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RELIABILITY STATISTICAL ANALYSIS OF TWO-PARAMETER EXPONENTIAL DISTRIBUTION UNDER CONSTANT STRESS ACCELERATED LIFE TEST WITH INVERSE POWER LAW MODEL

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Abstract Based on the inverse power law model, the maximum likelihood estimation and interval estimation of two-parameter Exponential distribution are derived in detail under constant stress accelerated life test. Secondly, the accuracy of point estimation and interval estimation is investigated by a large number of Monte Carlo simulations. Finally, examples and simulation examples are given to illustrate the application of the proposed method.

Keywords Two-parameter exponential distribution, constant stress accelerated life test, inverse power law model, maximum likelihood estimation, interval estimation.

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1. Introduction

There has been a great deal of literature on the reliability of accelerated life tests, which is not listed here. It should be pointed out that the research of accelerated life tests has achieved fruitful research results and applied in practice in which the failure distribution of the products involved is usually a location-scale parameter family distribution such as a one-parameter exponential distribution, a two-parameter Weibull distribution or a two-parameter lognormal distribution and etc. In terms of actual specific conditions, there are indeed a large number of product life subject to the two-parameter exponential distribution while the reliability analysis of the two-parameter exponential distribution product accelerated life test results are few, the main reason is that there is no good understanding of the basic assumption "failure mechanism unchanged".

Failure mechanisms are the cause of changes in the physical, chemical and material properties of a product. Studying the failure mechanism of the product is crucial to improving the quality of the product, and also helps to improve the reliability analysis results of some products, such as accelerated life test statistical analysis

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problems. It is based on three basic assumptions, one of which is to require the failure mechanism of the product to remain unchanged in different stress environments. Its reason has been given by Nelson [6–8]. Another example is the problem of statistical analysis of environmental factors which plays an important role in the conversion of reliability data in different environments. Therefore, the problem of statistical inference of environmental factors has attracted a lot of attention. It is also common to require that the failure mechanism of the product remain unchanged in different environments while using environmental factors for reliability analysis. For the life distribution, Sun [11] proved that an invariant coefficient of variation is a necessary condition for the invariance of the failure mechanism. The coefficient of variation is a widely used parameter, and since this parameter is a good reflection of the degree of dispersion of variable values, it is an important indicator of product quality stability. Sun [12] proposed a method to test whether the coefficients of variation are equal to test whether the failure mechanism is changed, and gives a hypothesis test method to test whether the coefficients of variation are equal in which the life distribution is lognormal distribution, Gamma distribution, Weibull distribution, Gumbel distribution and so on. Yang [13] and Zhou [15] have studied the properties of the acceleration coefficients in depth, pointing out the essential relationship between these properties and the invariance of the failure mechanism. The expressions of the conditions and acceleration coefficients of the failure mechanisms of various common life distributions and the failure distributions suitable for accelerated life tests are given, the time conversion formula and the basic assumptions of the accelerated life test are discussed as well. But in fact, the condition given by Yang [13] and Zhou [15] that the failure mechanism is unchanged is that the coefficient of variation remains constant.

Let the population X follow a two-parameter exponential distribution $\mathrm{Exp}(\mu, \theta)$, i.e. $X \sim \mathrm{Exp}(\mu, \theta)$, and its distribution function and density function are

$$F(x) = 1 - \exp\left(-\frac{x-\mu}{\theta}\right), f(x) = \frac{1}{\theta}\exp\left(-\frac{x-\mu}{\theta}\right), x \geq \mu \geq 0, \theta > 0,$$

where θ is the scale parameter and μ is the location parameter.

Zheng and Fang [14] proposed a new approach to obtain the exact lower and upper confidence limits for the mean life of the exponential distribution in the accelerated life tests with type-I censoring data. El-Raheem [2] studied the optimal allocation problem in multiple constant- stress accelerated life testing for the extension of the exponential distribution under type-II censored data and obtained the exact and asymptotic optimal allocations for small and large sample sizes under three optimizations criteria associated with Fisher information matrix. Hassan [4] studied the estimation of the stress-strength reliability model when the stress and the strength variables are modeled by two independent but not identically distributed random variables with generalized inverse exponential distributions. El-Raheem [3] derived point and interval estimations of parameters for the modified Kies exponential distribution by using the maximum likelihood and Bayes method under multiple constant-stress testing for progressive type-II censored data with binomial removal. Alotaibil [1] introduced the Gull alpha power exponentiated exponential distribution and studied its statistical properties and parameter estimations as well as a bivariate step-stress accelerated life test based on progressive type-I censoring. Shi [9,10] pointed out that if the product life follows the two-parameter exponential distribution, it is assumed that $\ln\theta = a + b\phi(S)$, $\ln\mu = c + d\phi(S)$ where a, b, c, d

are the parameters. When the stress S is temperature, $\phi(S) = \frac{1}{S}$, the model is the Arrhenins model. When the stress S is voltage, $\phi(S) = \ln(S)$, the model is an inverse power law model. In addition, the estimation method of product reliability index under the constant stress acceleration life test site of fixed truncation is given.

2. The fundamental assumptions of the constant stress accelerated life test and the inverse power law model

The statistical analysis of the constant stress accelerated life test of the two-parameter exponential distribution product under the inverse power law model is based on the following three basic assumptions.

Assumption 2.1. *Suppose that the life of the product X at any stress level V obeys the two-parameter exponential distribution with the scale parameter θ and the position parameter μ.*

Assumption 2.2. *The failure mechanism of the product does not change at various stress levels, i.e. the coefficients of variation $CV = \frac{\theta}{\mu+\theta}$ are the same for the product life distribution at each stress level. It also means that the ratios of parameters $\frac{\mu}{\theta} \hat{=} \tau$ are the same.*

Assumption 2.3. *The scale parameter θ and the acceleration stress level V satisfy the inverse power law model.*

According to the physical principle and experimental experience summary, the inverse power law model means that when the voltage is accelerated stress, the inverse power law relationship between the scale parameter θ (unit: hours) and voltage (unit: volts) of some products (such as insulation materials, capacitors, micro motors and some electronic devices) is $\theta = \frac{1}{dV^c}$, where $d > 0$, $c > 0$ are constants.

After taking the logarithm on both sides of the above equation, the logarithmic linear relationship is satisfied for θ:

$$\ln \theta = a + b\phi(V),$$

where $a = -\ln d$, $b = -c$, and $\phi(V) = \ln V$ is a function of stress V.

3. Point and interval estimates for parameters

A constant stress acceleration life test is performed on n products whose life distribution is two-parameter exponential distribution $\text{Exp}(\mu, \theta)$ with the location parameter μ and the scale parameter θ. The products are divided into k parts, that is $n_1, n_2, \cdots, n_k$, while the constant stress is divided into k stress, i.e. $V_1 < V_2 < \cdots < V_k$. The n_i products are subjected to a life test until all the products fail under the constant stress V_i, $i = 1, 2, \cdots, k$, and the corresponding order failure times $x_{(i1)}, x_{(i2)}, \cdots, x_{(in_i)}$ of the n_i products are recorded.

Under constant stress V_i, $i = 1, 2, \cdots, k$, the life X_i of the product follows a two-parameter exponential distribution $\text{Exp}(\mu_i, \theta_i)$, and its distribution function and density function are

$$F_{V_i}(x) = 1 - \exp\left(-\frac{x-\mu_i}{\theta_i}\right) = 1 - \exp\left[-\left(\frac{x}{\theta_i} - \tau\right)\right],$$
$$f_{V_i}(x) = \frac{1}{\theta_i}\exp\left[-\left(\frac{x}{\theta_i} - \tau\right)\right], x \geq \mu_i > 0, \theta_i > 0, i = 1, 2, \cdots, k.$$

3.1. Maximum likelihood estimations of parameters

The likelihood function is:

$$\begin{aligned} L(c,d,\tau) &= \prod_{i=1}^{k}\prod_{j=1}^{n_i}\left\{dV_i^c \exp\left(-dV_i^c x_{(ij)} + \tau\right)\right\} \\ &= d^n e^{n\tau}\left(\prod_{i=1}^{k} V_i^{n_i c}\right)\exp\left(-d\sum_{i=1}^{k} V_i^c \sum_{j=1}^{n_i} x_{(ij)}\right), \\ \ln L(c,d,\tau) &= n\ln d + n\tau + c\sum_{i=1}^{k} n_i \ln V_i - d\sum_{i=1}^{k} V_i^c \sum_{j=1}^{n_i} x_{(ij)} \\ &= n\ln d + n\tau + c\sum_{i=1}^{k} n_i \ln V_i - d\sum_{i=1}^{k} n_i \bar{x}_i V_i^c, \end{aligned}$$

where $\bar{x}_i = \frac{1}{n_i}\sum\limits_{j=1}^{n_i} x_{(ij)}$.

$$\frac{\partial L(c,d,\tau)}{\partial c} = \sum_{i=1}^{k} n_i \ln V_i - d\sum_{i=1}^{k} n_i \bar{x}_i V_i^c \ln V_i, \;\; \frac{\partial L(c,d,\tau)}{\partial d} = \frac{n}{d} - \sum_{i=1}^{k} n_i \bar{x}_i V_i^c.$$

Let $\frac{\partial L(c,d,\tau)}{\partial c} = 0$, $\frac{\partial L(c,d,\tau)}{\partial d} = 0$, and the following system of equations are obtained:

$$\begin{cases} \sum\limits_{i=1}^{k} n_i \ln V_i - d\sum\limits_{i=1}^{k} n_i \bar{x}_i V_i^c \ln V_i = 0, \\ \dfrac{n}{d} - \sum\limits_{i=1}^{k} n_i \bar{x}_i V_i^c = 0. \end{cases}$$

Simplify to obtain the following univariate transcendence equation with the only parameter c:

$$\frac{\sum\limits_{i=1}^{k} n_i \bar{x}_i V_i^c \ln V_i}{\sum\limits_{i=1}^{k} n_i \bar{x}_i V_i^c} = \frac{1}{n}\sum_{i=1}^{k} n_i \ln V_i. \tag{3.1}$$

Lemma 3.1. *If* $\frac{1}{n}\sum\limits_{i=1}^{k} n_i \ln V_i > \frac{\sum_{i=1}^{k} n_i \bar{x}_i \ln V_i}{\sum_{i=1}^{k} n_i \bar{x}_i}$, *then the equation* (3.1) *of the parameter* c *has a unique positive real root.*

Proof. Denote
$$g(c) = \frac{\sum_{i=1}^{k} n_i \bar{x}_i V_i^c \ln V_i}{\sum_{i=1}^{k} n_i \bar{x}_i V_i^c}.$$

It is easy to see $\lim_{c\to 0} g(c) = \frac{\sum_{i=1}^{k} n_i \bar{x}_i \ln V_i}{\sum_{i=1}^{k} n_i \bar{x}_i}$, $\lim_{c\to+\infty} g(c) = \lim_{c\to+\infty} \frac{\sum_{i=1}^{k} n_i \bar{x}_i \left(\frac{V_i}{V_k}\right)^c \ln V_i}{\sum_{i=1}^{k} n_i \bar{x}_i \left(\frac{V_i}{V_k}\right)^c} = \ln V_k$ and
$$g'(c) = \frac{\sum_{i=1}^{k} n_i \bar{x}_i V_i^c (\ln V_i)^2 \cdot \sum_{j=1}^{k} n_j \bar{x}_j V_j^c - \sum_{i=1}^{k} n_i \bar{x}_i V_i^c \ln V_i \cdot \sum_{j=1}^{k} n_j \bar{x}_j V_j^c \ln V_j}{\left(\sum_{i=1}^{k} n_i \bar{x}_i V_i^c\right)^2}.$$

It is noted that the numerator $\sum_{i=1}^{k} n_i \bar{x}_i V_i^c (\ln V_i)^2 \cdot \sum_{j=1}^{k} n_j \bar{x}_j V_j^c - \sum_{i=1}^{k} n_i \bar{x}_i V_i^c \ln V_i \cdot \sum_{j=1}^{k} n_j \bar{x}_j V_j^c \ln V_j$ of $g'(c)$ contains the term in terms of $V_i^c V_j^c$:
$$\begin{aligned} & n_i \bar{x}_i V_i^c (\ln V_i)^2 \cdot n_j \bar{x}_j V_j^c + n_j \bar{x}_j V_j^c (\ln V_j)^2 \cdot n_i \bar{x}_i V_i^c - 2 n_i \bar{x}_i V_i^c \ln V_i \cdot n_j \bar{x}_j V_j^c \ln V_j \\ =& n_i n_j \bar{x}_i \bar{x}_j V_i^c V_j^c (\ln V_i - \ln V_j)^2 \\ >& 0, \end{aligned}$$
i.e. $g(c)$ is a strictly monotonic increasing function. Since
$$\frac{\sum_{i=1}^{k} n_i \bar{x}_i \ln V_i}{\sum_{i=1}^{k} n_i \bar{x}_i} < \frac{1}{n} \sum_{i=1}^{k} n_i \ln V_i < \ln V_k,$$
the equation $g(c) = \frac{1}{n} \sum_{i=1}^{k} n_i \ln V_i$ has a unique positive real root. □

If $\frac{1}{n} \sum_{i=1}^{k} n_i \ln V_i > \frac{\sum_{i=1}^{k} n_i \bar{x}_i \ln V_i}{\sum_{i=1}^{k} n_i \bar{x}_i}$ is satisfied, then the maximum likelihood estimate $\hat{c}$ of the parameter c can be obtained by solving the equation (3.1). Moreover the maximum likelihood estimation $\hat{d} = \frac{n}{\sum_{i=1}^{k} n_i \bar{x}_i V_i^{\hat{c}}}$ of the parameter d can be obtained.

At the same time, the maximum likelihood estimates of the parameters $\theta_i, i = 1, 2, \cdots, k$ are derived by $\hat{\theta}_i = \frac{1}{\hat{d} V_i^{\hat{c}}}$.

Since $\frac{\partial L(c,d,\tau)}{\partial \tau} = n > 0$, the maximum likelihood estimate of τ is
$$\hat{\tau} = \min\left\{\frac{x_{(i1)}}{\hat{\theta}_i}, i = 1, 2, \cdots, k\right\}.$$

Set $n = 70$, $k = 4$, $n_1 = 5$, $n_2 = 10$, $n_3 = 20$, $n_4 = 35$ and their corresponding constant stress levels to be $V_1 = 10V$, $V_2 = 20V$, $V_3 = 30V$, $V_4 = 40V$, respectively.

The number of times that the condition $\frac{1}{n}\sum_{i=1}^{k} n_i \ln V_i > \frac{\sum_{i=1}^{k} n_i \bar{x}_i \ln V_i}{\sum_{i=1}^{k} n_i \bar{x}_i}$ satisfied is calculated from 1000 Monte Carlo simulations. As well as the mean and mean square error (MSE) of the maximum likelihood estimates of the parameter c, d, τ are calculated when this condition is satisfied. The results are shown in Table 1, from which it can be seen that: Firstly, almost all the sample data meet the conditions, that is, the maximum likelihood estimation of the parameter c can be obtained through equation (3.1). Secondly, the maximum likelihood estimations of the parameters c, d, τ also have high accuracy.

Table 1. Simulation results for the maximum likelihood estimations of parameters

truth values of parameters			times	$\hat{c}$		$\hat{d}$		$\hat{\tau}$	
c	d	τ		mean	MSE	mean	MSE	mean	MSE
1	0.001	0.1	999	0.9756	0.0832	0.0016	4.7795×10^{-6}	0.1032	0.0003
		0.2	1000	0.9731	0.0723	0.0014	3.2756×10^{-6}	0.1758	0.00112
		0.3	1000	0.9894	0.0539	0.0011	1.1011×10^{-6}	0.2370	0.00474
	0.01	0.1	1000	0.9614	0.0811	0.0172	0.000567	0.1034	0.00031
		0.2	1000	0.9727	0.0662	0.0134	0.000224	0.1755	0.00114
		0.3	1000	0.9875	0.0552	0.0112	0.000110	0.2372	0.00474
	0.1	0.1	998	0.9747	0.0751	0.1553	0.035784	0.1047	0.00036
		0.2	1000	0.9692	0.0616	0.1328	0.018108	0.1768	0.00103
		0.3	1000	0.966	0.0585	0.1227	0.013921	0.2371	0.00475
2	0.001	0.1	1000	1.9830	0.0793	0.0016	3.6217×10^{-6}	0.1047	0.00036
		0.2	1000	1.9774	0.0673	0.0014	2.4764×10^{-6}	0.1761	0.00115
		0.3	1000	1.9964	0.0565	0.0011	1.2694×10^{-6}	0.2356	0.00497
	0.01	0.1	1000	1.9693	0.0833	0.0172	0.000707	0.1035	0.00036
		0.2	1000	1.9868	0.0629	0.0130	0.000228	0.1770	0.00106
		0.3	1000	1.9837	0.0594	0.0116	0.000140	0.2371	0.00480
	0.1	0.1	1000	1.9727	0.0773	0.1628	0.054764	0.1033	0.00033
		0.2	1000	1.9850	0.0712	0.1360	0.029058	0.1764	0.00109
		0.3	1000	1.9937	0.0519	0.1073	0.008869	0.2375	0.00470
4	0.001	0.1	1000	3.9741	0.0878	0.0017	6.7213×10^{-6}	0.1039	0.00034
		0.2	1000	3.9704	0.0647	0.0014	2.2084×10^{-6}	0.1761	0.00112
		0.3	1000	3.9780	0.0591	0.0012	1.3617×10^{-6}	0.2367	0.00475
	0.01	0.1	1000	3.9755	0.0788	0.0163	0.000481	0.1038	0.00035
		0.2	1000	3.9818	0.0671	0.0135	0.000234	0.1758	0.00113
		0.3	1000	3.9911	0.0562	0.0112	0.000119	0.2364	0.00489
	0.1	0.1	1000	3.9898	0.0773	0.1551	0.045374	0.1044	0.00035
		0.2	1000	3.9593	0.0673	0.1436	0.025530	0.1759	0.00114
		0.3	1000	3.9989	0.0551	0.1083	0.011681	0.2366	0.00482

3.2. Interval estimations of parameters c, τ

According to the property of the order statistics of exponential distribution, it is easy to know that $n_i\left(\frac{X_{(i1)}}{\theta_i} - \tau\right), (n_i - 1)\frac{X_{(i2)} - X_{(i1)}}{\theta_i}, \cdots, \frac{X_{(in_i)} - X_{(i(n_i-1))}}{\theta_i}$ are independent and obey the standard exponential distribution Exp(1) under stress $V_i, i = 1, 2, \cdots, k$.

Besides, $2n_i\left(\frac{X_{(i1)}}{\theta_i} - \tau\right), 2(n_i - 1)\frac{X_{(i2)} - X_{(i1)}}{\theta_i}, \cdots, 2\frac{X_{(in_i)} - X_{(i(n_i-1))}}{\theta_i}$, are independent and follow $\chi^2(2)$. Moreover, $\frac{2}{\theta_i}\sum_{j=2}^{n_i}(n_i - j + 1)(X_{(ij)} - X_{(i(j-1))}) \sim \chi^2(2(n_i - 1))$.

If $l = \left[\frac{k}{2}\right]$, then we have

$$2\sum_{i=1}^{l}\frac{1}{\theta_i}\sum_{j=2}^{n_i}(n_i-j+1)(X_{(ij)}-X_{(i(j-1))}) \sim \chi^2\left(2\left(\sum_{i=1}^{l}n_i-l\right)\right),$$
$$2\sum_{i=l+1}^{k}\frac{1}{\theta_i}\sum_{j=2}^{n_i}(n_i-j+1)(X_{(ij)}-X_{(i(j-1))}) \sim \chi^2\left(2\left(\sum_{i=l+1}^{k}n_i-k+l\right)\right),$$

which means

$$2d\sum_{i=1}^{l}V_i^c\sum_{j=2}^{n_i}(n_i-j+1)(X_{(ij)}-X_{(i(j-1))}) \sim \chi^2\left(2\left(\sum_{i=1}^{l}n_i-l\right)\right),$$
$$2d\sum_{i=l+1}^{k}V_i^c\sum_{j=2}^{n_i}(n_i-j+1)(X_{(ij)}-X_{(i(j-1))}) \sim \chi^2\left(2\left(\sum_{i=l+1}^{k}n_i-k+l\right)\right).$$

Therefore,

$$\frac{\dfrac{2d\sum_{i=l+1}^{k}V_i^c\sum_{j=2}^{n_i}(n_i-j+1)(X_{(ij)}-X_{(i(j-1))})}{2\left(\sum_{i=l+1}^{k}n_i-k+l\right)}}{\dfrac{2d\sum_{i=1}^{l}V_i^c\sum_{j=2}^{n_i}(n_i-j+1)(X_{(ij)}-X_{(i(j-1))})}{2\left(\sum_{i=1}^{l}n_i-l\right)}} \sim F\left(2\left(\sum_{i=l+1}^{k}n_i-k+l\right), 2\left(\sum_{i=1}^{l}n_i-l\right)\right).$$

Denote

$$\mathcal{T}(c) = \frac{\sum_{i=1}^{l}n_i-l}{\sum_{i=l+1}^{k}n_i-k+l}\,\frac{\sum_{i=l+1}^{k}V_i^c\sum_{j=2}^{n_i}(n_i-j+1)(X_{(ij)}-X_{(i(j-1))})}{\sum_{i=1}^{l}V_i^c\sum_{j=2}^{n_i}(n_i-j+1)(X_{(ij)}-X_{(i(j-1))})},$$

then

$$\mathcal{T}(c) \sim F\left(2\left(\sum_{i=l+1}^{k}n_i-k+l\right), 2\left(\sum_{i=1}^{l}n_i-l\right)\right).$$

Lemma 3.2. *If $\lim_{c\to 0}\mathcal{T}(c) < a$, then the equation $\mathcal{T}(c) = a$ of the parameter c has a unique positive real root.*

Proof.

$$\lim_{c\to 0}\mathcal{T}(c) = \frac{\sum_{i=1}^{l}n_i-l}{\sum_{i=l+1}^{k}n_i-k+l}\,\frac{\sum_{i=l+1}^{k}\sum_{j=2}^{n_i}(n_i-j+1)(X_{(ij)}-X_{(i(j-1))})}{\sum_{i=1}^{l}\sum_{j=2}^{n_i}(n_i-j+1)(X_{(ij)}-X_{(i(j-1))})},$$
$$\lim_{c\to+\infty}\mathcal{T}(c) = +\infty,$$

$$[\mathcal{T}(c)]' = \frac{\sum_{i=1}^{l} n_i - l}{\sum_{i=l+1}^{k} n_i - k + l} \frac{G(c)}{\left[\sum_{i=1}^{l} V_i^c \sum_{j=2}^{n_i} (n_i - j + 1)(X_{(ij)} - X_{(i(j-1))})\right]^2},$$

where

$$\begin{aligned}
G(c) = & \sum_{i=l+1}^{k} V_i^c \ln V_i \sum_{j=2}^{n_i} (n_i - j + 1)(X_{(ij)} - X_{(i(j-1))}) \\
& \times \sum_{i=1}^{l} V_i^c \sum_{j=2}^{n_i} (n_i - j + 1)(X_{(ij)} - X_{(i(j-1))}) \\
& - \sum_{i=l+1}^{k} V_i^c \sum_{j=2}^{n_i} (n_i - j + 1)(X_{(ij)} - X_{(i(j-1))}) \\
& \times \sum_{i=1}^{l} V_i^c \ln V_i \sum_{j=2}^{n_i} (n_i - j + 1)(X_{(ij)} - X_{(i(j-1))}) \\
= & \sum_{i_1=l+1}^{k} \sum_{j_1=2}^{n_{i_1}} \sum_{i_2=1}^{l} \sum_{j_2=2}^{n_{i_2}} V_{i_1}^c V_{i_2}^c \ln V_{i_1} (n_{i_1} - j_1 + 1)(n_{i_2} - j_2 + 1) \\
& \times (X_{(i_1 j_1)} - X_{(i_1(j_1-1))})(X_{(i_2 j_2)} - X_{(i_2(j_2-1))}) \\
& - \sum_{i_1=l+1}^{k} \sum_{j_1=2}^{n_{i_1}} \sum_{i_2=1}^{l} \sum_{j_2=2}^{n_{i_2}} V_{i_1}^c V_{i_2}^c \ln V_{i_2} (n_{i_1} - j_1 + 1)(n_{i_2} - j_2 + 1) \\
& \times (X_{(i_1 j_1)} - X_{(i_1(j_1-1))})(X_{(i_2 j_2)} - X_{(i_2(j_2-1))}) \\
= & \sum_{i_1=l+1}^{k} \sum_{j_1=2}^{n_{i_1}} \sum_{i_2=1}^{l} \sum_{j_2=2}^{n_{i_2}} V_{i_1}^c V_{i_2}^c (\ln V_{i_1} - \ln V_{i_2})(n_{i_1} - j_1 + 1)(n_{i_2} - j_2 + 1) \\
& \times (X_{(i_1 j_1)} - X_{(i_1(j_1-1))})(X_{(i_2 j_2)} - X_{(i_2(j_2-1))}) \\
> & 0.
\end{aligned}$$

Thus the function $\mathcal{T}(c)$ is a strictly monotonic increasing function of the parameter c. Furthermore, if $\lim_{c \to 0} \mathcal{T}(c) < a$, then the equation $\mathcal{T}(c) = a$ of the parameter c has a unique positive real root. □

Given the confidence level $1 - \alpha$, if

$$\lim_{c \to 0} \mathcal{T}(c) < F_{1-\alpha/2} \left(2 \left(\sum_{i=l+1}^{k} n_i - k + l \right), 2 \left(\sum_{i=1}^{l} n_i - l \right) \right)$$

is satisfied, the confidence interval for the parameter c is $[\hat{c}_1, \hat{c}_2]$, where $\hat{c}_1$, $\hat{c}_2$ are the roots of the following equations, respectively:

$$\mathcal{T}(c) = F_{1-\alpha/2} \left(2 \left(\sum_{i=l+1}^{k} n_i - k + l \right), 2 \left(\sum_{i=1}^{l} n_i - l \right) \right),$$

$$\mathcal{T}(c) = F_{\alpha/2} \left(2 \left(\sum_{i=l+1}^{k} n_i - k + l \right), 2 \left(\sum_{i=1}^{l} n_i - l \right) \right).$$

According to Li [5], the minimum variance unbiased estimators of the parameters θ_i, μ_i under stress V_i, $i = 1, 2, \cdots, k$ are respectively denoted as $\tilde{\theta}_i = \frac{n_i}{n_i-1}(\bar{X}_i - X_{(i1)})$, $\tilde{\mu}_i = X_{(i1)} - \frac{\tilde{\theta}_i}{n_i}$, and

$$D(\tilde{\theta}_i) = \frac{\theta_i^2}{n_i - 1}, D(\tilde{\mu}_i) = \frac{\theta_i^2}{n_i(n_i-1)}, \mathrm{cov}(\tilde{\mu}_i, \tilde{\theta}_i) = -\frac{\theta_i^2}{n_i(n_i-1)}.$$

The point estimate of the parameter τ can be taken as

$$\tilde{\tau}_i = \frac{X_{(i1)}}{\hat{\theta}_i} - \frac{1}{n_i} = \frac{n_i - 1}{n_i}\frac{X_{(i1)}}{\bar{X}_i - X_{(i1)}} - \frac{1}{n_i}.$$

Lemma 3.3. *(i) The expected value of $\tilde{\tau}_i$ is $E(\tilde{\tau}_i) = \frac{n_i-1}{n_i-2}\tau + \frac{1}{n_i(n_i-2)}$;*

(ii) $\hat{\tau}_i = \frac{n_i-2}{n_i-1}\tilde{\tau}_i - \frac{1}{(n_i-1)n_i} = \frac{n_i-2}{n_i}\frac{X_{(i1)}}{\bar{X}_i - X_{(i1)}} - \frac{1}{n_i}$ is an unbiased and consistent estimate of τ, and its corresponding variance is $D(\hat{\tau}_i) = \frac{1}{n_i-3}\left(\tau^2 + \frac{2}{n_i}\tau + \frac{n_i-1}{n_i^2}\right)$.

Proof. Let $Z_{(j)} = \frac{X_{(ij)} - \mu_i}{\theta_i}, j = 1, 2, \cdots, n_i$, and we have

$$X_{(ij)} = \mu_i + \theta_i Z_{(j)} = \theta_i(Z_{(j)} + \tau), j = 1, 2, \cdots, n_i.$$

(i)

$$\begin{aligned}
\tilde{\tau}_i =& \frac{(n_i - 1)X_{(i1)}}{n_i\bar{X}_i - n_i X_{(i1)}} - \frac{1}{n_i} \\
=& \frac{2(n_i-1)(Z_{(1)} + \tau)}{2\sum_{j=2}^{n_i}(n_i - j + 1)(Z_{(j)} - Z_{(j-1)})} - \frac{1}{n_i} \\
=& \frac{n_i - 1}{n_i}\frac{2n_i Z_{(1)}}{2\sum_{j=2}^{n_i}(n_i - j + 1)(Z_{(j)} - Z_{(j-1)})} \\
&+ \frac{2(n_i-1)\tau}{2\sum_{j=2}^{n_i}(n_i - j + 1)(Z_{(i)} - Z_{(i-1)})} - \frac{1}{n_i}.
\end{aligned}$$

Denote $U = 2\sum_{j=2}^{n_i}(n_i - j + 1)(Z_{(j)} - Z_{(j-1)}), V = 2n_i Z_{(1)}$, then $\tilde{\tau}_i = \frac{n_i-1}{n_i}\frac{V}{U} + \frac{2(n_i-1)\tau}{U} - \frac{1}{n_i}$. Noting that U, V are independent of each other and $U \sim \chi^2(2(n_i - 1)), V \sim \chi^2(2)$. Then

$$\begin{aligned}
E(\tilde{\tau}_i) =& \frac{n_i - 1}{n_i}E(V)E(U^{-1}) + 2(n_i - 1)\tau E(U^{-1}) - \frac{1}{n_i} \\
=& 2\frac{n_i - 1}{n_i}\frac{1}{2(n_i - 2)} + 2(n_i - 1)\tau\frac{1}{2(n_i - 2)} - \frac{1}{n_i} \\
=& \frac{n_i - 1}{n_i - 2}\tau + \frac{1}{n_i(n_i - 2)}.
\end{aligned}$$

(ii) It is easy to see that $\hat{\tau}_i$ is an unbiased estimation of τ, and

$$\hat{\tau}_i = \frac{n_i - 2}{n_i - 1}\tilde{\tau} - \frac{1}{(n_i - 1)n_i} = \frac{n_i - 2}{n_i}\frac{V}{U} + \frac{2(n_i - 2)\tau}{U} - \frac{1}{n_i},$$

$$D(\hat{\tau}_i) = D\left[\frac{n_i-2}{n_i}\frac{V}{U} + \frac{2(n_i-2)\tau}{U} - \frac{1}{n_i}\right] = (n_i-2)^2 D\left(\frac{1}{n_i}\frac{V}{U} + \frac{2\tau}{U}\right)$$
$$= (n_i-2)^2\left[\frac{1}{n_i^2}D(VU^{-1}) + 4\tau^2 D(U^{-1}) + \frac{4\tau}{n_i}\mathrm{cov}(VU^{-1}, U^{-1})\right],$$

$$D(VU^{-1}) = E(V^2U^{-2}) - [E(VU^{-1})]^2 = E(V^2)E(U^{-2}) - [E(V)E(U^{-1})]^2$$
$$= 8\frac{1}{4(n_i-2)(n_i-3)} - \left[2\frac{1}{2(n_i-2)}\right]^2$$
$$= \frac{2}{(n_i-2)(n_i-3)} - \frac{1}{(n_i-2)^2} = \frac{n_i-1}{(n_i-2)^2(n_i-3)},$$

$$\mathrm{cov}(VU^{-1}, U^{-1}) = E(VU^{-2}) - E(VU^{-1})E(U^{-1})$$
$$= E(V)E(U^{-2}) - E(V)[E(U^{-1})]^2$$
$$= E(V)D(U^{-1}) = \frac{1}{2(n_i-2)^2(n_i-3)},$$

$$D(\hat{\tau}_i) = (n_i-2)^2\left[\frac{1}{n_i^2}\frac{n_i-1}{(n_i-2)^2(n_i-3)} + 4\tau^2\frac{1}{4(n_i-2)^2(n_i-3)}\right.$$
$$\left.+\frac{4\tau}{n_i}\frac{1}{2(n_i-2)^2(n_i-3)}\right]$$
$$= \frac{1}{n_i-3}\left(\tau^2 + \frac{2}{n_i}\tau + \frac{n_i-1}{n_i^2}\right).$$

□

By denoting

$$w_i = D(\hat{\tau}_i) = \frac{1}{n_i-3}\left(\tau_i^2 + \frac{2}{n_i}\tau_i + \frac{n_i-1}{n_i^2}\right),$$
$$\hat{w}_i = \frac{1}{n_i-3}\left(\hat{\tau}_i^2 + \frac{2}{n_i}\hat{\tau}_i + \frac{n_i-1}{n_i^2}\right), i = 1, 2, \cdots, k,$$

the unbiased estimation of τ can be $\hat{\tau} = \frac{\sum_{i=1}^k w_i^{-1}\hat{\tau}_i}{\sum_{i=1}^k w_i^{-1}}$, and

$$D(\hat{\tau}) = \frac{\sum_{i=1}^k w_i^{-2}D(\hat{\tau}_i)}{\left(\sum_{i=1}^k w_i^{-1}\right)^2} = \left(\sum_{i=1}^k w_i^{-1}\right)^{-1}.$$

When n_i, $i = 1, 2, \cdots, k$ is extremely large, we have the following approximation

$$\sqrt{\sum_{i=1}^k w_i^{-1}}(\hat{\tau} - \tau)\dot{\sim}N(0,1), \quad \sqrt{\sum_{i=1}^k \hat{w}_i^{-1}}(\hat{\tau} - \tau)\dot{\sim}N(0,1).$$

At the confidence level $1-\alpha$, denote that the upper $\frac{\alpha}{2}$ quantile of the standard normal distribution $N(0,1)$ is $U_{\alpha/2}$, and the approximate confidence interval for the parameter τ is

$$\left[\hat{\tau}-\sqrt{\left(\sum_{i=1}^{k}\hat{w}_i^{-1}\right)^{-1}}U_{\alpha/2}, \hat{\tau}+\sqrt{\left(\sum_{i=1}^{k}\hat{w}_i^{-1}\right)^{-1}}U_{\alpha/2}\right].$$

Let $n=70$, $k=4$, $n_1=5$, $n_2=10$, $n_3=20$, $n_4=35$, and constant stress levels be $V_1=10V$, $V_2=20V$, $V_3=30V$, $V_4=40V$, respectively. The number of times that the condition $\lim_{c\to 0}\mathcal{T}(c) < F_{1-\alpha/2}\left(2\left(\sum_{i=l+1}^{k} n_i-k+l\right), 2\left(\sum_{i=1}^{l} n_i-l\right)\right)$ met is calculated by 1000 Monte Carlo simulations. Besides, among 1000 simulations that satisfy above condition, we calculate the average lower bound, the average upper bound, the average interval length, and the number of times that the interval contains the true value c when the confidence level is $1-\alpha=0.95$. The results are shown in Table 2. At the same time, the average lower bound, the average upper bound, the average interval length, and the number of times that the interval contains the true value τ are calculated by 1000 Monte Carlo simulations when the confidence level is and the corresponding results are shown in Table 3. It can be seen from Tables 2 and 3 that most of the observations meet the conditions, and the number of times satisfied the conditions shows an increasing trend as the value of c increases, and the interval contains more than 950 times of the true value of the parameter as well.

Table 2. Simulation results for interval estimation of parameter c

truth values of parameters			times	average lower bound	average upper bound	average interval length	number of times that the interval contains the true value c
c	d	τ					
1	0.001	0.1	789	0.4228	1.9692	1.5464	963
		0.2	793	0.4264	1.9896	1.5633	974
		0.3	805	0.4429	2.0012	1.5583	963
	0.01	0.1	811	0.4341	1.9938	1.5597	969
		0.2	761	0.4353	2.0011	1.5658	966
		0.3	788	0.4389	1.9925	1.5536	959
	0.1	0.1	770	0.4418	2.0057	1.5639	969
		0.2	784	0.4373	1.9944	1.5570	970
		0.3	798	0.4440	2.0022	1.5582	957
2	0.001	0.1	997	1.2824	2.8392	1.5568	957
		0.2	998	1.2946	2.8420	1.5475	952
		0.3	998	1.2956	2.8541	1.5585	956
	0.01	0.1	998	1.2788	2.8310	1.5522	955
		0.2	1000	1.2814	2.8438	1.5624	955
		0.3	999	1.2987	2.8601	1.5614	958
	0.1	0.1	997	1.3149	2.8680	1.5531	951
		0.2	998	1.2993	2.8488	1.5495	954
		0.3	998	1.2745	2.8231	1.5486	961
4	0.001	0.1	1000	3.2927	4.8458	1.5531	951
		0.2	1000	3.2948	4.8496	1.5548	952
		0.3	1000	3.2598	4.8080	1.5483	954
	0.01	0.1	1000	3.2928	4.8446	1.5518	957
		0.2	1000	3.2799	4.8449	1.5651	956
		0.3	1000	3.2859	4.8460	1.5601	957
	0.1	0.1	1000	3.3063	4.8624	1.5561	956
		0.2	1000	3.2724	4.8269	1.5545	953
		0.3	1000	3.3008	4.8600	1.5592	963

Table 3. Simulation results for interval estimation of parameter τ

truth values of parameters			average lower bound	average upper bound	average interval length	number of times that the interval contains the true value τ
c	d	τ				
1	0.001	0.1	0.0336	0.1547	0.1211	986
		0.2	0.1102	0.2677	0.1575	973
		0.3	0.1831	0.3831	0.2000	951
	0.01	0.1	0.0332	0.1543	0.1211	989
		0.2	0.1107	0.2685	0.1578	976
		0.3	0.1835	0.3837	0.2002	950
	0.1	0.1	0.0336	0.1548	0.1212	988
		0.2	0.1104	0.2683	0.1578	969
		0.3	0.1833	0.3837	0.2004	954
2	0.001	0.1	0.0318	0.1519	0.1202	978
		0.2	0.1089	0.2654	0.1564	965
		0.3	0.1816	0.3797	0.1981	955
	0.01	0.1	0.0325	0.1529	0.1204	993
		0.2	0.1089	0.2652	0.1563	969
		0.3	0.1787	0.3754	0.1967	955
	0.1	0.1	0.0316	0.1517	0.1201	988
		0.2	0.1103	0.2675	0.1572	967
		0.3	0.1808	0.3783	0.1975	951
4	0.001	0.1	0.0319	0.1521	0.1202	989
		0.2	0.1072	0.2627	0.1555	952
		0.3	0.1808	0.3788	0.1979	952
	0.01	0.1	0.0323	0.1529	0.1206	987
		0.2	0.1071	0.2628	0.1556	964
		0.3	0.1815	0.3799	0.1984	950
	0.1	0.1	0.0319	0.1522	0.1203	988
		0.2	0.1091	0.2656	0.1566	961
		0.3	0.1828	0.3820	0.1992	954

3.3. Examples of point estimations and interval estimations of parameters

Example 3.1. Nelson [8] gives the time for the oil to break through the insulating liquid at high test voltages. Assuming that the voltage is used as the stress level and the acceleration model is an inverse power law model, the normal stress level of the test is $V_0 = 20KV$, and the corresponding failure data (unit: minutes) is obtained as shown in Table 4 when the constant stress acceleration life test is performed at various acceleration stress levels.

Table 4. Failure data for Example 3.1

28 kV	68.85, 108.29, 110.29, 426.07, 1067.60
30 kV	7.74, 17.05, 20.46, 21.02, 22.66, 43.4, 47.3, 139.07, 144.12, 175.88, 194.9
32 kV	0.27, 0.40, 0.69, 0.79, 2.75, 3.91, 9.88, 13.95, 15.93, 27.80, 53.24, 82.85, 89.29, 100.58, 215.10
34 kV	0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, 72.89
36 kV	0.35, 0.59, 0.96, 0.99, 1.69, 1.97, 2.07, 2.58, 2.71, 2.90, 3.67, 3.99, 5.35, 13.77, 25.50
38 kV	0.09, 0.39, 0.47, 0.73, 0.74, 1.13, 1.40, 2.38

Using the method in this paper, the maximum likelihood estimates of the parameters c, d, τ can be obtained $\hat{c} = 17.7996$, $\hat{d} = 4.59894 \times 10^{-29}$, $\hat{\tau} = 0.007675$.

At a confidence level of 0.95, the confidence interval for the parameter c is $[13.5938, 21.3561]$, and the upper confidence bound for the parameter τ is 0.0234.

Example 3.2. Let $n = 80$, $k = 5$, $n_1 = 5$, $n_2 = 10$, $n_3 = 15$, $n_4 = 20$, $n_5 = 30$, and the constant stress levels are $V_1 = 10V$, $V_2 = 20V$, $V_3 = 30V$, $V_4 = 40V$, $V_5 = 50V$, respectively. The truth values of parameters are $c = 3$, $d = 0.01$, $\tau = 0.2$. The Monte Carlo simulation generates the two-parameter exponential distribution of constant stress accelerated life test data as shown in Table 5.

Table 5. Simulated sample data for Example 3.2

$V_1 = 10V$	0.0314205, 0.0359857, 0.059218, 0.146946, 0.165314
$V_2 = 20V$	0.00259428, 0.00374237, 0.00382828, 0.00514573, 0.00758829, 0.0113899, 0.0136778, 0.0351629, 0.0377542, 0.0516479
$V_3 = 30V$	0.000958262, 0.00194195, 0.0020529, 0.00211367, 0.00212201, 0.00214035, 0.00244233, 0.00252733, 0.00317994, 0.00330975, 0.00357749, 0.00382611, 0.00384307, 0.003916, 0.00422746
$V_4 = 40V$	0.000343732, 0.000457334, 0.000966568, 0.00113147, 0.00139871, 0.00160937, 0.00175745, 0.00177247, 0.00183533, 0.00184875, 0.00303843, 0.00304948, 0.00308138, 0.00326221, 0.00333005, 0.0035067, 0.0038699, 0.00661915, 0.00696539, 0.00752688
$V_5 = 50V$	0.000178625, 0.000187964, 0.000207027, 0.000305789, 0.000315769, 0.000359765, 0.000362945, 0.000456046, 0.000466002, 0.000487773, 0.000527234, 0.000552236, 0.000568894, 0.000627443, 0.000669463, 0.000732436, 0.000751225, 0.000770397, 0.000815137, 0.000864102, 0.00100472, 0.00100706, 0.00101873, 0.00102434, 0.00122256, 0.00159513, 0.0015996, 0.00190227, 0.00267328, 0.00293717

Using the method in this paper, the maximum likelihood estimates of the parameters c, d, τ are obtained $\hat{c} = 2.84809$, $\hat{d} = 0.01387$, $\hat{\tau} = 0.17094$ respectively.

The confidence interval for the parameter c is $[2.4475, 3.8325]$, and the confidence interval for the parameter τ is $[0.0742, 0.2211]$ at a confidence level of 0.95.

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