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A PARAMETERIZED SHIFT-SPLITTING PRECONDITIONER FOR SADDLE POINT PROBLEMS

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Abstract Recently, Chen and Ma [*Journal of Computational and Applied Mathematics*, 344(2018): 691–700] constructed the generalized shift-splitting (GSS) preconditioner, and gave the corresponding theoretical analysis and numerical experiments. In this paper, based on the generalized shift-splitting (GSS) preconditioner, we generalize their algorithms and further study the parameter shift-splitting (PSS) preconditioner for complex symmetric linear systems. Moreover, by similar theoretical analysis, we obtain that the parameter shift-splitting iterative method is unconditionally convergent. In finally, one example is provided to confirm the effectiveness.

Keywords Complex symmetric linear systems, parameter shift-splitting, convergence, preconditioner, eigenvalue.

MSC(2010) 65F10, 65F15, 65F50.

1. Introduction

Consider the linear equations of the form

$$Au = b, \tag{1.1}$$

where $u, b \in \mathcal{C}^n$ and $A \in \mathcal{C}^{n\times n}$ is a complex symmetric matrix, whose form is

$$A = W + iT, \tag{1.2}$$

and $W, T \in \mathcal{R}^{n\times n}$ are real symmetric matrices, with W being positive definite and T being positive semidefinite. Here and in the sequel we use $i = \sqrt{-1}$ to denote the imaginary unit. We assume $T \neq 0$, which implies that A is non-Hermitian. Such kind of linear systems arise in many problems in scientific computing and engineering applications. For more detailed descriptions, we refer to [2, 7, 17, 24] and the references therein.

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The Hermitian and skew-Hermitian parts of the complex symmetric matrix $A \in \mathcal{C}^{n\times n}$ are given by

$$H = \frac{1}{2}(A + A^*) = W \text{ and } \mathrm{S} = \frac{1}{2}(\mathrm{A} - \mathrm{A}^*) = \mathrm{iT}$$

respectively, hence, $A \in \mathcal{C}^{n\times n}$ is non-Hermitian, but positive definite matrix. Here A^* is used to denote the conjugate transpose of the matrix A. Based on the Hermitian and skew-Hermitian splitting (HSS)

$$A = H + S$$

of the matrix $A \in \mathcal{C}^{n\times n}$, Bai et al. [10] gave HSS iteration method, which is as follows:

The HSS Iteration Method [10]. Let $x^{(0)} \in \mathcal{C}^n$ be arbitrary initial guess. For $k = 0, 1, 2, \ldots$ until the sequence of iterates $\{x^{(k)}\}_{k=0}^{\infty} \subset \mathcal{C}^n$ converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$\begin{cases} (\alpha I + W)x^{(k+\frac{1}{2})} = (\alpha I - iT)x^{(k)} + b, \\ (\alpha I + iT)x^{(k+1)} = (\alpha I - W)x^{(k+\frac{1}{2})} + b. \end{cases} \tag{1.3}$$

where α is a given positive constant and I is the identity matrix.

However, a potential difficulty with the HSS iteration method is the need to solve the shifted skew-Hermitian sub-system of linear equations at each iteration step, which is as difficult as that of the original problem; see [1, 3, 5, 9–13, 16, 20, 26–43] for more detailed descriptions about the HSS iteration method and its variants. Recently, by making use of the special structure of the coefficient matrix $A \in \mathcal{C}^{n\times n}$, Bai et al. established the following modified HSS iteration (MHSS) method and a preconditioned MHSS (PMHSS) method for solving the complex symmetric linear system (1.2) in an analogous fashion to the HSS iteration scheme in [7] and [6], respectively. Concerning the convergence of the stationary MHSS iteration method and PMHSS iteration method, Bai et al. [6, 7] analyzed the convergence. In 2013, Based on the ideas of [6] and [27], Li et al. presented a new approach named as the lopsided PMHSS (LPMHSS) iteration method to solve the complex symmetric linear system of linear equation (1.2). In 2015, Wu concerned with several variants of the HSS iterative method in [30]. In 2015, Cao et al. studied two variants of the PMHSS iterative method for a class of complex symmetric indefinite linear systems in [24]. In 2018, Chen and Ma constructed the generalized shift-splitting (GSS) preconditioner, and gave the corresponding theoretical analysis and numerical experiments in [23].

Let $u = x + iy$ and $b = p + iq$ where $x, y, p, q \in \mathcal{R}^n$. Then from [2, 33] we know that the complex linear system (1) can be recast as the following two-by-two block real equivalent formulation

$$\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} W & -T \\ T & W \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}. \tag{1.4}$$

The system of linear equations (1.4) can be seen as a special case of the generalized saddle point problems [22].

To further generalize the GSS iteration method and accelerate its convergence rate, based on the generalized shift-splitting (GSS) preconditioner, we generalize their algorithms and further study the parameter shift-splitting (PSS) preconditioner for complex symmetric linear systems.

The organization of the paper is as follows. In Section 2 we provide the parameter shift-splitting (PSS) preconditioner for complex symmetric linear system (1.2). In Section 3, we establish the convergence of the parameter shift-splitting iteration method. Finally, in section 4, one example is provided to demonstrate the feasibility and effectiveness of PSS preconditioner.

2. The parameter shift-splitting preconditione

In 2018, based on the iterative methods studied in [21, 24, 32], Chen and Ma [23] constructed the generalized shift-splitting of the matrix $\mathcal{A}$, which is as follows:

$$\mathcal{A} = \frac{1}{2}\begin{pmatrix} \alpha I + W & -T \\ T & \beta I + W \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \alpha I - W & T \\ -T & \beta I - W \end{pmatrix}, \tag{2.1}$$

where $\alpha > 0$ and $\beta > 0$ are two real constants and I is the identity matrix (with appropriate dimension). By this special splitting, the following generalized shift-splitting iterative method can be defined for solving the generalized saddle point problems (1.4):

Algorithm 1: The generalized shift-splitting iterative method [23]. Given an initial guess u^0, for $k = 0, 1, 2, ...,$ until $\{u^k\}$ converges, compute

$$\frac{1}{2}\begin{pmatrix} \alpha I + W & -T \\ T & \beta I + W \end{pmatrix} u^{k+1} = \frac{1}{2}\begin{pmatrix} \alpha I - W & T \\ -T & \beta I - W \end{pmatrix} u^k + \begin{pmatrix} p \\ q \end{pmatrix}, \tag{2.2}$$

where $\alpha > 0$ and $\beta > 0$ are two given positive constants.

In this paper, to further generalize the GSS iteration method and accelerate its convergence rate, we propose the parameter shift-splitting iterative method, which is as follows:

Algorithm 2: The parameter shift-splitting iterative (PSS) method. Given an initial guess u^0, for $k = 0, 1, 2, ...,$ until $\{u^k\}$ converges,

$$\frac{1}{2\xi}\begin{pmatrix} \alpha I + \xi W & -\xi T \\ \xi T & \beta I + \xi W \end{pmatrix} u^{k+1} = \frac{1}{2\xi}\begin{pmatrix} \alpha I - \xi W & \xi T \\ -\xi T & \beta I - \xi W \end{pmatrix} u^k + \begin{pmatrix} p \\ q \end{pmatrix}, \tag{2.3}$$

where $\alpha > 0, \beta > 0$ and $\xi > 0$ are three given constants.

Remark 2.1. We may remove the previous factor $\frac{1}{2\xi}$ because of making no difference on the preconditioned system. For large, sparse or structure matrices, iterative methods are an attractive option. In particular, Krylov subspace methods apply preconditioner PPS techniques that involve orthogonal projections onto subspaces of the form

$$\mathcal{K}(\mathcal{A}, b) \equiv \operatorname{span}\left\{b, \mathcal{A}b, \mathcal{A}^2 b, ..., \mathcal{A}^{n-1}b, ...\right\}.$$

Remark 2.2. Obviously, when $\xi = 1$, the parameter shift-splitting iterative (PSS) method reduces to the generalized shift-splitting iterative (GSS) method. So, PSS method is the extension of GSS method. When choosing appropriate parameter ξ, PSS method will have fast convergence speed.

By simple calculation, the iteration format of the two-sweep shift-splitting iteration is

$$u^{k+1} = \mathcal{T}u^k + 2\xi \begin{pmatrix} \alpha I + \xi W & -\xi T \\ \xi T & \beta I + \xi W \end{pmatrix}^{-1} \begin{pmatrix} p \\ q \end{pmatrix} \tag{2.4}$$

where

$$\mathcal{T} = \begin{pmatrix} \alpha I + \xi W & -\xi T \\ \xi T & \beta I + \xi W \end{pmatrix}^{-1} \begin{pmatrix} \alpha I - \xi W & \xi T \\ -\xi T & \beta I - \xi W \end{pmatrix}. \tag{2.5}$$

Since the parameter ξ do not affect the splitting preconditioner, the corresponds to the two-sweep shift-splitting iteration (2.5) is given by

$$\mathcal{P}_{\mathcal{PSS}} = \begin{pmatrix} \alpha I + \xi W & -\xi T \\ \xi T & \beta I + \xi W \end{pmatrix}$$

which is called the two-sweep shift-splitting preconditioner for the generalized saddle point matrix $\mathcal{A}$.

Algorithm 3: For a given vector $r = [r_1^T, r_2^T]^T$, the vector $z = [z_1^T, z_2^T]^T$ can be computed similar to the analysis in [21] by the following steps:

Step 1: Solve $(\beta I + \xi W)\omega = r_2$ for ω;
Step 2: Compute $\omega_1 = r_1 + T\omega$;
Step 3: Solve $(\alpha I + \xi W + T(\beta I + \xi W)^{-1})z_1 = \omega_1$ for z_1;
Step 4: Solve $(\beta I + \xi W)\nu = Tz_1$ for ν;
Step 5: $z_2 = \omega - \nu$.

Remark 2.3. Through similar analysis about Algorithm 1 in [23], the authors can find we need to solve a linear system with the coefficient matrix $\alpha I + \xi W + T(\beta I + \xi W)^{-1}T$ and two linear systems with the coefficient matrix $\beta I + \xi W$. Moreover, these linear systems are symmetric positive definite for $\alpha > 0, \beta > 0$ and $\xi > 0$. So, we use CG method, Cholesky or LU factorization to solve the sub-systems through selecting appropriate parameters.

3. Convergence of PSS method

In this section, we will study the convergence of the parameter shift-splitting iteration method, which is motivated by the corresponding results in [32]. Let $\rho(\mathcal{T})$ denote the spectral radius of the matrix $\mathcal{T}$. Then the two-sweep shift-splitting iteration converges if and only if $\rho(\mathcal{T}) < 1$. Let λ be an eigenvalue of $\mathcal{T}$ and $[\phi^*, \psi^*]^T$ be the corresponding eigenvector. Then we have

$$\begin{cases} (\alpha I - \xi W)\phi + \xi T\psi = \lambda(\alpha I + \xi W)\phi - \lambda\xi T\psi, \\ -\xi T\phi + (\beta I - \xi W)\psi = \lambda\xi T\phi + \lambda(\beta I + \xi W)\psi. \end{cases} \tag{3.1}$$

To study the convergence of the two-sweep shift-splitting iteration method, two lemmas are given.

Lemma 3.1. *Let $W \in \mathcal{R}^{n\times n}$ be a symmetric positive definite matrix, and $T \in \mathcal{R}^{n\times n}$ be a symmetric positive semidefinite matrix. Let $\mathcal{T}$ be defined as in (2.5) with $\alpha > 0, \beta > 0$ and $\xi > 0$. If λ is an eigenvalue of the iteration matrix $\mathcal{T}$, then $\lambda \neq \pm 1$.*

Proof. If $\lambda = 1$, then from Eq. (3.1), we can obtain

$$\xi W\phi - \xi T\psi = 0, \tag{3.2}$$

and

$$\xi T\phi + \xi W\psi = 0. \tag{3.3}$$

By similar proving process to Lemma 2.1 in [23], we can get $\lambda \neq 1$. Through similar proving, we can also get $\lambda \neq -1$. □

Lemma 3.2. *Let $W \in \mathcal{R}^{n\times n}$ be a symmetric positive definite matrix, and $T \in \mathcal{R}^{n\times n}$ be a symmetric positive semidefinite matrix. Let λ be an eigenvalue of the iteration matrix $\mathcal{T}$ (with $\alpha > 0, \beta > 0, \xi > 0$) and $[\phi^*, \psi^*]^T$ be the corresponding eigenvector with $\phi, \psi \in \mathcal{C}^{n\times n}$. Then if $\psi = 0$, we have $|\lambda| < 1$.*

Proof. If $\psi = 0$, then from (3.1) we get

$$(\alpha I + \xi W)^{-1}(\alpha I - \xi W)\phi = \lambda\phi. \tag{3.4}$$

Since W is symmetric positive definite, then by [9] we can obtain

$$|\lambda| \leq \| (\alpha I + \xi W)^{-1}(\alpha I - \xi W) \|_2 < 1. \tag{3.5}$$

□

Theorem 3.3. *Let $W \in \mathcal{R}^{n\times n}$ be a symmetric positive definite matrix, and $T \in \mathcal{R}^{n\times n}$ be a symmetric positive semidefinite matrix. Let $\rho(\mathcal{T})$ denote the spectral radius of the parameter shift-splitting iteration matrix $\mathcal{T}$. Then it holds that*

$$\rho(\mathcal{T}) < 1, \forall \alpha > 0, \beta > 0, \xi > 0 \tag{3.6}$$

i.e., the parameter shift-splitting iterative method converges to the unique solution of the generalized saddle point problems (1.4).

Proof. Let λ be an eigenvalue of the iteration matrix $\mathcal{T}$ (with $\alpha > 0, \beta > 0, \xi > 0$) and $[\phi^*, \psi^*]^T$ be the corresponding eigenvector with $\phi, \psi \in \mathcal{C}^{n\times n}$.

If $\psi = 0$, then from Lemma 3.2 we can obtain $|\lambda| < 1$.

If $\psi \neq 0$, without loss of generality let $||\psi||_2 = 1$. Multiplying both sides of the second equation in Eq. (3.1) by ψ^* yields

$$-\xi(T\psi)^*\phi + \beta - \xi\psi^* W\psi = \xi\lambda(T\psi)^*\phi + \lambda(\beta + \xi\psi^* W\psi). \tag{3.7}$$

If $T\psi = 0$, then Eq. (3.7) implies

$$|\lambda| = \left|\frac{\beta - \xi\psi^* W\psi}{\beta + \xi\psi^* W\psi}\right| < 1. \tag{3.8}$$

If $T\psi \neq 0$, by Lemma 3.1 we have $\lambda \neq -1$. Then we can get from the first equation in Eq. (3.1) that $\phi \neq 0$ and

$$T\psi = \frac{\alpha(\lambda - 1)}{\xi(1 + \lambda)}\phi + W\phi. \tag{3.9}$$

Substituting (3.9) into (3.7), we can obtain

$$(1-\lambda)\beta-(1+\lambda)\xi\psi^{*}W\psi=\xi(\lambda+1)(\alpha\frac{\bar{\lambda}-1}{(1+\bar{\lambda})\xi}\phi^{*}\phi+\phi^{*}W\phi). \tag{3.10}$$

Here, $\bar{\lambda}$ denotes the conjugate of λ. Let $\varsigma=\psi^{*}W\psi,\varphi=\phi^{*}\phi,\chi=\phi^{*}W\phi$, we can obtain from Eq. (3.10)

$$\omega\beta+\alpha\bar{\omega}\varphi=\xi(\varsigma+\chi), \tag{3.11}$$

where $\omega=\frac{1-\lambda}{\lambda+1}$. Since $\alpha,\beta,\varsigma,\varphi,\chi,\xi>0$, from Eq. (3.11) we have

$$Re(\omega)=\frac{\xi(\varsigma+\chi)}{\beta+\alpha\varphi}>0, \tag{3.12}$$

where $Re(\omega)$ denotes real part of ω. So, we can obtain

$$|\lambda|=\frac{1-\omega}{1+\omega}=\sqrt{\frac{[1-Re(\omega)]^2+[Im(\omega)]^2}{[1+Re(\omega)]^2+[Im(\omega)]^2}}<1, \tag{3.13}$$

where $Re(\omega)$ and $Im(\omega)$ denote real part and imaginary part of ω, respectively. □

Remark 3.1. [20, 21, 28] From Theorem 3.3, we know that the parameter shift-splitting iterative method is convergent unconditionally. However, the convergence of the stationary iteration is typically too slow for the method to be competitive. For this reason, we propose using the Krylov subspace method to accelerate the convergence of the iteration. In particular, Krylov subspace methods apply techniques that involve orthogonal projections onto subspaces of the form

$$\mathcal{K}(\mathcal{A},b)\equiv\operatorname{span}\{b,\mathcal{A}b,\mathcal{A}^2b,...,\mathcal{A}^{n-1}b,...\}.$$

The conjugate gradient method (CG), minimum residual method (MINRES) and generalized minimal residual method (GMRES) are all common iterative Krylov subspace methods. The CG method is used for symmetric, positive definite matrices, MINRES for symmetric and possibly indefinite matrices and GMRES for unsymmetric matrices.

4. Numerical examples

In this section, we present one example [3] to illustrate the effectiveness of the parameter shift-splitting preconditioner for GMRES(m) method and MINRES to solve the linear systems (1.3) in the sense of iteration step (denoted as It_{GMRES}), elapsed CPU time in seconds (denoted as CPU), and relative residual error (denoted as Res_{GMRES}). All numerical examples are carried out in Matlab 7.0. In our experiments, all runs with respect to both GSS method and PSS method are started from initial vector $((x^{(0)})^T,(y^{(0)})^T)^T=0$, and terminated if the current iteration satisfy RES $:=\frac{\|b-Au^{(k)}\|_2}{\|b\|_2}<10^{-6}$.

Consider the linear system of equations (1.1) with

$$T=I\otimes V+V\otimes I \text{ and } \mathrm{W}=10(\mathrm{I}\otimes\mathrm{V_C}+\mathrm{V_C}\otimes\mathrm{I})+9(\mathrm{e_1e_l^T}+\mathrm{e_le_1^T})\otimes\mathrm{I},$$

where $V=\operatorname{tridiag}(-1,2,-1)\in\mathcal{R}^{l\times l},\mathrm{V_C}=\mathrm{V}-\mathrm{e_1e_l^T}-\mathrm{e_le_1^T}\in\mathcal{R}^{l\times l}$ and e_1 and e_l are the first and last unit vectors in $\mathcal{R}^l$, respectively. Here T and K correspond

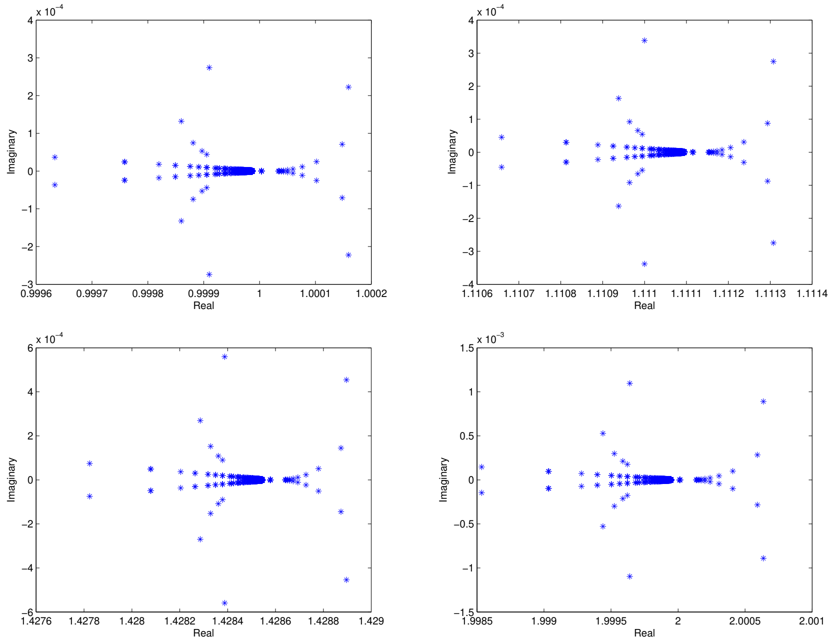

Figure 1. The eigenvalue distribution for the generalized shift-splitting preconditioned matrix $\mathcal{T}_{GSS}^{-1}\mathcal{A}$ when $\alpha = \beta = 0.001$(the first), the parameter shift-splitting preconditioned matrix $\mathcal{T}_{PSS}^{-1}\mathcal{A}$ when $\alpha = \beta = 0.001, \xi = 0.8$(the second),$\alpha = \beta = 0.001, \xi = 0.7$(the third) and $\alpha = \beta = 0.001, \xi = 0.5$(the fourth), respectively. Here, $l = 16$.

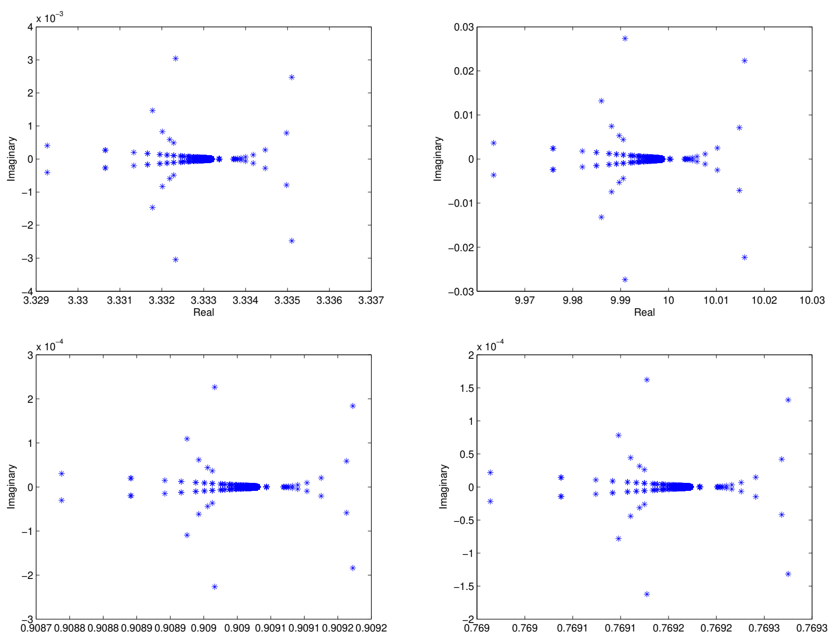

Figure 2. The eigenvalue distribution for the parameter shift-splitting preconditioned matrix $\mathcal{T}_{PSS}^{-1}\mathcal{A}$ when $\alpha = \beta = 0.001, \xi = 0.5$(the first),$\alpha = \beta = 0.001, \xi = 0.3$(the second), $\alpha = \beta = 0.001, \xi = 0.1$(the third) and $\alpha = \beta = 0.001, \xi = 1.3$(the fourth), respectively. Here, $l = 16$.

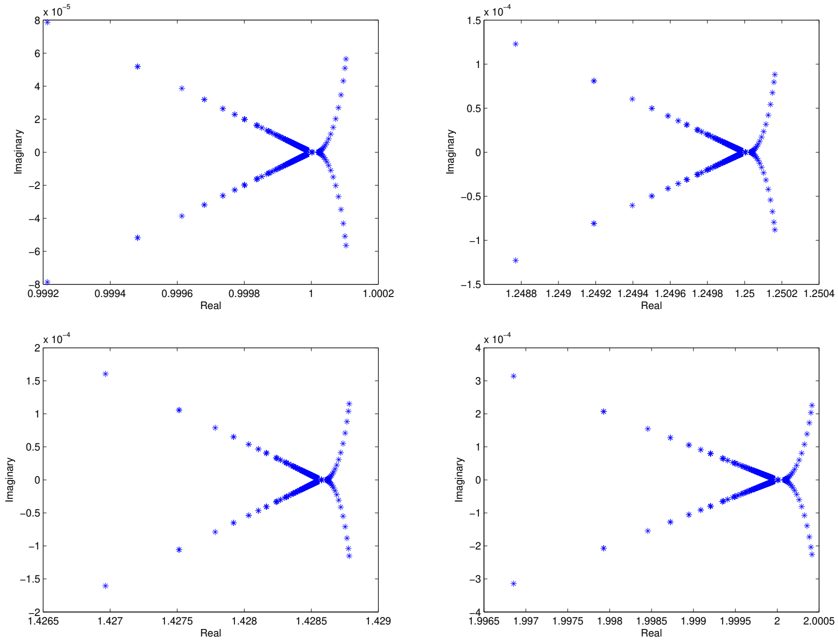

Figure 3. The eigenvalue distribution for the generalized shift-splitting preconditioned matrix $\mathcal{T}_{GSS}^{-1}\mathcal{A}$ when $\alpha = \beta = 0.001$(the first), the parameter shift-splitting preconditioned matrix $\mathcal{T}_{PSS}^{-1}\mathcal{A}$ when $\alpha = \beta = 0.001, \xi = 0.8$(the second),$\alpha = \beta = 0.001, \xi = 0.7$(the third) and $\alpha = \beta = 0.001, \xi = 0.5$(the fourth), respectively. Here, $l = 24$.

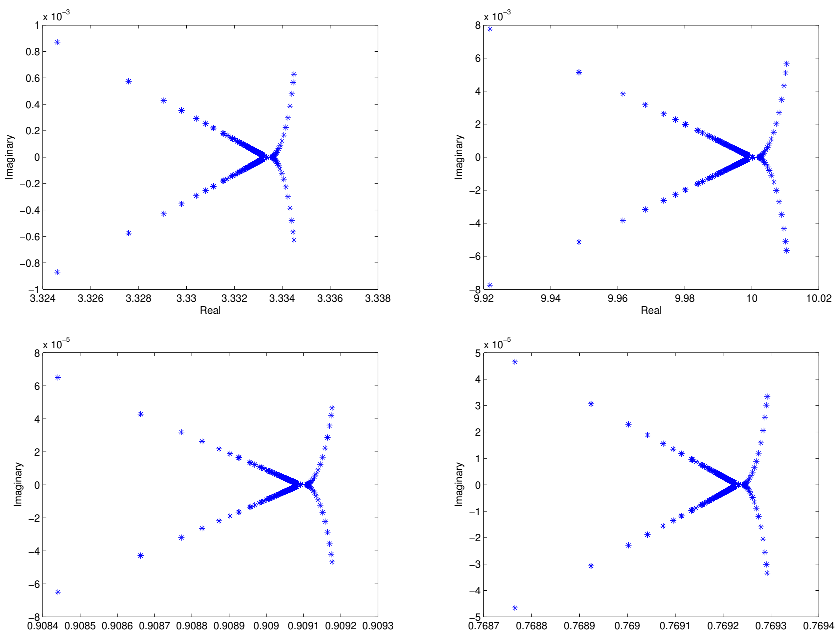

Figure 4. The eigenvalue distribution for the parameter shift-splitting preconditioned matrix $\mathcal{T}_{PSS}^{-1}\mathcal{A}$ when $\alpha = \beta = 0.001, \xi = 0.5$(the first),$\alpha = \beta = 0.001, \xi = 0.3$(the second), $\alpha = \beta = 0.001, \xi = 0.1$(the third) and $\alpha = \beta = 0.001, \xi = 1.3$(the fourth), respectively. Here, $l = 24$.

to the five-point centered difference matrices approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions and periodic boundary conditions, respectively, on a uniform mesh in the unit square $[0,1] \times [0,1]$ with the mesh-size $h = \frac{1}{l+1}$.

In Figs $1 \sim 4$, we report the eigenvalue distribution for the generalized shift-splitting preconditioned matrix $\mathcal{T}_{GSS}^{-1}\mathcal{A}$ and the parameter shift-splitting preconditioned matrix for different parameter, respectively. In Tables $1 \sim 2$, we report iteration counts, relative residual and cpu time about preconditioned matrices $\mathcal{T}_{GSS}^{-1}\mathcal{A}$ and $\mathcal{T}_{PSS}^{-1}\mathcal{A}$ with $l = 16$ and $l = 24$ when choosing different parameters. Figs $1 \sim 4$ and Tables $1 \sim 2$ show that the GSS preconditioner and PSS preconditioner have more clustered eigenvalue distribution when choosing different parameters.

Table 1. Iteration counts, relative residual and CPU time about preconditioned matrices $\mathcal{T}_{GSS}^{-1}\mathcal{A}$ and $\mathcal{T}_{PSS}^{-1}\mathcal{A}$ when choosing different parameters. Here, $l = 16$.

α	β	ξ	It_{GMRES}	Res_{GMRES}	CPU(s)
0.001	0.001	1	1(1)	4.0867×10^{-9}	0.105
0.001	0.001	0.8	1(1)	5.0454×10^{-9}	0.110
0.001	0.001	0.7	1(1)	8.3407×10^{-9}	0.104
0.001	0.001	0.5	1(1)	1.6350×10^{-8}	0.109
0.001	0.001	0.3	1(1)	4.5426×10^{-8}	0.103
0.001	0.001	0.1	1(1)	4.0932×10^{-7}	0.107
0.001	0.001	1.1	1(1)	3.3773×10^{-9}	0.103
0.001	0.001	1.3	1(1)	2.4180×10^{-9}	0.133

Table 2. Iteration counts, relative residual and CPU time about preconditioned matrices $\mathcal{T}_{GSS}^{-1}\mathcal{A}$ and $\mathcal{T}_{PSS}^{-1}\mathcal{A}$ when choosing different parameters. Here, $l = 24$.

α	β	ξ	It_{GMRES}	Res_{GMRES}	CPU(s)
0.001	0.001	1	2(1)	1.4192×10^{-8}	0.302
0.001	0.001	0.8	2(1)	1.774×10^{-8}	0.299
0.001	0.001	0.7	2(1)	2.0275×10^{-8}	0.306
0.001	0.001	0.5	2(1)	2.8385×10^{-8}	0.302
0.001	0.001	0.3	2(1)	4.7309×10^{-8}	0.306
0.001	0.001	0.1	2(1)	1.4195×10^{-7}	0.299
0.001	0.001	1.1	2(1)	1.2902×10^{-8}	0.301
0.001	0.001	1.3	2(1)	1.0917×10^{-8}	0.303

5. Conclusions

In this paper, based on the generalized shift-splitting (GSS) preconditioner [23], the author generalizes the corresponding algorithms and further studies the parameter shift-splitting (PSS) preconditioner for complex symmetric linear systems. Moreover, theoretical analysis shows the parameter shift-splitting iterative method is unconditionally convergent. In finally, one example is provided to confirm the effectiveness.

The authors declare that they have no competing interests.

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