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SOLVABILITY FOR COUPLED IMPULSIVE FRACTIONAL PROBLEMS OF THE KIRCHHOFF TYPE WITH $P\&Q$-LAPLACIAN

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Abstract In this paper, we investigate the existence and multiplicity of nontrivial solutions for the $p\&q$-Laplacian Kirchhoff impulsive fractional differential equations through variational methods. By utilizing the Nehari manifold and fibering maps, we establish the existence of at least one nontrivial solution to such equations for any $(\lambda, \mu) \in \Theta_*$. Furthermore, using the idea of truncation arguments and Krasnoselskii genus theory, we demonstrate the existence of infinitely many nontrivial solutions for the equation when Kirchhoff functions M_1 and M_2 are degenerate considering any $(\lambda, \mu) \in \Theta_{**}$.

Keywords Kirchhoff fractional differential equations, $p\&q$-Laplacian, impulsive problems, variational methods.

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1. Introduction

This paper focuses on investigating the existence and multiplicity of nontrivial solutions to the $p\&q$-Laplacian impulsive fractional differential equations involving Kirchhoff functions:

$$\begin{cases} \quad [M_1(\|(u,v)\|_\alpha^p)]^{p-1}\ {}_tD_T^\alpha \Phi_p({}_0D_t^\alpha u(t)) + [M_2(\|(u,v)\|_\beta^q)]^{q-1}\ {}_tD_T^\beta \Phi_q({}_0D_t^\beta u(t)) \\ = F_u(t,u(t),v(t)),\ t \neq t_j,\ \text{a.e. } t \in [0,T], \\ \quad [M_1(\|(u,v)\|_\alpha^p)]^{p-1}\ {}_tD_T^\alpha \Phi_p({}_0D_t^\alpha v(t)) + [M_2(\|(u,v)\|_\beta^q)]^{q-1}\ {}_tD_T^\beta \Phi_q({}_0D_t^\beta v(t)) \\ = F_v(t,u(t),v(t)),\ t \neq t_i',\ \text{a.e. } t \in [0,T], \\ \Delta([M_1(\|(u,v)\|_\alpha^p)]^{p-1}\ {}_tD_T^{\alpha-1}\Phi_p({}_0D_t^\alpha u))(t_j) \\ + \Delta([M_2(\|(u,v)\|_\beta^q)]^{q-1}\ {}_tD_T^{\beta-1}\Phi_q({}_0D_t^\beta u))(t_j) + \lambda\zeta(t_j)I_j(u(t_j)) = 0,\ j \in \Delta_1, \\ \Delta([M_1(\|(u,v)\|_\alpha^p)]^{p-1}\ {}_tD_T^{\alpha-1}\Phi_p({}_0D_t^\alpha v))(t_i') \\ + \Delta([M_2(\|(u,v)\|_\beta^q)]^{q-1}\ {}_tD_T^{\beta-1}\Phi_q({}_0D_t^\beta v))(t_i') + \mu\varrho(t_i')S_i(v(t_i')) = 0,\ i \in \Delta_2, \\ u(0) = u(T) = 0,\ v(0) = v(T) = 0, \end{cases} \tag{1.1}$$

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where $M_1, M_2 \in C^1(R_0^+, R_0^+)$, $1 < p, q < \infty$, $\alpha \in (\frac{1}{p}, 1]$, $\beta \in (\frac{1}{q}, 1]$, ${}_0D_t^\gamma$ and ${}_tD_T^\gamma$ denote left and right standard Riemann-Liouville fractional derivatives, respectively, $\gamma \in \{\alpha, \beta\}$, $\Phi_\upsilon(z) = |z|^{\upsilon-2}z$, $\upsilon > 1$, $\upsilon \in \{p, q\}$, λ, μ are real parameters, $\zeta(t)$, $\varrho(t) \in C([0,T], R)$ with $\zeta_\pm(t) = \pm\max\{\pm\zeta(t), 0\} \neq 0$, $\varrho_\pm(t) = \pm\max\{\pm\varrho(t), 0\} \neq 0$, $F : [0,T] \times R^2 \to R$ is continuous with respect to t, for all $(u, v) \in R^2$, continuously differentiable with respect to u and v for almost every $t \in [0,T]$, F_u and F_v denote the partial derivatives of F with respect to u and v, norms $\|\cdot\|_\alpha$, $\|\cdot\|_\beta$ are specified later, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $0 = t'_0 < t'_1 < \cdots < t'_n < t'_{n+1} = T$, $\Delta_1 = \{1, 2, \cdots, m\}$, $\Delta_2 = \{1, 2, \cdots, n\}$, I_j, $S_i \in C^1(R, R)$ for all $j \in \Delta_1$, $i \in \Delta_2$ and

$$\begin{aligned}
&\Delta([M_1(\|(u,v)\|_\alpha^p)]^{p-1}\ {}_tD_T^{\alpha-1}\Phi_p({}_0D_t^\alpha u))(t_j)\\
&\quad + \Delta([M_2(\|(u,v)\|_\beta^q)]^{q-1}\ {}_tD_T^{\beta-1}\Phi_q({}_0D_t^\beta u))(t_j)\\
=&[M_1(\|(u(t_j^+), v(t_j^+))\|_\alpha^p)]^{p-1}\ {}_tD_T^{\alpha-1}\Phi_p({}_0D_t^\alpha u(t_j^+))\\
&\quad + [M_2(\|(u(t_j^+), v(t_j^+))\|_\beta^q)]^{q-1}\ {}_tD_T^{\beta-1}\Phi_q({}_0D_t^\beta u(t_j^+))\\
&\quad - [M_1(\|(u(t_j^-), v(t_j^-))\|_\alpha^p)]^{p-1}\ {}_tD_T^{\alpha-1}\Phi_p({}_0D_t^\alpha u(t_j^-))\\
&\quad - [M_2(\|(u(t_j^-), v(t_j^-))\|_\beta^q)]^{q-1}\ {}_tD_T^{\beta-1}\Phi_q({}_0D_t^\beta u(t_j^-)),
\end{aligned}$$

and

$$\begin{aligned}
&\Delta([M_1(\|(u,v)\|_\alpha^p)]^{p-1}\ {}_tD_T^{\alpha-1}\Phi_p({}_0D_t^\alpha v))(t'_i)\\
&\quad + \Delta([M_2(\|(u,v)\|_\beta^q)]^{q-1}\ {}_tD_T^{\beta-1}\Phi_q({}_0D_t^\beta v))(t'_i)\\
=&[M_1(\|(u(t_i'^+), v(t_i'^+))\|_\alpha^p)]^{p-1}\ {}_tD_T^{\alpha-1}\Phi_p({}_0D_t^\alpha v(t_i'^+))\\
&\quad + [M_2(\|(u(t_i'^+), v(t_i'^+))\|_\beta^q)]^{q-1}\ {}_tD_T^{\beta-1}\Phi_q({}_0D_t^\beta v(t_i'^+))\\
&\quad - [M_1(\|(u(t_i'^-), v(t_i'^-))\|_\alpha^p)]^{p-1}\ {}_tD_T^{\alpha-1}\Phi_p({}_0D_t^\alpha v(t_i'^-))\\
&\quad - [M_2(\|(u(t_i'^-), v(t_i'^-))\|_\beta^q)]^{q-1}\ {}_tD_T^{\beta-1}\Phi_q({}_0D_t^\beta v(t_i'^-)),
\end{aligned}$$

where

$$\begin{aligned}
&[M_1(\|(u(t_j^\pm), v(t_j^\pm))\|_\alpha^p)]^{p-1}\ {}_tD_T^{\alpha-1}\Phi_p({}_0D_t^\alpha u(t_j^\pm))\\
&\quad + [M_2(\|(u(t_j^\pm), v(t_j^\pm))\|_\beta^q)]^{q-1}\ {}_tD_T^{\beta-1}\Phi_q({}_0D_t^\beta u(t_j^\pm))\\
=&\lim_{t\to t_j^\pm}[M_1(\|(u(t), v(t))\|_\alpha^p)]^{p-1}\ {}_tD_T^{\alpha-1}\Phi_p({}_0D_t^\alpha u(t))\\
&\quad + [M_2(\|(u(t), v(t))\|_\beta^q)]^{q-1}\ {}_tD_T^{\beta-1}\Phi_q({}_0D_t^\beta u(t)),
\end{aligned}$$

and

$$\begin{aligned}
&[M_1(\|(u(t_i'^\pm), v(t_i'^\pm))\|_\alpha^p)]^{p-1}\ {}_tD_T^{\alpha-1}\Phi_p({}_0D_t^\alpha v(t_i'^\pm))\\
&\quad + [M_2(\|(u(t_i'^\pm), v(t_i'^\pm))\|_\beta^q)]^{q-1}\ {}_tD_T^{\beta-1}\Phi_q({}_0D_t^\beta v(t_i'^\pm))\\
=&\lim_{t\to t_i'^\pm}[M_1(\|(u(t), v(t))\|_\alpha^p)]^{p-1}\ {}_tD_T^{\alpha-1}\Phi_p({}_0D_t^\alpha v(t))\\
&\quad + [M_2(\|(u(t), v(t))\|_\beta^q)]^{q-1}\ {}_tD_T^{\beta-1}\Phi_q({}_0D_t^\beta v(t)).
\end{aligned}$$

Fractional differential equations (FDEs) are an extension of ordinary differential equations and integration to arbitrarily noninteger orders that have risen in prominence as a result of their extensive applications in many disciplines of research and engineering. In actuality, several equations incorporating fractional order derivatives are used to simulate a wide range of phenomena, including fluid flow, diffusive transport comparable to diffusion, rheology, probability, and electrical networks; see previous studies (see [2, 7, 13]). It should be emphasized that the existence and multiplicity of solutions to nonlinear boundary value problems of FDEs have been extensively investigated using critical point theory and variational methods (see [11, 12, 16, 19, 29]). In recent years, many scholars have devoted themselves to the study of classical $p\&q$-Laplacian elliptic type equations over bounded and unbounded domains using variational methods and critical point theory (see [8,21,23]). At the same time, we are aware of very few contributions concerning FDEs with $p\&q$-Laplacian and impulsive terms (see [17,18,30]). For example, Li et al. [18] were the first to study the following impulsive fractional boundary value problem with $p\&q$-Laplacian operators:

$$\begin{cases} \quad {}_tD_T^{\alpha}\Phi_p({}_0D_t^{\alpha}u(t)) + |u(t)|^{p-2}u(t) + \ {}_tD_T^{\beta}\Phi_q({}_0D_t^{\beta}u(t)) + |u(t)|^{q-2}u(t) \\ = f(t, u(t),\ {}_0D_t^{\alpha}u(t),\ {}_0D_t^{\beta}u(t)),\ t \neq t_j,\ \text{a.e. } t \in [0, T], \\ \quad \Delta({}_tD_T^{\alpha-1}\Phi_p({}_0^cD_t^{\alpha}u) + \ {}_tD_T^{\beta-1}\Phi_q({}_0^cD_t^{\beta}u))(t_j) \\ = I_j(u(t_j)),\ j = 1, 2, \cdots, m, u(0) = u(T) = 0,\ \text{a.e. } t \in [0, T], \end{cases} \tag{1.2}$$

where ${}_0D_t^{\alpha}$, ${}_0D_t^{\beta}$ and ${}_tD_T^{\alpha}$, ${}_tD_T^{\beta}$ denote the left and right Riemann-Liouville fractional derivatives, respectively. Based on the Mountain pass theorem and the iterative technique, Li et al. obtained the existence of at least one nontrivial solution to problem (1.2).

On the other hand, great attention has recently been focused on the study of Kirchhoff-type differential equations which stand out due to the presence of a Kirchhoff function $M \in C(R_0^+, R_0^+)$. In recent years, there have been fruitful achievements in using variational methods to study the existence and multiplicity of solutions for Kirchhoff-type differential equations (see [1,25,26,28]). In addition, some new results have been obtained in the study of Kirchhoff-type FDEs by variational methods and critical point theory when the Kirchhoff function M is given by $M(s) = a + bs$ for all $s \in R_0^+$, where $a, b > 0$. For example, Kratou [15] studied the existence and multiplicity of solutions to the Kirchhoff fractional equation with singular nonlinearity:

$$\begin{cases} \left(a + b\int_0^T |{}_0D_t^{\alpha}u(t)|^p dt\right)^{p-1} {}_tD_T^{\alpha}\Phi_p({}_0D_t^{\alpha}u(t)) = \dfrac{\lambda g(t)}{u^{\gamma}(t)} + f(t, u(t)),\ t \in (0, T), \\ u(0) = u(T) = 0, \end{cases} \tag{1.3}$$

where $a \geq 1$, $b, \lambda > 0$, $p > 1$, $\alpha \in (\frac{1}{p}, 1]$, $0 < \gamma < 1 < p < r$, $g \in C([0, T])$. Using the idea of the Nehari manifold technique, Kratou proved the existence of at least two positive solutions to the problem (1.3). When $\frac{\lambda g(t)}{u^{\gamma}(t)} = 0$, Chen and Liu [5] used the Nehari manifold method to obtain that there exists at least one nontrivial ground state solution to problem (1.3); Chen et al. [6] obtained that problem (1.3) has at least one solution and infinitely many nontrivial weak solutions by using the mountain pass theorem and the genus properties. To the best of my knowledge, there are few related results on the existence and multiplicity of nontrivial solutions for a class of coupled $p\&q$-Laplacian Kirchhoff-type impulsive fractional problems.

Mainly inspired by the above works, we investigate the existence and multiplicity of solutions for problem (1.1) by using variational methods and critical point theory.

Problem (1.1) is more general in the form of the equation. For examples, when $M_1 = M_2 = 1$, problem (1.1) can be reduced to the $p\&q$-Laplacian impulsive fractional differential equations; when $M_1 = M_2 = 1$, $p = q = 2$, and $\alpha = \beta = 1$, problem (1.1) is transformed into the following second-order impulsive differential equations:

$$\begin{cases} -2u''(t) = F_u(t, u(t), v(t)), \ t \neq t_j, \ \text{a.e. } t \in [0, T], \\ -2v''(t) = F_v(t, u(t), v(t)), \ t \neq t'_i, \ \text{a.e. } t \in [0, T], \\ 2\Delta(u'(t_j)) + \lambda\zeta(t_j)I_j(u(t_j)) = 0, \ j \in \Delta_1, \\ 2\Delta(v'(t'_i)) + \mu\varrho(t'_i)S_i(v(t'_i)) = 0, \ i \in \Delta_2, \\ u(0) = u(T) = 0, \quad v(0) = v(T) = 0. \end{cases}$$

For that reason, problem (1.1) is an extension of the integer-order impulsive differential equation.

Another novel aspect of this paper is that we provide an existence result utilising the technique of Nehari manifold and fibering maps and a multiplicity result utilising the Krasnoselskii genus for problem (1.1). The main challenge in employing Nehari manifold analysis is that Kirchhoff functions M_1 and M_2 are not specified as $M_1(s) = M_2(s) = a + bs^{\vartheta-1}$ for all $s \in R_0^+$, where $a, b > 0$, $\vartheta \geq 1$, leading to an analysis completely different from Xie and Chen [28], Kratou [15], and Fiscella and Mishra [9]. For the purpose to obtain the critical point of the functional $J_{\lambda,\mu}$ associated with problem (1.1), we are devoted to analyze the behavior of functional $J_{\lambda,\mu}$ over its Nehari Nehari manifold $\mathcal{N}_{\lambda,\mu}$. Then, we introduce the decomposition $\mathcal{N}_{\lambda,\mu} = \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu}$. It is promptly seen that critical points of $J_{\lambda,\mu}|_{\mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu}}$ are critical points of $J_{\lambda,\mu}$. Accordingly, it is crucial to make sure that $\mathcal{N}^0_{\lambda,\mu} = \emptyset$. Considering the generality of the Kirchhoff functions M_1 and M_2, which lead to $\mathcal{N}^0_{\lambda,\mu} \neq \emptyset$, condition (M1) effectively avoids this. Moreover, condition (F1) has vital significance in that the homogeneity and Euler identity of F enable us to successfully establish the Nehari Nehari manifold $\mathcal{N}_{\lambda,\mu}$. Condition (F2) is also indispensable to ensure the compactness of functional $J_{\lambda,\mu}$. Motivated by Xiang et al. [27], we provide an multiplicity result for problem (1.1) by the Krasnoselskii genus. Xiang et al. [27] investigated the Kirchhoff function $M(s) = a + b\theta s^{\theta-1}$ for all $s \geq 0$, where $a \geq 0$, $b \geq 0$, $a + b > 0$ and $\theta > 1$. Nevertheless, M_1 and M_2 are more general, making the analysis more challenging. Moreover, taking into account the nonlocal nature of Kirchhoff functions M_1 and M_2 and the impact of sign-changing weight functions ζ and ϱ, we establish some new estimates under the perturbing influence of the pair of parameters (λ, μ) to gain truncated functional $\overline{J}_{\lambda,\mu}$. Finally, it is evident from conditions (M1′) and (M2′) that Kirchhoff functions M_1 and M_2 are degenerate, which forces us to discuss the (PS)-sequence compactness of truncated functional $\overline{J}_{\lambda,\mu}$ in three cases, as displayed in (ii) of Lemma 4.2 below.

In order to precisely establish the existence result for problem (1.1) in Section 3, we introduce the following assumptions conditions:

(M1) There exist constants $1 < p\vartheta_1, q\vartheta_2 < p, q \leq \kappa \leq p^2, q^2$, $\eta_1, \eta_2 > 1$, $0 < \xi_1 < 1 < \delta_1$, $(\sigma - p)\xi_1\eta_1 > (p^2 - p)\delta_1$, $0 < \xi_2 < 1 < \delta_2$, $(\sigma - q)\xi_2\eta_2 > (q^2 - q)\delta_2$, $\xi_1\sigma > \vartheta_1\delta_1 p$, $(2p + p^2)\delta_1\theta < \xi_1$, $\xi_2\sigma > \vartheta_2\delta_2 q$ and $(2q + q^2)\delta_2\theta < \xi_2$ such that

$$\xi_1 x^{\frac{\kappa}{p}} < \vartheta_1\overline{M}_1(x) \leq [M_1(x)]^{p-1}x \leq \eta_1[M_1(x)]^{p-2}M'_1(x)x^2 < \delta_1 x^{\frac{\kappa}{p}},$$
$$\xi_2 y^{\frac{\kappa}{q}} < \vartheta_2\overline{M}_2(y) \leq [M_2(y)]^{q-1}y \leq \eta_2[M_2(y)]^{q-2}M'_2(y)y^2 < \delta_2 y^{\frac{\kappa}{q}},$$

for all x, $y > 0$, where $\overline{M}_1(x) = \int_0^x [M_1(s)]^{p-1}ds$, $\overline{M}_2(y) = \int_0^y [M_2(s)]^{q-1}ds$;

(H1) There exist positive constants a_j, b_j with $\overline{a} = \max\{a_j\}$, $\overline{b} = \max\{b_i\}$ for all $j \in \Delta_1$, $i \in \Delta_2$, $0 < \theta < 1 < p, q$ such that

$$|I_j(z)z| \leq a_j|z|^\theta, \text{ and } |S_i(z)z| \leq b_i|z|^\theta, \ \forall z \in R;$$

(H2) For all $j \in \Delta_1$, $i \in \Delta_2$, I_j, $K_i : R \to R$ such that

$$I_j(su(t_j)) = s^{\theta-1}I_j(u(t_j)), \text{ and } S_i(sv(t_i')) = s^{\theta-1}S_i(v(t_i')),$$

for all $s > 0$ and $(u, v) \in R^2$;

(F1) $F(t, u, v)$ is σ-homogeneous and satisfies Euler identity with $\sigma\theta > 1$, $\sigma > p^2, q^2$, that is

$$F(t, su, sv) = s^\sigma F(t, u, v), \text{ and } uF_u(t, u, v) + vF_v(t, u, v) = \sigma F(t, u, v),$$

for all $(u, v) \in R^2$;

(F2) There exists constant $C_* > 0$ such that

$$|F(t, u, v)| \leq C_*|(u, v)|^\sigma, \text{ for all } (u, v) \in R^2,$$

where $|(u, v)| = (u^2 + v^2)^{\frac{1}{2}}$.

In order to obtain the multiplicity result by the Krasnoselskii genus for problem (1.1) in Section 4, we give the following assumptions conditions:

(M1′) There exist $\overline{\vartheta}_1, \overline{\vartheta}_2 \geq 1$ such that

$$[M_1(x)]^{p-1}x \leq \overline{\vartheta}_1\overline{M}_1(x), \text{ and } [M_2(y)]^{q-1}y \leq \overline{\vartheta}_2\overline{M}_2(y), \text{ for all } x, y \in R_0^+;$$

(M2′) $M_1(0) = M_2(0) = 0$, and for any $\tau, \overline{\tau} > 0$, there exist $m_1 = m_1(\tau) > 0$ and $m_2 = m_2(\overline{\tau}) > 0$ such that

$$M_1(x) \geq m_1, \text{ for all } x \geq \tau, \text{ and } M_2(y) \geq m_2, \text{ for all } y \geq \overline{\tau};$$

(F1′) There exists $\overline{\eta}$ with $\overline{\vartheta}_* = \max\{p\overline{\vartheta}_1, q\overline{\vartheta}_2\} < \overline{\eta}$ such that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ for which

$$|\nabla F(t, u, v)| \leq \overline{\vartheta}_*\varepsilon|(u, v)|^{\overline{\vartheta}_*-1} + \overline{\eta}C_\varepsilon|(u, v)|^{\overline{\eta}-1}, \text{ for all } (u, v) \in R^2,$$

holds, where $|(u, v)| = (u^2 + v^2)^{\frac{1}{2}}$ and $\nabla F = (F_u, F_v)$;

(F2′) There exists $0 < \overline{\varsigma} < \theta < 1$ such that

$$0 \leq \overline{\varsigma}F(t, u, v) \leq \nabla F(t, u, v) \cdot (u, v), \text{ for all } (u, v) \in R^2;$$

(F3′) $F(t, -u, -v) = F(t, u, v)$ for all $(t, u, v) \in [0, T] \times R^2$, $I_j(-z) = -I_j(z)$ and $S_i(-z) = -S_i(z)$ for all $z \in R$, $j \in \Delta_1$, $i \in \Delta_2$.

Remark 1.1. Kirchhoff problems are said to be non-degenerate if $\inf_{s \in R_0^+} M(s) > 0$ and degenerate if $M(0) = 0$ and $\inf_{s \in R^+} M(s) > 0$. some recent results concerning the degenerate case on Kirchhoff-type problems are referred to in [10, 25, 27].

Our main results are as follows:

Theorem 1.1. *Assume that (M1), (H1), (H2) and (F1), (F2) hold. Then, for any* $(\lambda, \mu) \in \Theta_*$*, problem* (1.1) *admits at least a nontrivial solution.*

Theorem 1.2. *Assume that (H1), (M1′), (M2′) and (F1′)-(F3′) hold. Then, for any* $(\lambda, \mu) \in \Theta_{**}$*, problem* (1.1) *admits infinitely many nontrivial solutions.*

The paper is structured as follows: In Section 2, we recall some definitions of fractional calculus and discuss the variational formulation of problem (1.1). In Section 3, we introduce the Nehari manifold structure and fibering maps analysis related to problem (1.1) to obtain the existence of the solution. In Section 4, we apply Krasnoselskii genus theory to investigate the infinitely many solutions for problem (1.1).

2. Preliminaries

In this section, we present some preliminary findings that will be utilized in the subsequent sections.

For $0 < \gamma \leq 1$ and $1 < \upsilon < \infty$, the functional space that incorporates this boundary condition will be denoted by $E_{\gamma,\upsilon}$ as the closure of $C_0^\infty([0,T], R)$, where

$$E_{\gamma,\upsilon} = \Big\{ u \in L^\upsilon([0,T], R) \mid {}_0D_t^\gamma u(t) \in L^\upsilon([0,T], R), u(0) = u(T) = 0 \Big\},$$

endowed with the norm

$$\|z\|_{\gamma,\upsilon} = (\int_0^T |{}_0D_t^\gamma z(t)|^\upsilon dt)^{\frac{1}{\upsilon}}, \ \forall z \in E_{\gamma,\upsilon}.$$

Lemma 2.1 ([14]). *Let* $0 < \gamma \leq 1$ *and* $1 < \upsilon < \infty$*. For any* $z \in E_{\gamma,\upsilon}$*, we have*

$$\|z\|_{L^\upsilon} \leq \frac{T^\gamma}{\Gamma(\gamma+1)} \|{}_0D_t^\gamma z\|_{L^\upsilon} = \mathcal{A}_{\gamma,\upsilon} \|z\|_{\gamma,\upsilon}, \tag{2.1}$$

furthermore, when $\gamma > \frac{1}{\upsilon}$, $\frac{1}{\upsilon} + \frac{1}{\upsilon'} = 1$*, we have*

$$\|z\|_\infty \leq \frac{T^{\gamma - \frac{1}{\upsilon}}}{\Gamma(\gamma)((\gamma-1)\upsilon'+1)^{\frac{1}{\upsilon'}}} \|{}_0D_t^\gamma z\|_{L^\upsilon} = \overline{\mathcal{A}}_{\gamma,\upsilon} \|z\|_{\gamma,\upsilon}, \tag{2.2}$$

where $\mathcal{A}_{\gamma,\upsilon} = \frac{T^\gamma}{\Gamma(\gamma+1)}$, $\overline{\mathcal{A}}_{\gamma,\upsilon} = \frac{T^{\gamma-\frac{1}{\upsilon}}}{\Gamma(\gamma)((\gamma-1)\upsilon'+1)^{\frac{1}{\upsilon'}}}$.

Let $E_\alpha = E_{\alpha,p} \times E_{\alpha,p}$ and $E_\beta = E_{\beta,q} \times E_{\beta,q}$, which are reflexive Banach spaces endowed with the norms

$$\|(u,v)\|_\alpha = (\|u\|_{\alpha,p}^p + \|v\|_{\alpha,p}^p)^{\frac{1}{p}}, \text{ and } \|(u,v)\|_\beta = (\|u\|_{\beta,q}^q + \|v\|_{\beta,q}^q)^{\frac{1}{q}}.$$

Then, set $E_{\alpha,\beta} = E_\alpha \cap E_\beta$ endowed with the norm

$$\|(u,v)\|_{\alpha,\beta} = \|(u,v)\|_\alpha + \|(u,v)\|_\beta, \ \forall (u,v) \in E_{\alpha,\beta}. \tag{2.3}$$

For all $\overline{\upsilon} \geq p, q$ and $(u,v) \in E_{\alpha,\beta}$, by (2.2), we have

$$\|(u,v)\|_{L^{\overline{\upsilon}}}^{\overline{\upsilon}} \leq 2^{\overline{\upsilon}-1} (\|u\|_{L^{\overline{\upsilon}}([0,T])}^{\overline{\upsilon}} + \|v\|_{L^{\overline{\upsilon}}([0,T])}^{\overline{\upsilon}})$$

$$\leq 2^{\overline{\upsilon}-1}T\overline{\mathcal{A}}^{\overline{\upsilon}}_{\alpha,p}(\|u\|^{\overline{\upsilon}}_{\alpha,p}+\|v\|^{\overline{\upsilon}}_{\alpha,p})$$
$$\leq \overline{\mathcal{M}}(\overline{\upsilon})\|(u,v)\|^{\overline{\upsilon}}_{\alpha,\beta}, \tag{2.4}$$

where $\overline{\mathcal{M}}(\overline{\upsilon})=2^{\overline{\upsilon}-1}T\overline{\mathcal{A}}^{\overline{\upsilon}}_{\alpha,p}$, the inequality $2^{1-\overline{\imath}}(a+b)^{\overline{\imath}}\leq a^{\overline{\imath}}+b^{\overline{\imath}}\leq(a+b)^{\overline{\imath}}$ for all $\overline{\imath}\geq 1$ and $a,\,b\geq 0$ is applied.

Lemma 2.2 ([14]). *Let $\frac{1}{p}<\alpha\leq 1$, $\frac{1}{q}<\beta\leq 1$, $1<p,q<\infty$. The fractional derivative spaces E_α and E_β are reflexive and separable Banach spaces.*

Lemma 2.3 ([14]). *Let $\frac{1}{p}<\alpha\leq 1$, $\frac{1}{q}<\beta\leq 1$, $1<p,q<\infty$. Assume that the sequence $\{z_n\}$ converges weakly to z in E_α or E_β , i.e., $z_n\rightharpoonup z$. Then $\{z_n\}$ converges strongly to z in $(C([0,T],R))^2$, i.e., $\|z_n-z\|_\infty\to 0$ as $n\to\infty$.*

Lemma 2.4. *Let $\{(u_n,v_n)\}_n$ and (u,v) be in $E_{\alpha,\beta}$ such that $(u_n,v_n)\rightharpoonup(u,v)$ weakly in $E_{\alpha,\beta}$ and $(u_n,v_n)\to(u,v)$ a.e. in R. Then*

$$\|(u_n,v_n)\|^p_\alpha-\|(u_n-u,v_n-v)\|^p_\alpha=\|(u,v)\|^p_\alpha+o_n(1),\ \text{as } n\to\infty,$$
$$\|(u_n,v_n)\|^q_\beta-\|(u_n-u,v_n-v)\|^q_\beta=\|(u,v)\|^q_\beta+o_n(1),\ \text{as } n\to\infty.$$

Proof. The basis of our argument is derived from Lemma 3.2 in [20]. However, we have chosen to omit the proof for brevity. □

Definition 2.1. We say that the couple $(u,v)\in E_{\alpha,\beta}$ is a weak solution to problem (1.1) if for all $(x,y)\in E_{\alpha,\beta}$, it satisfies

$$\begin{aligned}&[M_1(\|(u,v)\|^p_\alpha)]^{p-1}\int_0^T\Phi_p({}_0D^\alpha_t u(t)){}_0D^\alpha_t x(t)+\Phi_p({}_0D^\alpha_t v(t)){}_0D^\alpha_t y(t)dt\\&+[M_2(\|(u,v)\|^q_\beta)]^{q-1}\int_0^T\Phi_q({}_0D^\beta_t u(t)){}_0D^\beta_t x(t)+\Phi_q({}_0D^\beta_t v(t)){}_0D^\beta_t y(t)dt\\&-\sum_{j=1}^m\lambda\zeta(t_j)I_j(u(t_j))x(t_j)-\sum_{i=1}^n\mu\varrho(t'_i)S_i(v(t'_i))y(t'_i)\\=&\int_0^T F_u(t,u(t),v(t))x(t)+F_v(t,u(t),v(t))y(t)dt.\end{aligned}\tag{2.5}$$

Problem (1.1) possesses a variational structure, and its solutions can be regarded as critical points of the functional $J_{\lambda,\mu}$, where

$$\begin{aligned}J_{\lambda,\mu}(u,v)=&\frac{1}{p}\overline{M}_1(\|(u,v)\|^p_\alpha)+\frac{1}{q}\overline{M}_2(\|(u,v)\|^q_\beta)-\sum_{j=1}^m\int_0^{u(t_j)}\lambda\zeta(t_j)I_j(z)dz\\&-\sum_{i=1}^n\int_0^{v(t'_i)}\mu\varrho(t'_i)S_i(s)ds-\int_0^T F(t,u(t),v(t))dt,\end{aligned}\tag{2.6}$$

for any $(u,v)\in E_{\alpha,\beta}$. Then, $J_{\lambda,\mu}:E_{\alpha,\beta}\to R$ is of class $C^1(E_{\alpha,\beta},R)$ and

$$\begin{aligned}&\langle J'_{\lambda,\mu}(u,v),(x,y)\rangle\\=&[M_1(\|(u,v)\|^p_\alpha)]^{p-1}\int_0^T\Phi_p({}_0D^\alpha_t u(t)){}_0D^\alpha_t x(t)+\Phi_p({}_0D^\alpha_t v(t)){}_0D^\alpha_t y(t)dt\\&+[M_2(\|(u,v)\|^q_\beta)]^{q-1}\int_0^T\Phi_q({}_0D^\beta_t u(t)){}_0D^\beta_t x(t)+\Phi_q({}_0D^\beta_t v(t)){}_0D^\beta_t y(t)dt\end{aligned}$$

$$- \sum_{j=1}^{m} \lambda\zeta(t_j) I_j(u(t_j)) x(t_j) - \sum_{i=1}^{n} \mu\varrho(t'_i) S_i(v(t'_i)) y(t'_i)$$
$$- \int_0^T F_u(t, u(t), v(t)) x(t) + F_v(t, u(t), v(t)) y(t) dt, \tag{2.7}$$

for any (u,v), $(x,y) \in E_{\alpha,\beta}$.

Lemma 2.5. *If the function $(u,v) \in E_{\alpha,\beta}$ is a weak solution of problem* (1.1), *then (u,v) is a classical solution of problem* (1.1).

Proof. If $(u,v) \in E_{\alpha,\beta}$ is a weak solution of problem (1.1), then (2.5) holds for any $(x,y) \in E_{\alpha,\beta}$. Choose a function $(x,y) \in C_0^\infty(t_j, t_{j+1}) \times C_0^\infty(t'_i, t'_{i+1})$ for any $j \in \Delta_1$ and $i \in \Delta_2$. Then, (2.5) becomes

$$\begin{aligned}
&[M_1(\|(u,v)\|_\alpha^p)]^{p-1} \Big(\int_{t_j}^{t_{j+1}} \Phi_p({}_0D_t^\alpha u)\, {}_0D_t^\alpha x dt + \int_{t'_i}^{t'_{i+1}} \Phi_p({}_0D_t^\alpha v)\, {}_0D_t^\alpha y dt \Big) \\
&+ [M_2(\|(u,v)\|_\beta^q)]^{q-1} \Big(\int_{t_j}^{t_{j+1}} \Phi_q({}_0D_t^\beta u)\, {}_0D_t^\beta x dt + \int_{t'_i}^{t'_{i+1}} \Phi_q({}_0D_t^\beta v)\, {}_0D_t^\beta y dt \Big) \\
&= \int_{t_j}^{t_{j+1}} F_u(t,u,v) x dt + \int_{t'_i}^{t'_{i+1}} F_v(t,u,v) y dt.
\end{aligned} \tag{2.8}$$

By using Definitions 2.1, 2.2 and Proposition 2.4 of [29], we have

$$\begin{aligned}
&[M_1(\|(u,v)\|_\alpha^p)]^{p-1} \Big(\int_{t_j}^{t_{j+1}} \Phi_p({}_0D_t^\alpha u)\, {}_0D_t^\alpha x dt + \int_{t'_i}^{t'_{i+1}} \Phi_p({}_0D_t^\alpha v)\, {}_0D_t^\alpha y dt \Big) \\
&+ [M_2(\|(u,v)\|_\beta^q)]^{q-1} \Big(\int_{t_j}^{t_{j+1}} \Phi_q({}_0D_t^\beta u)\, {}_0D_t^\beta x dt + \int_{t'_i}^{t'_{i+1}} \Phi_q({}_0D_t^\beta v)\, {}_0D_t^\beta y dt \Big) \\
=&[M_1(\|(u,v)\|_\alpha^p)]^{p-1} \Big(\int_{t_j}^{t_{j+1}} {}_tD_T^\alpha(\Phi_p({}_0D_t^\alpha u)) x dt + \int_{t'_i}^{t'_{i+1}} {}_tD_T^\alpha(\Phi_p({}_0D_t^\alpha v)) y dt \Big) \\
&+ [M_2(\|(u,v)\|_\beta^q)]^{q-1} \Big(\int_{t_j}^{t_{j+1}} {}_tD_T^\beta(\Phi_q({}_0D_t^\beta u)) x dt + \int_{t'_i}^{t'_{i+1}} {}_tD_T^\beta(\Phi_q({}_0D_t^\beta v)) y dt \Big).
\end{aligned} \tag{2.9}$$

Hence, we can obtain from (2.8) and (2.9) that

$$\begin{aligned}
&[M_1(\|(u,v)\|_\alpha^p)]^{p-1}\ {}_tD_T^\alpha \Phi_p({}_0D_t^\alpha u(t)) + [M_2(\|(u,v)\|_\beta^q)]^{q-1}\ {}_tD_T^\beta \Phi_q({}_0D_t^\beta u(t)) \\
=&F_u(t, u(t), v(t)),\ \forall t \in (t_j, t_{j+1}),
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
&[M_1(\|(u,v)\|_\alpha^p)]^{p-1}\ {}_tD_T^\alpha \Phi_p({}_0D_t^\alpha v(t)) + [M_2(\|(u,v)\|_\beta^q)]^{q-1}\ {}_tD_T^\beta \Phi_q({}_0D_t^\beta v(t)) \\
=&F_v(t, u(t), v(t)),\ \forall t \in (t'_i, t'_{i+1}),
\end{aligned} \tag{2.11}$$

which shows that (u,v) satisfies the equation of problem (1.1) a.e. on $[0,T]\backslash\{t_1, \cdots, t_m\} \times [0,T]\backslash\{t'_1, \cdots, t'_n\}$. Moreover, according to Proposition 2.6 of [29] and $M_1, M_2 \in C^1(R_0^+, R_0^+)$, we have

$$[M_1(\|(u(t_j^\pm), v(t_j^\pm))\|_\alpha^p)]^{p-1}\ {}_tD_T^{\alpha-1} \Phi_p({}_0D_t^\alpha u(t_j^\pm))$$

$$
\begin{aligned}
&+[M_2(\|(u(t_j^{\pm}),v(t_j^{\pm}))\|_{\beta}^{q})]^{q-1}\ {}_tD_T^{\beta-1}\Phi_q({}_0D_t^{\beta}u(t_j^{\pm}))\\
=&\lim_{t\to t_j^{\pm}}[M_1(\|(u(t),v(t))\|_{\alpha}^{p})]^{p-1}\ {}_tD_T^{\alpha-1}\Phi_p({}_0D_t^{\alpha}u(t))\\
&+[M_2(\|(u(t),v(t))\|_{\beta}^{q})]^{q-1}\ {}_tD_T^{\beta-1}\Phi_q({}_0D_t^{\beta}u(t)),
\end{aligned}
$$

and

$$
\begin{aligned}
&[M_1(\|(u(t_i'^{\pm}),v(t_i'^{\pm}))\|_{\alpha}^{p})]^{p-1}\ {}_tD_T^{\alpha-1}\Phi_p({}_0D_t^{\alpha}v(t_i'^{\pm}))\\
&+[M_2(\|(u(t_i'^{\pm}),v(t_i'^{\pm}))\|_{\beta}^{q})]^{q-1}\ {}_tD_T^{\beta-1}\Phi_q({}_0D_t^{\beta}v(t_i'^{\pm}))\\
=&\lim_{t\to t_i'^{\pm}}[M_1(\|(u(t),v(t))\|_{\alpha}^{p})]^{p-1}\ {}_tD_T^{\alpha-1}\Phi_p({}_0D_t^{\alpha}v(t))\\
&+[M_2(\|(u(t),v(t))\|_{\beta}^{q})]^{q-1}\ {}_tD_T^{\beta-1}\Phi_q({}_0D_t^{\beta}v(t))
\end{aligned}
$$

exist. Multiplying (2.10) by x and multiplying (2.11) by y. Then, integrating between 0 and T, we have

$$
\begin{aligned}
&[M_1(\|(u,v)\|_{\alpha}^{p})]^{p-1}\int_0^T\ {}_tD_T^{\alpha}(\Phi_p({}_0D_t^{\alpha}u))x+\ {}_tD_T^{\alpha}(\Phi_p({}_0D_t^{\alpha}v))ydt\\
&+[M_2(\|(u,v)\|_{\beta}^{q})]^{q-1}\int_0^T\ {}_tD_T^{\beta}(\Phi_q({}_0D_t^{\beta}u))x+\ {}_tD_T^{\beta}(\Phi_q({}_0D_t^{\beta}v))ydt\\
=&\int_0^T F_u(t,u,v)x+F_v(t,u,v)ydt. \qquad (2.12)
\end{aligned}
$$

On the basis of Proposition 2.5 in [29], we have

$$
\begin{aligned}
&[M_1(\|(u,v)\|_{\alpha}^{p})]^{p-1}\int_0^T\ {}_tD_T^{\alpha}(\Phi_p({}_0D_t^{\alpha}u))x+\ {}_tD_T^{\alpha}(\Phi_p({}_0D_t^{\alpha}v))ydt\\
&+[M_2(\|(u,v)\|_{\beta}^{q})]^{q-1}\int_0^T\ {}_tD_T^{\beta}(\Phi_q({}_0D_t^{\beta}u))x+\ {}_tD_T^{\beta}(\Phi_q({}_0D_t^{\beta}v))ydt\\
=&\sum_{j=1}^{m}\Big[\Delta\Big([M_1(\|(u,v)\|_{\alpha}^{p})]^{p-1}\ {}_tD_T^{\alpha-1}\Phi_p({}_0D_t^{\alpha}u)\Big)(t_j)\\
&+\Delta\Big([M_2(\|(u,v)\|_{\beta}^{q})]^{q-1}\ {}_tD_T^{\beta-1}\Phi_q({}_0D_t^{\beta}u)\Big)(t_j)\Big]x(t_j)\\
&+\sum_{i=1}^{n}\Big[\Delta\Big([M_1(\|(u,v)\|_{\alpha}^{p})]^{p-1}\ {}_tD_T^{\alpha-1}\Phi_p({}_0D_t^{\alpha}v)\Big)(t_i')\\
&+\Delta\Big([M_2(\|(u,v)\|_{\beta}^{q})]^{q-1}\ {}_tD_T^{\beta-1}\Phi_q({}_0D_t^{\beta}v)\Big)(t_i')\Big]y(t_i')\\
&+[M_1(\|(u,v)\|_{\alpha}^{p})]^{p-1}\int_0^T\Phi_p({}_0D_t^{\alpha}u)_0D_t^{\alpha}x+\Phi_p({}_0D_t^{\alpha}v)_0D_t^{\alpha}ydt\\
&+[M_2(\|(u,v)\|_{\beta}^{q})]^{q-1}\int_0^T\Phi_q({}_0D_t^{\beta}u)_0D_t^{\beta}x+\Phi_q({}_0D_t^{\beta}v)_0D_t^{\beta}ydt. \qquad (2.13)
\end{aligned}
$$

Hence, combining with (2.5), (2.12) and (2.13), we have

$$
\sum_{j=1}^{m}\Big[\Delta\Big([M_1(\|(u,v)\|_{\alpha}^{p})]^{p-1}\ {}_tD_T^{\alpha-1}\Phi_p({}_0D_t^{\alpha}u)\Big)(t_j)
$$

$$
\begin{aligned}
&+\Delta\Big([M_2(\|(u,v)\|_\beta^q)]^{q-1}\ {}_tD_T^{\beta-1}\Phi_q({}_0D_t^\beta u)\Big)(t_j)+\lambda\zeta(t_j)I_j(u(t_j))\Big]x(t_j)\\
&+\sum_{i=1}^{n}\Big[\Delta\Big([M_1(\|(u,v)\|_\alpha^p)]^{p-1}\ {}_tD_T^{\alpha-1}\Phi_p({}_0D_t^\alpha v)\Big)(t_i')\\
&+\Delta\Big([M_2(\|(u,v)\|_\beta^q)]^{q-1}\ {}_tD_T^{\beta-1}\Phi_q({}_0D_t^\beta v)\Big)(t_i')+\mu\varrho(t_i')S_i(v(t_i'))\Big]y(t_i')\\
=&0.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\Delta([M_1(\|(u,v)\|_\alpha^p)]^{p-1}\ {}_tD_T^{\alpha-1}\Phi_p({}_0D_t^\alpha u))(t_j)\\
&+\Delta([M_2(\|(u,v)\|_\beta^q)]^{q-1}\ {}_tD_T^{\beta-1}\Phi_q({}_0D_t^\beta u))(t_j)+\lambda\zeta(t_j)I_j(u(t_j))=0,\ j\in\Delta_1,
\end{aligned}
$$

and

$$
\begin{aligned}
&\Delta([M_1(\|(u,v)\|_\alpha^p)]^{p-1}\ {}_tD_T^{\alpha-1}\Phi_p({}_0D_t^\alpha v))(t_i')\\
&+\Delta([M_2(\|(u,v)\|_\beta^q)]^{q-1}\ {}_tD_T^{\beta-1}\Phi_q({}_0D_t^\beta v))(t_i')+\mu\varrho(t_i')S_i(v(t_i'))=0,\ i\in\Delta_2,
\end{aligned}
$$

which means that (u,v) satisfies the impulsive conditions of problem (1.1). Meanwhile, since $(u,v)\in E_{\alpha,\beta}$, it follows that (u,v) also satisfies the boundary conditions of problem (1.1). We complete the proof. □

Definition 2.2. Let $c\in R$ and $J_{\lambda,\mu}\in C^1(E_{\alpha,\beta},R)$. The functional $J_{\lambda,\mu}$ satisfies the $(PS)_c$ condition if any sequence $\{(u_n,v_n)\}\subset E_{\alpha,\beta}$ such that $J_{\lambda,\mu}(u_n,v_n)\to c$ and $J'_{\lambda,\mu}(u_n,v_n)\to 0$ as $n\to\infty$ in $E^*_{\alpha,\beta}$, where $E^*_{\alpha,\beta}$ is the dual space of $E_{\alpha,\beta}$, has a convergent subsequence in $E_{\alpha,\beta}$.

3. Existence of nontrivial solutions

In this section, we introduce the Nehari manifold for problem (1.1) to analyze the behavior of $J_{\lambda,\mu}$. It is assumed, without further mention, that the structural assumptions required in Theorem 1.1 are satisfied.

Now, we define the Nehari manifold associated with functional problem $J_{\lambda,\mu}$ as

$$\mathcal{N}_{\lambda,\mu}=\Big\{(u,v)\in E_{\alpha,\beta}\backslash\{(0,0)\}:\langle J'_{\lambda,\mu}(u,v),(u,v)\rangle=0\Big\}.$$

It is easy to see that $(u,v)\in\mathcal{N}_{\lambda,\mu}$ if and only if

$$
\begin{aligned}
&[M_1(\|(u,v)\|_\alpha^p)]^{p-1}\|(u,v)\|_\alpha^p+[M_2(\|(u,v)\|_\beta^q)]^{q-1}\|(u,v)\|_\beta^q\\
&-\sum_{j=1}^{m}\lambda\zeta(t_j)I_j(u(t_j))u(t_j)-\sum_{i=1}^{n}\mu\varrho(t_i')S_i(v(t_i'))v(t_i')\\
&-\sigma\int_0^T F(t,u(t),v(t))dt=0.
\end{aligned}
\tag{3.1}
$$

Specifically, we derive several topological properties of the $\mathcal{N}_{\lambda,\mu}$, which are expressed as

$$\mathcal{N}_{\lambda,\mu}=\Big\{(u,v)\in E_{\alpha,\beta}\backslash\{(0,0)\}:\mathcal{J}'_{u,v}(1)=0\Big\}$$

$$=\Big\{(su,sv)\in E_{\alpha,\beta}\backslash\{(0,0)\}:\mathcal{J}'_{u,v}(s)=0\Big\},$$

where $\mathcal{J}_{u,v}:[0,+\infty)\to R$ is the fibering map given by $\mathcal{J}_{u,v}(s)=J_{\lambda,\mu}(su,sv)$ for any $s\geq 0$ and $(u,v)\in E_{\alpha,\beta}$, i.e.

$$\begin{aligned}\mathcal{J}_{u,v}(s)=&\frac{1}{p}\overline{M}_1(s^p\|(u,v)\|_\alpha^p)+\frac{1}{q}\overline{M}_2(s^q\|(u,v)\|_\beta^q)\\&-\sum_{j=1}^m\int_0^{su(t_j)}\lambda\zeta(t_j)I_j(z)dz-\sum_{i=1}^n\int_0^{sv(t'_i)}\mu\varrho(t'_i)S_i(z)dz\\&-s^\sigma\int_0^T F(t,u(t),v(t))dt.\end{aligned}\tag{3.2}$$

Therefore, for any $(u,v)\in E_{\alpha,\beta}$, we have

$$\begin{aligned}\mathcal{J}'_{u,v}(s)=&[M_1(s^p\|(u,v)\|_\alpha^p)]^{p-1}s^{p-1}\|(u,v)\|_\alpha^p+[M_2(s^q\|(u,v)\|_\beta^q)]^{q-1}s^{q-1}\|(u,v)\|_\beta^q\\&-s^{\theta-1}\sum_{j=1}^m\lambda\zeta(t_j)I_j(u(t_j))u(t_j)-s^{\theta-1}\sum_{i=1}^n\mu\varrho(t'_i)S_i(v(t'_i))v(t'_i)\\&-\sigma s^{\sigma-1}\int_0^T F(t,u(t),v(t))dt,\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}\mathcal{J}''_{u,v}(s)=&s^{p-2}(p-1)[M_1(s^p\|(u,v)\|_\alpha^p)]^{p-1}\|(u,v)\|_\alpha^p\\&+s^{q-2}(q-1)[M_2(s^q\|(u,v)\|_\beta^q)]^{q-1}\|(u,v)\|_\beta^q\\&+s^{2p-2}p(p-1)[M_1(s^p\|(u,v)\|_\alpha^p)]^{p-2}M'_1(s^p\|(u,v)\|_\alpha^p)\|(u,v)\|_\alpha^{2p}\\&+s^{2q-2}q(q-1)[M_2(s^q\|(u,v)\|_\beta^q)]^{q-2}M'_2(s^q\|(u,v)\|_\beta^q)\|(u,v)\|_\beta^{2q}\\&-(\theta-1)s^{\theta-2}\Big(\sum_{j=1}^m\lambda\zeta(t_j)I_j(u(t_j))u(t_j)+\sum_{i=1}^n\mu\varrho(t'_i)S_i(v(t'_i))v(t'_i)\Big)\\&-\sigma(\sigma-1)s^{\sigma-2}\int_0^T F(t,u(t),v(t))dt.\end{aligned}\tag{3.4}$$

Then, for any $(u,v)\in\mathcal{N}_{\lambda,\mu}$, by (3.3), we have

$$\begin{aligned}\mathcal{J}'_{u,v}(1)=&[M_1(\|(u,v)\|_\alpha^p)]^{p-1}\|(u,v)\|_\alpha^p+[M_2(\|(u,v)\|_\beta^q)]^{q-1}\|(u,v)\|_\beta^q\\&-\sum_{j=1}^m\lambda\zeta(t_j)I_j(u(t_j))u(t_j)-\sum_{i=1}^n\mu\varrho(t'_i)S_i(v(t'_i))v(t'_i)\\&-\sigma\int_0^T F(t,u(t),v(t))dt,\end{aligned}$$

and by (3.1) and (3.4), we have

$$\begin{aligned}\mathcal{J}''_{u,v}(1)=&(p-1)[M_1(\|(u,v)\|_\alpha^p)]^{p-1}\|(u,v)\|_\alpha^p+(q-1)[M_2(\|(u,v)\|_\beta^q)]^{q-1}\|(u,v)\|_\beta^q\\&+p(p-1)[M_1(\|(u,v)\|_\alpha^p)]^{p-2}M'_1(\|(u,v)\|_\alpha^p)\|(u,v)\|_\alpha^{2p}\\&+q(q-1)[M_2(\|(u,v)\|_\beta^q)]^{q-2}M'_2(\|(u,v)\|_\beta^q)\|(u,v)\|_\beta^{2q}\end{aligned}$$

$$
\begin{aligned}
&-(\theta-1)\Big(\sum_{j=1}^{m}\lambda\zeta(t_j)I_j(u(t_j))u(t_j)+\sum_{i=1}^{n}\mu\varrho(t_i')S_i(v(t_i'))v(t_i')\Big)\\
&-\sigma(\sigma-1)\int_0^T F(t,u(t),v(t))dt\\
=&(p-\theta)[M_1(\|(u,v)\|_\alpha^p)]^{p-1}\|(u,v)\|_\alpha^p+(q-\theta)[M_2(\|(u,v)\|_\beta^q)]^{q-1}\|(u,v)\|_\beta^q\\
&+p(p-1)[M_1(\|(u,v)\|_\alpha^p)]^{p-2}M_1'(\|(u,v)\|_\alpha^p)\|(u,v)\|_\alpha^{2p}\\
&+q(q-1)[M_2(\|(u,v)\|_\beta^q)]^{q-2}M_2'(\|(u,v)\|_\beta^q)\|(u,v)\|_\beta^{2q}\\
&-\sigma(\sigma-\theta)\int_0^T F(t,u(t),v(t))dt\\
=&(p-\sigma)[M_1(\|(u,v)\|_\alpha^p)]^{p-1}\|(u,v)\|_\alpha^p+(q-\sigma)[M_2(\|(u,v)\|_\beta^q)]^{q-1}\|(u,v)\|_\beta^q\\
&+p(p-1)[M_1(\|(u,v)\|_\alpha^p)]^{p-2}M_1'(\|(u,v)\|_\alpha^p)\|(u,v)\|_\alpha^{2p}\\
&+q(q-1)[M_2(\|(u,v)\|_\beta^q)]^{q-2}M_2'(\|(u,v)\|_\beta^q)\|(u,v)\|_\beta^{2q}\\
&-(\theta-\sigma)\Big(\sum_{j=1}^{m}\lambda\zeta(t_j)I_j(u(t_j))u(t_j)+\sum_{i=1}^{n}\mu\varrho(t_i')S_i(v(t_i'))v(t_i')\Big).
\end{aligned}
\tag{3.5}
$$

Following the methodology developed by [24], $\mathcal{N}_{\lambda,\mu}$ is partitioned into the following three constituent elements:

$$
\mathcal{N}_{\lambda,\mu}^{\pm}=\Big\{(u,v)\in\mathcal{N}_{\lambda,\mu}|\mathcal{J}_{u,v}''(1)\gtrless 0\Big\},\ \text{and}\ \mathcal{N}_{\lambda,\mu}^{0}=\Big\{(u,v)\in\mathcal{N}_{\lambda,\mu}|\mathcal{J}_{u,v}''(1)=0\Big\}.
$$

Lemma 3.1. *For any $(\lambda,\mu)\in\Theta^0$, we have $\mathcal{N}_{\lambda,\mu}^0=\emptyset$, where*

$$
\Theta^0=\Big\{(\lambda,\mu)\in R^2\backslash\{(0,0)\}:0<(m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}}+(n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\le Q_0\Big\},
$$

and

$$
Q_0=\Big(\frac{\min\{\mathcal{B}_\alpha^*,\mathcal{B}_\beta^*\}}{2^\kappa(\sigma-\theta)\mathcal{A}_{\alpha,p}^\theta}\Big)^{\frac{p}{p-\theta}}\Big(\frac{\min\{\mathcal{B}_\alpha,\mathcal{B}_\beta\}2^{1-\kappa}}{\sigma(\sigma-\theta)C_*\overline{\mathcal{M}}(\sigma)}\Big)^{\frac{\kappa-\theta}{\sigma-\kappa}\cdot\frac{p}{p-\theta}}.
$$

Proof. We debate through contradiction. Suppose there exists a pair $(\lambda,\mu)\in\Theta^0$ such that $\mathcal{N}_{\lambda,\mu}^0\neq\emptyset$; in this scenario, we can distinguish between two cases.

Case I. $(u,v)\in\mathcal{N}_{\lambda,\mu}^0$ and $\sum\limits_{j=1}^{m}\lambda\zeta(t_j)I_j(u(t_j))u(t_j)+\sum\limits_{i=1}^{n}\mu\varrho(t_i')S_i(v(t_i'))v(t_i')=0$. Evidently, considering (3.5) and (M1), it can be inferred that

$$
\begin{aligned}
0<&\Big((p-\sigma)\xi_1+p(p-1)\frac{\delta_1}{\eta_1}\Big)\|(u,v)\|_\alpha^\kappa+\Big((q-\sigma)\xi_2+q(q-1)\frac{\delta_2}{\eta_2}\Big)\|(u,v)\|_\beta^\kappa\\
=&-\mathcal{B}_\alpha^*\|(u,v)\|_\alpha^\kappa-\mathcal{B}_\beta^*\|(u,v)\|_\beta^\kappa\\
<&0,
\end{aligned}
$$

which is a contradiction, where $\mathcal{B}_\alpha^*=(\sigma-p)\xi_1+p(1-p)\frac{\delta_1}{\eta_1}>0$, $\mathcal{B}_\beta^*=(\sigma-q)\xi_2+q(1-q)\frac{\delta_2}{\eta_2}>0$, due to $\sigma>p^2,q^2$, $(\sigma-p)\xi_1\eta_1>(p^2-p)\delta_1$ and $(\sigma-q)\xi_2\eta_2>(q^2-q)\delta_2$.

Case II. $(u,v)\in\mathcal{N}_{\lambda,\mu}^0$ and $\sum\limits_{j=1}^{m}\lambda\zeta(t_j)I_j(u(t_j))u(t_j)+\sum\limits_{i=1}^{n}\mu\varrho(t_i')S_i(v(t_i'))v(t_i')\neq0$.

Considering (M1) and (3.5), it can be concluded that

$$\begin{aligned}\sigma(\sigma-\theta)\int_0^T F(t,u(t),v(t))dt >&\Big((p-\theta)\xi_1+p(p-1)\frac{\xi_1}{\eta_1}\Big)\|(u,v)\|_\alpha^\kappa\\ &+\Big((q-\theta)\xi_2+q(q-1)\frac{\xi_2}{\eta_2}\Big)\|(u,v)\|_\beta^\kappa\\ \geq&\min\{\mathcal{B}_\alpha,\mathcal{B}_\beta\}2^{1-\kappa}\|(u,v)\|_{\alpha,\beta}^\kappa, \end{aligned}\tag{3.6}$$

where $\mathcal{B}_\alpha=(p-\theta)\xi_1+p(p-1)\frac{\xi_1}{\eta_1}$, $\mathcal{B}_\beta=(q-\theta)\xi_2+q(q-1)\frac{\xi_2}{\eta_2}$. Moreover, by (F2) and (2.4), we have

$$\begin{aligned}\sigma(\sigma-\theta)\int_0^T F(t,u(t),v(t))dt \leq&\sigma(\sigma-\theta)C_*\|(u,v)\|_\sigma^\sigma\\ \leq&\sigma(\sigma-\theta)C_*\overline{\mathcal{M}}(\sigma)\|(u,v)\|_{\alpha,\beta}^\sigma. \end{aligned}\tag{3.7}$$

Consequently, we can deduce from (3.6) and (3.7) that

$$\|(u,v)\|_{\alpha,\beta}>\Big(\frac{\min\{\mathcal{B}_\alpha,\mathcal{B}_\beta\}2^{1-\kappa}}{\sigma(\sigma-\theta)C_*\overline{\mathcal{M}}(\sigma)}\Big)^{\frac{1}{\sigma-\kappa}}.\tag{3.8}$$

On the other hand, by (M1) and (3.5), we have

$$\begin{aligned}&(\sigma-\theta)(\sum_{j=1}^m\lambda\zeta(t_j)I_j(u(t_j))u(t_j)+\sum_{i=1}^n\mu\varrho(t_i')S_i(v(t_i'))v(t_i'))\\ &>\min\{\mathcal{B}_\alpha^*,\mathcal{B}_\beta^*\}2^{1-\kappa}\|(u,v)\|_{\alpha,\beta}^\kappa. \end{aligned}\tag{3.9}$$

In addition, according to (H1) and Hölder inequality, we have

$$\begin{aligned}&\sum_{j=1}^m\lambda\zeta(t_j)I_j(u(t_j))u(t_j)+\sum_{i=1}^n\mu\varrho(t_i')S_i(v(t_i'))v(t_i')\\ \leq&\Big(m\bar{a}|\lambda|\ \|\zeta\|_\infty+n\bar{b}|\mu|\ \|\varrho\|_\infty\Big)\mathcal{A}_{\alpha,p}^\theta(\|u\|_{\alpha,p}^\theta+\|v\|_{\alpha,p}^\theta)\\ \leq&\mathcal{A}_{\alpha,p}^\theta\Big((m\bar{a}|\lambda|\ \|\zeta\|_\infty+n\bar{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}\Big((\|u\|_{\alpha,p}^\theta+\|v\|_{\alpha,p}^\theta)^{\frac{p}{\theta}}\Big)^{\frac{\theta}{p}}\\ \leq&2\mathcal{A}_{\alpha,p}^\theta\Big((m\bar{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}}+(n\bar{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}\|(u,v)\|_{\alpha,\beta}^\theta. \end{aligned}\tag{3.10}$$

Apparently, from (3.9) and (3.10), we have

$$\|(u,v)\|_{\alpha,\beta}<\Big(\frac{2^\kappa(\sigma-\theta)\mathcal{A}_{\alpha,p}^\theta((m\bar{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}}+(n\bar{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}})^{\frac{p-\theta}{p}}}{\min\{\mathcal{B}_\alpha^*,\mathcal{B}_\beta^*\}}\Big)^{\frac{1}{\kappa-\theta}}.\tag{3.11}$$

From (3.8) and (3.11), we get

$$\begin{aligned}&(m\bar{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}}+(n\bar{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\\ >&\Big(\frac{\min\{\mathcal{B}_\alpha^*,\mathcal{B}_\beta^*\}}{2^\kappa(\sigma-\theta)\mathcal{A}_{\alpha,p}^\theta}\Big)^{\frac{p}{p-\theta}}\Big(\frac{\min\{\mathcal{B}_\alpha,\mathcal{B}_\beta\}2^{1-\kappa}}{\sigma(\sigma-\theta)C_*\overline{\mathcal{M}}(\sigma)}\Big)^{\frac{\kappa-\theta}{\sigma-\kappa}\cdot\frac{p}{p-\theta}}, \end{aligned}$$

which contradicts $(\lambda,\mu)\in\Theta^0$. This completes the proof. □

Lemma 3.2. *For any $(\lambda,\mu)\in\Theta^0$, the functional $J_{\lambda,\mu}(u,v)$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$.*

Proof. For any $(u,v)\in\mathcal{N}_{\lambda,\mu}$, we can deduce from (2.6), (3.1), (M1) and (3.10) that

$$\begin{aligned}J_{\lambda,\mu}(u,v)=&J_{\lambda,\mu}(u,v)-\frac{1}{\sigma}\langle J'_{\lambda,\mu}(u,v),(u,v)\rangle\\ >&(\frac{1}{p}\frac{\xi_1}{\vartheta_1}-\frac{1}{\sigma}\delta_1)\|(u,v)\|_\alpha^\kappa+(\frac{1}{q}\frac{\xi_2}{\vartheta_2}-\frac{1}{\sigma}\delta_2)\|(u,v)\|_\beta^\kappa\\ &-(\frac{1}{\theta}+\frac{1}{\sigma})2\mathcal{A}_{\alpha,p}^\theta\Big((m\bar{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}}+(n\bar{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}\|(u,v)\|_{\alpha,\beta}^\theta\\ \geq&\min\{\mathcal{C}_\alpha,\mathcal{C}_\beta\}2^{1-\kappa}\|(u,v)\|_{\alpha,\beta}^\kappa\\ &-(\frac{1}{\theta}+\frac{1}{\sigma})2\mathcal{A}_{\alpha,p}^\theta\Big((m\bar{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}}+(n\bar{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}\|(u,v)\|_{\alpha,\beta}^\theta,\end{aligned}$$

where $\mathcal{C}_\alpha=\frac{1}{p}\frac{\xi_1}{\vartheta_1}-\frac{1}{\sigma}\delta_1>0$, $\mathcal{C}_\beta=\frac{1}{q}\frac{\xi_2}{\vartheta_2}-\frac{1}{\sigma}\delta_2>0$, considering $\xi_1\sigma>\vartheta_1\delta_1 p$, $\xi_2\sigma>\vartheta_2\delta_2 q$ and $\theta<\kappa$. This completes the proof. □

In the following result, we illustrate that $\mathcal{N}_{\lambda,\mu}^+$ and $\mathcal{N}_{\lambda,\mu}^-$ are non-empty. In order to deal with sign-changing functions ζ and ϱ, we define

$$\mathfrak{A}^\pm=\Big\{(u,v)\in E_{\alpha,\beta}|\sum_{j=1}^m\lambda\zeta(t_j)I_j(u(t_j))u(t_j)+\sum_{i=1}^n\mu\varrho(t'_i)S_i(v(t'_i))v(t'_i)\gtrless 0\Big\},$$

$$\mathfrak{B}^\pm=\Big\{(u,v)\in E_{\alpha,\beta}|\int_0^T F(t,u(t),v(t))dt\gtrless 0\Big\}.$$

Lemma 3.3. *(i) For any $(u,v)\in\mathfrak{A}^+\cap\mathfrak{B}^+$ and $(\lambda,\mu)\in\Theta^1$, there exist positive constants $s_*=s_*(u,v)$, $s_{**}=s_{**}(u,v)$, $s^+=s^+(u,v)$ and $s^-=s^-(u,v)$ with $0<s^+<s_*<s^-<s_{**}$ such that $(s^+u,s^+v)\in\mathcal{N}_{\lambda,\mu}^+$, $(s^-u,s^-v)\in\mathcal{N}_{\lambda,\mu}^-$ and*

$$J_{\lambda,\mu}(s^+u,s^+v)=\min_{0\leq s\leq s^-}J_{\lambda,\mu}(su,sv),\ J_{\lambda,\mu}(s^-u,s^-v)=\max_{s_*\leq s\leq s_{**}}J_{\lambda,\mu}(su,sv);$$

(ii) For any $(u,v)\in\mathfrak{A}^+\cap\mathfrak{B}^-$, there exists a unique $s^+=s^+(u,v)>0$ such that $(s^+u,s^+v)\in\mathcal{N}_{\lambda,\mu}^+$ and

$$J_{\lambda,\mu}(s^+u,s^+v)=\min_{s\geq 0}J_{\lambda,\mu}(su,sv).$$

Proof. For given $(u,v)\in E_{\alpha,\beta}\backslash\{(0,0)\}$, let us consider the function $\mathfrak{L}_{u,v}(s):R^+\to R$ as

$$\begin{aligned}\mathfrak{L}_{u,v}(s)=&s^{p-\theta}[M_1(s^p\|(u,v)\|_\alpha^p)]^{p-1}\|(u,v)\|_\alpha^p+s^{q-\theta}[M_2(s^q\|(u,v)\|_\beta^q)]^{q-1}\|(u,v)\|_\beta^q\\ &-\sigma s^{\sigma-\theta}\int_0^T F(t,u(t),v(t))dt.\end{aligned}\tag{3.12}$$

Then, for any $s>0$, if $(su,sv)\in\mathcal{N}_{\lambda,\mu}$, we can infer that

$$\mathfrak{L}_{u,v}(s)=\sum_{j=1}^m\lambda\zeta(t_j)I_j(u(t_j))u(t_j)+\sum_{i=1}^n\mu\varrho(t'_i)S_i(v(t'_i))v(t'_i).$$

By (3.12), we have

$$\begin{aligned}\mathfrak{L}'_{u,v}(s) =&(p-\theta)s^{p-\theta-1}[M_1(s^p\|(u,v)\|^p_\alpha)]^{p-1}\|(u,v)\|^p_\alpha\\&+p(p-1)s^{2p-\theta-1}[M_1(s^p\|(u,v)\|^p_\alpha)]^{p-2}M_1'(s^p\|(u,v)\|^p_\alpha)\|(u,v)\|^{2p}_\alpha\\&+(q-\theta)s^{q-\theta-1}[M_2(s^q\|(u,v)\|^q_\beta)]^{q-1}\|(u,v)\|^q_\beta\\&+q(q-1)s^{2q-\theta-1}[M_2(s^q\|(u,v)\|^q_\beta)]^{q-2}M_2'(s^q\|(u,v)\|^q_\beta)\|(u,v)\|^{2q}_\beta\\&-\sigma(\sigma-\theta)s^{\sigma-\theta-1}\int_0^T F(t,u(t),v(t))dt,\end{aligned}\tag{3.13}$$

and so if $(su,sv)\in\mathcal{N}_{\lambda,\mu}$, then

$$\mathcal{J}''_{su,sv}(1)=s^{\theta+1}\mathfrak{L}'_{u,v}(s).\tag{3.14}$$

Hence, $(su,sv)\in\mathcal{N}^+_{\lambda,\mu}$ (or $\mathcal{N}^-_{\lambda,\mu}$) if and only if $\mathfrak{L}'_{u,v}(s)>0$ (or <0).
(i) Let $(u,v)\in\mathfrak{A}^+\cap\mathfrak{B}^+$. We note that $\mathfrak{L}_{u,v}(0)=0$, $\mathfrak{L}_{u,v}(s)\to-\infty$ as $s\to+\infty$. By (3.7), (M1) and (3.13), we have

$$\begin{aligned}\mathfrak{L}'_{u,v}(s)>&\min\{\mathcal{B}_\alpha,\mathcal{B}_\beta\}2^{1-\kappa}\|(u,v)\|^\kappa_{\alpha,\beta}s^{\kappa-\theta-1}\\&-(\sigma(\sigma-\theta)C_*\overline{\mathcal{M}}(\sigma)\|(u,v)\|^\sigma_{\alpha,\beta})s^{\sigma-\theta-1},\end{aligned}\tag{3.15}$$

which implies $\lim\limits_{s\to0^+}\mathfrak{L}'_{u,v}(s)>0$ and $\lim\limits_{s\to\infty}\mathfrak{L}'_{u,v}(s)=-\infty$. Then, define the function $\mathcal{P}(s):R^+\to R$ as

$$\begin{aligned}\mathcal{P}(s)=&\min\{\mathcal{B}_\alpha,\mathcal{B}_\beta\}2^{1-\kappa}\|(u,v)\|^\kappa_{\alpha,\beta}s^{\kappa-\theta-1}\\&-(\sigma(\sigma-\theta)C_*\overline{\mathcal{M}}(\sigma)\|(u,v)\|^\sigma_{\alpha,\beta})s^{\sigma-\theta-1}.\end{aligned}$$

Due to $\kappa<\sigma$, there exists a unique $s_{\max}$, where

$$s_{\max}=\Big(\frac{\min\{\mathcal{B}_\alpha,\mathcal{B}_\beta\}2^{1-\kappa}(\kappa-\theta-1)}{\sigma(\sigma-\theta)C_*\overline{\mathcal{M}}(\sigma)\|(u,v)\|^{\sigma-\kappa}_{\alpha,\beta}(\sigma-\theta-1)}\Big)^{\frac{1}{\sigma-\kappa}},$$

such that $\mathcal{P}(s)$ is increasing on $(0,s_{\max})$, decreasing on $(s_{\max},+\infty)$. Hence, there exists a unique $s_0>s_{\max}$ such that $\mathcal{P}(s_0)=0$. Then, we can deduce that $\mathfrak{L}'_{u,v}(s)$ exists a minimal root $s_*\geq s_0$ such that $\mathfrak{L}'_{u,v}(s_*)=0$, $\mathfrak{L}''_{u,v}(s_*)<0$. Therefore, there exists $s_{**}>s_*$ such that $\mathfrak{L}_{u,v}$ is increasing on $(0,s_*)$, decreasing on (s_*,s_{**}). Taking into account the sign of $\mathfrak{L}_{u,v}(s_{**})$, we discuss it in the following two cases.

Case I. $\mathfrak{L}_{u,v}(s_{**})\leq0$. By (3.7), (3.12) and (M1), we have

$$\mathfrak{L}_{u,v}(s)\geq\min\{\xi_1,\xi_2\}2^{1-\kappa}s^{\kappa-\theta}\|(u,v)\|^\kappa_{\alpha,\beta}-\sigma s^{\sigma-\theta}C_*\overline{\mathcal{M}}(\sigma)\|(u,v)\|^\sigma_{\alpha,\beta}.$$

Let the function $\mathcal{P}_1(s):R^+\to R$ be defined by

$$\mathcal{P}_1(s)=\min\{\xi_1,\xi_2\}2^{1-\kappa}\|(u,v)\|^\kappa_{\alpha,\beta}s^{\kappa-\theta}-\sigma C_*\overline{\mathcal{M}}(\sigma)\|(u,v)\|^\sigma_{\alpha,\beta}s^{\sigma-\theta}.$$

Then, there exists unique

$$s_1=\Big(\frac{\min\{\xi_1,\xi_2\}2^{1-\kappa}(\kappa-\theta)}{\sigma C_*\overline{\mathcal{M}}(\sigma)\|(u,v)\|^{\sigma-\kappa}_{\alpha,\beta}(\sigma-\theta)}\Big)^{\frac{1}{\sigma-\kappa}},$$

such that

$$\begin{aligned}\max_{s>0}\mathcal{P}_1(s) =&\mathcal{P}_1(s_1)\\ =&\min\{\xi_1,\xi_2\}2^{1-\kappa}\|(u,v)\|^{\theta}_{\alpha,\beta}\frac{\sigma-\kappa}{\sigma-\theta}\Big(\frac{\min\{\xi_1,\xi_2\}2^{1-\kappa}(\kappa-\theta)}{\sigma C_*\overline{\mathcal{M}}(\sigma)(\sigma-\theta)}\Big)^{\frac{\kappa-\theta}{\sigma-\kappa}}.\end{aligned}$$

Hence, we have $\mathfrak{L}_{u,v}(s_*) > \max\limits_{s>0}\mathcal{P}_1(s) = \mathcal{P}_1(s_1) > 0$. Moreover, for any $(\lambda,\mu)\in\Theta^0_1$, where

$$\Theta^0_1 = \Big\{(\lambda,\mu)\in R^2\backslash\{(0,0)\} : 0 < (m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}} \le Q_1\Big\},$$

and

$$Q_1 =\Big(\frac{\min\{\xi_1,\xi_2\}2^{-\kappa}(\sigma-\kappa)}{\mathcal{A}^\theta_{\alpha,\beta}(\sigma-\theta)}\Big)^{\frac{p}{p-\theta}}\Big(\frac{\min\{\xi_1,\xi_2\}2^{1-\kappa}(\kappa-\theta)}{\sigma C_*\overline{\mathcal{M}}(\sigma)(\sigma-\theta)}\Big)^{\frac{\kappa-\theta}{\sigma-\kappa}\cdot\frac{p}{p-\theta}}.$$

Then, for any $(\lambda,\mu)\in\Theta^0_1$, we have

$$\begin{aligned}0 <&\sum_{j=1}^{m}\lambda\zeta(t_j)I_j(u(t_j))u(t_j) + \sum_{i=1}^{n}\mu\varrho(t'_i)S_i(v(t'_i))v(t'_i)\\ \le&2\mathcal{A}^\theta_{\alpha,p}\Big((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}\|(u,v)\|^\theta_{\alpha,\beta}\\ \le&\mathcal{P}_1(s_1).\end{aligned}$$

Up to now, there exist unique $s^+ < s_*$ and $s_* < s^- < s_{**}$ such that

$$\mathfrak{L}_{u,v}(s^+) = \sum_{j=1}^{m}\lambda\zeta(t_j)I_j(u(t_j))u(t_j) + \sum_{i=1}^{n}\mu\varrho(t'_i)S_i(v(t'_i))v(t'_i) = \mathfrak{L}_{u,v}(s^-).$$

That is, (s^+u, s^+v), $(s^-u, s^-v)\in\mathcal{N}_{\lambda,\mu}$. Moreover, by (3.14), we have $(s^+u, s^+v)\in\mathcal{N}^+_{\lambda,\mu}$, $(s^-u, s^-v)\in\mathcal{N}^-_{\lambda,\mu}$. Further, considering

$$\mathcal{J}'_{u,v}(s) = s^{\theta-1}\Big(\mathfrak{L}_{u,v}(s) - \sum_{j=1}^{m}\lambda\zeta(t_j)I_j(u(t_j))u(t_j) - \sum_{i=1}^{n}\mu\varrho(t'_i)S_i(v(t'_i))v(t'_i)\Big),$$

we have $\mathcal{J}'_{u,v}(s) < 0$ for all $s\in[0,s^+)$, $\mathcal{J}'_{u,v}(s) > 0$ for all $s\in(s^+,s^-)$ and $\mathcal{J}'_{u,v}(s) < 0$ for all $s\in[s^-,s_{**})$. Hence, we have

$$J_{\lambda,\mu}(s^+u,s^+v) = \min_{0\le s\le s^-}J_{\lambda,\mu}(su,sv),\ \text{and}\ J_{\lambda,\mu}(s^-u,s^-v) = \max_{s_*\le s\le s_{**}}J_{\lambda,\mu}(su,sv).$$

Case II. $0 < \mathfrak{L}_{u,v}(s_{**}) < \mathfrak{L}_{u,v}(s_*)$. Then, there exists sufficiently small $\varepsilon\in(0, 2\mathcal{A}^\theta_{\alpha,\beta}\|(u,v)\|^\theta_{\alpha,\beta}Q_1^{\frac{p-\theta}{p}})$ such that for any $(\lambda,\mu)\in\Theta^1$, where

$$\begin{aligned}\Theta^1 =&\Big\{(\lambda,\mu)\in R^2\backslash\{(0,0)\} : (Q_1^{\frac{p-\theta}{p}} - \frac{\varepsilon}{2\mathcal{A}^\theta_{\alpha,\beta}\|(u,v)\|^\theta_{\alpha,\beta}})^{\frac{p}{p-\theta}}\\ &\le (m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}} \le Q_1\Big\},\end{aligned}$$

we have

$$\begin{aligned}\mathfrak{L}_{u,v}(s_{**}) \leq & \mathcal{P}_1(s_1) - \varepsilon \\ \leq & \sum_{j=1}^{m} \lambda\zeta(t_j)I_j(u(t_j))u(t_j) + \sum_{i=1}^{n} \mu\varrho(t_i')S_i(v(t_i'))v(t_i') \\ \leq & 2\mathcal{A}_{\alpha,p}^{\theta}\Big((m\overline{a}|\lambda| \ \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu| \ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}} \|(u,v)\|_{\alpha,\beta}^{\theta} \\ \leq & \mathcal{P}_1(s_1).\end{aligned}$$

As analyzed in case1, there exist unique $s^+ < s_*$ and $s_* < s^- < s_{**}$ such that $(s^+u, s^+v) \in \mathcal{N}_{\lambda,\mu}^+$ and $(s^-u, s^-v) \in \mathcal{N}_{\lambda,\mu}^-$. moreover, we have

$$J_{\lambda,\mu}(s^+u, s^+v) = \min_{0\leq s\leq s^-} J_{\lambda,\mu}(su, sv), \text{ and } J_{\lambda,\mu}(s^-u, s^-v) = \max_{s_*\leq s\leq s_{**}} J_{\lambda,\mu}(su, sv).$$

(ii) Let $(u, v) \in \mathfrak{A}^+ \cap \mathfrak{B}^-$. Apparently, $\mathfrak{L}_{u,v}(0) = 0$, $\mathfrak{L}_{u,v}(s) \to +\infty$ as $s \to +\infty$. According to (3.13), we can deduce that $\mathfrak{L}_{u,v}'(s) > 0$ for all $s > 0$, that is $\mathfrak{L}_{u,v}$ is increasing on $(0, +\infty)$. Since, $(u, v) \in \mathfrak{A}^+$, there exists unique $s^+ > 0$ such that

$$\mathfrak{L}_{u,v}(s^+) = \sum_{j=1}^{m} \lambda\zeta(t_j)I_j(u(t_j))u(t_j) + \sum_{i=1}^{n} \mu\varrho(t_i')S_i(v(t_i'))v(t_i').$$

That is, $(s^+u, s^+v) \in \mathcal{N}_{\lambda,\mu}$. In addition, (3.14) implies that $(s^+u, s^+v) \in \mathcal{N}_{\lambda,\mu}^+$ and $J_{\lambda,\mu}(s^+u, s^+v) = \min_{s\geq 0} J_{\lambda,\mu}(su, sv)$. □

Lemma 3.4. *For any* $(\lambda, \mu) \in \Theta^2$*, then* $c_{\lambda,\mu}^+ = \inf_{(u,v)\in\mathcal{N}_{\lambda,\mu}^+} J_{\lambda,\mu}(u, v) < 0$*, where*

$$\Theta^2 = \Big\{(\lambda, \mu) \in R^2\backslash\{(0, 0)\} : 0 < (m\overline{a}|\lambda| \ \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu| \ \|\varrho\|_\infty)^{\frac{p}{p-\theta}} \leq Q_2\Big\},$$

and

$$Q_2 = \Big(\frac{\max\{\mathcal{Q}_1, \mathcal{Q}_2\}2^{1-\kappa}}{4\mathcal{A}_{\alpha,p}^{\theta}}\Big)^{\frac{p}{p-\theta}} \|(u, v)\|_{\alpha,\beta}^{\frac{p(\kappa-\theta)}{p-\theta}}.$$

Proof. Let $(u, v) \in \mathcal{N}_{\lambda,\mu}^+$. By (3.5), we have

$$\begin{aligned}&\sigma(\sigma - \theta)\int_0^T F(t, u(t), v(t))dt \\ &<\Big((p - \theta)\delta_1 + p(p - 1)\frac{\delta_1}{\eta_1}\Big)\|(u, v)\|_\alpha^\kappa + \Big((q - \theta)\delta_2 + p(p - 1)\frac{\delta_2}{\eta_2}\Big)\|(u, v)\|_\beta^\kappa.\end{aligned}$$

Then, from (2.6), (3.1) and (M1), we have

$$\begin{aligned}c_{\lambda,\mu}^+ \leq & J_{\lambda,\mu}(u, v) \\ < & \Big(\frac{1}{p}\frac{\delta_1}{\vartheta_1} + \frac{1}{\theta\sigma}((p - \theta)\delta_1 + p(p - 1)\frac{\delta_1}{\eta_1}) - \frac{1}{\theta}\xi_1\Big)\|(u, v)\|_\alpha^\kappa \\ & + \Big(\frac{1}{q}\frac{\delta_2}{\vartheta_2} + \frac{1}{\theta\sigma}((q - \theta)\delta_1 + q(q - 1)\frac{\delta_2}{\eta_2}) - \frac{1}{\theta}\xi_2\Big)\|(u, v)\|_\beta^\kappa\end{aligned}$$

$$+\frac{4}{\theta}\mathcal{A}_{\alpha,p}^{\theta}\Big((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}}+(n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}\|(u,v)\|_{\alpha,\beta}^{\theta}. \quad (3.16)$$

Taking into account that $p\vartheta_1, q\vartheta_2 > 1$, $\theta\sigma > 1$, $(2p+p^2)\delta_1\theta < \xi_1$, and $(2q+q^2)\delta_2\theta < \xi_2$, it follows from (3.16) that

$$\begin{aligned}c_{\lambda,\mu}^{+} \leq & J_{\lambda,\mu}(u,v)\\ < & -\frac{1}{\theta}\max\{\mathcal{Q}_1,\mathcal{Q}_2\}2^{1-\kappa}\|(u,v)\|_{\alpha,\beta}^{\kappa}\\ & +\frac{4}{\theta}\mathcal{A}_{\alpha,p}^{\theta}\Big((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}}+(n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}\|(u,v)\|_{\alpha,\beta}^{\theta}\\ < & 0,\end{aligned}$$

where $\mathcal{Q}_1 = \xi_1-(2p+p^2)\delta_1\theta$ and $\mathcal{Q}_2 = \xi_2-(2q+q^2)\delta_2\theta$. This completes the proof. □

Lemma 3.5. *Let $(u,v) \in \mathcal{N}_{\lambda,\mu}^{+}$ and $(\lambda,\mu) \in \Theta^0$, there is a number $\varepsilon > 0$ and a continuous function $\varphi : \Theta_\varepsilon(0,0) \subseteq E_{\alpha,\beta} \to R^+$ such that*

$$\varphi(0,0) = 1, \text{ and } \varphi(z,\overline{z})((u,v)-(z,\overline{z})) \in \mathcal{N}_{\lambda,\mu}^{+},$$

where $\Theta_\varepsilon(0,0) = \{(z,\overline{z}) \in E_{\alpha,\beta} : \|(z,\overline{z})\|_{\alpha,\beta} < \varepsilon\}$.

Proof. To begin, we define a function $\Upsilon : R^+ \times E_{\alpha,\beta} \to R$ as follows:

$$\begin{aligned}\Upsilon(s,(z,\overline{z})) = & s^p[M_1(s^p\|(u,v)-(z,\overline{z})\|_\alpha^p)]^{p-1}\|(u,v)-(z,\overline{z})\|_\alpha^p\\ & + s^q[M_2(s^q\|(u,v)-(z,\overline{z})\|_\beta^q)]^{q-1}\|(u,v)-(z,\overline{z})\|_\beta^q\\ & - s^\theta\Big(\sum_{j=1}^{m}\lambda\zeta(t_j)I_j(u(t_j)-z(t_j))(u(t_j)-z(t_j))\\ & - \sum_{i=1}^{n}\mu\varrho(t_i')S_i(v(t_i')-\overline{z}(t_i'))(v(t_i')-\overline{z}(t_i'))\Big)\\ & - \sigma s^\sigma\int_0^T F(t,u-z,v-\overline{z})dt.\end{aligned}$$

Thanks to $(u,v) \in \mathcal{N}_{\lambda,\mu}^{+}$, we have $\Upsilon(1,(0,0)) = 0$ and $\frac{\partial\Upsilon}{\partial s}(1,(0,0)) > 0$. Applying the implicit function theorem at $(1,(0,0))$, we obtain that there exists $\varepsilon_0 > 0$ such that for any $(z,\overline{z}) \in E_{\alpha,\beta}$ with $\|(z,\overline{z})\|_{\alpha,\beta} < \varepsilon_0$, the equation $\Upsilon(s,(z,\overline{z})) = 0$ has a unique continuous solution $s = \varphi(z,\overline{z}) > 0$ and $\varphi(0,0) = 1$. Furthermore, we have

$$\langle\varphi'(0,0),(z,\overline{z})\rangle = \frac{W_*}{\mathcal{J}_{u,v}''(1)}, \quad (3.17)$$

where

$$\begin{aligned}W_* = & \Big((p-1)[M_1(\|(u,v)\|_\alpha^p)]^{p-2}M_1'(\|(u,v)\|_\alpha^p)\|(u,v)\|_\alpha^p+[M_1(\|(u,v)\|_\alpha^p)]^{p-1}\Big)\\ & \times p\int_0^T(\Phi_p({}_0D_t^\alpha u){}_0D_t^\alpha z+\Phi_p({}_0D_t^\alpha v){}_0D_t^\alpha\overline{z})dt\\ & +\Big((q-1)[M_2(\|(u,v)\|_\beta^q)]^{q-2}M_2'(\|(u,v)\|_\beta^q)\|(u,v)\|_\beta^q+[M_2(\|(u,v)\|_\beta^q)]^{q-1}\Big)\end{aligned}$$

$$
\begin{aligned}
&\times q\int_0^T (\Phi_q({}_0D_t^\beta u)_0D_t^\beta z + \Phi_q({}_0D_t^\beta v)_0D_t^\beta \overline{z})dt \\
&-\sum_{j=1}^{m} \lambda\zeta(t_j)(I_j'(u(t_j))z(t_j)u(t_j) + I_j(u(t_j))z(t_j)) \\
&-\sum_{i=1}^{n} \mu\varrho(t_i')(S_i'(v(t_i'))\overline{z}(t_i')v(t_i') + S_i(v(t_i'))\overline{z}(t_i')) \\
&-\sigma\int_0^T F_u(t,u,v)z + F_v(t,u,v)\overline{z}dt,
\end{aligned}
$$

and $\Upsilon(\varphi(z,\overline{z}),(z,\overline{z})) = 0$ for all $(z,\overline{z}) \in B_{\varepsilon_0}(0,0)$, which means that $\varphi(z,\overline{z})((u,v) - (z,\overline{z})) \in \mathcal{N}_\lambda$ for all $(z,\overline{z}) \in \Theta_{\varepsilon_0}(0,0)$. Moreover, we can choose $0 < \varepsilon < \varepsilon_0$ such that for any $(z,\overline{z}) \in E_{\alpha,\beta}$ with $\|(z,\overline{z})\|_{\alpha,\beta} \le \varepsilon$ we have $\frac{\partial \Upsilon}{\partial s}(\varphi(z,\overline{z}),(z,\overline{z})) > 0$, which implies that $\varphi(z,\overline{z})((u,v) - (z,\overline{z})) \in \mathcal{N}_\lambda^+$ for all $(z,\overline{z}) \in \Theta_\varepsilon(0,0)$. □

Lemma 3.6. *If $(\lambda,\mu) \in \Theta^0$, then there exists a minimizing sequence $\{(u_n,v_n)\} \subset \mathcal{N}_{\lambda,\mu}$ such that*

$$
J_{\lambda,\mu}(u_n,v_n) = c_{\lambda,\mu} + o_n(1), \text{ and } J'_{\lambda,\mu}(u_n,v_n) = o_n(1), \tag{3.18}
$$

where $c_{\lambda,\mu} = \inf_{(u,v)\in\mathcal{N}_{\lambda,\mu}} J_{\lambda,\mu}(u,v)$.

Proof. By Lemma 3.2 and Ekeland's variational principle, there exists a minimizing sequence $\{(u_n,v_n)\} \subset \mathcal{N}_{\lambda,\mu}$ for $J_{\lambda,\mu}$ such that

$$
c_{\lambda,\mu} < J_{\lambda,\mu}(u_n,v_n) < c_{\lambda,\mu} + \frac{1}{n}, \tag{3.19}
$$

and

$$
J_{\lambda,\mu}(u_n,v_n) < J_{\lambda,\mu}(u,v) + \frac{1}{n}\|(u,v) - (u_n,v_n)\|_{\alpha,\beta},\ \forall (u,v) \in \mathcal{N}_{\lambda,\mu}, \tag{3.20}
$$

apparently, by (3.19), we get $(3.18)_1$. Now, we show that $\|J'_{\lambda,\mu}(u_n,v_n)\|_{\alpha,\beta} \to 0$ as $n \to \infty$. By Lemma 3.5, we obtain the function $\varphi_n : \Theta_{\varepsilon_n}(0,0) \to R$ such that $\varphi_n(z,\overline{z})((u_n,v_n) - (z,\overline{z})) \in \mathcal{N}_{\lambda,\mu}$ for all $(z,\overline{z}) \in \Theta_{\varepsilon_n}(0,0)$. For fixed n, choose $0 < \epsilon < \varepsilon_n$ and define $(z_\epsilon,\overline{z}_\epsilon) = \epsilon\frac{(u,v)}{\|(u,v)\|_{\alpha,\beta}}$ with $(u,v) \in E_{\alpha,\beta}\backslash(0,0)$. Set $(z_{1,\epsilon},z_{2,\epsilon}) = \varphi_n(z_\epsilon,\overline{z}_\epsilon)((u_n,v_n) - (z_\epsilon,\overline{z}_\epsilon))$, then it is clear that $(z_{1,\epsilon},z_{2,\epsilon}) \in \mathcal{N}_{\lambda,\mu}$. According to (3.20), we have

$$
J_{\lambda,\mu}(z_{1,\epsilon},z_{2,\epsilon}) - J_{\lambda,\mu}(u_n,v_n) \ge -\frac{1}{n}\|(z_{1,\epsilon},z_{2,\epsilon}) - (u_n,v_n)\|_{\alpha,\beta}.
$$

Applying the mean value theorem, we have

$$
\begin{aligned}
&\langle J'_{\lambda,\mu}(u_n,v_n),(z_{1,\epsilon},z_{2,\epsilon}) - (u_n,v_n)\rangle + o_n(\|(z_{1,\epsilon},z_{2,\epsilon}) - (u_n,v_n)\|_{\alpha,\beta}) \\
\ge& -\frac{1}{n}\|(z_{1,\epsilon},z_{2,\epsilon}) - (u_n,v_n)\|_{\alpha,\beta}.
\end{aligned}
$$

Then

$$
-\langle J'_{\lambda,\mu}(u_n,v_n),(z_\epsilon,\overline{z}_\epsilon)\rangle + (\varphi_n(z_\epsilon,\overline{z}_\epsilon) - 1)\langle J'_{\lambda,\mu}(u_n,v_n),(u_n,v_n) - (z_\epsilon,\overline{z}_\epsilon)\rangle
$$

$$\geq -\frac{1}{n}\|(z_{1,\epsilon},z_{2,\epsilon})-(u_n,v_n)\|_{\alpha,\beta}+o_n(\|(z_{1,\epsilon},z_{2,\epsilon})-(u_n,v_n)\|_{\alpha,\beta}).$$

Considering $\langle J'_{\lambda,\mu}(z_{1,\epsilon},z_{2,\epsilon}),(u_n,v_n)-(z_\epsilon,\overline{z}_\epsilon)\rangle=0$, we get

$$\begin{aligned}&(\varphi_n(z_\epsilon,\overline{z}_\epsilon)-1)\langle J'_{\lambda,\mu}(u_n,v_n)-J'_{\lambda,\mu}(z_{1,\epsilon},z_{2,\epsilon}),(u_n,v_n)-(z_\epsilon,\overline{z}_\epsilon)\rangle\\&-\epsilon\langle J'_{\lambda,\mu}(u_n,v_n),\frac{(u,v)}{\|(u,v)\|_{\alpha,\beta}}\rangle\\\geq&-\frac{1}{n}\|(z_{1,\epsilon},z_{2,\epsilon})-(u_n,v_n)\|_{\alpha,\beta}+o_n(\|(z_{1,\epsilon},z_{2,\epsilon})-(u_n,v_n)\|_{\alpha,\beta}).\end{aligned}$$

Hence, we have

$$\begin{aligned}&\langle J'_{\lambda,\mu}(u_n,v_n),\frac{(u,v)}{\|(u,v)\|_{\alpha,\beta}}\rangle\\\leq&\frac{1}{\epsilon n}\|(z_{1,\epsilon},z_{2,\epsilon})-(u_n,v_n)\|_{\alpha,\beta}+\frac{1}{\epsilon}o_n(\|(z_{1,\epsilon},z_{2,\epsilon})-(u_n,v_n)\|_{\alpha,\beta})\\&+\frac{1}{\epsilon}(\varphi_n(z_\epsilon,\overline{z}_\epsilon)-1)\langle J'_{\lambda,\mu}(u_n,v_n)-J'_{\lambda,\mu}(z_{1,\epsilon},z_{2,\epsilon}),(u_n,v_n)-(z_\epsilon,\overline{z}_\epsilon)\rangle.\end{aligned}\tag{3.21}$$

Since

$$\|(z_{1,\epsilon},z_{2,\epsilon})-(u_n,v_n)\|_{\alpha,\beta}\leq\epsilon|\varphi_n(z_\epsilon,\overline{z}_\epsilon)|+|\varphi_n(z_\epsilon,\overline{z}_\epsilon)-1|\ \|(u_n,v_n)\|_{\alpha,\beta},$$

and

$$\lim_{\epsilon\to0}\frac{\varphi_n(z_\epsilon,\overline{z}_\epsilon)-1}{\epsilon}\leq\|\varphi'_n(0,0)\|_{\alpha,\beta},$$

passing to the limit $\epsilon\to0^+$ in (3.21), there exists $K_0>0$ such that

$$\langle J'_{\lambda,\mu}(u_n,v_n),\frac{(u,v)}{\|(u,v)\|_{\alpha,\beta}}\rangle\leq\frac{K_0}{n}(1+\|\varphi'_n(0,0)\|_{\alpha,\beta}).\tag{3.22}$$

Below, we demonstrate that $\|\varphi'_n(0,0)\|_{\alpha,\beta}$ is bounded. Arguing by contradiction, we assume that $\langle\varphi'_n(0,0),(z,\overline{z})\rangle\to\infty$ as $n\to\infty$. By (3.19) and Lemma 3.2, we have $\sup_n\|(u_n,v_n)\|_{\alpha,\beta}<\infty$. Then, from (3.17), the boundedness of $\{(u_n,v_n)\}$ and Hölder inequality, there exists positive constant L_0 such that

$$\langle\varphi'_n(0,0),(z,\overline{z})\rangle=\frac{L_0(\|z\|_{\alpha,p}+\|\overline{z}\|_{\alpha,p}+\|z\|_{\beta,q}+\|\overline{z}\|_{\beta,q})}{\mathcal{J}''_{u_n,v_n}(1)},$$

which implies that there exists a subsequence $\{(u_n,v_n)\}$ such that $\mathcal{J}''_{u_n,v_n}(1)=o_n(1)$ as $n\to\infty$. Following the proof of Lemma 3.1, we have $(\lambda,\mu)\notin\Theta^0$, which gives a contradiction. □

Lemma 3.7. *Let* $\{(u_n,v_n)\}\subset\mathcal{N}_{\lambda,\mu}$. $J_{\lambda,\mu}$ *satisfies the* (PS)*-condition at level* $c_{\lambda,\mu}$.

Proof. It follows from Lemma 3.6 that there exists $\{(u_n,v_n)\}$ be a $(PS)_{c_{\lambda,\mu}}$ sequence for $J_{\lambda,\mu}$, i.e.

$$J_{\lambda,\mu}(u_n,v_n)\to c_{\lambda,\mu},\ \text{and}\ J'_{\lambda,\mu}(u_n,v_n)\to0\ \text{in}\ E^*_{\alpha,\beta},\ \text{as}\ n\to\infty.$$

Lemma 3.2 shows that $\{(u_n,v_n)\}$ is bounded in $E_{\alpha,\beta}$. Hence, there exists $(u,v)\in E_{\alpha,\beta}$ and a subsequence of $\{(u_n,v_n)\}_n$, still denoted by $\{(u_n,v_n)\}_n$ such that

$(u_n, v_n) \rightharpoonup (u, v)$ in $E_{\alpha,\beta}$. Additionally, by Lemma 2.3, we have $(u_n, v_n) \to (u, v)$ uniformly in $(C([0,T], R))^2$, i.e. $\|(u_n, v_n) - (u, v)\|_\infty \to 0$ as $n \to \infty$.

Now, let $(\mathfrak{a}, \mathfrak{b}) \in E_{\gamma,\upsilon}$ be fixed, $0 < \gamma \le 1$, $1 < \upsilon < \infty$, and denote by $\mathcal{S}_{\gamma,\upsilon}(\mathfrak{a}, \mathfrak{b})$ the linear functional on $E_{\gamma,\upsilon}$

$$\langle \mathcal{S}_{\gamma,\upsilon}(\mathfrak{a}, \mathfrak{b}), (\mathfrak{c}_1, \mathfrak{c}_2)\rangle = \int_0^T \Phi_\upsilon({}_0D_t^\gamma \mathfrak{a}(t))_0 D_t^\gamma \mathfrak{c}_1(t) + \Phi_\upsilon({}_0D_t^\gamma \mathfrak{b}(t))_0 D_t^\gamma \mathfrak{c}_2(t)dt,$$

for any $(\mathfrak{c}_1, \mathfrak{c}_2) \in E_{\gamma,\upsilon}$. By the Hölder inequality, we have

$$|\langle \mathcal{S}_{\gamma,\upsilon}(\mathfrak{a}, \mathfrak{b}), (\mathfrak{c}_1, \mathfrak{c}_2)\rangle| \le \|\mathfrak{a}\|_{\gamma,\upsilon}^{\upsilon-1}\|\mathfrak{c}_1\|_{\gamma,\upsilon} + \|\mathfrak{b}\|_{\gamma,\upsilon}^{\upsilon-1}\|\mathfrak{c}_2\|_{\gamma,\upsilon}.$$

Thus, for any $(\mathfrak{c}_1, \mathfrak{c}_2) \in E_{\gamma,\upsilon}$, the linear functional $\mathcal{S}_{\gamma,\upsilon}(\mathfrak{a}, \mathfrak{b})$ is also continuous. Hence, combined with $(u_n, v_n) \rightharpoonup (u, v)$ in $E_{\alpha,\beta}$, we have

$$\langle \mathcal{S}_{\gamma,\upsilon}(u, v), (u_n - u, v_n - v)\rangle \to 0, \text{ as } n \to \infty. \tag{3.23}$$

Then, we can deduce that

$$\begin{aligned}
o_n(1) =&\langle J'_{\lambda,\mu}(u_n, v_n) - J_\lambda(u, v), (u_n - u, v_n - v)\rangle \\
=&[M_1(\|(u_n, v_n)\|_\alpha^p)]^{p-1}\langle \mathcal{S}_{\alpha,p}(u_n, v_n) - \mathcal{S}_{\alpha,p}(u, v), (u_n - u, v_n - v)\rangle \\
&+ [M_2(\|(u_n, v_n)\|_\beta^q)]^{q-1}\langle \mathcal{S}_{\beta,q}(u_n, v_n) - \mathcal{S}_{\beta,q}(u, v), (u_n - u, v_n - v)\rangle \\
&+ ([M_1(\|(u_n, u_n)\|_\alpha^p)]^{p-1} - [M_1(\|(u, v)\|_\alpha^p)]^{p-1})\langle \mathcal{S}_{\alpha,p}(u, v), (u_n - u, v_n - v)\rangle \\
&+ ([M_2(\|(u_n, u_n)\|_\beta^q)]^{q-1} - [M_2(\|(u, v)\|_\beta^q)]^{q-1})\langle \mathcal{S}_{\beta,q}(u, v), (u_n - u, v_n - v)\rangle \\
&- \sum_{j=1}^m \lambda\zeta(t_j)(I_j(u_n(t_j)) - I_j(u(t_j)))(u_n(t_j) - u(t_j)) \\
&- \sum_{i=1}^n \mu\varrho(t'_i)(S_i(v_n(t'_i)) - S_i(v(t'_i)))(v_n(t'_i) - v(t'_i)) \\
&- \int_0^T (F_{u_n}(t, u_n, v_n) - F_u(t, u, v))(u_n - u)dt \\
&- \int_0^T (F_{v_n}(t, u_n, v_n) - F_v(t, u, v))(v_n - v)dt.
\end{aligned} \tag{3.24}$$

Moreover

$$\begin{aligned}
&\Big|\sum_{j=1}^m \lambda\zeta(t_j)(I_j(u_n(t_j)) - I_j(u(t_j)))(u_n(t_j) - u(t_j))\Big| \\
\le& m\bar{a}|\lambda|\ \|\zeta\|_\infty(\sup_{n\in N}|u_n|^{\theta-1} + \|u\|_\infty^{\theta-1})\|u_n - u\|_\infty \to 0, \text{ as } n \to \infty,
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
&\Big|\sum_{i=1}^n \mu\varrho(t'_i)(K_i(v_n(t'_i)) - K_i(v(t'_i)))(v_n(t'_i) - v(t'_i))\Big| \\
\le& n\bar{b}|\mu|\ \|\varrho\|_\infty(\sup_{n\in N}|v_n|^{\theta-1} + \|v\|_\infty^{\theta-1})\|v_n - v\|_\infty \to 0, \text{ as } n \to \infty,
\end{aligned} \tag{3.26}$$

and

$$\Big|\int_0^T (F_{u_n}(t, u_n, v_n) - F_u(t, u, v))(u_n - u)$$

$$
\begin{aligned}
&+ (F_{v_n}(t,u_n,v_n) - F_v(t,u,v))(v_n - v)dt\Big| \\
\leq & T|F_{u_n}(t,u_n,v_n) - F_u(t,u,v)|\ \|u_n - u\|_\infty \\
&+ T|F_{v_n}(t,u_n,v_n) - F_v(t,u,v)|\ \|v_n - v\|_\infty \\
\to & 0, \text{ as } n \to \infty. \qquad (3.27)
\end{aligned}
$$

From (3.23)-(3.27), we have

$$\langle \mathcal{S}_{\alpha,p}(u_n,v_n) - \mathcal{S}_{\alpha,p}(u,v), (u_n - u, v_n - v)\rangle \to 0, \text{ as } n \to \infty, \qquad (3.28)$$

and

$$\langle \mathcal{S}_{\beta,q}(u_n,v_n) - \mathcal{S}_{\beta,q}(u,v), (u_n - u, v_n - v)\rangle \to 0, \text{ as } n \to \infty. \qquad (3.29)$$

To illustrate our findings, we require the following Simon inequalities:

$$|d_1 - d_2|^h \leq \begin{cases} \overline{\mathcal{D}}(|d_1|^{h-2}d_1 - |d_2|^{h-2}d_2)(d_1 - d_2), h \geq 2, \\ \overline{\mathcal{D}}((|d_1|^{h-2}d_1 - |d_2|^{h-2}d_2)(d_1 - d_2))^{\frac{h}{2}}(|d_1|^h + |d_2|^h)^{\frac{2-h}{2}}, 1 < h < 2, \end{cases}$$

for all d_1, $d_2 \in R$, where $\overline{\mathcal{D}}$ is positive constant.

When $1 < p < 2$, by (3.28) and the Hölder inequality, we have

$$
\begin{aligned}
\|(u_n - u, v_n - v)\|_\alpha^p =& \|u_n - u\|_{\alpha,p}^p + \|v_n - v\|_{\alpha,p}^p \\
\leq & \overline{\mathcal{D}}\Big(\int_0^T (\Phi_p({}_0D_t^\alpha u_n) - \Phi_p({}_0D_t^\alpha u))({}_0D_t^\alpha u_n - {}_0D_t^\alpha u)dt\Big)^{\frac{p}{2}} \\
&\times \Big(\int_0^T (|{}_0D_t^\alpha u_n|^p + |{}_0D_t^\alpha u|^p)dt\Big)^{\frac{2-p}{2}} \\
&+ \overline{\mathcal{D}}\Big(\int_0^T (\Phi_p({}_0D_t^\alpha v_n) - \Phi_p({}_0D_t^\alpha v))({}_0D_t^\alpha v_n - {}_0D_t^\alpha v)dt\Big)^{\frac{p}{2}} \\
&\times \Big(\int_0^T (|{}_0D_t^\alpha v_n|^p + |{}_0D_t^\alpha v|^p)dt\Big)^{\frac{2-p}{2}} \\
\to & 0, \text{ as } n \to \infty. \qquad (3.30)
\end{aligned}
$$

When $p \geq 2$, by (3.28), we have

$$
\begin{aligned}
\|(u_n - u, v_n - v)\|_\alpha^p =& \|u_n - u\|_{\alpha,p}^p + \|v_n - v\|_{\alpha,p}^p \\
\leq & \overline{\mathcal{D}}\langle \mathcal{S}_{\alpha,p}(u_n,v_n) - \mathcal{S}_{\alpha,p}(u,v), (u_n - u, v_n - v)\rangle \\
\to & 0, \text{ as } n \to \infty. \qquad (3.31)
\end{aligned}
$$

Hence, we have $\|(u_n - u, v_n - v)\|_\alpha^p \to 0$ as $n \to \infty$. Similarly, we can obtain $\|(u_n - u, v_n - v)\|_\beta^q \to 0$, as $n \to \infty$. Then, we have $(u_n, v_n) \to (u, v)$ as $n \to \infty$ in $E_{\alpha,\beta}$. This ends the proof. □

Proof of Theorem 1.1. Let $(\lambda,\mu) \in \Theta_* = \Theta^0 \cap \Theta^1 \cap \Theta^2$. Thanks to the Lemma 3.6, we can consider a $(PS)_{c^+_{\lambda,\mu}}$ sequence $\{(u_n,v_n)\} \subset \mathcal{N}^+_{\lambda,\mu} \subset \mathcal{N}_{\lambda,\mu}$ such that

$$J_{\lambda,\mu}(u_n,v_n) \to c^+_{\lambda,\mu}, \text{ and } J'_{\lambda,\mu}(u_n,v_n) \to 0, \text{ as } n \to \infty.$$

Then, by Lemma 3.4 and 3.7, there exists $(u_{\lambda,\mu}, v_{\lambda,\mu}) \in \mathcal{N}_{\lambda,\mu}$ such that

$$J'_{\lambda,\mu}(u_{\lambda,\mu}, v_{\lambda,\mu}) = 0, \text{ and } J_{\lambda,\mu}(u_{\lambda,\mu}, v_{\lambda,\mu}) = c^+_{\lambda,\mu} < 0.$$

Further, we show that $(u_{\lambda,\mu}, v_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$. We debate through contradiction. Assume that $(u_{\lambda,\mu}, v_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$. Considering (3.5), we have $(u_{\lambda,\mu}, v_{\lambda,\mu}) \in \mathfrak{B}^+$. Moreover, $(u_{\lambda,\mu}, v_{\lambda,\mu}) \in \mathcal{N}_{\lambda,\mu}$ and $J_{\lambda,\mu}(u_{\lambda,\mu}, v_{\lambda,\mu}) = c^+_{\lambda,\mu} < 0$ imply $(u_{\lambda,\mu}, v_{\lambda,\mu}) \in \mathfrak{A}^+$. Then, by Lemma 3.3, there exist $s^-(u_{\lambda,\mu}, v_{\lambda,\mu}) > s^+(u_{\lambda,\mu}, v_{\lambda,\mu}) > 0$ such that $(s^- u_{\lambda,\mu}, s^- v_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$ and $(s^+ u_{\lambda,\mu}, s^+ v_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$, which implies $s^- = 1$ and $s^+ < 1$. Hence, there exists $s_0 \in (s^+, s^-)$ such that

$$\begin{aligned} J_{\lambda,\mu}(s^+ u_{\lambda,\mu}, s^+ v_{\lambda,\mu}) = &\min_{0\leq s\leq s^-} J_{\lambda,\mu}(s u_{\lambda,\mu}, s v_{\lambda,\mu}) < J_{\lambda,\mu}(s_0 u_{\lambda,\mu}, s_0 v_{\lambda,\mu}) \\ &< J_{\lambda,\mu}(s^- u_{\lambda,\mu}, s^- v_{\lambda,\mu}) = J_{\lambda,\mu}(u_{\lambda,\mu}, v_{\lambda,\mu}) = c^+_{\lambda,\mu}. \end{aligned}$$

That's a contradiction. Thus, $(u_{\lambda,\mu}, v_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$. Upon next, we show that $(u_{\lambda,\mu}, v_{\lambda,\mu})$ is a local minimizer of $J_{\lambda,\mu}$ in $E_{\alpha,\beta}$, which is a critical point of $J_{\lambda,\mu}$ in $E_{\alpha,\beta}$ (refer to reference [3]). Since $(u_{\lambda,\mu}, v_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$, by Lemma 3.3, we have $s^+(u_{\lambda,\mu}, v_{\lambda,\mu}) = 1 < s_*(u_{\lambda,\mu}, v_{\lambda,\mu})$. Thus, by continuity of $(u, v) \mapsto s_*(u, v)$, for fixed $\epsilon_0 > 0$, there exists $\rho_0 = \rho_0(\epsilon_0) > 0$ such that $1 + \epsilon_0 < s_*((u_{\lambda,\mu}, v_{\lambda,\mu}) - (u, v))$ for all $\|(u, v)\|_{\alpha,\beta} < \rho_0$. Also, from Lemma 3.5, it is easy to see that for a given $\rho_1 > 0$, there exists a C^1 map $\varphi : \Theta_{\rho_1}(0, 0) \subseteq E_{\alpha,\beta} \to R^+$ such that $\varphi(u, v)((u_{\lambda,\mu}, v_{\lambda,\mu}) - (u, v)) \in \mathcal{N}^+_{\lambda,\mu}$ and $\varphi(0, 0) = 1$. Consequently, for any $0 < \rho_* = \min\{\rho_0, \rho_1\}$ and the uniqueness of zeros of fibering map, we have $s^+((u_{\lambda,\mu}, v_{\lambda,\mu}) - (u, v)) = \varphi(u, v) < 1 + \epsilon_0 < s_*((u_{\lambda,\mu}, v_{\lambda,\mu}) - (u, v))$ for all $\|(u, v)\|_{\alpha,\beta} < \rho_*$. Since $s_*((u_{\lambda,\mu}, v_{\lambda,\mu}) - (u, v)) > 1$, we obtain

$$\begin{aligned} J_{\lambda,\mu}(u_{\lambda,\mu}, v_{\lambda,\mu}) \leq &J_{\lambda,\mu}(s^+((u_{\lambda,\mu}, v_{\lambda,\mu}) - (u, v))((u_{\lambda,\mu}, v_{\lambda,\mu}) - (u, v))) \\ \leq &J_{\lambda,\mu}((u_{\lambda,\mu}, v_{\lambda,\mu}) - (u, v)), \end{aligned}$$

for all $\|(u, v)\|_{\alpha,\beta} < \rho_*$. This shows that $(u_{\lambda,\mu}, v_{\lambda,\mu})$ is a local minimizer of $J_{\lambda,\mu}$ in $E_{\alpha,\beta}$. Moreover, since $(0, 0) \notin \mathcal{N}^+_{\lambda,\mu}$, we have $(u_{\lambda,\mu}, v_{\lambda,\mu}) \neq (0, 0)$, Hence, $(u_{\lambda,\mu}, v_{\lambda,\mu})$ is a nontrivial solution of problem (1.1). □

4. Existence of infinity many nontrivial solutions

In this section, we explore the multiplicity of solutions for problem (1.1) by applying the Krasnoselskii genus theory. First of all, we refer to [22] to recall some basic notions about Krasnoselskii genus.

Let X be a Banach space, and Ξ denotes the family of sets $A \subset X\backslash\{0\}$, where A is closed in X and symmetric with respect to 0, that is if $a \in A$ implies $-a \in A$.

Definition 4.1. Let $A \in \Xi$. The Krasnoselskii genus $\overline{\gamma}(A)$ of A is defined as the least positive integer n such that there is an odd mapping $\hbar \in C(A, R^n\backslash\{0\})$ for all $a \in A$. If n does not exist, $\overline{\gamma}(A) = \infty$. Moreover, $\overline{\gamma}(\emptyset) = 0$.

Note that condition (M1′) implies that

$$\overline{M}_1(x) \geq \overline{M}_1(1)x^{\overline{\vartheta}_1},\ \forall x \in [0, 1],\ \text{and}\ \overline{M}_2(y) \geq \overline{M}_2(1)y^{\overline{\vartheta}_2},\ \forall y \in [0, 1]. \tag{4.1}$$

Moreover, for any $\varepsilon > 0$, condition (F1′) gives the existence of $C_\varepsilon > 0$ such that for a.e. $t \in [0, T]$

$$|F(t, u, v)| \leq \varepsilon|(u, v)|^{\overline{\vartheta}_*} + C_\varepsilon|(u, v)|^{\overline{\eta}},\ \text{for all}\ (u, v) \in R^2. \tag{4.2}$$

Considering the nonlocal characteristics of M_1 and M_2, let us assume that $0 < \|(u,v)\|_{\alpha,\beta} \le 1$. Consequently, based on (2.4), (2.6), (3.10), (4.1) and (4.2), we can deduce that

$$\begin{aligned} J_{\lambda,\mu}(u,v) \geq& \frac{1}{p}\overline{M}_1(1)\|(u,v)\|_{\alpha}^{p\overline{\vartheta}_1} + \frac{1}{q}\overline{M}_2(1)\|(u,v)\|_{\beta}^{q\overline{\vartheta}_2} \\ &- \frac{2}{\theta}\mathcal{A}_{\alpha,p}^{\theta}\Big((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}\|(u,v)\|_{\alpha,\beta}^{\theta} \\ &- \varepsilon\overline{\mathcal{M}}(\overline{\vartheta}_*)\|(u,v)\|_{\alpha,\beta}^{\overline{\vartheta}_*} - C_\varepsilon\overline{\mathcal{M}}(\overline{\eta})\|(u,v)\|_{\alpha,\beta}^{\overline{\eta}} \\ \geq& \overline{M}_{1,2}\|(u,v)\|_{\alpha,\beta}^{\overline{\vartheta}_*} - \mathcal{M}_*\|(u,v)\|_{\alpha,\beta}^{\overline{\eta}} \\ &- \mathcal{A}_*\Big((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}\|(u,v)\|_{\alpha,\beta}^{\theta}. \end{aligned} \tag{4.3}$$

Then, we can choose $\varepsilon > 0$ sufficiently small such that $\overline{M}_{1,2} > 0$, where

$$\overline{M}_{1,2} = \min\{\frac{1}{p}\overline{M}_1(1), \frac{1}{q}\overline{M}_2(1)\}2^{1-\overline{\vartheta}_*} - \varepsilon\overline{\mathcal{M}}(\overline{\vartheta}_*),\ \mathcal{A}_* = \frac{2}{\theta}\mathcal{A}_{\alpha,p}^{\theta},\ \mathcal{M}_* = C_\varepsilon\overline{\mathcal{M}}(\overline{\eta}).$$

Define a function $\mathcal{T}_{\lambda,\mu} : (0,+\infty) \to R$ as

$$\begin{aligned} \mathcal{T}_{\lambda,\mu}(X) =& \overline{M}_{1,2}X^{\overline{\vartheta}_*} - \mathcal{M}_*X^{\overline{\eta}} \\ &- \mathcal{A}_*\Big((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}X^{\theta}, \end{aligned}$$

then $J_{\lambda,\mu}(u,v) \ge \mathcal{T}_{\lambda,\mu}(\|(u,v)\|_{\alpha,\beta})$ for all $(u,v) \in E_{\alpha,\beta}$ with $0 < \|(u,v)\|_{\alpha,\beta} \le 1$. Noticeably, $\mathcal{T}'_{\lambda,\mu}(X) = X^{\theta-1}\overline{\mathcal{T}}_{\lambda,\mu}(X)$, where

$$\begin{aligned} \overline{\mathcal{T}}_{\lambda,\mu}(X) =& \overline{M}_{1,2}\overline{\vartheta}_*X^{\overline{\vartheta}_*-\theta} - \mathcal{M}_*\overline{\eta}X^{\overline{\eta}-\theta} \\ &- \mathcal{A}_*\Big((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}\theta,\ \forall X > 0. \end{aligned}$$

Then, there exists a unique

$$X_0 = \Big(\frac{\overline{M}_{1,2}\overline{\vartheta}_*(\overline{\vartheta}_* - \theta)}{\mathcal{M}_*\overline{\eta}(\overline{\eta} - \theta)}\Big)^{\frac{1}{\overline{\eta}-\overline{\vartheta}_*}} > 0,$$

such that $\overline{\mathcal{T}}'_{\lambda,\mu}(X_0) = 0$. Namely, $\overline{\mathcal{T}}_{\lambda,\mu}(X)$ is increasing on $(0, X_0)$, decreasing on $(X_0, +\infty)$. Let $(\lambda,\mu) \in \Theta_0$, where

$$\Theta_0 = \Big\{(\lambda,\mu) \in R^2\backslash\{(0,0)\} : 0 < (m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}} < \Omega_0\Big\},$$

and

$$\Omega_0 = \Big(\frac{\overline{M}_{1,2}\overline{\vartheta}_*\frac{\overline{\eta}-\overline{\vartheta}_*}{\overline{\eta}-\theta}}{\mathcal{A}_*\theta}\Big)^{\frac{p}{p-\theta}}\Big(\frac{\overline{M}_{1,2}\overline{\vartheta}_*(\overline{\vartheta}_* - \theta)}{\mathcal{M}_*\overline{\eta}(\overline{\eta} - \theta)}\Big)^{\frac{\overline{\vartheta}_*-\theta}{\overline{\eta}-\overline{\vartheta}_*}\cdot\frac{p}{p-\theta}},$$

then we have

$$\max\overline{\mathcal{T}}_{\lambda,\mu}(X) = \overline{\mathcal{T}}_{\lambda,\mu}(X_0)$$

$$=\overline{M}_{1,2}\overline{\vartheta}_*\frac{\overline{\eta}-\overline{\vartheta}_*}{\overline{\eta}-\theta}\Big(\frac{\overline{M}_{1,2}\overline{\vartheta}_*(\overline{\vartheta}_*-\theta)}{\mathcal{M}_*\overline{\eta}(\overline{\eta}-\theta)}\Big)^{\frac{\overline{\vartheta}_*-\theta}{\overline{\eta}-\overline{\vartheta}_*}}$$
$$-\mathcal{A}_*\Big((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}}+(n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}\theta>0.$$

Hence, there exist $X_1\in(0,X_0)$ and $X_2\in(X_0,+\infty)$ such that $\mathcal{T}'_{\lambda,\mu}(X_1)=\mathcal{T}'_{\lambda,\mu}(X_2)$ $=0$. Taking into account that $\lim\limits_{X\to 0^+}\mathcal{T}_{\lambda,\mu}(X)<0$ and $\lim\limits_{X\to\infty}\mathcal{T}_{\lambda,\mu}(X)=-\infty$, we have $\mathcal{T}_{\lambda,\mu}(X)$ is decreasing on $(0,X_1)$, increasing on (X_1,X_2), decreasing on $(X_2,+\infty)$. Moreover, let $(\lambda,\mu)\in\Theta_1$, where

$$\Theta_1=\Big\{(\lambda,\mu)\in R^2\backslash\{(0,0)\}:0<(m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}}+(n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}<\Omega_1\Big\},$$

and

$$\Omega_1=\Big(\frac{\overline{M}_{1,2}\frac{\overline{\eta}-\overline{\vartheta}_*}{\overline{\eta}-\theta}}{\mathcal{A}_*}\Big)^{\frac{p}{p-\theta}}\Big(\frac{\overline{M}_{1,2}(\overline{\vartheta}_*-\theta)}{\mathcal{M}_*(\overline{\eta}-\theta)}\Big)^{\frac{\overline{\vartheta}_*-\theta}{\overline{\eta}-\overline{\vartheta}_*}\cdot\frac{p}{p-\theta}}.$$

Then, if we define a function $\mathcal{T}^1_{\lambda,\mu}:(0,+\infty)\to R$ as

$$\mathcal{T}^1_{\lambda,\mu}(X)=\overline{M}_{1,2}X^{\overline{\vartheta}_*-\theta}-\mathcal{M}_*X^{\overline{\eta}-\theta},$$

we can deduce that there exists unique $X_*=(\frac{\overline{M}_{1,2}(\overline{\vartheta}_*-\theta)}{\mathcal{M}_*(\overline{\eta}-\theta)})^{\frac{1}{\overline{\eta}-\overline{\vartheta}_*}}$ such that

$$\max\mathcal{T}^1_{\lambda,\mu}(X)=\mathcal{T}^1_{\lambda,\mu}(X_*)=\overline{M}_{1,2}\frac{\overline{\eta}-\overline{\vartheta}_*}{\overline{\eta}-\theta}\Big(\frac{\overline{M}_{1,2}(\overline{\vartheta}_*-\theta)}{\mathcal{M}_*(\overline{\eta}-\theta)}\Big)^{\frac{\overline{\vartheta}_*-\theta}{\overline{\eta}-\overline{\vartheta}_*}}>0.$$

Hence, for any $(\lambda,\mu)\in\Theta_1$, we have

$$\begin{aligned}\mathcal{T}_{\lambda,\mu}(X_2)=&\max_{X>0}\mathcal{T}_{\lambda,\mu}(X)\\ \geq& X_*^\theta\Big(\mathcal{T}^1_{\lambda,\mu}(X_*)-\mathcal{A}_*((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}}+(n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}})^{\frac{p-\theta}{p}}\Big)\\ >&0.\end{aligned}$$

At this point, we arrive at there exist $0<K_0(\lambda,\mu)<K_1(\lambda,\mu)$ with $\mathcal{T}_{\lambda,\mu}(K_0(\lambda,\mu))=\mathcal{T}_{\lambda,\mu}(K_1(\lambda,\mu))=0$. From the structure of $\mathcal{T}_{\lambda,\mu}(X)$, we have $\mathcal{T}_{\lambda,\mu}(X)\leq 0$ if $X\in(0,K_0(\lambda,\mu)]$, $\mathcal{T}_{\lambda,\mu}(X)>0$ if $X\in(K_0(\lambda,\mu),K_1(\lambda,\mu))$, and $\mathcal{T}_{\lambda,\mu}(X)\leq 0$ if $X\in[K_1(\lambda,\mu),+\infty)$.

Lemma 4.1. $\lim\limits_{(\lambda,\mu)\to(0,0)}K_0(\lambda,\mu)=0$.

Proof. From $\mathcal{T}_{\lambda,\mu}(K_0(\lambda,\mu))=0$ and $\mathcal{T}'_{\lambda,\mu}(K_0(\lambda,\mu))>0$, we have

$$\begin{aligned}&\overline{M}_{1,2}K_0(\lambda,\mu)^{\overline{\vartheta}_*}-\mathcal{A}_*\Big((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}}+(n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}K_0(\lambda,\mu)^\theta\\ &-\mathcal{M}_*K_0(\lambda,\mu)^{\overline{\eta}}=0,\end{aligned}\tag{4.4}$$

and

$$\overline{M}_{1,2}\overline{\vartheta}_*K_0(\lambda,\mu)^{\overline{\vartheta}_*-1}-\mathcal{A}_*\Big((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}}+(n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}\theta K_0(\lambda,\mu)^{\theta-1}$$

$$- \mathcal{M}_*\overline{\eta}K_0(\lambda,\mu)^{\overline{\eta}-1} > 0. \tag{4.5}$$

Subsequently, based on (4.4) and (4.5), we have

$$K_0(\lambda,\mu) < \Big(\frac{\overline{M}_{1,2}(\overline{\vartheta}_* - \theta)}{\mathcal{M}_*(\overline{\eta}-\theta)}\Big)^{\frac{1}{\overline{\eta}-\overline{\vartheta}_*}}.$$

The uniform boundedness of $K_0(\lambda,\mu)$ with respect to (λ,μ) can be inferred. Consequently, we choose a sequnence $\{(\lambda_n,\mu_n)\}_{n=1}^{\infty}$ with $(\lambda_n,\mu_n) \to (0,0)$ as $n\to\infty$. This choice ensures that $K_0(\lambda_n,\mu_n) \to K_{0*} \geq 0$ as $n\to\infty$. Then, (4.4) and (4.5) imply that

$$\overline{M}_{1,2}K_{0*}^{\overline{\vartheta}_*} - \mathcal{M}_*K_{0*}^{\overline{\eta}} = 0, \tag{4.6}$$

and

$$\overline{M}_{1,2}\overline{\vartheta}_*K_{0*}^{\overline{\vartheta}_*-1} - \mathcal{M}_*\overline{\eta}K_{0*}^{\overline{\eta}-1} \geq 0. \tag{4.7}$$

Combining (4.6) and (4.7) , we have

$$\mathcal{M}_*(\overline{\eta}-\overline{\vartheta}_*)K_{0*}^{\overline{\eta}} \leq 0,$$

which implies that $K_{0*}=0$. Considering the arbitrary nature of $\{(\lambda_n,\mu_n)\}$, we can deduce that $\lim\limits_{(\lambda,\mu)\to(0,0)} K_0(\lambda,\mu) = 0$. □

Take $\aleph : R^+ \to [0,1]$, nonincreasing and C^∞ with $\aleph(z) = 1$ if $z\in[0,K_0(\lambda,\mu)]$, $\aleph(z) = 0$ if $z \in [\min\{K_1(\lambda,\mu),1\},\infty)$. From Lemma 4.1, there exists $\Omega_2 > 0$ sufficiently small when $(\lambda,\mu)\in\Theta_2$ where

$$\Theta_2 = \Big\{(\lambda,\mu)\in R^2\backslash\{(0,0)\} : 0 < (m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}} < \Omega_2\Big\},$$

such that $K_0(\lambda,\mu) < \min\{K_1(\lambda,\mu),1\}$. Subsequently, for any $(\lambda,\mu)\in\overline{\Theta}$, where $\overline{\Theta} = \Theta_0\cap\Theta_1\cap\Theta_2$, we hereby introduce the truncated functional

$$\begin{aligned}\overline{J}_{\lambda,\mu}(u,v) =& \frac{1}{p}\overline{M}_1(\|(u,v)\|_\alpha^p) + \frac{1}{q}\overline{M}_2(\|(u,v)\|_\beta^q) \\ & - \sum_{j=1}^m\int_0^{u(t_j)}\lambda\zeta(t_j)I_j(z)dz - \sum_{i=1}^n\int_0^{v(t_i')}\mu\varrho(t_i')S_i(s)ds \\ & - \aleph(\|(u,v)\|_{\alpha,\beta})\int_0^T F(t,u(t),v(t))dt. \end{aligned} \tag{4.8}$$

According to (4.3), for any $0 < \|(u,v)\|_{\alpha,\beta} \leq 1$, we can observe that

$$\begin{aligned}\overline{J}_{\lambda,\mu}(u,v) \geq& \overline{M}_{1,2}\|(u,v)\|_{\alpha,\beta}^{\overline{\vartheta}_*} - \mathcal{M}_*\aleph(\|(u,v)\|_{\alpha,\beta})\|(u,v)\|_{\alpha,\beta}^{\overline{\eta}} \\ & - \mathcal{A}_*\Big((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}\|(u,v)\|_{\alpha,\beta}^{\theta}.\end{aligned}$$

Define a function $\mathcal{S}_{\lambda,\mu} : (0,+\infty) \to R$ as

$$\mathcal{S}_{\lambda,\mu}(X) = \overline{M}_{1,2}X^{\overline{\vartheta}_*} - \mathcal{M}_*\aleph(X)X^{\overline{\eta}}$$

$$- \mathcal{A}_*\Big((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}} X^\theta,$$

then $\overline{J}_{\lambda,\mu}(u,v) \geq \mathcal{S}_{\lambda,\mu}(\|(u,v)\|_{\alpha,\beta})$ for all $(u,v) \in E_{\alpha,\beta}$ with $0 < \|(u,v)\|_{\alpha,\beta} \leq 1$. Moreover, we can deduce that $\mathcal{S}_{\lambda,\mu}(X) \geq \mathcal{T}_{\lambda,\mu}(X)$ for all $X \geq 0$, $\mathcal{S}_{\lambda,\mu}(X) = \mathcal{T}_{\lambda,\mu}(X)$ for all $0 \leq X \leq K_0(\lambda,\mu)$.

Lemma 4.2. *For all $(\lambda,\mu) \in \Theta_{**}$, the following hold:*

(i) If $\overline{J}_{\lambda,\mu}(u,v) < 0$, then $\|(u,v)\|_{\alpha,\beta} < K_0(\lambda,\mu)$ and $J_{\lambda,\mu}(\overline{u},\overline{v}) = \overline{J}_{\lambda,\mu}(\overline{u},\overline{v})$ for all $(\overline{u},\overline{v})$ in a sufficiently small neighborhood of (u,v);

(ii) $\overline{J}_{\lambda,\mu}(u,v)$ satisfies the local $(PS)_c$ condition for all $c < 0$.

Proof. (i) We distinguish two different cases.

Case 1. $\min\{K_1(\lambda,\mu),1\} = 1$. If $K_0(\lambda,\mu) \leq \|(u,v)\|_{\alpha,\beta} < 1$, then $\overline{J}_{\lambda,\mu}(u,v) \geq \mathcal{S}_{\lambda,\mu}(\|(u,v)\|_{\alpha,\beta}) \geq \mathcal{T}_{\lambda,\mu}(\|(u,v)\|_{\alpha,\beta}) \geq 0$. If $\|(u,v)\|_{\alpha,\beta} \geq 1$, by the definition of $\aleph$ we know that $\aleph(\|(u,v)\|_{\alpha,\beta}) = 0$. Due to the impacts of M_1 and M_2, we have classified Case 1 into four subcases.

Case 1_1. $\|(u,v)\|_{\alpha,\beta} \geq 1$ with $\|(u,v)\|_\alpha \geq 1$ and $\|(u,v)\|_\beta \geq 1$. Then, by (M1′), (M2′) with $\tau = \overline{\tau} = 1$ and (3.10), we have

$$\begin{aligned}\overline{J}_{\lambda,\mu}(u,v) \geq &\min\Big\{\frac{1}{p\vartheta_1}m_1^{p-1}, \frac{1}{q\vartheta_2}m_2^{q-1}\Big\}2^{1-\varsigma}\|(u,v)\|_{\alpha,\beta}^\varsigma \\ &- \mathcal{A}_*\Big((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}} \|(u,v)\|_{\alpha,\beta}^\theta,\end{aligned}$$

where $\varsigma = \min\{p,q\}$. Define the function $\mathcal{S}_1 : (0,+\infty) \to R$ as

$$\mathcal{S}_1(X) = \Lambda_{1*}X^\varsigma - \mathcal{A}_*\Big((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}} X^\theta,\ \forall X \geq 0,$$

where $\Lambda_{1*} = \min\{\frac{1}{p\vartheta_1}m_1^{p-1}, \frac{1}{q\vartheta_2}m_2^{q-1}\}2^{1-\varsigma}$. Undoubtedly, $\mathcal{S}_1(X)$ has a global minimum point at

$$X_{\min} = \Big(\frac{\mathcal{A}_*((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}})^{\frac{p-\theta}{p}}\theta}{\Lambda_{1*}\varsigma}\Big)^{\frac{1}{\varsigma-\theta}},$$

with

$$\begin{aligned}\min_{X>0} \mathcal{S}_1(X) =&\mathcal{S}_1(X_{\min}) \\ =&X_{\min}^\theta \mathcal{A}_*\Big((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}(\frac{\theta}{\varsigma} - 1) \\ <&0.\end{aligned}$$

We point out that $\mathcal{S}_1(X) \geq 0$ if and only if $X \geq X^1_{\lambda,\mu}$, where

$$X^1_{\lambda,\mu} = \Big(\frac{\mathcal{A}_*((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}})^{\frac{p-\theta}{p}}}{\Lambda_{1*}}\Big)^{\frac{1}{\varsigma-\theta}}.$$

Hence, choosing $(\lambda,\mu)\in\Theta_{1*}$, where

$$\Theta_{1*}=\Big\{(\lambda,\mu)\in R^2\backslash\{(0,0)\}:0<(m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}}+(n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\leq(\frac{\Lambda_{1*}}{\mathcal{A}_*})^{\frac{p}{p-\theta}}\Big\},$$

we have $\overline{J}_{\lambda,\mu}(u,v)\geq\mathcal{S}_1(\|(u,v)\|_{\alpha,\beta})\geq 0$ for all $\|(u,v)\|_{\alpha,\beta}\geq 1$.

Case 1_2**.** $\|(u,v)\|_{\alpha,\beta}\geq 1$ with $0<\|(u,v)\|_\alpha<1$ and $0<\|(u,v)\|_\beta<1$. Then, by (3.10) and (4.1), we have

$$\begin{aligned}\overline{J}_{\lambda,\mu}(u,v)\geq&\min\{\frac{1}{p}\overline{M}_1(1),\frac{1}{q}\overline{M}_2(1)\}2^{1-\overline{\vartheta}_*}\|(u,v)\|_{\alpha,\beta}^{\overline{\vartheta}_*}\\&-\mathcal{A}_*\Big((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}}+(n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}\|(u,v)\|_{\alpha,\beta}^{\theta}.\end{aligned}$$

Define the function $\mathcal{S}_2:(0,+\infty)\to R$ as

$$\mathcal{S}_2(X)=\Lambda_{2*}X^{\overline{\vartheta}_*}-\mathcal{A}_*\Big((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}}+(n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}X^{\theta},\ \forall X\geq 0,$$

where $\Lambda_{2*}=\min\{\frac{1}{p}\overline{M}_1(1),\frac{1}{q}\overline{M}_2(1)\}2^{1-\overline{\vartheta}_*}$. Likewise, $\mathcal{S}_2(X)\geq 0$ if and only if $X\geq X^2_{\lambda,\mu}$, where

$$X^2_{\lambda,\mu}=\Big(\frac{\mathcal{A}_*((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}}+(n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}})^{\frac{p-\theta}{p}}}{\Lambda_{2*}}\Big)^{\frac{1}{\overline{\vartheta}_*-\theta}}.$$

Hence, choosing $(\lambda,\mu)\in\Theta_{2*}$, where

$$\Theta_{2*}=\Big\{(\lambda,\mu)\in R^2\backslash\{(0,0)\}:0<(m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}}+(n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\leq(\frac{\Lambda_{2*}}{\mathcal{A}_*})^{\frac{p}{p-\theta}}\Big\},$$

we have $\overline{J}_{\lambda,\mu}(u,v)\geq\mathcal{S}_2(\|(u,v)\|_{\alpha,\beta})\geq 0$ for all $\|(u,v)\|_{\alpha,\beta}\geq 1$.

Case 1_3**.** $\|(u,v)\|_{\alpha,\beta}\geq 1$ with $\|(u,v)\|_\alpha\geq 1$ and $0<\|(u,v)\|_\beta<1$. Then, by (M1′), (M2′) with $\tau=1$ and (3.10), we have

$$\begin{aligned}&\overline{J}_{\lambda,\mu}(u,v)\\\geq&\frac{1}{p\vartheta_1}m_1^{p-1}\|(u,v)\|_\alpha^p-\mathcal{A}_*\Big((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}}+(n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}\|(u,v)\|_\alpha^{\theta}.\end{aligned}$$

Define the function $\mathcal{S}_3:(0,+\infty)\to R$ as

$$\mathcal{S}_3(X)=\Lambda_{3*}X^{p}-\mathcal{A}_*\Big((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}}+(n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}X^{\theta},\ \forall X\geq 0,$$

where $\Lambda_{3*}=\frac{1}{p\vartheta_1}m_1^{p-1}$. Then, $\mathcal{S}_3(X)\geq 0$ if and only if $X\geq X^3_{\lambda,\mu}$, where

$$X^3_{\lambda,\mu}=\Big(\frac{\mathcal{A}_*((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}}+(n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}})^{\frac{p-\theta}{p}}}{\Lambda_{3*}}\Big)^{\frac{1}{p-\theta}}.$$

Hence, choosing $(\lambda,\mu)\in\Theta_{3*}$, where

$$\Theta_{3*}=\Big\{(\lambda,\mu)\in R^2\backslash\{(0,0)\}:0<(m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}}+(n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\leq(\frac{\Lambda_{3*}}{\mathcal{A}_*})^{\frac{p}{p-\theta}}\Big\},$$

we have $\overline{J}_{\lambda,\mu}(u,v) \geq \mathcal{S}_3(\|(u,v)\|_{\alpha,\beta}) \geq 0$ for all $\|(u,v)\|_{\alpha,\beta} \geq 1$.

Case 1_4**.** $\|(u,v)\|_{\alpha,\beta} \geq 1$ with $0 < \|(u,v)\|_\alpha < 1$ and $\|(u,v)\|_\beta \geq 1$. Then, by (M1′), (M2′) with $\overline{\tau} = 1$ and (3.10), we have

$$\begin{aligned}&\overline{J}_{\lambda,\mu}(u,v)\\ \geq&\frac{1}{q\overline{\vartheta}_2}m_2^{q-1}\|(u,v)\|_\beta^q - \mathcal{A}_*\Big((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}\|(u,v)\|_\beta^\theta.\end{aligned}$$

Define the function $\mathcal{S}_4 : (0,+\infty) \to R$ as

$$\mathcal{S}_4(X) = \Lambda_{4*}X^q - \mathcal{A}_*\Big((|\lambda|\ \|\zeta\|_{\gamma_*})^{\frac{p}{p-\theta}} + (|\mu|\ \|\varrho\|_{\gamma_*})^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}X^\theta,\ \forall X \geq 0,$$

where $\Lambda_{4*} = \frac{1}{q\overline{\vartheta}_2}m_2^{q-1}$. Then, $\mathcal{S}_4(X) \geq 0$ if and only if $X \geq X^4_{\lambda,\mu}$, where

$$X^4_{\lambda,\mu} = \Big(\frac{\mathcal{A}_*((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}})^{\frac{p-\theta}{p}}}{\Lambda_{4*}}\Big)^{\frac{1}{q-\theta}}.$$

Hence, choosing $(\lambda,\mu) \in \Theta_{4*}$, where

$$\begin{aligned}&\Theta_{4*}\\ =&\Big\{(\lambda,\mu) \in R^2\backslash\{(0,0)\} : 0 < (m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}} \leq (\frac{\Lambda_{4*}}{\mathcal{A}_*})^{\frac{p}{p-\theta}}\Big\},\end{aligned}$$

we have $\overline{J}_{\lambda,\mu}(u,v) \geq \mathcal{S}_4(\|(u,v)\|_{\alpha,\beta}) \geq 0$ for all $\|(u,v)\|_{\alpha,\beta} \geq 1$.

Case 2. $\min\{K_1(\lambda,\mu),1\} = K_1(\lambda,\mu)$. If $K_0(\lambda,\mu) \leq \|(u,v)\|_{\alpha,\beta} < K_1(\lambda,\mu)$, then $\overline{J}_{\lambda,\mu}(u,v) \geq \mathcal{S}_{\lambda,\mu}(\|(u,v)\|_{\alpha,\beta}) \geq \mathcal{T}_{\lambda,\mu}(\|(u,v)\|_{\alpha,\beta}) \geq 0$. If $K_1(\lambda,\mu) \leq \|(u,v)\|_{\alpha,\beta} < 1$, considering $\aleph(\|(u,v)\|_{\alpha,\beta}) = 0$, we have

$$\begin{aligned}\overline{J}_{\lambda,\mu}(u,v) \geq&\overline{M}_{1,2}\|(u,v)\|_{\alpha,\beta}^{\overline{\vartheta}_*}\\ &- \mathcal{A}_*\Big((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}\|(u,v)\|_{\alpha,\beta}^\theta.\end{aligned}$$

Due to $\mathcal{T}_{\lambda,\mu}(K_1(\lambda,\mu)) = 0$, that is

$$\begin{aligned}&\overline{M}_{1,2}K_1(\lambda,\mu)^{\overline{\vartheta}_*} - \mathcal{A}_*\Big((m\overline{a}|\lambda|\ \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu|\ \|\varrho\|_\infty)^{\frac{p}{p-\theta}}\Big)^{\frac{p-\theta}{p}}K_1(\lambda,\mu)^\theta\\ =&\mathcal{M}_*K_1(\lambda,\mu)^{\overline{\eta}} > 0,\end{aligned}$$

we have $\overline{J}_{\lambda,\mu}(u,v) > 0$ for all $K_1(\lambda,\mu) \leq \|(u,v)\|_{\alpha,\beta} < 1$. If $\|(u,v)\|_{\alpha,\beta} \geq 1$, arguing similarly to Case1_1-Case1_4, we have $\mathcal{J}_\lambda(u,v) \geq 0$.

Hence, from **Case 1** and **Case 2**, for all $(\lambda,\mu) \in \Theta_{**}$, where $\Theta_{**} = \overline{\Theta} \cap \Theta_{1*} \cap \Theta_{2*} \cap \Theta_{3*} \cap \Theta_{4*}$, we obtain $\overline{J}_{\lambda,\mu}(u,v) < 0$, then $\|(u,v)\|_{\alpha,\beta} < K_0(\lambda,\mu)$. Moreover, we observe that $J_{\lambda,\mu}(\overline{u},\overline{v}) = \overline{J}_{\lambda,\mu}(\overline{u},\overline{v})$ for all $\|(\overline{u},\overline{v}) - (u,v)\|_{\alpha,\beta} < K_1(\lambda,\mu) - \|(u,v)\|_{\alpha,\beta}$.

(ii) Let $\{(u_n,v_n)\}$ be a (PS)-sequence for $\overline{J}_{\lambda,\mu}$ on the level $c < 0$, that is

$$\overline{J}_{\lambda,\mu}(u_n,v_n) \to c,\ \text{and}\ \overline{J}'_{\lambda,\mu}(u_n,v_n) \to 0\ \text{in}\ E^*_{\alpha,\beta},\ \text{as}\ n \to \infty.$$

This implies that $\overline{J}_{\lambda,\mu}(u_n, v_n) < 0$ for $n \in N$ large enough. Then, by (i) of Lemma 4.2, we have $\|(u, v)\|_{\alpha,\beta} < K_0(\lambda, \mu)$ and $J_{\lambda,\mu}(u_n, v_n) = \overline{J}_{\lambda,\mu}(u_n, v_n) \to c < 0$ and $J'_{\lambda,\mu}(u_n, v_n) = \overline{J}'_{\lambda,\mu}(u_n, v_n) \to 0$ as $n \to \infty$. By virtue of the coerciveness of $\overline{J}_{\lambda,\mu}$ in $E_{\alpha,\beta}$, it can be inferred that $\{(u_n, v_n)\}$ is bounded in $E_{\alpha,\beta}$. Hence, there exists $(\overline{u}_{\lambda,\mu}, \overline{v}_{\lambda,\mu}) \in E_{\alpha,\beta}$, $e_\alpha \geq 0$, $e_\beta \geq 0$ and a subsequence of $\{(u_n, v_n)\}_n$, still denoted by $\{(u_n, v_n)\}_n$ such that

$$(u_n, v_n) \rightharpoonup (\overline{u}_{\lambda,\mu}, \overline{v}_{\lambda,\mu}) \text{ in } E_{\alpha,\beta},\ \|(u_n, v_n)\|_\alpha \to e_\alpha,\ \|(u_n, v_n)\|_\beta \to e_\beta,\ \text{as } n \to \infty.$$

As a result of (F1′), we have

$$\begin{aligned}&\Big| \int_0^T (F_{u_n}(t, u_n, v_n) - F_{\overline{u}_{\lambda,\mu}}(t, \overline{u}_{\lambda,\mu}, \overline{v}_{\lambda,\mu}))(u_n - \overline{u}_{\lambda,\mu}) \\ &\quad + (F_{v_n}(t, u_n, v_n) - F_{\overline{v}_{\lambda,\mu}}(t, \overline{u}_{\lambda,\mu}, \overline{v}_{\lambda,\mu}))(v_n - \overline{v}_{\lambda,\mu}) dt \Big| \\ \to& 0, \quad \text{as } n \to \infty.\end{aligned}$$

Following a similar line of argument as presented in (3.23)-(3.26), we can deduce that

$$\begin{aligned}o_n(1) =& \Big\langle J'_{\lambda,\mu}(u_n, v_n) - J_\lambda(\overline{u}_{\lambda,\mu}, \overline{v}_{\lambda,\mu}), (u_n - \overline{u}_{\lambda,\mu}, v_n - \overline{v}_{\lambda,\mu}) \Big\rangle \\ =& [M_1(\|(u_n, v_n)\|_\alpha^p)]^{p-1} \\ & \times \Big\langle \mathcal{S}_{\alpha,p}(u_n, v_n) - \mathcal{S}_{\alpha,p}(\overline{u}_{\lambda,\mu}, \overline{v}_{\lambda,\mu}), (u_n - \overline{u}_{\lambda,\mu}, v_n - \overline{v}_{\lambda,\mu}) \Big\rangle \\ & + [M_2(\|(u_n, v_n)\|_\beta^q)]^{q-1} \\ & \times \Big\langle \mathcal{S}_{\beta,q}(u_n, v_n) - \mathcal{S}_{\beta,q}(\overline{u}_{\lambda,\mu}, \overline{v}_{\lambda,\mu}), (u_n - \overline{u}_{\lambda,\mu}, v_n - \overline{v}_{\lambda,\mu}) \Big\rangle. \end{aligned} \tag{4.9}$$

Due to the degeneracy of M_1 and M_2 , we discuss them in the following three cases and conclude that only the first one holds.

Case i. $e_\alpha > 0$ and $e_\beta > 0$. It is apparent that condition (M2′) implies that $M_1(e_\alpha^p) > 0$, $M_2(e_\beta^q) > 0$. Then, we have

$$\langle \mathcal{S}_{\alpha,p}(u_n, v_n) - \mathcal{S}_{\alpha,p}(\overline{u}_{\lambda,\mu}, \overline{v}_{\lambda,\mu}), (u_n - \overline{u}_{\lambda,\mu}, v_n - \overline{v}_{\lambda,\mu}) \rangle \to 0,\ \text{as } n \to \infty, \tag{4.10}$$
$$\langle \mathcal{S}_{\beta,q}(u_n, v_n) - \mathcal{S}_{\beta,q}(\overline{u}_{\lambda,\mu}, \overline{v}_{\lambda,\mu}), (u_n - \overline{u}_{\lambda,\mu}, v_n - \overline{v}_{\lambda,\mu}) \rangle \to 0,\ \text{as } n \to \infty. \tag{4.11}$$

Hence, arguing similarly to (3.30) and (3.31), we have

$$\|(u_n - \overline{u}_{\lambda,\mu}, v_n - \overline{v}_{\lambda,\mu})\|_\alpha^p \to 0,\ \text{and } \|(u_n - \overline{u}_{\lambda,\mu}, v_n - \overline{v}_{\lambda,\mu})\|_\beta^q \to 0,\ \text{as } n \to \infty.$$

Additionally, by employing Lemma 2.4, we derive that

$$\|(u_n, v_n)\|_\alpha^p = \|(u_n - \overline{u}_{\lambda,\mu}, v_n - \overline{v}_{\lambda,\mu})\|_\alpha^p + \|(\overline{u}_{\lambda,\mu}, \overline{v}_{\lambda,\mu})\|_\alpha^p + o_n(1), \tag{4.12}$$
$$\|(u_n, v_n)\|_\beta^q = \|(u_n - \overline{u}_{\lambda,\mu}, v_n - \overline{v}_{\lambda,\mu})\|_\beta^q + \|(\overline{u}_{\lambda,\mu}, \overline{v}_{\lambda,\mu})\|_\beta^q + o_n(1). \tag{4.13}$$

Then, we have

$$\|(u_n, v_n)\|_\alpha \to \|(\overline{u}_{\lambda,\mu}, \overline{v}_{\lambda,\mu})\|_\alpha,\ \text{and } \|(u_n, v_n)\|_\beta \to \|(\overline{u}_{\lambda,\mu}, \overline{v}_{\lambda,\mu})\|_\beta,\ \text{as } n \to \infty.$$

Therefore, we get $(u_n, v_n) \to (\overline{u}_{\lambda,\mu}, \overline{v}_{\lambda,\mu})$ as $n \to \infty$ in $E_{\alpha,\beta}$.

Case ii. $e_\alpha = 0$ and $e_\beta > 0$. The condition (M2′) implies $M_1(e_\alpha^p) = 0$, $M_2(e_\beta^q) > 0$. Then, (4.11) is valid. Hence, we obtain $\|(u_n - \overline{u}_{\lambda,\mu}, v_n - \overline{v}_{\lambda,\mu})\|_\beta^q \to 0$ as $n \to \infty$.

Moreover, it follows from (4.12) that $(\overline{u}_{\lambda,\mu}, \overline{v}_{\lambda,\mu}) = (0,0)$. Then, according to equation (4.13), we can deduce that the norm of $\|(u_n, v_n)\|_\beta \to e_\beta = \|(\overline{u}_{\lambda,\mu}, \overline{v}_{\lambda,\mu})\|_\beta = 0 > 0$, which leads to a contradiction.

Case iii. $e_\alpha > 0$ and $e_\beta = 0$. The condition (M2′) implies $M_1(e_\alpha^p) > 0$, $M_2(e_\beta^q) = 0$. Consequently, (4.10) holds. Repeating the analysis procedure as in case Caseii yields $\|(u_n, v_n)\|_\alpha \to e_\alpha = \|(\overline{u}_{\lambda,\mu}, \overline{v}_{\lambda,\mu})\|_\alpha = 0 > 0$, which is a contradiction. □

Lemma 4.3. *For each $n \in N$, there exists a real number $\overline{\varepsilon} = \overline{\varepsilon}(n) > 0$ such that $\overline{\gamma}(\overline{J}_{\lambda,\mu}^{-\overline{\varepsilon}}) \geq n$, where $\overline{J}_{\lambda,\mu}^{-\overline{\varepsilon}} = \{(u,v) \in E_{\alpha,\beta} : \overline{J}_{\lambda,\mu}(u,v) \leq -\overline{\varepsilon}\}$.*

Proof. By (F2′), there exist W_0, $W_1 > 0$ such that

$$F(t,u,v) \geq W_0(|u|^{\overline{\varsigma}} + |v|^{\overline{\varsigma}}) - W_1, \text{ for any } (u,v) \in R^2.$$

Consider a fixed positive integer n, and let $E_{\alpha,\beta}^n$ denote an n-dimensional subspace of $E_{\alpha,\beta}$. The fact that $E_{\alpha,\beta}^n$ is a space of finite dimension implies that all its norms in $E_{\alpha,\beta}^n$ are mutually equivalent. Subsequently, we define

$$\mathcal{C}_*^1 = \inf\left\{ \int_0^T (|u|^{\overline{\varsigma}} + |v|^{\overline{\varsigma}}) dt \,|\, (u,v) \in E_{\alpha,\beta}^n, \|(u,v)\|_{\alpha,\beta} = 1 \right\} > 0.$$

Then, taking $(u,v) \in E_{\alpha,\beta}$ with $\|(u,v)\|_{\alpha,\beta} = 1$, for $0 < \xi < K_0(\lambda,\mu)$, we can deduce that

$$\begin{aligned}\overline{J}_{\lambda,\mu}(\xi u, \xi v) \leq & \frac{1}{p} \max_{0<\iota_1<1} [M_1(\iota_1)]^{p-1} \xi^p + \frac{1}{q} \max_{0<\iota_2<1} [M_2(\iota_2)]^{q-1} \xi^q \\ & + \frac{\xi^\theta}{\theta} 2\mathcal{A}_{\alpha,p}^\theta \Big((m\overline{a}|\lambda| \; \|\zeta\|_\infty)^{\frac{p}{p-\theta}} + (n\overline{b}|\mu| \; \|\varrho\|_\infty)^{\frac{p}{p-\theta}} \Big)^{\frac{p-\theta}{p}} \\ & - \xi^{\overline{\varsigma}} W_0 \mathcal{C}_*^1 + W_1 T.\end{aligned}$$

Given $0 < \overline{\varsigma} < \theta < 1$, for any $\overline{\varepsilon} > 0$, there exists a sufficiently small $\xi \in (0, K_0(\lambda,\mu))$ such that $\overline{J}_{\lambda,\mu}(\xi u, \xi v) \leq -\overline{\varepsilon}$ holds ture for all $(u,v) \in E_{\alpha,\beta}$ with $\|(u,v)\|_{\alpha,\beta} = 1$. Let $K_\xi = \{(u,v) \in E_{\alpha,\beta} | \; \|(u,v)\|_{\alpha,\beta} = \xi\}$, then $K_\xi \cap E_{\alpha,\beta}^n \subset \{(u,v) \in E_{\alpha,\beta} : \overline{J}_{\lambda,\mu}(u,v) \leq -\overline{\varepsilon}\}$. By the mapping property of the genus [22, Proposition 7.5], we have $\overline{\gamma}(\overline{J}_{\lambda,\mu}^{-\overline{\varepsilon}}) \geq \overline{\gamma}(K_\xi \cap E_{\alpha,\beta}^n) = n$. □

Proof of Theorem 1.2. Let us commence by introducing some notation.

$$\Sigma_n = \Big\{ A \subset \Xi \backslash \{(0,0)\}, \overline{\gamma}(A) \geq n \Big\}, \; c_n = \inf_{A \in \Sigma_n} \sup_{(u,v) \in A} \overline{J}_{\lambda,\mu}(u,v),$$

$$L_c = \Big\{ (u,v) \in E_{\alpha,\beta} | \overline{J}_{\lambda,\mu}(u,v) = c, \; \overline{J}'_{\lambda,\mu}(u,v) = 0 \Big\}.$$

Apparently, condition (F3′) ensures that $\overline{J}_{\lambda,\mu}(u,v)$ is even and Lemma 4.3 reveals that $\overline{J}_{\lambda,\mu}^{-\overline{\varepsilon}} \in \Sigma_n$. Moreover, as $\overline{J}_{\lambda,\mu}(u,v)$ is bounded from below, it follows that $-\infty < c_n \leq -\overline{\varepsilon}(n) < 0$. The $(PS)_{c_n}$-condition of $\overline{J}_{\lambda,\mu}(u,v)$ can be inferred from (ii) of Lemma 4.2. Thus, c_n is a critical value of $\overline{J}_{\lambda,\mu}(u,v)$ for any $n \in N$.

The inclusion $\Sigma_{n+1} \subset \Sigma_n$ implies $c_n \leq c_{n+1}$. We proceed by considering two distinct scenarios.

Case I. If $-\infty < c_1 < c_2 < \cdots < c_n < c_{n+1} < \cdots$, then $\overline{\gamma}(L_{c_n}) \geq 1$, indicating that $\{c_n\}$ represents a sequence of distinct negative critical values of $\overline{J}_{\lambda,\mu}(u,v)$.

Case II. If there exists a positive integer n_0 such that $c = c_{n_0} = c_{n_0+1} = \cdots = c_{n_0+N_*}$ for some $N_* \geq 1$, then it follows that $\overline{\gamma}(L_c) \geq N_* + 1$. Assuming a contradiction, we hypothesize that $c = c_{n_0} = c_{n_0+1} = \cdots = c_{n_0+N_*} < 0$, then $\overline{\gamma}(L_c) \leq N_*$. Considering that $\overline{J}_{\lambda,\mu}$ satisfies $(PS)_c$ condition according to Lemma 4.2, we obtain that L_c is a compact set and $\overline{\gamma}(L_c) \leq N_* < \infty$. According to the continuity property of the genus, there exists $\delta^* > 0$ such that $\overline{\gamma}(U_{\delta^*}) = \overline{\gamma}(L_c) \leq N_* < \infty$. Since $\overline{J}_{\lambda,\mu}$ is even, by the Deformation Theorem [22], there exists an odd homeomorphism $\mathcal{T} : E_{\alpha,\beta} \to E_{\alpha,\beta}$ such that $\mathcal{T}(\overline{J}_{\lambda,\mu}^{c+\overline{\varepsilon}_0} \backslash U_{\delta^*}) \subset \overline{J}_{\lambda,\mu}^{c-\overline{\varepsilon}_0}$ for some $0 < \overline{\varepsilon}_0 < -c$. In addition, owing to $c = c_{n_0+N_*} = \inf_{A\in\Sigma_{n_0+N_*}} \sup_{(u,v)\in A} \overline{J}_{\lambda,\mu}(u,v)$, there exists an $A \in \Sigma_{n_0+N_*}$ such that $\sup_{(u,v)\in A} \overline{J}_{\lambda,\mu}(u,v) = c < c + \overline{\varepsilon}_0$, i.e. $A \subset \overline{J}_{\lambda,\mu}^{c+\overline{\varepsilon}_0}$. Then, $\mathcal{T}(A\backslash U_{\delta^*}) \subset \mathcal{T}(\overline{J}_{\lambda,\mu}^{c+\overline{\varepsilon}_0} \backslash U_{\delta^*}) \subset \overline{J}_{\lambda,\mu}^{c-\overline{\varepsilon}_0}$, i.e.

$$\sup_{(u,v)\in\mathcal{T}(A\backslash U_{\delta^*})} \overline{J}_{\lambda,\mu}(u,v) \leq c - \overline{\varepsilon}_0. \tag{4.14}$$

Hence, we have $\overline{\gamma}(\overline{A\backslash U_{\delta^*}}) \geq \overline{\gamma}(A) - \overline{\gamma}(U_{\delta^*}) \geq n_0 + N_* - N_* = n_0$, i.e. $\mathcal{T}(\overline{A\backslash U_{\delta^*}}) \in \Sigma_{n_0}$ and $\sup_{(u,v)\in\mathcal{T}(A\backslash U_{\delta^*})} \overline{J}_{\lambda,\mu}(u,v) \geq c_{n_0} = c$, which contradicts (4.14). Consequently, it follows that $\overline{\gamma}(L_c) \geq N_* + 1 > 2$, which shows that L_c contains infinitely many point. Furthermore, $\overline{J}_{\lambda,\mu}(u,v) = c < 0 = \overline{J}_{\lambda,\mu}(0,0)$, then $(u,v) \neq (0,0)$. And also, it is established that $\overline{J}_{\lambda,\mu}(u,v) = J_{\lambda,\mu}(u,v)$ if $\overline{J}_{\lambda,\mu}(u,v) < 0$, then there are infinitely many nontrivial critical points of $J_{\lambda,\mu}(u,v)$. Therefore, problem (1.1) has infinitely many nontrivial solutions. □

5. Examples

In this section we give two examples to illustrate the application of our results.

Example 5.1. We consider the following impulsive fractional differential equation:

$$\begin{cases} \quad 15\|(u,v)\|_\alpha^2 \ {}_tD_T^\alpha({}_0D_t^\alpha u(t)) + 15.5\|(u,v)\|_\beta^2 \ {}_tD_T^\beta({}_0D_t^\beta u(t)) \\ = (t+1)|(u,v)|^{500}u, \ t \neq t_1, \ \text{a.e. } t \in [0,T], \\ \quad 15\|(u,v)\|_\alpha^2 \ {}_tD_T^\alpha({}_0D_t^\alpha v(t)) + 15.5\|(u,v)\|_\beta^2 \ {}_tD_T^\beta({}_0D_t^\beta v(t)) \\ = (t+1)|(u,v)|^{500}v, \ t \neq t_1', \ \text{a.e. } t \in [0,T], \\ \Delta(15\|(u,v)\|_\alpha^2 \ {}_tD_T^{\alpha-1}({}_0D_t^\alpha u))(t_1) \\ \ + \Delta(15.5\|(u,v)\|_\beta^2 \ {}_tD_T^{\beta-1}({}_0D_t^\beta u))(t_1) + \lambda\zeta(t_1)\dfrac{|u(t_1)|^{0.002}}{u(t_1)} = 0, \\ \Delta(15\|(u,v)\|_\alpha^2 \ {}_tD_T^{\alpha-1}({}_0D_t^\alpha v))(t_1') \\ \ + \Delta(15.5\|(u,v)\|_\beta^2 \ {}_tD_T^{\beta-1}({}_0D_t^\beta v))(t_1') + \mu\varrho(t_1')\dfrac{|v(t_1')|^{0.002}}{v(t_1')} = 0, \\ u(0) = u(T) = 0, \ v(0) = v(T) = 0. \end{cases} \tag{5.1}$$

For this case, $M_1(x) = 15x$, $M_2(y) = 15.5y$, $p = q = 2$, $I_1(z) = S_1(z) = \frac{|z|^{0.002}}{z}$, $z \in R$, $F(t,u,v) = (t+1)\frac{|(u,v)|^{502}}{502}$.

We verify that all the conditions of Theorem 1.1 are satisfied. Obviously, M_1 and M_2 satisfy (M1) with

$$0.5x^2 < 0.8 \times 7.5x^2 \leq 15x^2 \leq 1.2 \times 15x^2 < 20x^2,$$

$$0.4y^2 < 0.9 \times 7.75y^2 \le 15.5y^2 \le 1.4 \times 15.5y^2 < 22y^2,$$

where $\xi_1 = 0.5$, $\vartheta_1 = 0.8$, $\eta_1 = 1.2$, $\delta_1 = 20$, $\sigma = 502$, $\theta = 0.002$, $\xi_2 = 0.4$, $\vartheta_2 = 0.9$, $\eta_2 = 1.4$, $\delta_2 = 22$, $\kappa = 4$; I_1 and S_1 satisfy (H1) and (H2) with

$$\begin{aligned} &|I_1(z)z| \le 1.2|z|^{0.002}, \text{ and } |S_1(z)z| \le 2.4|z|^{0.002}, \\ &I_1(su) = s^{0.002-1}I_1(u), \text{ and } S_1(sv) = s^{0.002-1}S_1(v), \end{aligned}$$

where $a_1 = 1.2$, $b_1 = 2.4$; F satisfies (F1) and (F2) with

$$\begin{aligned} &F(t, su, sv) = s^{502}F(t, u, v), \text{ and } uF_u(t, u, v) + vF_v(t, u, v) = 502F(t, u, v), \\ &|F(t, u, v)| \le \frac{T+1}{502}|(u, v)|^{502}, \end{aligned}$$

where $C_* = \frac{T+1}{502}$. Thus, all the conditions of Theorem 1.1 are satisfied. Then, Theorem 1.1 implies that there exists $\Theta_* > 0$ such that for any $(\lambda, \mu) \in \Theta_*$, problem (5.1) admits at least a nontrivial solution.

Example 5.2. We consider the following impulsive fractional differential equation:

$$\begin{cases} \quad [a\|(u,v)\|_\alpha^3]^2 \ {}_tD_T^\alpha \Phi_3({}_0D_t^\alpha u(t)) + [b\|(u,v)\|_\beta^3]^2 \ {}_tD_T^\beta \Phi_3({}_0D_t^\beta u(t)) \\ = h(t)|(u,v)|^{12}u, \ t \ne t_1, \ \text{a.e. } t \in [0,T], \\ \quad [a\|(u,v)\|_\alpha^3]^2 \ {}_tD_T^\alpha \Phi_3({}_0D_t^\alpha v(t)) + [b\|(u,v)\|_\beta^3]^2 \ {}_tD_T^\beta \Phi_3({}_0D_t^\beta v(t)) \\ = h(t)|(u,v)|^{12}v, \ t \ne t_1', \ \text{a.e. } t \in [0,T], \\ \Delta([a\|(u,v)\|_\alpha^3]^2 \ {}_tD_T^{\alpha-1}\Phi_3({}_0D_t^\alpha u))(t_1) \\ + \Delta([b\|(u,v)\|_\beta^3]^2 \ {}_tD_T^{\beta-1}\Phi_3({}_0D_t^\beta u))(t_1) + \lambda\zeta(t_1)\overline{a}\dfrac{|u(t_1)|^{0.9}}{u(t_1)} = 0, \\ \Delta([a\|(u,v)\|_\alpha^3]^2 \ {}_tD_T^{\alpha-1}\Phi_3({}_0D_t^\alpha v))(t_1') \\ + \Delta([b\|(u,v)\|_\beta^3]^2 \ {}_tD_T^{\beta-1}\Phi_3({}_0D_t^\beta v))(t_1') + \mu\varrho(t_1')\overline{b}\dfrac{|v(t_1')|^{0.9}}{v(t_1')} = 0, \\ u(0) = u(T) = 0, \ v(0) = v(T) = 0, \end{cases} \tag{5.2}$$

where $a > 0$, $b > 0$, $\overline{a} > 0$, $\overline{b} > 0$, $0 \le h \in C([0,T], R^+)$. For this case, $p = q = 3$, $M_1(x) = ax$, $M_2(y) = by$, $F(t,u,v) = \frac{1}{13}h(t)|(u,v)|^{13}$, $I_1(s) = \overline{a}\frac{|s|^{0.9}}{s}$, $S_1(s) = \overline{b}\frac{|s|^{0.9}}{s}$.

We verify that all the conditions of Theorem 1.2 are satisfied. Obviously, M_1 and M_2 satisfy (M1′), (M2′) with

$$[M_1(x)]^2x \le 3\overline{M}_1(x), \text{ and } [M_2(y)]^2y \le 4\overline{M}_2(y),$$

where $\overline{\vartheta}_1 = 3$, $\overline{\vartheta}_2 = 4$; F satisfies (F1′)-(F3′) with

$$\begin{aligned} &|\nabla F(t,u,v)| \le 12\varepsilon|(u,v)|^{11} + 13C_\varepsilon|(u,v)|^{12}, \\ &0 \le 0.8F(t,u,v) \le \nabla F(t,u,v) \cdot (u,v), \end{aligned}$$

where $\overline{\vartheta}_* = 12$, $\overline{\eta} = 13$, $\overline{\varsigma} = 0.8$, $\theta = 0.9$; I_1 and S_1 satisfy (H1), (F3′) with

$$|I_1(z)z| \le 2\overline{a}|z|^{0.9}, \text{ and } |S_1(z)z| \le 2\overline{b}|z|^{0.9},$$

where $a_1 = 2\overline{a}$, $b_1 = 2\overline{b}$. Then, by Theorem 1.2, there exists $\Theta_{**} > 0$ such that for any $(\lambda, \mu) \in \Theta_{**}$, problem (5.2) admits infinitely many nontrivial solutions.

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