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# HYERS-ULAM-RASSIAS STABILITY OF $\kappa$-CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

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**Abstract** The paper is connected with the existence of solutions and Hyers-Ulam stability for a class of nonlinear fractional differential equations with $\kappa$-Caputo fractional derivative in boundary value problems. The existence and uniqueness results are obtained by utilizing the Banach fixed point theorem and Leray-Schauder nonlinear alternative theorem. In addition, two sufficient conditions to guarantee the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability of boundary value problems of fractional differential equations are also presented. Finally, theoretical results are illustrated by two numerical examples.

**Keywords** Fractional differential equations, fixed point theorem, existence, Hyers-Ulam-Rassias stability.

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## 1. Introduction

In recent years, the theory of fractional differential equations has become an important reaserch topic, see [10,12,14,17,29]. Boundary value problems of fractional differential equations have various applications in science such as physics, chemistry, mechanics and engineering (see, e.g., [6,9,13,15]). Meanwhile, based on different kinds of analytical techniques, various results about the existence of solutions have been obtained, which can be found in [8,20,25,27]. In [30], the authors considered the existence of solutions to boundary value problems for fractional differential equations

$$\begin{cases} -D^{\alpha}w(t) = p(t)h(t,w(t)) - q(t), \quad 0 < t < 1, \\ w(0) = w'(0) = w(1) = 0, \end{cases}$$

where $D^{\alpha}$ is the standard Riemann-Liouville derivative, $2 < \alpha \leq 3$ is a real number, $q : (0,1) \to [0,\infty)$ is Lebesgue integrable and does not vanish identically on any subinterval of $(0,1)$. Also, they established the existence results by Krasnoselskii's

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fixed point theorem in a cone. Cui [5] studied the following boundary value problems

$$\begin{cases} D^{\alpha}w(t) + p(t)h(t, w(t)) + q(t) = 0, \quad 0 < t < 1, \\ w(0) = w'(0) = 0, \quad w(1) = 0, \end{cases}$$

where $D^{\alpha}$ and $\alpha$ satisfy the above conditions, $p : (0, 1) \to [0, \infty)$ is continuous and does not vanish identically on any subinterval of $(0, 1)$, $q : (0, 1) \to \mathbb{R}$ is continuous and Lebesgue integrable. Under the assumption that $h(t, w)$ is a Lipschitz continuous function, Cui [5] deduced that the Lipschitz constant is related to the first eigenvalues corresponding to the relevant operators. Badawi etc [3] studied the following boundary value problems

$$\begin{cases} {}^{c}D^{\alpha}w(t) + p(t)h(t, w(t)) + q(t) = 0, \quad 0 < t < 1, \\ w(0) = a, \quad w'(0) = b, \quad w(1) = d, \end{cases}$$

where ${}^{c}D^{\alpha}$ is a Caputo fractional derivative with $2 < \alpha \leq 3$, $a, b, d \in \mathbb{R}$ are constants, $p : (0, 1) \to [0, \infty)$ is continuous and does not vanish identically on any subinterval of $(0, 1)$, $q : (0, 1) \to \mathbb{R}$ is continuous and Lebesgue integrable. Under the assumption that the bounded conditions are constants, by means of the Banach contraction mapping principle and Larry-Schauder alternative theorem, Badawi etc [3] investigated the existence and uniqueness of solutions for the boundary value problems of the nonlinear fractional differential equations with a variable coefficient.

Recently, the Hyers-Ulam stability of differential equations has received much attention because it is quite significant in numerical analysis, economics, biology and other practical problems which are not easy to find exact solutions, see [4, 16, 18, 21, 22, 28]. Hyers-Ulam-Rassias stability is an extension of Hyers-Ulam stability, it relaxes the linear assumption used in Hyers-Ulam stability and allows for nonlinear perturbations. The study of Hyers-Ulam stability and Hyers-Ulam-Rassias stability contribute to the development of new mathematical tools and concepts that have proved useful in solving some otherwise thorny problems in these fields.

In this paper we study the following boundary value problems for fractional differential equations (BVP in short)

$$\begin{cases} {}^{c}D^{\alpha;\kappa}w(t) + p(t)h(t, w(t)) + q(t) = 0, \quad 0 < t < 1, \\ w(0) = a, \quad w'(0) = b, \quad w(1) = d, \end{cases} \tag{1.1}$$

where ${}^{c}D^{\alpha;\kappa}$ is a $\kappa$-Caputo fractional derivative with $2 < \alpha \leq 3$ (it is also called the Caputo-type fractional derivatives with respect to the function $\kappa$, see Definition 2.2 below), $a, b, d \in \mathbb{R}$ are constants, $q : (0, 1) \to \mathbb{R}$ is continuous and Lebesgue integrable, $p : (0, 1) \to [0, \infty)$ is continuous and does not vanish identically on any subinterval of $(0, 1)$.

The rest of the present paper is organized as follows. In section 2, we recall some useful fixed point theorems and give the solution to boundary value problems by virtue of Green's function. In section 3, we use Leray-Schauder alterative theorem and Banach fixed point theorem to explore the existence and uniqueness of the solution. In section 4, we study the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability of nonlinear fractional differential equations. Our theoretical results are illustrated by two numerical examples in section 5.

## 2. Preliminaries

We recall some concepts concerning to the fractional integrals (FI) and fractional derivatives (FD) of a function $w$ with respect to another function $\kappa$. Let $[a,T] \subset \mathbb{R}$ such that $0 \leq a < T$. Denote by $C([a,T],\mathbb{R})$ the space of continuous functions $w$ on $[a,T]$ with the norm defined by $\|w\| = \sup_{t\in[a,T]} |w(t)|$. For convenience, throughout the paper, we choose the notation $\mathcal{K}_\kappa$ to express the set of the functions $\kappa : [a,T] \to \mathbb{R}_+$ satisfying the properties as follows: $\kappa$ is an increasing, positive, and differentiable function such that $\kappa'(t) \neq 0$, for all $t \in (a,T)$.

**Definition 2.1.** [1] Let $\alpha > 0$ and $n \in \mathbb{N} = [\alpha] + 1$. The Riemann-Liouville FD with respect to $\kappa$ (or the $\kappa$-Riemann-Liouville FD) with the order $\alpha$ of $w$ is defined by

$$ {}^{RL}D^{\alpha;\kappa}_{a^+} w(t) := \left(\frac{1}{\kappa'(t)}\frac{d}{dt}\right)^n J^{n-\alpha;\kappa}_{a^+} w(t), \tag{2.1}$$

where $J^{\alpha;\kappa}_{a^+} w(t)$ denotes the fractional integral of $w$ with respect to $\kappa$ with the order $\alpha$ on $[a,T]$ defined as follows

$$J^{\alpha;\kappa}_{a^+} w(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1} w(s)ds,$$

where

$$n = \begin{cases} [\alpha]+1, & \text{if } \alpha \notin \mathbb{N}, \\ \alpha, & \text{if } \alpha \in \mathbb{N}. \end{cases} \tag{2.2}$$

In this paper, the fractional derivative that relates to our work is a Caputo-type operator defined as the below.

**Definition 2.2.** [2] Let $\alpha > 0$ and $n \in \mathbb{N}$. The Caputo-type FD with respect to $\kappa$ (or the $\kappa$-Caputo FD) with the order $\alpha$ of $w$ is defined by

$$ {}^{c}D^{\alpha;\kappa}_{a^+} w(t) := {}^{RL}D^{\alpha;\kappa} \left[ w(t) - \sum_{i=0}^{n-1} \frac{w^{[i]}_\kappa(a)}{i!} (\kappa(t)-\kappa(a))^i \right], \tag{2.3}$$

where $w \in C^{n-1}([a,T],\mathbb{R})$, ${}^{RL}D^{\alpha;\kappa}_{a^+} w(t)$ exists and

$$w^{[i]}_\kappa(t) := \left(\frac{1}{\kappa'(t)}\frac{d}{dt}\right)^i w(t).$$

It follows from Theorem 3 in Almeida [2] that for any $w \in C^n([a,T],\mathbb{R})$, the $\kappa$-Caputo FD of $w$ is given by

$$\begin{aligned} {}^{c}D^{\alpha;\kappa}_{a^+} w(t) &= J^{n-\alpha;\kappa}_{a^+} \left(\frac{1}{\kappa'(t)}\frac{d}{dt}\right)^n w(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1} w^{[n]}_\kappa(s)ds. \end{aligned}$$

Some properties of the FI and the $\kappa$-Caputo FD are provided below.

**Theorem 2.1.** [23] *Let $\alpha > 0$. The following assertions hold: (i) If the function $w : [a,T] \to \mathbb{R}$ is continuous, then*

$$ {}^{c}D_{a^+}^{\alpha;\kappa} J_{a^+}^{\alpha;\kappa} w(t) = w(t). $$

*(ii) If $w \in C^{n-1}([a,T],\mathbb{R})$ and ${}^{RL}D_{a^+}^{\alpha;\kappa} w(t)$ exists, then*

$$ J_{a^+}^{\alpha;\kappa} D_{a^+}^{\alpha;\kappa} w(t) = w(t) - \sum_{i=0}^{n-1} \frac{w_k^{[i]}(a)}{i!} (\kappa(t)-\kappa(a))^i. $$

*(iii) If $\mu(t) = (\kappa(t)-\kappa(a))^{\beta}$ and $v(t) = E_{\alpha,1}\left(\lambda(\kappa(t)-\kappa(a))^{\alpha}\right)$, then*

$$ \begin{aligned} J_{a^+}^{\alpha;\kappa} \mu(t) &= \frac{\Gamma(1+\beta)}{\Gamma(\beta+\alpha+1)} (\kappa(t)-\kappa(a))^{\beta+\alpha}, \\ J_{a^+}^{\alpha;\kappa} v(t) &= \frac{1}{\lambda}\left[E_{\alpha,1}\left(\lambda(\kappa(t)-\kappa(a))^{\alpha}\right) - 1\right], \\ {}^{c}D_{a^+}^{\alpha;\kappa} \mu(t) &= \frac{\Gamma(1+\beta)}{\Gamma(\beta-\alpha+1)} (\kappa(t)-\kappa(a))^{\beta-\alpha}, \\ {}^{c}D_{a^+}^{\alpha;\kappa} v(t) &= \lambda E_{\alpha,1}\left(\lambda(\kappa(t)-\kappa(a))^{\alpha}\right). \end{aligned} $$

Next we introduce Leray-Schauder nonlinear alternative theorem to prove the existence of the solution of (1.1).

**Lemma 2.1.** ( [19] Leray-Schauder nonlinear alternative). *Let $F$ be a Banach space and $C$ be a closed, convex subset of $F$. $\Omega$ is an open subset of $C$ and $0 \in \Omega$. Suppose $T : \bar{\Omega} \to C$ is a continuous, compact map (that is, $T(\Omega)$ is a relatively compact subset of $C$). Then, either*
*(i) $T$ has a fixed point in $\Omega$, or*
*(ii) there is a function $w \in \partial\Omega$ and $\lambda \in (0,1)$ such that $w = \lambda T(w)$.*

Now, we give the expresssion for the solution of the corresponding linear fractional differential equations of (1.1) by means of Green's function.

**Lemma 2.2.** *Let $a, b, d \in \mathbb{R}$, $y \in C[0,1]$ and $2 < \alpha \leq 3$. The unique solution of the boundary value problems*

$$ \begin{cases} {}^{c}D^{\alpha;\kappa} w(t) + y(t) = 0, \quad 0 < t < 1, \\ w(0) = a, \quad w'(0) = b, \quad w(1) = d, \end{cases} \tag{2.4} $$

*is given by*

$$ w(t) = A(t) + \int_0^1 G(t,s) y(s) ds, \tag{2.5} $$

*where $A(t) = a + \frac{\kappa(t)-\kappa(0)}{\kappa'(0)} b + \frac{(\kappa(t)-\kappa(0))^2}{(\kappa(1)-\kappa(0))^2}\left[d - a - \frac{\kappa(1)-\kappa(0)}{\kappa'(0)} b\right]$, $G(t,s)$ is a Green's function given by*

$$ G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} m(t)\kappa'(s)(\kappa(1)-\kappa(s))^{\alpha-1} - \kappa'(s)\left(\kappa(t)-\kappa(s)\right)^{\alpha-1}, 0 \leq s \leq t \leq 1, \\ m(t)\kappa'(s)(\kappa(1)-\kappa(s))^{\alpha-1}, \quad 0 \leq t \leq s \leq 1, \end{cases} $$

*where $m(t) = \frac{(\kappa(t)-\kappa(0))^2}{(\kappa(1)-\kappa(0))^2}$.*

**Proof.** Applying Theorem 2.1, the equation (2.4) is equivalent to the integral equation

$$w(t) = w(0) + \frac{w'(0)}{\kappa'(0)}(\kappa(t) - \kappa(0)) + C_1(\kappa(t) - \kappa(0))^2 - J^{\alpha;\kappa} y(t). \tag{2.6}$$

Substituting $w(0) = a, w'(0) = b$ and $w(1) = d$ into equation (2.6), we obtain

$$d = a + \frac{b}{\kappa'(0)}(\kappa(1) - \kappa(0)) + C_1(\kappa(1) - \kappa(0))^2 - J^{\alpha;\kappa} y(1),$$

and then

$$C_1 = \frac{d - a - \frac{b}{\kappa'(0)}(\kappa(1) - \kappa(0)) + J^{\alpha;\kappa} y(1)}{(\kappa(1) - \kappa(0))^2}.$$

Substituting $C_1$ into equation (2.6), denote $m(t) = \frac{(\kappa(t)-\kappa(0))^2}{(\kappa(1)-\kappa(0))^2}$, we obtain

$$\begin{aligned} w(t) &= a + \frac{\kappa(t) - \kappa(0)}{\kappa'(0)} b + m(t) \left[ d - a - \frac{\kappa(1) - \kappa(0)}{\kappa'(0)} b + J^{\alpha;\kappa} y(1) \right] - J^{\alpha;\kappa} y(t) \\ &= a + \frac{\kappa(t) - \kappa(0)}{\kappa'(0)} b + m(t) \left[ d - a - \frac{\kappa(1) - \kappa(0)}{\kappa'(0)} b \right] \\ &\quad + m(t) J^{\alpha;\kappa} y(1) - J^{\alpha;\kappa} y(t). \end{aligned} \tag{2.7}$$

Thus, we have

$$w(t) = A(t) + \int_0^1 G(t, s) y(s) ds.$$

The proof is completed. □

We recall the definitions of the Hyers-Ulam stability (HU-stability) and the Hyers-Ulam-Rassias stability (HUR-stability) of BVP (1.1).

**Definition 2.3.** [24] BVP (1.1) is called HU-stable if there exists a constant $C_h > 0$ such that for each $\varepsilon > 0$ and for each solution $\hat{w} \in C^1([a, T], \mathbb{R})$ of the following inequality

$$|{}^c D^{\alpha;\kappa} \hat{w}(t) + p(t) h(t, \hat{w}(t)) + q(t)| \le \varepsilon, \quad \forall t \in [a, T],$$

there exists a solution $w \in C^1([a, T], \mathbb{R})$ of BVP (1.1) satisfying

$$|w(t) - \hat{w}(t)| \le C_h \varepsilon, \quad \forall t \in [a, T].$$

**Definition 2.4.** [26] BVP (1.1) is called HUR-stable with respect to $\varphi \in C([a, T], \mathbb{R})$ if there exists a constant $C_{h,\varphi} > 0$ such that for each $\varepsilon > 0$ and each solution $\hat{w} \in C^1([a, T], \mathbb{R})$ of the inequality

$$|{}^c D^{\alpha;\kappa} \hat{w}(t) + p(t) h(t, \hat{w}(t)) + q(t)| \le \varepsilon \varphi(t), \quad \forall t \in [a, T], \tag{2.8}$$

there exits a solution $w \in C^1([a, T], \mathbb{R})$ of BVP (1.1) satisfying the estimate

$$|w(t) - \hat{w}(t)| \le C_{h,\varphi} \varepsilon \varphi(t), \quad \forall t \in [a, T].$$

Here, we observe that HUR-stable in Definition 2.4 implies HU-stable in Definition 2.3 if $\varphi(t) = 1$. Similar to Remark 4.6 in [7], Remark 2.11 and Remark 2.9 in [26], we also receive some remarks.

**Remark 2.1.** A function $w \in C^1([a, T], \mathbb{R})$ is a solution of BVP (1.1) if and only if there exists a function $\bar{w} \in C^1([a, T], \mathbb{R})$ such that:
(i) $|\bar{w}(t)| \le \varepsilon$ (or $|\tilde{w}(t)| \le \varepsilon \varphi(t)$);
(ii) ${}^c D^{\alpha;\kappa} w(t) + p(t) h(t, w(t)) + q(t) = \bar{w}(t), \forall t \in [a, T]$.

# 3. Existence results

In this section, we study the existence of solutions to BVP (1.1). Let us denote by $C([a,T],\mathbb{R})$ the Banach space of all continuous functions $w:[0,1]\to\mathbb{R}$ endowed with supremum norm $\|w\| = \max_{t\in[0,1]}|w(t)|$. According to equation (2.5), we can easily give the definition of the solution of BVP (1.1).

**Definition 3.1.** A function $w:[0,1]\to\mathbb{R}$ is said to be a solution to BVP (1.1), if $w$ satisfies

$$w(t) = A(t) + \int_0^1 G(t,s)[p(s)h(s,w(s)) + q(s)]ds \tag{3.1}$$

for $t\in[0,1]$.

Define an operator $T: C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ by

$$Tw(t) = A(t) + \int_0^1 G(t,s)[p(s)h(s,w(s)) + q(s)]ds$$

for $w\in C([0,1],\mathbb{R})$ and $t\in[0,1]$. Then, we transform the existence of solutions to the fixed point problem. We first list the following hypotheses.
(H1) $h:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous.
(H2) There exists a nonnegative function $g\in L^1([0,1],\mathbb{R}_+)$ such that

$$|h(t,w) - h(t,v)| \le g(t)|w-v|$$

for all $w,v\in\mathbb{R}$ and $t\in[0,1]$.
(H3) There exists a nonnegative function $\phi\in L^p([0,1],\mathbb{R}_+)$ where $p>1$, and a continuous nondecreasing function $\psi:[0,\infty)\to(0,\infty)$ such that

$$|h(t,w)| \le \phi(t)\psi(|w|)$$

for all $(t,w)\in[0,1]\times\mathbb{R}$.

**Theorem 3.1.** *Suppose that the conditions (H1) and (H2) are satisfied. If*

$$N = \max_{0\le t\le 1}\left|\int_0^1 G(t,s)p(s)g(s)ds\right| < 1, \tag{3.2}$$

*then the BVP* (1.1) *has a unique solution in* $C([0,1],\mathbb{R})$.

**Proof.** Consider the operator $T: C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ defined by (3). Then, $w\in C([0,1],\mathbb{R})$ is a solution to the BVP (1.1) if and only if $w$ is a fixed point of $T$. For $w,v\in C([0,1],\mathbb{R})$, applying condition (H2), we obtain

$$\begin{aligned}|Tw(t) - Tv(t)| &\le \int_0^1 G(t,s)|p(s)[h(s,w(s)) - h(s,v(s))]|ds \\ &\le \max_{0\le t\le 1}|w(t)-v(t)|\int_0^1 G(t,s)|p(s)g(s)|ds \\ &\le \|w-v\|\int_0^1 G(t,s)|p(s)g(s)|ds.\end{aligned}$$

Hence, $\|Tw - Tv\| \le N\|w-v\|$. The assumption (3.2) shows that $T$ is a contraction. By Banach fixed point theorem, $T$ has a unique fixed point in $C([0,1],\mathbb{R})$, which is the solution to the BVP (1.1). The proof is completed. □

Next, we prove an existence result by using Larry-Schouder nonlinear alternative theorem. For simplicity, let

$$\begin{aligned} k &= \max_{0\le t\le 1}|A(t)| \\ &= \left|a + \frac{\kappa(1)-\kappa(0)}{\kappa'(0)}b + \frac{(\kappa(1)-\kappa(0))^2}{(\kappa(1)-\kappa(0))^2}\left(d - a - \frac{\kappa(1)-\kappa(0)}{\kappa'(0)}b\right)\right| \\ &= \left|a + \frac{\kappa(1)-\kappa(0)}{\kappa'(0)}b + d - a - \frac{\kappa(1)-\kappa(0)}{\kappa'(0)}b\right| \\ &= |d|, \end{aligned}$$

$M_1 = \max_{0\le t\le 1}\left|\int_0^1 G(t,s)p(s)\phi(s)ds\right|$, $M_2 = \max_{0\le t\le 1}\left|\int_0^1 G(t,s)q(s)ds\right| + k$, and $M = \max\{M_1, M_2\}$, where $\phi$ is the function appearing in condition (H3).

We first verify a lemma for the sake of the proof of our theorem.

**Lemma 3.1.** *For all $t_1, t_2 \in [0,1]$, $t_1 < t_2$, $\int_0^1 |G(t_1,s)-G(t_2,s)|ds \to 0$ as $t_1 - t_2 \to 0$.*

**Proof.** For all $t_1, t_2 \in [0,1], t_1 < t_2$, we have

$$\begin{aligned} &\int_0^1 |G(t_1,s)-G(t_2,s)|ds \\ =&\int_0^{t_1} |G(t_1,s)-G(t_2,s)|ds + \int_{t_1}^{t_2} |G(t_1,s)-G(t_2,s)|ds \\ &+ \int_{t_2}^1 |G(t_1,s)-G(t_2,s)|ds \\ =&\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|\frac{(\kappa(t_1)-\kappa(0))^2}{(\kappa(1)-\kappa(0))^2}(\kappa(1)-\kappa(s))^{\alpha-1} - \frac{(\kappa(t_2)-\kappa(0))^2}{(\kappa(1)-\kappa(0))^2}(\kappa(1)-\kappa(s))^{\alpha-1}\right. \\ &\left. -(\kappa(t_1)-\kappa(s))^{\alpha-1} + (\kappa(t_2)-\kappa(s))^{\alpha-1}\right| d\kappa(s) \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left|\frac{(\kappa(t_1)-\kappa(0))^2}{(\kappa(1)-\kappa(0))^2} - \frac{(\kappa(t_2)-\kappa(0))^2}{(\kappa(1)-\kappa(0))^2}\right|(\kappa(1)-\kappa(s))^{\alpha-1}d\kappa(s) \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(\kappa(t_2)-\kappa(s))^{\alpha-1}d\kappa(s) \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_2}^{1}\left|\frac{(\kappa(t_1)-\kappa(0))^2}{(\kappa(1)-\kappa(0))^2} - \frac{(\kappa(t_2)-\kappa(0))^2}{(\kappa(1)-\kappa(0))^2}\right|(\kappa(1)-\kappa(s))^{\alpha-1}d\kappa(s) \\ =&\frac{1}{\Gamma(\alpha)}\int_0^{1}\left|\frac{(\kappa(t_1)-\kappa(0))^2}{(\kappa(1)-\kappa(0))^2} - \frac{(\kappa(t_2)-\kappa(0))^2}{(\kappa(1)-\kappa(0))^2}\right|(\kappa(1)-\kappa(s))^{\alpha-1}d\kappa(s) \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^{t_2}(\kappa(t_2)-\kappa(s))^{\alpha-1}d\kappa(s) - \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(\kappa(t_1)-\kappa(s))^{\alpha-1}d\kappa(s) \\ =&\frac{1}{\Gamma(\alpha+1)}\left|\frac{(\kappa(t_1)-\kappa(0))^2}{(\kappa(1)-\kappa(0))^2} - \frac{(\kappa(t_2)-\kappa(0))^2}{(\kappa(1)-\kappa(0))^2}\right|[\kappa(1)]^\alpha \\ &+ \frac{1}{\Gamma(\alpha+1)}\left|(\kappa(t_2)-\kappa(0))^\alpha - (\kappa(t_1)-\kappa(0))^\alpha\right|. \end{aligned}$$

It is easy to see that

$$\left|\frac{(\kappa(t_1)-\kappa(0))^2}{(\kappa(1)-\kappa(0))^2}-\frac{(\kappa(t_2)-\kappa(0))^2}{(\kappa(1)-\kappa(0))^2}\right|\to 0, \qquad |(\kappa(t_2)-\kappa(0))^\alpha-(\kappa(t_1)-\kappa(0))^\alpha|\to 0,$$

as $t_1-t_2\to 0$. Therefore, we have $\int_0^1|G(t_1,s)-G(t_2,s)|ds\to 0$. □

**Theorem 3.2.** *Suppose that (H1) and (H3) are satisfied. If*

$$\limsup_{r\to+\infty} M\frac{\psi(r)}{r}<1,$$

*then the BVP* (1.1) *has at least one solution in* $C([0,1],\mathbb{R})$.

**Proof.** Firstly, let us prove that $T$ is completely continuous. It is obvious that $T$ is continuous since $h$ and $G$ are continuous. There exists a number $r>0$ such that $M(\psi(r)+1)<r$ since $\limsup_{r\to+\infty} M\frac{\psi(r)}{r}<1$. Let $B_r=\{w\in C([0,1],\mathbb{R}):\|w\|\le r\}$. Then $B_r$ is a bounded subset in $C([0,1],\mathbb{R})$. For any $w\in B_r$, we have

$$\begin{aligned}
|Tw(t)| &= \left|\int_0^1 G(t,s)[p(s)h(s,w(s))+q(s)]ds+A(t)\right|\\
&\le \int_0^1 |G(t,s)|[|p(s)|\phi(s)\psi(||w||)+|q(s)|]ds+|A(t)|\\
&\le \psi(r)\int_0^1 |G(t,s)|ds|p(s)|\phi(s)+\int_0^1|G(t,s)|ds|q(s)|+|A(t)|\\
&\le \psi(r)\int_0^1 |G(t,s)|ds|p(s)|\phi(s)+\int_0^1|G(t,s)|ds|q(s)|+k\\
&\le (\psi(r)M_1+M_2)\\
&\le M(\psi(r)+1).
\end{aligned}$$

Hence $T(B_r)$ is uniformly bounded. For all $w\in B_r$ and $t_1,t_2\in[0,1], t_1<t_2$, we have

$$\begin{aligned}
|Tw(t_1)-Tw(t_2)| \le& \int_0^1 |G(t_1,s)-G(t_2,s)||p(s)h(s,w(s))+q(s)|ds\\
&+|A(t_1)-A(t_2)|\\
\le& \int_0^1 |G(t_1,s)-G(t_2,s)||p(s)|\phi(s)\psi(\|w\|)ds\\
&+\int_0^1 |G(t_1,s)-G(t_2,s)||q(s)|ds+|A(t_1)-A(t_2)|\\
\le&\|p\|\psi(r)\int_0^1 |G(t_1,s)-G(t_2,s)|\phi(s)ds\\
&+\|q\|\int_0^1 |G(t_1,s)-G(t_2,s)|ds+|A(t_1)-A(t_2)|.
\end{aligned}$$

We can get the following equality by use of Hölder inequality

$$\int_0^1 |G(t_1,s)-G(t_2,s)|\phi(s)ds$$

$$
\begin{aligned}
&\leq \int_0^1 |(m(t_1)-m(t_2))(\kappa(1)-\kappa(s))^{\alpha-1}\phi(s)|d\kappa(s)\\
&\quad + \int_0^{t_2} \left|(\kappa(t_2)-\kappa(s))^{\alpha-1}\phi(s)\right| d\kappa(s) - \int_0^{t_1} \left|(\kappa(t_1)-\kappa(s))^{\alpha-1}\phi(s)\right| d\kappa(s)\\
&\leq |m(t_1)-m(t_2)| \left(\int_0^1 (\kappa(1)-\kappa(s))^{(\alpha-1)q} d\kappa(s)\right)^{\frac{1}{q}} \left(\int_0^1 \phi^p(\kappa(s))d\kappa(s)\right)^{\frac{1}{p}}\\
&\quad + \left(\int_0^{t_2} (\kappa(t_2)-\kappa(s))^{(\alpha-1)q} d\kappa(s)\right)^{\frac{1}{q}} \left(\int_0^{t_2} \phi^p(\kappa(s))d\kappa(s)\right)^{\frac{1}{p}}\\
&\quad - \left(\int_0^{t_1} (\kappa(t_2)-\kappa(s))^{(\alpha-1)q} d\kappa(s)\right)^{\frac{1}{q}} \left(\int_0^{t_1} \phi^p(\kappa(s))d\kappa(s)\right)^{\frac{1}{p}}\\
&\leq \frac{\|\phi\|_p |m(t_1)-m(t_2)|}{(1+(\alpha-1)q)^{\frac{1}{q}}} (\kappa(1)-\kappa(0))^{\frac{1+(\alpha-1)q}{q}}\\
&\quad + \frac{\|\phi\|_p}{(1+(\alpha-1)q)^{\frac{1}{q}}} \left((\kappa(t_2)-\kappa(0))^{\frac{1+(\alpha-1)q}{q}} - (\kappa(t_1)-\kappa(0))^{\frac{1+(\alpha-1)q}{q}}\right),
\end{aligned}
$$

where $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $m(t) = \frac{(\kappa(t)-\kappa(0))^2}{(\kappa(1)-\kappa(0))^2}$. It is easy to see that $|Tw(t_1) - Tw(t_2)| \to 0$ as $t_1 - t_2 \to 0$ due to Lemma 3.1, and the convergence is independent of $w \in B_r$. This show that $T(B_r)$ is equicontinuous. By Arzela-Ascolli theorem, we deduce that $T$ is completely continuous.

Now, let $\Omega = \{w \in B_r : \|w\| < r\}$. Then, $\Omega$ is an open and bounded subset in $B_r$ and $0 \in \Omega$. If there is a $w \in \partial\Omega$ such that $w = \lambda Tw$ for some $\lambda \in (0, 1)$ and for each $t \in [0, 1]$, then we have

$$|w(t)| = \lambda|Tw(t)| \leq |Tw(t)| \leq M(\psi(r) + 1) < r.$$

This is contradict to the fact that $w \in \partial\Omega$. Hence Lemma 2.1 (Leray-Schauder nonlinear alternative) allows us to conclude that $T$ has a fixed point $w^* \in \bar{\Omega}$. Therefore the BVP (1.1) has at least a solution $w^* \in B_r$. This completes the proof. □

In our next theorem, we replace condition (H3) with anothor condition.

(H4) There exists positive functions $a_1, a_2 \in C[0, 1]$ such that

$$|h(t, w)| \leq a_1(t) + a_2(t)|w|$$

for all $t \in [0, 1]$.

Similar to Theorem 3.2 above, we let $B = \max_{0\leq t\leq 1} \left|\int_0^1 G(t, s)p(s)a_2(s)ds\right|$, $A = \max_{0\leq t\leq 1} \left|\int_0^1 G(t, s)a_1(s)ds\right| + \max_{0\leq t\leq 1} \left|\int_0^1 G(t, s)q(s)ds\right| + k$, and $k = ||A(t)|| = |d|$, where $a_1$ and $a_2$ are the functions appearing in condition (H4).

**Theorem 3.3.** *Assume that conditions (H1) and (H4) hold. Suppose that* $0 < B < 1$. *Then, there exists a solution of BVP* (1.1).

**Proof.** Let us prove that $T$ is completely continuous firstly. It is clear that $T$ is continuous since $h$ and $G$ are continuous. Let $U = \{w \in C([0, 1], \mathbb{R}) : \|w\| < R\}$, where $R = \frac{A}{1-B}$ or $R = RB + A$. Then,

$$\begin{aligned}
|Tw(t)| &= \left| \int_0^1 G(t,s)[p(s)h(s,w(s)) + q(s)]ds + A(t) \right| \\
&\leq \int_0^1 |G(t,s)|[|p(s)|(a_1(s) + a_2(s)|w|) + |q(s)|]ds + |A(t)| \\
&\leq \|w\| \int_0^1 |G(t,s)p(s)a_2(s)|ds + \int_0^1 |G(t,s)a_1(s)|ds \\
&\quad + \int_0^1 |G(t,s)q(s)|ds + k \\
&\leq \|w\|B + A \\
&\leq RB + A.
\end{aligned}$$

Hence $TU$ is uniformly bounded. For all $t_1, t_2 \in [0,1], t_1 < t_2$ and $w \in U$, we have

$$\begin{aligned}
&|Tw(t_1) - Tw(t_2)| \\
\leq &\int_0^1 |G(t_1,s) - G(t_2,s)||p(s)h(s,w(s)) + q(s)|ds + |A(t_1) - A(t_2)| \\
\leq &\int_0^1 |G(t_1,s) - G(t_2,s)||p(s)|(a_1(s) + a_2(s)\|w\|)ds \\
&+ \int_0^1 |G(t_1,s) - G(t_2,s)||q(s)|ds + |A(t_1) - A(t_2)| \\
\leq &\|p\|R \int_0^1 |G(t_1,s) - G(t_2,s)|a_2(s)ds + \|p\| \int_0^1 |G(t_1,s) - G(t_2,s)|a_1(s)ds \\
&+ \int_0^1 |G(t_1,s) - G(t_2,s)||q(s)|ds + |A(t_1) - A(t_2)| \\
\leq &(\|pa_2\|R + \|pa_1\| + \|q\|) \int_0^1 |G(t_1,s) - G(t_2,s)|ds + |A(t_1) - A(t_2)|.
\end{aligned}$$

It is easy to see that $|Tw(t_1) - Tw(t_2)| \to 0$ as $t_1 - t_2 \to 0$ due to Lemma 3.1. Thus $|Tw(t_1) - Tw(t_2)| \to 0$, and the convergence is independent of $w \in U$. This show that $TU$ is equicontinuous. By Arzela-Ascolli theorem, we deduce that $T$ is completely continuous.

Now let $\Omega = \{w \in B : \|w\| < R\}$. Then, $\Omega$ is an open and bounded subset in $B$ and $0 \in \Omega$. If there is a $w \in \partial\Omega$ such that $w = \lambda Tw$ for some $\lambda \in (0,1)$ and each $t \in [0,1]$, then for this $w$ and $\lambda$ we have

$$R = \|w\| = \lambda\|Tw\| < \|Tw\| \leq A + B\|w\| \leq RB + A = R,$$

which is a contradiction. By Lemma 2.1 (Leray-Schauder nonlinear alternative), there exists a fixed point $w \in \bar{\Omega}$ of $T$. This fixed point is a solution of BVP (1.1) and the proof is complete. □

As a special case of $\kappa$-type fractional integrals and fractional derivatives, Hadamard type fractional integrals and fractional derivatives are given.

**Definition 3.2.** [11] Let $(a,b)(0 \leq a < b \leq \infty)$ be a finite or infinite interval of the half-axis $\mathbb{R}^+$, and the left-sided and right-sided Hadamard fractional integral of order $\alpha > 0$ is defined by

$$J_{a+}^{\alpha;H} w(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{w(s)ds}{s} \quad (a < t < b),$$

and

$$J_{b-}^{\alpha;H} w(t) := \frac{1}{\Gamma(\alpha)} \int_t^b \left(\log \frac{s}{t}\right)^{\alpha-1} \frac{w(s)ds}{s} \quad (a < t < b).$$

**Definition 3.3.** [11] Provided that the integral in Definition 3.2 exists. Let $\delta = td/dt$. The left-sided and right-sided Hadamard fractional derivatives of order $\alpha > 0$ on $(a,b)$ are defined by

$${}^cD_{a+}^{\alpha;H} w(t) := \delta^n J_{a+}^{n-\alpha;H} w(t) = \left(t\frac{d}{dt}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha+1} \frac{w(s)ds}{s},$$

and

$${}^cD_{b-}^{\alpha;H} w(t) := (-\delta)^n J_{b-\alpha}^{n-\alpha;H} w(t) = \left(-t\frac{d}{dt}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_t^b \left(\log \frac{s}{t}\right)^{n-\alpha+1} \frac{w(s)ds}{s},$$

where $n = [\alpha] + 1$ and $a < t < b$.

**Remark 3.1.** Let $\kappa(t) = log(t)$, then $\kappa$-type fractional integrals and fractional derivatives turn into Hadamard type fractional integrals and fractional derivatives. We obtain the following boundary value problems for fractional differential equation

$$\begin{cases} {}^cD^{\alpha;H} w(t) + p(t)h(t,w(t)) + q(t) = 0, \quad 0 < t < 1, \\ w(0) = a, \quad w'(0) = b, \quad w(1) = d, \end{cases} \tag{3.3}$$

where ${}^cD^{\alpha;H}$ is Hadamard fractional derivatives with $2 < \alpha \leq 3$, $a, b, d \in \mathbb{R}$ are constants, $q : (0,1) \to \mathbb{R}$ is continuous and Lebesgue integrable and $p : (0,1) \to [0,\infty)$ is continuous and does not vanish identically on any subinterval of $(0,1)$. It is obvious that we can get corollarys below.

**Corollary 3.1.** *Suppose that the condition (H1) and (H2) are satisfied. If*

$$N = \max_{0 \leq t \leq 1} \left| \int_0^1 G(t,s)p(s)ds \right| < 1$$

*then the BVP* (3.3) *has a unique solution in* $C([0,1],\mathbb{R})$.

**Corollary 3.2.** *Suppose that (H1) and (H3) are satisfied. If*

$$\limsup_{r \to +\infty} M \frac{\psi(r)}{r} < 1,$$

*then the BVP* (3.3) *has at least one solution in* $C([0,1],\mathbb{R})$.

**Corollary 3.3.** *Assume that conditions (H1) and (H4) hold. Suppose that* $0 < B < 1$. *Then, there exists a solution of BVP* (3.3).

# 4. HU-stability and HUR-stability

In this section, the analysis of HU-stability and HUR-stability of the fractional differential equation (1.1) is presented.

**Theorem 4.1.** *Assume that the conditions of Theorem 3.1 are satisfied and the inequality* (2.3) *has at least one solution. Then, the BVP* (1.1) *is HU-stable.*

**Proof.** For each $\varepsilon > 0$, and each function $\hat{w}$ that satisfies the following inequalities

$$|{}^cD^{\alpha;\kappa}\hat{w}(t) + p(t)h(t,\hat{w}(t)) + q(t)| \le \varepsilon, \quad \forall t \in [0,1],$$

a function $\bar{w}(t) = {}^cD^{\alpha;\kappa}\hat{w}(t) + p(t)h(t,\hat{w}(t)) + q(t)$ can be found. Then, we have $|\bar{w}(t)| \le \varepsilon$, which implies that

$$J^{\alpha;\kappa}\bar{w}(t) = \hat{w}(t) - A(t) - \int_0^1 G(t,s)[p(s)h(s,\hat{w}(s)) + q(s)]ds, \quad \forall t \in [0,1],$$

where $A(t)$ is a polynomial function which is given in Lemma 2.2. Then,

$$\hat{w}(t) = A(t) + \int_0^1 G(t,s)[p(s)h(s,\hat{w}(s)) + q(s)]ds + J^{\alpha;\kappa}\bar{w}(t), \quad \forall t \in [0,1].$$

According to Theorem 3.1, it has been verified that there is a unique solution $w(t)$ of BVP (1.1), and $w$ can be expressed as

$$w(t) = A(t) + \int_0^1 G(t,s)[p(s)h(s,w(s)) + q(s)]ds.$$

So, we have

$$\begin{aligned}
&|w(t) - \hat{w}(t)| \\
&\le \bigg|\int_0^1 G(t,s)p(s)(h(s,w(s)) - h(s,\hat{w}(s)))ds \\
&\quad - \frac{1}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t) - \kappa(s))^{\alpha-1}\bar{w}(s)ds\bigg| \\
&\le \int_0^1 G(t,s)p(s)|h(s,w(s)) - h(s,\hat{w}(s))|ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t) - \kappa(s))^{\alpha-1}|\bar{w}(s)|ds.
\end{aligned}$$

Since $|h(s,w) - h(s,\hat{w})| \le g(t)|w - \hat{w}|$, it indicates that

$$\begin{aligned}
&|w(t) - \hat{w}(t)| \\
&\le \int_0^1 G(t,s)p(s)g(s)\|w - \hat{w}\|ds + \frac{1}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t) - \kappa(s))^{\alpha-1}|\bar{w}(s)|ds \\
&\le \|w - \hat{w}\|\int_0^1 G(t,s)p(s)g(s)ds + \varepsilon\frac{1}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t) - \kappa(s))^{\alpha-1}ds
\end{aligned}$$

$$\leq \|w-\hat{w}\|N + \frac{(\kappa(t)-\kappa(0))^{\alpha}}{\Gamma(\alpha+1)}\varepsilon,$$

where N is as in Theorem 3.1. Let $N_1 = \sup_{t\in[0,1]}(\frac{(\kappa(t)-\kappa(0))^{\alpha}}{\Gamma(\alpha+1)}) = \frac{(\kappa(1)-\kappa(0))^{\alpha}}{\Gamma(\alpha+1)}$, we have

$$\|w-\hat{w}\| \leq \|w-\hat{w}\|N + \varepsilon N_1, \quad \forall t \in [0,1],$$

or

$$\|w-\hat{w}\| \leq \frac{N_1}{1-N}\varepsilon, \quad \forall t \in [0,1].$$

If we take $C_h = \frac{N_1}{1-N}$, then we can deduce that

$$\|w-\hat{w}\| \leq C_h\varepsilon, \quad \forall t \in [0,1],$$

which leads to the HU-stability of BVP (1.1). $\square$

**Theorem 4.2.** *Assume that the conditions of Theorem 3.1 are satisfied, the inequality* (2.8) *has at least one solution and there exists a constant $C_\varphi > 0$ such that*

$$\frac{1}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}\varphi(s)ds \leq C_\varphi\varphi(t), \quad \forall t \in [0,1],$$

*where $\varphi \in C([0,1],\mathbb{R}_+)$ is an increasing function. Then, the BVP* (1.1) *is HUR-stable.*

**Proof.** For each $\varepsilon > 0$, and for each function $\hat{w}$ that satisfies the following inequality

$$|{}^cD^{\alpha;\kappa}\hat{w}(t) + p(t)h(t,\hat{w}) + q(t)| \leq \varepsilon\varphi(t), \quad \forall t \in [0,1],$$

a function $\tilde{w}(t) = {}^cD^{\alpha;\kappa}\hat{w}(t) + p(t)h(t,\hat{w}) + q(t)$ can be found. Then, we have $|\tilde{w}(t)| \leq \varepsilon\varphi(t)$, which implies that

$$J^{\alpha;\kappa}\tilde{w}(t) = \hat{w}(t) - A(t) - \int_0^1 G(t,s)[p(s)h(s,\hat{w}(s)) + q(s)]ds, \quad \forall t \in [0,1],$$

where $A(t)$ is a polynomial function which is given in Lemma 2.2. Then,

$$\hat{w}(t) = A(t) + \int_0^1 G(t,s)[p(s)h(s,\hat{w}(s)) + q(s)]ds + J^{\alpha;\kappa}\tilde{w}(t), \quad \forall t \in [0,1].$$

According to Theorem 3.1, it has been verified that there is a unique solution $w(t)$ of BVP (1.1), and $w$ can be expressed as

$$w(t) = A(t) + \int_0^1 G(t,s)[p(s)h(s,w(s)) + q(s)]ds,$$

then, we have

$$\begin{aligned}
&|w(t)-\hat{w}(t)| \\
&\leq \bigg|\int_0^1 G(t,s)p(s)(h(s,w(s)) - h(s,\hat{w}(s)))ds \\
&\quad -\frac{1}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}\tilde{w}(s)ds\bigg|
\end{aligned}$$

$$\leq \int_0^1 G(t,s)p(s)|h(s,w(s)) - h(s,\hat{w}(s))|ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}|\tilde{w}(s)|ds.$$

Since $|h(s,w) - h(s,\hat{w})| \leq g(t)|w-\hat{w}|$, it indicates that

$$\begin{aligned}
&|w(t) - \hat{w}(t)| \\
&\leq \int_0^1 G(t,s)p(s)g(s)\|w-\hat{w}\|ds + \frac{1}{\Gamma(\alpha)} \int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}|\tilde{w}(s)|ds \\
&\leq \|w-\hat{w}\| \int_0^1 G(t,s)p(s)g(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t \varepsilon\kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}\varphi(s)ds \\
&\leq \|w-\hat{w}\|N + C_\varphi\varepsilon\varphi(t),
\end{aligned}$$

where N is as in Theorem 3.1. Then we can get

$$\|w-\hat{w}\| \leq \|w-\hat{w}\|N + C_\varphi\varepsilon\varphi(t), \quad \forall t \in [0,1],$$

or

$$\|w-\hat{w}\| \leq \frac{C_\varphi}{1-N}\varepsilon\varphi(t), \quad \forall t \in [0,1].$$

If we take $C_{h,\varphi} = \frac{C_\varphi}{1-N}$, then we can deduce

$$\|w-\hat{w}\| \leq C_{h,\varphi}\varepsilon\varphi(t), \quad \forall t \in [0,1],$$

which leads to the HUR-stability of BVP (1.1). □

# 5. Example

**Example 5.1.** Consider the following fractional boundary value problem:

$$\begin{cases} {}^cD^{\frac{7}{3};\kappa}w(t) + \dfrac{t\sin(w)}{3(1+t)} + t = 0, \quad 0 \leq t \leq 1, \\ w(0) = a, \quad w'(0) = b, \quad w(1) = d, \end{cases} \tag{5.1}$$

where $\kappa(t) = \sqrt{1+t}$. In this case, we have

$$h(t,w) = \frac{t\sin(w)}{3}, \quad 2 < \alpha = \frac{7}{3} < 3, \quad p(t) = \frac{1}{1+t}, \quad q(t) = t,$$
$$|h(t,w) - h(t,v)| \leq \frac{1+t}{3}|2\cos(\frac{w+v}{2})\sin(\frac{w-v}{2})| \leq \frac{1+t}{3}|w-v|, \quad g(t) = \frac{1+t}{3}.$$

Clearly, the problem (5.1) satisfies Theorem 3.1 due to

$$N = \max_{0\leq t\leq 1}\left|\int_0^1 G(t,s)p(s)g(s)ds\right|$$

$$
\begin{aligned}
&= \max_{0\le t\le 1} \frac{1}{\Gamma(\frac{7}{3})} \left| \frac{\left(\sqrt{1+t}-1\right)^2}{\left(\sqrt{2}-1\right)^2} \int_0^1 \frac{1}{2\sqrt{1+s}} (\sqrt{2}-\sqrt{1+s})^{\frac{4}{3}} p(s)g(s)ds \right. \\
&\quad \left. - \int_0^1 \frac{1}{2\sqrt{1+s}} \left(\sqrt{1+t}-\sqrt{1+s}\right)^{\frac{4}{3}} p(s)g(s)ds \right| \\
&\le \max_{0\le t\le 1} \frac{1}{3\Gamma(\frac{7}{3})} \left( \left| \frac{\left(\sqrt{1+t}-1\right)^2}{\left(\sqrt{2}-1\right)^2} \int_0^1 \frac{1}{2\sqrt{1+s}} (\sqrt{2}-\sqrt{1+s})^{\frac{4}{3}} ds \right| \right. \\
&\quad \left. + \left| \int_0^1 \frac{1}{2\sqrt{1+s}} \left(\sqrt{1+t}-\sqrt{1+s}\right)^{\frac{4}{3}} ds \right| \right) \\
&\le \max_{0\le t\le 1} \frac{1}{\Gamma(\frac{7}{3})} \left( \frac{\left(\sqrt{1+t}-1\right)^2}{\left(\sqrt{2}-1\right)^2} \frac{1}{7} (\sqrt{2}-1)^{\frac{7}{3}} + \frac{1}{7} (\sqrt{2}-1)^{\frac{7}{3}} \right) \\
&\approx 0.0307 \\
&< 1.
\end{aligned}
$$

In addition, we have

$$
\begin{aligned}
|h(t,w)| &= \frac{t\sin(w)}{3} \le \phi(t)\psi(|w|), \\
M_1 &= \max_{0\le t\le 1} \left| \int_0^1 G(t,s)p(s)\phi(s)ds \right| \approx 0.0307, \\
M_2 &= \max_{0\le t\le 1} \left| \int_0^1 G(t,s)q(s)ds \right| + k \approx 0.0298,
\end{aligned}
$$

where $\phi(t) = 1+t$, $\psi(|w|) = \frac{\sin(|w|)}{3}$, $k = d = 0$, $M = \max\{M_1, M_2\} \approx 0.0298$. Then,

$$M(\psi(r)+1) - r \approx 0.0298(\frac{r}{3}+1) - r < 0, \text{ for } r = 2.$$

The assumptions (H1) and (H3) of Theorem 3.2 hold. Therefore, it can be verified that the problem (5.1) has a unique solution.

Furthermore, we discuss the Hyers-Ulam stability of the problem (5.1). For each $\varepsilon > 0$ and each function $\hat{w}$ that satisfies

$$|{}^cD^{\frac{7}{3};\kappa}\hat{w}(t) + \frac{t\sin(\hat{w})}{3(1+t)} + t| \le \varepsilon, \quad \forall t \in [0,1],$$

a function $\bar{w}(t) = {}^cD^{\frac{7}{3};\kappa}\hat{w}(t) + \frac{t\sin(\hat{w})}{3(1+t)} + t$ can be found. Additionally, let $w(t)$ be the unique solution of the problem (5.1). It follows from that

$$\|w - \hat{w}\| \le \frac{N_1}{1-N}\varepsilon = C_h\varepsilon,$$

where $N_1 = \frac{(\kappa(1)-\kappa(0))^\alpha}{\Gamma(\alpha+1)} \approx 0.0460$, $N \approx 0.0307$, and $C_h \approx 0.0475$. In terms of Theorem 4.1, it is obvious that problem (5.1) is Hyers-Ulam stable.

**Example 5.2.** Set $\kappa(t) = \exp(t)$, $h(t,w) = \frac{tw+t}{4+w}$, $\alpha = \frac{11}{4}$, $p(t) = \frac{1}{1+t}$, $q(t) = t$.

Consider the following fractional boundary value problem

$$\begin{cases} {}^cD^{\frac{11}{4};\kappa}w(t) + \dfrac{tw+t}{(1+t)(4+w)} + t = 0, \quad 0 \le t \le 1, \\ w(0) = a, \quad w'(0) = b, \quad w(1) = d. \end{cases} \tag{5.2}$$

Obviously,

$$|h(t,w) - h(t,v)| \le \left|\frac{4t(w-v)}{(4+w)(4+v)}\right| \le \frac{1+t}{4}|w-v|, \quad g(t) = \frac{1+t}{4},$$

$$\begin{aligned} N &= \max_{0\le t\le 1}\left|\int_0^1 G(t,s)p(s)g(s)ds\right| \\ &= \max_{0\le t\le 1}\frac{1}{\Gamma(\frac{11}{4})}\left|\frac{(e^t-1)^2}{(e-1)^2}\int_0^1 e^s(e-e^s)^{\frac{7}{4}}p(s)g(s)ds - \int_0^1 e^s(e^t-e^s)^{\frac{7}{4}}p(s)g(s)ds\right| \\ &\le \max_{0\le t\le 1}\frac{1}{4\Gamma(\frac{11}{4})}\left(\left|\frac{(e^t-1)^2}{(e-1)^2}\int_0^1 e^s(e-e^s)^{\frac{7}{4}}ds\right| + \left|\int_0^1 e^s(e^t-e^s)^{\frac{7}{4}}ds\right|\right) \\ &\le \max_{0\le t\le 1}\frac{1}{\Gamma(\frac{11}{4})}\left(\frac{(e^t-1)^2}{(e-1)^2}\frac{1}{11}(e-1)^{\frac{11}{4}} + \frac{1}{11}(e-1)^{\frac{11}{4}}\right) \\ &\approx 0.5009 \\ &< 1. \end{aligned}$$

Next, we examine the assumptions of Theorem 3.3. We have

$$|h(t,w)| = \frac{tw+t}{4+w} \le a_1(t) + a_2(t)|w|,$$

$$B = \max_{0\le t\le 1}\left|\int_0^1 G(t,s)p(s)a_2(s)ds\right| \approx 0.5009 < 1.$$

Therefore, the assumptions (H1) and (H4) hold. Thus, combining Theorem 3.1 and Theorem 3.3, it can be verified that the problem (5.2) has a unique solution.

Now, we consider the Hyers-Ulam-Rassias stability of the problem (5.2). Set $\varphi(t) = E_{\alpha,1}\left(5(k(t)-k(0))^{\alpha}\right)$. Then, for each $\varepsilon > 0$ and each function $\hat{w}$ that satisfies

$$\left|{}^cD^{\frac{11}{4};\kappa}\hat{w}(t) + \frac{t\hat{w}+t}{(1+t)(4+\hat{w})} + t\right| \le \varepsilon\varphi(t), \quad \forall t \in [0,1],$$

a function $\tilde{w}(t) = {}^cD^{\frac{11}{4};\kappa}\hat{w}(t) + \frac{t\hat{w}+t}{(1+t)(4+\hat{w})} + t$ can be found. Additionally, let $w(t)$ be the unique solution of the problem (5.2). We find

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)}\int_0^t k'(s)(k(t)-k(s))^{\alpha-1}\varphi(s)ds \\ =&J^{\alpha;\kappa}E_{\alpha,1}\left(5\left(k(s)-k(0)\right)^{\alpha}\right) \end{aligned}$$

$$
\begin{aligned}
&=\frac{1}{5}\left(E_{\alpha,1}\left(5(k(t)-k(0))^{\alpha}\right)-1\right)\\
&\leq\frac{1}{5}E_{\alpha,1}\left(5(k(t)-k(0))^{\alpha}\right)\\
&=\frac{1}{5}\varphi(t),
\end{aligned}
$$

where the formula of fractional integrals of the Mittag-Leffler function in Theorem 2.1 is used. It follows from $C_\varphi=\frac{1}{5}$ and Theorem 4.2 that

$$\|w-\hat{w}\|\leq\frac{C_\varphi}{1-N}\varepsilon\varphi(t)=C_{h,\varphi}\varepsilon\varphi(t),$$

where $N\approx 0.5009$, $C_{h,\varphi}=\frac{C_\varphi}{1-N}\approx 0.4007$, which means that problem (5.2) is Hyers-Ulam-Rassias stable.

# 6. Conclusion

In this paper, we first studied the existence and uniqueness of solutions for a class of boundary value problem of $\kappa$-Caputo fractional differential equations. The Banach fixed point theorem and the Larry-Schauder nonlinear alternative theorem are taken up as the major methods for investigating the existence and uniqueness of solutions. Then, we proposed two sufficient conditions for ensuring the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of nonlinear fractional differential equations, respectively. Finally, two examples were presented to verify the numerical applications of the result. In addition, a possible continuation of this work might be studying the stability of existence results and the optimization of proof conditions.

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