}$, where A and B are two nonzero constants and $a(z)$ is a polynomial with $\deg a \leq n - 1$.

Thus Theorem 1.6 is proved.

4. Proof of Theorem 1.10

Set

$$\varphi(z) = \frac{f^{(k)}(z)}{f(z+\eta)}. \tag{4.1}$$

By Lemma 2.2 and Lemma 2.3 we have

$$m(r, \varphi) = S(r, f). \tag{4.2}$$

Since $E\left(0, f(z+\eta)\right) \subset E\left(0, f^{(k)}(z)\right)$, $E\left(\infty, f^{(k)}(z)\right) \subset E\left(\infty, f(z+\eta)\right)$, then by (4.1) we deduce that $N\left(r, \varphi\right) = S(r, f)$ and $\varphi(z) \not\equiv 0$. Hence by (4.2) we have

$$T(r, \varphi) = S(r, f). \tag{4.3}$$

We claim $\varphi(z) \equiv 1$. Otherwise we suppose that $\varphi(z) \not\equiv 1$.

From $E\left(\infty, f^{(k)}(z)\right) \subset E\left(\infty, f(z+\eta)\right)$, we have

$$N(r, f^{(k)}(z)) \leq N\left(r, f(z+\eta)\right). \tag{4.4}$$

It follows that $N(r, f^{(k)}) = N(r, f) + k\overline{N}(r, f)$, Lemma 2.4 and (4.4) that

$$\overline{N}(r, f) = S(r, f). \tag{4.5}$$

By (4.1), (4.3), $E_{2)}\left(1, f^{(k)}(z)\right) = E_{2)}\left(1, f(z+\eta)\right)$ and Nevanlinna's first fundamnetal theorem we have

$$\overline{N}_{2)}\left(r, \frac{1}{f^{(k)}-1}\right) = \overline{N}_{2)}\left(r, \frac{1}{f(z+\eta)-1}\right) \le N\left(r, \frac{1}{\varphi-1}\right) \le S(r,f). \tag{4.6}$$

By (4.1) we have

$$f^{(k)} - \varphi = \varphi\left[f(z+\eta)-1\right]. \tag{4.7}$$

It follows from (4.3) and (4.7) that

$$T\left(r, f^{(k)}\right) = T(r,f) + S(r,f). \tag{4.8}$$

Thus, we have $S(r,f) = S\left(r, f^{(k)}\right)$.

By (4.3), (4.6), (4.7), Lemma 2.4 and Nevanlinna's first fundamental theorem we obtain

$$\begin{aligned} \overline{N}\left(r, \frac{1}{f^{(k)}-\varphi}\right) &= \overline{N}\left(r, \frac{1}{\varphi}\right) + \overline{N}\left(r, \frac{1}{f(z+\eta)-1}\right) \\ &\le \overline{N}_{2)}\left(r, \frac{1}{f(z+\eta)-1}\right) + \overline{N}_{(3}\left(r, \frac{1}{f(z+\eta)-1}\right) + S(r,f) \\ &\le \frac{1}{3} N_{(3}\left(r, \frac{1}{f(z+\eta)-1}\right) + S(r,f) \\ &\le \frac{1}{3} T(r,f) + S(r,f). \end{aligned} \tag{4.9}$$

Hence, by (4.5), (4.6), (4.8), (4.9) and Lemma 2.6 we have

$$\begin{aligned} & T\left(r, f^{(k)}\right) \\ \le & \overline{N}\left(r, f^{(k)}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}-1}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}-\varphi}\right) + S\left(r, f^{(k)}\right) \\ \le & \overline{N}(r,f) + \overline{N}_{2)}\left(r, \frac{1}{f^{(k)}-1}\right) + \overline{N}_{(3}\left(r, \frac{1}{f^{(k)}-1}\right) + \frac{1}{3}T(r,f) + S\left(r, f^{(k)}\right) \\ \le & \frac{1}{3} T\left(r, f^{(k)}\right) + \frac{1}{3}T(r,f) + S\left(r, f^{(k)}\right) \\ \le & \frac{2}{3} T\left(r, f^{(k)}\right) + S\left(r, f^{(k)}\right). \end{aligned}$$

It follows that $T\left(r, f^{(k)}\right) \le S\left(r, f^{(k)}\right)$, a contradiction.

Thus Theorem 1.10 is proved.

Acknowledgements

We thank the referee for his/her valuable comments and suggestions made to this paper.

References

[1] A. Banerjee and S. Maity, *Meromorphic function partially shares small functions or values with its linear c-shift operator*, Bull. Korean Math. Soc., 2021, 58(5), 1175–1192.

[2] B. Q. Chen and S. Li, *Uniqueness problems on entire functions that share a small function with their difference operators*, Adv. Difference Equ., 2014, 311, 11 pp.

[3] C. X. Chen and Z. X. Chen, *Uniqueness of meromorphic functions and their differences*, Acta. Math. Sinica (Chin. Ser.), 2016, 59(6), 821–834.

[4] C. X. Chen and R. R. Zhang, *Uniqueness theorems related to difference operators of entire functions*, Ann. Math. Ser. A, 2021, 42(1), 11–22.

[5] S. J. Chen and A. Z. Xu, *Uniqueness of derivatives and shifts of meromorphic functions*, Comput. Methods Funct. Theory, 2022, 22(2), 197–205.

[6] Y. M. Chiang and S. J. Feng, *On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plan*, Ramanujan J., 2008, 16(1), 105–129.

[7] Y. M. Chiang and S. J. Feng, *On the growth of logarithmic differences difference quotients and logarithmic derivatives of meromorphic functions*, Trans. Amer. Math. Soc., 2009, 361(7), 3767–3791.

[8] M. L. Fang, *Uniqueness of meromorphic functions connected with differential polynomials*, Adv. in Math. (China), 1995, 24(3), 244–249.

[9] R. Halburd, R. Korhonen and K. Tohge, *Holomorphic curves with shift-invariant hyperplane preimages*, Trans. Am. Math. Soc., 2014, 366(8), 4267–4298.

[10] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

[11] Z. Y. He, J. B. Xiao and M. L. Fang, *Unicity of meromorphic functions concerning differences and small functions*, Open Math., 2022, 20(1), 447–459.

[12] J. Heittokangas, R. Korhonen, I. Laine and J. Rieppo, *Uniqueness of meromorphic functions sharing values with their shifts*, Complex Var. Elliptic Equ., 2011, 56(1–4), 81–92.

[13] X. H. Huang and D. Liu, *Uniqueness of entire functions that share small function with their difference polynomials*, Pure Math., 2019, 9(3), 362–369.

[14] I. Lahiri, *An entire function that shares a small function with its derivative and linear differential polynomial*, Comput. Methods Funct. Theory, 2023, 23(3), 393–416.

[15] Y. H. Li and J. Y. Qiao, *The uniqueness of meromorphic functions concerning small functions*, Sci. China Ser. A, 2000, 43(6), 581–590.

[16] X. M. Liu and J. F. Chen, *Uniqueness related to higher order difference operators of entire functions*, J. Korean Soc. Math. Educ. Ser. B: Pure Apple. Math., 2023, 30(1), 43–65.

[17] S. Majumder and R. K. Sarkar, *Power of entire function sharing non-zero polynomials with it's linear differential polynomial*, Comput. Methods Funct. Theory, 2023, 23(1), 23–48.

[18] S. Mallick and M. B. Ahamed, *On uniqueness of a meromorphic function and its higher difference operators sharing two sets*, Anal. Math. Phys., 2022, 12(3), 24pp.

[19] X. G. Qi and L. Z. Yang, *Uniqueness of meromorphic functions concerning their shifts and derivatives*, Comput. Methods Funct. Theory, 2020, 20(1), 159–178.

[20] A. Roy and A. Banerjee, *Entire function and its linear $C-$shift operator sharing a set of small functions CM*, Complex Var. Elliptic Equ., 2023, 68(7), 1093–1118.

[21] D. Q. Si, *Meromorphic functions on annuli sharing finite sets with truncated multiplicity*, J. Math. Anal. Appl., 2023, 520(2), 16 pp.

[22] P. L. Wang, D. Liu and M. L. Fang, *The deficiency and value distribution of meromorphic functions concerning difference*, Acta. Math. Sinica (Chin. Ser.), 2016, 59(3), 357–362.

[23] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers Group, Dordrecht, 2003.

[24] L. Yang, *Value Distribution Theory*, Springer-Verlag, Berlin, 1993.