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FURTHER RESULTS OF M-EIGENVALUE LOCALIZATION THEOREM FOR FOURTH-ORDER PARTIALLY SYMMETRIC TENSORS AND THEIR APPLICATIONS

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Abstract In this paper, we give some new M-eigenvalue inclusion theorems for fourth-order partially symmetric tensors, which are more tighter than some existing inclusion sets. On the basis, some new upper bounds of the M-spectral radius are presented. Further, as applications, we propose sufficient conditions for the strong ellipticity condition in the elastic materials. Numerical examples are shown to illustrate validity and superiority of our results.

Keywords Partially symmetric tensors, M-eigenvalue, strong ellipticity condition.

MSC(2010) 15A69, 15A72.

1. Introduction

1.1. Background

Let $\mathbb{R}$ be the set of all real numbers, $\mathbb{R}^n$ be the set of all dimension n real vectors, and $[n] = \{1, 2, ..., n\}$. A fourth-order real tensor, denoted by $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[n_1]\times[n_2]\times[n_3]\times[n_4]}$, consists of $n_1 \times n_2 \times n_3 \times n_4$ components:

$$a_{ijkl} \in \mathbb{R}, \quad i \in [n_1], j \in [n_2], k \in [n_3], l \in [n_4].$$

Specifically, $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ is called partially symmetric tensors, if its components are invariant under the following permutation of subscripts:

$$a_{ijkl} = a_{kjil} = a_{ilkj} = a_{klij}, \quad i, k \in [m],\ j, l \in [n].$$

In fact, the tensor of elastic moduli for elastic materials exactly is partially symmetric [10], and the components of such tensor are regarded as the coefficients of the bi-quadratic polynomial optimization problem defined by

$$\begin{cases} \max \quad f(x, y) = \sum\limits_{i,k=1}^{m} \sum\limits_{j,l=1}^{n} a_{ijkl} x_i y_j x_k y_l, \\ s.t. \quad x^{\mathrm{T}} x = 1, y^{\mathrm{T}} y = 1,\ x \in \mathbb{R}^m, y \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

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and

$$\begin{cases} \min \quad f(x,y) = \sum_{i,k=1}^{m} \sum_{j,l=1}^{n} a_{ijkl} x_i y_j x_k y_l, \\ s.t. \quad\ x^{\mathrm{T}}x = 1, y^{\mathrm{T}}y = 1,\ x \in \mathbb{R}^m, y \in \mathbb{R}^n. \end{cases} \tag{1.2}$$

This optimization problem arises from the strong ellipticity condition problem in solid mechanics [10] and the entanglement problem in quantum physics [6, 9]. The entanglement problem is to determine whether a quantum state is separable or inseparable (entangled) [7]. It is known that both the strong ellipticity and ordinary ellipticity play an important roles in nonlinear elastic material analysis [4, 17, 24, 26, 31]. Qi et al. [22] pointed out that strong ellipticity condition holds if and only if the optimal value of the above global polynomial optimization problem is positive. In polynomial optimization theory [16,28,33], the biquadratic optimization problem is NP-hard to solve [19, 32]. In order to better study the optimization problems, through the theory of tensor eigenvalues [21,23], Han et al. [10] in 2009 for the first time transformed this optimization problem into the M-eigenvalue problem of a fourth-order partially symmetric tensor.

Recently, the research on M-eigenvalues of partially symmetric tensors has become popular [2, 12, 14, 18, 30]. However, due to the complexity of the tensor eigenvalue problem [14], it is difficult to directly calculate. To solve this problem, an inclusive set of M-eigenvalues of a partially symmetric tensor similar to the Geršgorin disc theorem of matrix eigenvalues can be given by analogy. He et al. [2] proposed the M-eigenvalue interval theorem. Li et al [18] gave the M-eigenvalue inclusion intervals. He et al. [12] proposed new S-type inclusion theorems for the M-eigenvalues of a fourth-order partially symmetric tensor.

The M-eigenvalue inclusive set can be used to solve the actual calculation of the largest M-eigenvalue and the strong ellipticity condition of elastic materials. In order to solve the NP-hard problem of M-eigenvalue, Wang et al. [27] presented a practical algorithm, denoted by WQZ-algorithm, to compute the largest M-eigenvalue of a fourth-order partially symmetric tensor. As an application, Li et al. used the M-spectral radius obtained by the M-eigenvalue inclusion intervals as a parameter in the WQZ-algorithm in [12]. Qi et al. [22] have shown that the necessary and sufficient condition for the establishment of the strong ellipticity condition is that the smallest M-eigenvalue of partially symmetric tensor is positive, called M-positive definite [3, 15, 21, 23]. Further, Wang et al. [21] provided some checkable sufficient conditions for the positive definiteness of fourth-order partially symmetric nonnegative tensors. Based on the M-eigenvalue with the strong ellipticity [1,5,8,12,13,20,25,29,34], the research in [11] provided some checkable sufficient conditions for the strong ellipticity, called M-positive definiteness.

Based on this, when studying the inclusion set of M-eigenvalues, we should consider the M-eigenvalue containing set whose center is at the origin or not, and get the inclusion interval as small as possible. Moreover, when the strong ellipticity condition holds, it is necessary to judge the positive definiteness of the partial symmetric tensor. Therefore, the rest of the paper is organized as follows. In Section 2, we give some new M-eigenvalue inclusion sets centered at the origin, and prove that the results are more accurate than some existing conclusions. In Section 3, we give a new M-eigenvalue containment set whose center is not at the origin, and prove it is tighter than some existing conclusions. In Section 4, we first recall the WQZ-algorithm. As an application, we apply the upper bound of the M-eigenvalue to the WQZ-algorithm as a parameter. In Section 5, we propose some existing sufficient conditions for the positive definiteness of the fourth-order partially symmetric tensor. Additionally, we apply the derived sufficient conditions to the strong ellipticity condition in the elastic materials.

1.2. Definition and proposition

Definition 1.1. [22] Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor(PST) and $\lambda \in \mathbb{R}$. Then λ is called an M-eigenvalue of $\mathcal{A}$, if there are vectors $x \in \mathbb{R}^m\backslash\{0\}$ and $y \in \mathbb{R}^n\backslash\{0\}$ such that

$$\begin{cases} \mathcal{A}\cdot yxy = \lambda x, \\ \mathcal{A}xyx\cdot = \lambda y, \\ x^{\mathrm{T}}x = 1, \\ y^{\mathrm{T}}y = 1, \end{cases} \tag{1.3}$$

where $\mathcal{A}\cdot yxy$ and $\mathcal{A}xyx\cdot$ are real vectors with i-th and l-th components defined by

$$(\mathcal{A}\cdot yxy)_i = \sum_{k=1}^{m}\sum_{j,l=1}^{n} a_{ijkl}y_jx_ky_l, \quad (\mathcal{A}xyx\cdot)_l = \sum_{i,k=1}^{m}\sum_{j=1}^{n} a_{ijkl}x_iy_jx_k.$$

x and y are called the corresponding left and right M-eigenvectors. If x and y are left and right M-eigenvectors of $\mathcal{A}$, associated with an M-eigenvalue λ, then $\lambda = \mathcal{A}xyxy$.

Definition 1.2. [21] We call $\mathcal{F}_{\mathcal{M}} \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ an M-identity tensor if its entries satisfy

$$(\mathcal{F}_{\mathcal{M}})_{ijkl} = \begin{cases} 1, & \text{if } i = k, j = l, \\ 0, & \text{otherwise}, \end{cases} \tag{1.4}$$

where $i, k \in [m], j, l \in [n]$.

Obviously, $\mathcal{F}_{\mathcal{M}} \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ is a partially symmetric tensor and has the following property:

$$\begin{cases} \mathcal{F}_{\mathcal{M}}\cdot yxy = x, \\ \mathcal{F}_{\mathcal{M}}xyx\cdot = y, \end{cases} \tag{1.5}$$

with $x^{\mathrm{T}}x = 1, y^{\mathrm{T}}y = 1$ for all $x \in \mathbb{R}^m, y \in \mathbb{R}^n$.

Definition 1.3. [25] The M-spectral radius $\rho(\mathcal{A})$ of $\mathcal{A}$ is defined as

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\},$$

where $\sigma(\mathcal{A})$ is M-spectrum of $\mathcal{A}$, the set of all M-eigenvalues of $\mathcal{A}$.

The largest M-eigenvalue of $\mathcal{A}$ is

$$\lambda_{\max}(\mathcal{A}) = \max\{\lambda : \lambda \in \sigma(\mathcal{A})\}.$$

The M-spectral radius of $\mathcal{A}$ is the largest M-eigenvalue. Furthermore, there is a pair of nonnegative M-eigenvectors corresponding to the M-spectral radius.

2. M-eigenvalue inclusion theorems centered at the origin

In this section, we discuss several new M-eigenvalue inclusion theorems of fourth-order partially symmetric tensors and establish the corresponding inclusion relationships. First, we introduce relative results given in [2].

Theorem 2.1. [2] *Suppose $\mathcal{A} = (a_{ijkl})$ is a partially symmetric tensor with $i, k \in [m], j, l \in [n]$. Then*

$$\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) = \bigcup_{i\in[m]} \Gamma_i(\mathcal{A}),$$

where $\Gamma_i(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| \leq R_i(\mathcal{A})\}$, and $R_i(\mathcal{A}) = \sum\limits_{k\in[m],\ j,l\in[n]} |a_{ijkl}|$.

Theorem 2.2. [2] *Suppose $\mathcal{A} = (a_{ijkl})$ is a partially symmetric tensor with $i, k \in [m], j, l \in [n]$. Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}) = \bigcup_{i\in[m]} \left(\bigcap_{k\in[m],k\neq i} \mathcal{L}_{i,k}(\mathcal{A}) \right),$$

where

$$\mathcal{L}_{i,k}(\mathcal{A}) = \{\lambda \in \mathbb{R} : (|\lambda| - (R_i(\mathcal{A}) - R_i^k(\mathcal{A})))|\lambda| \leq R_i^k(\mathcal{A})R_k(\mathcal{A})\},$$

and $R_i^k(\mathcal{A}) = \sum\limits_{j,l\in[n]} |a_{ijkl}|$.

Theorem 2.3. [2] *Suppose $\mathcal{A} = (a_{ijkl})$ is a partially symmetric tensor with $i, k \in [m], j, l \in [n]$. Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A}) = \bigcup_{i,k\in[m],\ k\neq i} \left(\mathcal{M}_{i,k}(\mathcal{A}) \bigcup \mathcal{H}_{i,k}(\mathcal{A}) \right),$$

where

$$\mathcal{M}_{i,k}(\mathcal{A}) = \{\lambda \in \mathbb{R} : (|\lambda| - (R_i(\mathcal{A}) - R_i^k(\mathcal{A})))(|\lambda| - R_k^k(\mathcal{A})) \leq R_i^k(\mathcal{A})(R_k(\mathcal{A}) - R_k^k(\mathcal{A}))\},$$

and

$$\mathcal{H}_{i,k}(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| - (R_i(\mathcal{A}) - R_i^k(\mathcal{A})) \leq 0,\ |\lambda| - R_k^k(\mathcal{A}) < 0\}.$$

Theorem 2.4. [2] *Suppose $\mathcal{A} = (a_{ijkl})$ is a partially symmetric tensor with $i, k \in [m], j, l \in [n]$. Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A}) = \bigcup_{i,k\in[m],\ k\neq i} \mathcal{N}_{i,k}(\mathcal{A}),$$

where $\mathcal{N}_{i,k}(\mathcal{A}) = \{\lambda \in \mathbb{R} : (|\lambda| - R_i^i(\mathcal{A}))|\lambda| \leq (R_i(\mathcal{A}) - R_i^i(\mathcal{A}))R_k(\mathcal{A})\}$.

Remark 2.1. According to [2], we know $\mathcal{L}(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$, $\mathcal{M}(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$ and $\mathcal{N}(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$. That is $\mathcal{L}(\mathcal{A})$, $\mathcal{M}(\mathcal{A})$ and $\mathcal{N}(\mathcal{A})$ are more accurate than $\Gamma(\mathcal{A})$.

Now, we give two new M-eigenvalue inclusion theorems and establish the corresponding inclusion relationships.

Theorem 2.5. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor. Then*

$$\sigma(\mathcal{A}) \subseteq \Upsilon(\mathcal{A}) = \bigcup_{i,k\in[m],k\neq i} \left(\widehat{r}_{i,k}(\mathcal{A}) \bigcup \widetilde{r}_{i,k}(\mathcal{A})\right),$$

where

$$\widehat{r}_{i,k}(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| - R_i(\mathcal{A}) + R_i^k(\mathcal{A}) \leq 0,\ |\lambda| - R_k(\mathcal{A}) + R_k^i(\mathcal{A}) < 0\},$$

and

$$\widetilde{r}_{i,k}(\mathcal{A}) = \{\lambda \in \mathbb{R} : [|\lambda| - R_i(\mathcal{A}) + R_i^k(\mathcal{A})][|\lambda| - R_k(\mathcal{A}) + R_k^i(\mathcal{A})] \leq R_i^k(\mathcal{A})R_k^i(\mathcal{A})\}.$$

Proof. Assume that λ is an M-eigenvalue of $\mathcal{A}$, $x = (x_1, x_2, ..., x_m)^T \in \mathbb{R}^m\backslash\{0\}$ and $y=(y_1, y_2, ..., y_n)^T\in\mathbb{R}^n\backslash\{0\}$ are the corresponding left and right M-eigenvectors, then

$$\mathcal{A}\cdot yxy = \lambda x, \mathcal{A}xyx\cdot = \lambda y, x^{\mathrm{T}}x = 1 \quad \text{and} \quad y^{\mathrm{T}}y = 1.$$

Let

$$|x_t| \geq |x_s| = \max_{i\in[m],i\neq t} |x_i|, \quad 0 < |x_t| \leq 1.$$

From $\lambda x = \mathcal{A}\cdot yxy$, it holds

$$\begin{aligned}\lambda x_t &= (\mathcal{A}\cdot yxy)_t \\ &= \sum_{k\in[m],\ j,l\in[n]} a_{tjkl}y_jx_ky_l \\ &= \sum_{k\in[m],k\neq s,\ j,l\in[n]} a_{tjkl}y_jx_ky_l + \sum_{j,l\in[n]} a_{tjsl}y_jx_sy_l.\end{aligned}$$

Then

$$\begin{aligned}|\lambda| &\leq \sum_{k\in[m],k\neq s,\ j,l\in[n]} |a_{tjkl}||y_j|\frac{|x_k|}{|x_t|}|y_l| + \sum_{j,l\in[n]} |a_{tjsl}||y_j|\frac{|x_s|}{|x_t|}|y_l| \\ &\leq \sum_{k\in[m],k\neq s,\ j,l\in[n]} |a_{tjkl}| + \sum_{j,l\in[n]} |a_{tjsl}|\frac{|x_s|}{|x_t|}.\end{aligned}$$

Therefore,

$$|\lambda| - \sum_{k\in[m],k\neq s,\ j,l\in[n]} |a_{tjkl}| \leq \sum_{j,l\in[n]} |a_{tjsl}|\frac{|x_s|}{|x_t|}. \tag{2.1}$$

(1) If $|x_s| = 0$, then $|\lambda| - (R_t(\mathcal{A}) - R_t^s(\mathcal{A})) \leq 0$.
  (i) If $|\lambda| - R_s(\mathcal{A}) + R_s^t(\mathcal{A}) \geq 0$, then $\lambda \in \widetilde{r}_{t,s}(\mathcal{A}) \subseteq \Upsilon(\mathcal{A})$.
  (ii) If $|\lambda| - R_s(\mathcal{A}) + R_s^t(\mathcal{A}) < 0$, then $\lambda \in \widehat{r}_{t,s}(\mathcal{A}) \subseteq \Upsilon(\mathcal{A})$.
(2) If $|x_s| > 0$, we have

$$\lambda x_s = (\mathcal{A}\cdot yxy)_s$$

$$
\begin{aligned}
&= \sum_{k\in[m],\ j,l\in[n]} a_{sjkl} y_j x_k y_l \\
&= \sum_{k\in[m],k\neq t,\ j,l\in[n]} a_{sjkl} y_j x_k y_l + \sum_{j,l\in[n]} a_{sjtl} y_j x_t y_l.
\end{aligned}
$$

Then

$$
\begin{aligned}
|\lambda| &\leq \sum_{k\in[m],k\neq t,\ j,l\in[n]} |a_{sjkl}||y_j|\frac{|x_k|}{|x_s|}|y_l| + \sum_{j,l\in[n]} |a_{sjtl}||y_j|\frac{|x_t|}{|x_s|}|y_l| \\
&\leq \sum_{k\in[m],k\neq t,\ j,l\in[n]} |a_{sjkl}| + \sum_{j,l\in[n]} |a_{sjtl}|\frac{|x_t|}{|x_s|}.
\end{aligned}
$$

Therefore,

$$
|\lambda| - \sum_{k\in[m],k\neq t,\ j,l\in[n]} |a_{sjkl}| \leq \sum_{j,l\in[n]} |a_{sjtl}|\frac{|x_t|}{|x_s|}. \tag{2.2}
$$

(i) If $|\lambda| - R_t(\mathcal{A}) + R_t^s(\mathcal{A}) \geq 0$ *or* $|\lambda| - R_s(\mathcal{A}) + R_s^t(\mathcal{A}) \geq 0$, multiplying (6) with (7) yields

$$
[|\lambda| - R_t(\mathcal{A}) + R_t^s(\mathcal{A})][|\lambda| - R_s(\mathcal{A}) + R_s^t(\mathcal{A})] \leq R_t^s(\mathcal{A})R_s^t(\mathcal{A}).
$$

That is

$$
\lambda \in \widetilde{r}_{t,s}(\mathcal{A}) \subseteq \Upsilon(\mathcal{A}).
$$

(ii) If $|\lambda| - R_t(\mathcal{A}) + R_t^s(\mathcal{A}) < 0$ and $|\lambda| - R_s(\mathcal{A}) + R_s^t(\mathcal{A}) < 0$, then $\lambda \in \widehat{r}_{t,s}(\mathcal{A}) \subseteq \Upsilon(\mathcal{A})$.

Thus $\sigma(\mathcal{A}) \subseteq \Upsilon(\mathcal{A})$. The proof is completed. $\square$

On the basis of Theorem 2.1 and Theorem 2.5, we can establish the following inclusion relationship between $\Gamma(\mathcal{A})$ and $\Upsilon(\mathcal{A})$.

Corollary 2.1. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor. Then*

$$
\sigma(\mathcal{A}) \subseteq \Upsilon(\mathcal{A}) \subseteq \Gamma(\mathcal{A}).
$$

Proof. For any $\lambda \in \Upsilon(\mathcal{A})$, we complete the proof by two cases.
Case 1. If $\lambda \in \widehat{r}_{i,k}(\mathcal{A})$, then

$$
|\lambda| - R_i(\mathcal{A}) + R_i^k(\mathcal{A}) \leq 0 \quad \text{and} \quad |\lambda| - R_k(\mathcal{A}) + R_k^i(\mathcal{A}) < 0.
$$

Therefore,

$$
|\lambda| \leq R_i(\mathcal{A}) \quad \text{and} \quad |\lambda| < R_k(\mathcal{A}),
$$

which implies $\lambda \in \Gamma(\mathcal{A})$.
Case 2. If $\lambda \in \widetilde{r}_{i,k}(\mathcal{A})$, then

$$
[|\lambda| - R_i(\mathcal{A}) + R_i^k(\mathcal{A})][|\lambda| - R_k(\mathcal{A}) + R_k^i(\mathcal{A})] \leq R_i^k(\mathcal{A})R_k^i(\mathcal{A}).
$$

(i) If $R_i^k(\mathcal{A})R_k^i(\mathcal{A}) = 0$, then

$$
|\lambda| - R_i(\mathcal{A}) + R_i^k(\mathcal{A}) \leq 0 \quad \text{or} \quad |\lambda| - R_k(\mathcal{A}) + R_k^i(\mathcal{A}) \leq 0.
$$

Therefore,

$$|\lambda| \le R_i(\mathcal{A}) \quad \text{or} \quad |\lambda| \le R_k(\mathcal{A}),$$

which implies $\lambda \in \Gamma(\mathcal{A})$.
(ii) If $R_i^k(\mathcal{A})R_k^i(\mathcal{A}) > 0$, then

$$\frac{|\lambda| - R_i(\mathcal{A}) + R_i^k(\mathcal{A})}{R_i^k(\mathcal{A})} \cdot \frac{|\lambda| - R_k(\mathcal{A}) + R_k^i(\mathcal{A})}{R_k^i(\mathcal{A})} \le 1.$$

This is

$$\frac{|\lambda| - R_i(\mathcal{A}) + R_i^k(\mathcal{A})}{R_i^k(\mathcal{A})} \le 1 \quad \text{or} \quad \frac{|\lambda| - R_k(\mathcal{A}) + R_k^i(\mathcal{A})}{R_k^i(\mathcal{A})} \le 1.$$

Therefore,

$$|\lambda| \le R_i(\mathcal{A}) \quad \text{or} \quad |\lambda| \le R_k(\mathcal{A}),$$

which implies $\lambda \in \Gamma(\mathcal{A})$. Thus $\Upsilon(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$. □

Theorem 2.6. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor. Then*

$$\sigma(\mathcal{A}) \subseteq \Theta(\mathcal{A}) = \bigcup_{i,k\in[m],\ k\neq i} \left(u_{i,k}(\mathcal{A}) \bigcup \widetilde{u}_i(\mathcal{A})\right),$$

where
$u_{i,k}(\mathcal{A}) = \{\lambda \in \mathbb{R} : [|\lambda| - R_i^i(\mathcal{A})][|\lambda| - R_k^k(\mathcal{A})] \le (R_i(\mathcal{A}) - R_i^i(\mathcal{A}))(R_k(\mathcal{A}) - R_k^k(\mathcal{A}))\}$,
$\widetilde{u}_{i,k}(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| - R_i^i(\mathcal{A}) \le 0, |\lambda| - R_k^k(\mathcal{A}) < 0\}$, $R_i^i(\mathcal{A}) = \sum_{j,l\in[m]} |a_{ijil}|$.

Proof. Assume that λ is an M-eigenvalue of $\mathcal{A}$, $x = (x_1, x_2, ..., x_m)^T \in \mathbb{R}^m\backslash\{0\}$ and $y=(y_1, y_2, ..., y_n)^T \in \mathbb{R}^n\backslash\{0\}$ are the corresponding left and right M-eigenvectors, then

$$\mathcal{A}\cdot yxy = \lambda x, \mathcal{A}xyx\cdot = \lambda y, x^{\mathrm{T}}x = 1 \quad \text{and} \quad y^{\mathrm{T}}y = 1.$$

Let

$$|x_t| \ge |x_s| = \max_{i\in[m], i\neq t} |x_i|, \quad 0 < |x_t| \le 1.$$

From $\lambda x = \mathcal{A}\cdot yxy$, it holds

$$\begin{aligned}
\lambda x_t &= (\mathcal{A}\cdot yxy)_t \\
&= \sum_{k\in[m],\ j,l\in[n]} a_{tjkl} y_j x_k y_l \\
&= \sum_{k\in[m], k\neq t,\ j,l\in[n]} a_{tjkl} y_j x_k y_l + \sum_{j,l\in[n]} a_{tjtl} y_j x_t y_l.
\end{aligned}$$

Then

$$\begin{aligned}
|\lambda| &\le \sum_{k\in[m], k\neq t,\ j,l\in[n]} |a_{tjkl}||y_j|\frac{|x_k|}{|x_t|}|y_l| + \sum_{j,l\in[n]} |a_{tjtl}||y_j||y_l| \\
&\le \sum_{k\in[m], k\neq t,\ j,l\in[n]} |a_{tjkl}|\frac{|x_s|}{|x_t|} + \sum_{j,l\in[n]} |a_{tjtl}|.
\end{aligned}$$

Therefore,

$$|\lambda| - \sum_{j,l\in[n]} |a_{tjtl}| \leq \sum_{k\in[m],k\neq t,\ j,l\in[n]} |a_{tjkl}| \frac{|x_s|}{|x_t|}. \tag{2.3}$$

(1) If $|x_s| = 0$, then $|\lambda| - R_t^t(\mathcal{A})) \leq 0$, which implies $\lambda \in \widetilde{u}_t(\mathcal{A}) \subseteq \Theta(\mathcal{A})$.
(2) If $|x_s| > 0$, we have

$$\begin{aligned}\lambda x_s &= (\mathcal{A}\cdot yxy)_s \\ &= \sum_{k\in[m],\ j,l\in[n]} a_{sjkl} y_j x_k y_l \\ &= \sum_{k\in[m],k\neq s,\ j,l\in[n]} a_{sjkl} y_j x_k y_l + \sum_{j,l\in[n]} a_{sjsl} y_j x_s y_l.\end{aligned}$$

Then

$$\begin{aligned}|\lambda| &\leq \sum_{k\in[m],k\neq s,\ j,l\in[n]} |a_{sjkl}||y_j| \frac{|x_k|}{|x_s|} |y_l| + \sum_{j,l\in[n]} |a_{sjsl}||y_j||y_l| \\ &\leq \sum_{k\in[m],k\neq s,\ j,l\in[n]} |a_{sjkl}| \frac{|x_t|}{|x_s|} + \sum_{j,l\in[n]} |a_{sjsl}|.\end{aligned}$$

Therefore,

$$|\lambda| - \sum_{j,l\in[n]} |a_{sjsl}| \leq \sum_{k\in[m],k\neq s,\ j,l\in[n]} |a_{sjkl}| \frac{|x_t|}{|x_s|}. \tag{2.4}$$

(i) If $|\lambda| - R_t^t(\mathcal{A})) \geq 0$ *or* $|\lambda| - R_s^s(\mathcal{A})) \geq 0$, multiplying (8) with (9) yields

$$[|\lambda| - R_t^t(\mathcal{A})][|\lambda| - R_s^s(\mathcal{A})] \leq (R_t(\mathcal{A}) - R_t^t(\mathcal{A}))(R_s(\mathcal{A}) - R_s^s(\mathcal{A})).$$

That is

$$\lambda \in u_{t,s}(\mathcal{A}) \subseteq \Theta(\mathcal{A}).$$

(ii) If $|\lambda| - R_t^t(\mathcal{A})) < 0$ *and* $|\lambda| - R_s^s(\mathcal{A})) < 0$, then $\lambda \in \widetilde{u}_{t,s}(\mathcal{A}) \subseteq \Theta(\mathcal{A})$. This shows that $\sigma(\mathcal{A}) \subseteq \Theta(\mathcal{A})$. □

On the basis of Theorem 2.1 and Theorem 2.6, we can establish the following inclusion relationship between $\Gamma(\mathcal{A})$ and $\Theta(\mathcal{A})$.

Corollary 2.2. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor. Then*

$$\sigma(\mathcal{A}) \subseteq \Theta(\mathcal{A}) \subseteq \Gamma(\mathcal{A}).$$

Proof. For any $\lambda \in \Theta(\mathcal{A})$, we break the proof into two cases.

Case 1. If $\lambda \in \widetilde{u}_i(\mathcal{A})$, then

$$|\lambda| - R_i^i(\mathcal{A}) \leq 0.$$

Therefore,

$$|\lambda| \leq R_i(\mathcal{A}),$$

which implies $\lambda \in \Gamma(\mathcal{A})$.

Case 2. If $\lambda \in u_{i,k}(\mathcal{A})$, then

$$[|\lambda| - R_i^i(\lambda A)][|\lambda| - R_k^k(\mathcal{A})] \leq (R_i(\mathcal{A}) - R_i^i(\mathcal{A}))(R_k(\mathcal{A}) - R_k^k(\mathcal{A}).$$

(i) If $(R_i(\mathcal{A}) - R_i^i(\mathcal{A}))(R_k(\mathcal{A}) - R_k^k(\mathcal{A}) = 0$, then

$$[|\lambda| - R_i^i(\lambda\mathcal{A})][|\lambda| - R_k^k(\mathcal{A})] \leq 0.$$

Therefore,

$$|\lambda| \leq R_i(\mathcal{A}) \quad \text{or} \quad |\lambda| \leq R_k(\mathcal{A}),$$

which implies $\lambda \in \Gamma(\mathcal{A})$.
(ii) If $(R_i(\mathcal{A}) - R_i^i(\mathcal{A}))(R_k(\mathcal{A}) - R_k^k(\mathcal{A}) > 0$, then

$$\frac{|\lambda| - R_i^i(\mathcal{A})}{R_i(\mathcal{A}) - R_i^i(\mathcal{A})} \cdot \frac{|\lambda| - R_k^k(\mathcal{A})}{R_k(\mathcal{A}) - R_k^k(\mathcal{A})} \leq 1.$$

This is

$$\frac{|\lambda| - R_i^i(\mathcal{A})}{R_i(\mathcal{A}) - R_i^i(\mathcal{A})} \leq 1 \quad \text{or} \quad \frac{|\lambda| - R_k^k(\mathcal{A})}{R_k(\mathcal{A}) - R_k^k(\mathcal{A})} \leq 1.$$

Therefore,

$$|\lambda| \leq R_i(\mathcal{A}) \quad \text{or} \quad |\lambda| \leq R_k(\mathcal{A}),$$

which implies $\lambda \in \Gamma(\mathcal{A})$. Thus $\Theta(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$. □

Example 2.1. [2] Consider the fourth-order partially symmetric tensor with

$$a_{ijkl} = \begin{cases} a_{1111} = 1, a_{1112} = 2, a_{1121} = 2, a_{1212} = 3, \\ a_{1222} = 5, a_{1211} = 2, a_{1122} = 4, a_{1221} = 4, \\ a_{2111} = 2, a_{2112} = 4, a_{2121} = 3, a_{2122} = 5, \\ a_{2211} = 4, a_{2212} = 5, a_{2221} = 5, a_{2222} = 6. \end{cases}$$

By Theorem 2.1 to Theorem 2.4, we have

$$\Gamma(\mathcal{A}) = \bigcup_{i\in[m]} \Gamma_i(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| \leq 34\},$$

$$\mathcal{L}(\mathcal{A}) = \bigcup_{i\in[m]} \left(\bigcap_{k\in[m],k\neq i} \mathcal{L}_{i,k}(\mathcal{A}) \right) = \{\lambda \in \mathbb{R} : |\lambda| \leq \frac{19 + \sqrt{1741}}{2}\},$$

$$\mathcal{M}(\mathcal{A}) = \bigcup_{i,k\in[m],\ k\neq i} \left(\mathcal{M}_{i,k}((A)) \bigcup \mathcal{H}_{i,k}(\mathcal{A}) \right) = \{\lambda \in \mathbb{R} : |\lambda| \leq \frac{27 + \sqrt{1021}}{2}\},$$

$$\mathcal{N}(\mathcal{A}) = \bigcup_{i,k\in[m],\ k\neq i} \mathcal{N}_{i,k}(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| \leq \frac{19 + \sqrt{1741}}{2}\}.$$

From Theorem 2.5, we obtain

$$\Upsilon(\mathcal{A}) = \bigcup_{i,k\in[m],k\neq i} \left(\widehat{r}_{i,k}(\mathcal{A}) \bigcup \widetilde{r}_{i,k}(\mathcal{A}) \right) = \{\lambda \in \mathbb{R} : |\lambda| \leq \frac{27 + \sqrt{1021}}{2}\},$$

where

$$\widehat{r}_{1,2}(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| \leq 8\},\ \widehat{r}_{2,1}(\mathcal{A}) = \{\lambda \in \mathbb{C} : |\lambda| < 8\},$$

$$\widetilde{r}_{1,2}(\mathcal{A}) = \widetilde{r}_{2,1}(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| \leq \frac{27 + \sqrt{1021}}{2}\}.$$

From Theorem 2.6, we obtain

$$\Theta(\mathcal{A}) = \bigcup_{i,k\in[m],\ k\neq i} \left(u_{i,k}(\mathcal{A})\bigcup \widetilde{u}_i(\mathcal{A})\right) = \{\lambda\in\mathbb{R} : |\lambda| \leq \frac{27+\sqrt{1021}}{2}\},$$

where

$$\begin{aligned} u_{1,2}(\mathcal{A}) &= u_{2,1}(\mathcal{A}) = \{\lambda\in\mathbb{R} : |\lambda| \leq \frac{27+\sqrt{1021}}{2}\},\\ \widetilde{u}_1(\mathcal{A}) &= \{\lambda\in\mathbb{R} : |\lambda| \leq 8\},\\ \widetilde{u}_2(\mathcal{A}) &= \{\lambda\in\mathbb{R} : |\lambda| \leq 19\}. \end{aligned}$$

Further, we use Figure 1 to show the above calculation results. From Figure 1, $\Upsilon(\mathcal{A})$ and $\Theta(\mathcal{A})$ are more accurate than $\Gamma(\mathcal{A})$ and $\mathcal{L}(\mathcal{A})$.

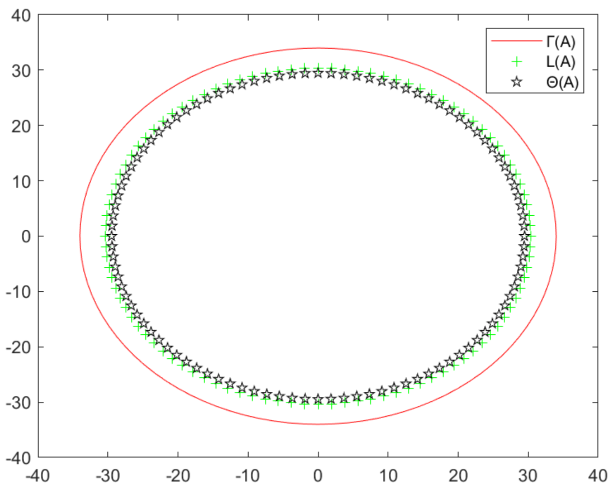

Figure 1. Comparison of inclusion sets of Example 2.1.

Example 2.2. [2] Consider the fourth-order partially symmetric tensor with

$$a_{ijkl} = \begin{cases} a_{1111}=-1, a_{1112}=2, a_{1131}=3, a_{1121}=-1, a_{1211}=2, a_{1221}=1, a_{1122}=1,\\ a_{2111}=-1, a_{2211}=1, a_{2112}=1, a_{2131}=-2, a_{2222}=2,\\ a_{3111}=3, a_{3232}=-1, a_{3131}=-2,\\ a_{ijkl}=0,\ \text{otherwise}. \end{cases}$$

By Theorem 2.1 to Theorem 2.4, we have

$$\Gamma(\mathcal{A}) = \bigcup_{i\in[m]} \Gamma_i(\mathcal{A}) = \{\lambda\in\mathbb{R} : |\lambda| \leq 11\},$$

$$\mathcal{L}(\mathcal{A}) = \bigcup_{i\in[m]} \left(\bigcap_{k\in[m],k\neq i} \mathcal{L}_{i,k}(\mathcal{A})\right) = \{\lambda\in\mathbb{R} : |\lambda| \leq 4+\sqrt{34}\},$$

$$\mathcal{M}(\mathcal{A}) = \bigcup_{i,k\in[m],\ k\neq i} \left(\mathcal{M}_{i,k}((A))\bigcup \mathcal{H}_{i,k}(\mathcal{A})\right) = \{\lambda\in\mathbb{R} : |\lambda| \leq 5+2\sqrt{6}\},$$

$$\mathcal{N}(\mathcal{A}) = \bigcup_{i,k\in[m],\ k\neq i} \mathcal{N}_{i,k}(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| \leq \frac{5+\sqrt{193}}{2}\}.$$

From Theorem 2.5, we obtain

$$\Upsilon(\mathcal{A}) = \bigcup_{i,k\in[m],k\neq i} \left(\widehat{r}_{i,k}(\mathcal{A}) \bigcup \widetilde{r}_{i,k}(\mathcal{A})\right) = \{\lambda \in \mathbb{R} : |\lambda| \leq 6+\sqrt{13}\},$$

where

$$\begin{aligned}
&\widehat{r}_{1,2}(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| < 4\},\ \widehat{r}_{1,3}(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| < 3\},\\
&\widehat{r}_{2,1}(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| \leq 4\},\ \widehat{r}_{2,3}(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| \leq 5\},\\
&\widehat{r}_{3,1}(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| \leq 3\},\ \widehat{r}_{3,2}(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| < 5\},\\
&\widetilde{r}_{1,2}(\mathcal{A}) = \widetilde{r}_{2,1}(\mathcal{A}) = \{\lambda \in \mathbb{R} : 6-\sqrt{13} \leq |\lambda| \leq 6+\sqrt{13}\},\\
&\widetilde{r}_{1,3}(\mathcal{A}) = \widetilde{r}_{3,1}(\mathcal{A}) = \{\lambda \in \mathbb{R} : \frac{11-\sqrt{61}}{2} \leq |\lambda| \leq \frac{11+\sqrt{61}}{2}\},\\
&\widetilde{r}_{2,3}(\mathcal{A}) = \widetilde{r}_{3,2}(\mathcal{A}) = \{\lambda \in \mathbb{R} : 5 \leq |\lambda| \leq 6\}.
\end{aligned}$$

From Theorem 2.6, we obtain

$$\Theta(\mathcal{A}) = \bigcup_{i,k\in[m],\ k\neq i} \left(u_{i,k}(\mathcal{A}) \bigcup \widetilde{u}_i(\mathcal{A})\right) = \{\lambda \in \mathbb{R} : |\lambda| \leq \frac{7+\sqrt{129}}{2}\},$$

where

$$\begin{aligned}
&u_{1,2}(\mathcal{A}) = u_{2,1}(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| \leq \frac{7+\sqrt{129}}{2}\},\\
&u_{1,3}(\mathcal{A}) = u_{3,1}(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| \leq 4+\sqrt{19}\},\\
&u_{2,3}(\mathcal{A}) = u_{3,2}(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| \leq \frac{5+\sqrt{61}}{2}\},\\
&\widetilde{u}_1(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| \leq 5\},\\
&\widetilde{u}_2(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| \leq 2\},\\
&\widetilde{u}_3(\mathcal{A}) = \{\lambda \in \mathbb{R} : |\lambda| \leq 3\}.
\end{aligned}$$

Moreover, we use Figure 2 to show the above calculation results. From Figure 2, it can be seen that the new M-eigenvalue inclusion set $\Upsilon(\mathcal{A})$ and $\Theta(\mathcal{A})$ are more accurate than $\Gamma(\mathcal{A})$, $\mathcal{L}(\mathcal{A})$ and $\mathcal{M}(\mathcal{A})$.

3. M-eigenvalue inclusion theorems

In this section, we first introduce some existing M-eigenvalue inclusion theorems in [25] whose center point is not at the origin. Then we give some new M-eigenvalue inclusion theorems where the center point is not at the origin. Further, we show that they are more tighter than some existing conclusions.

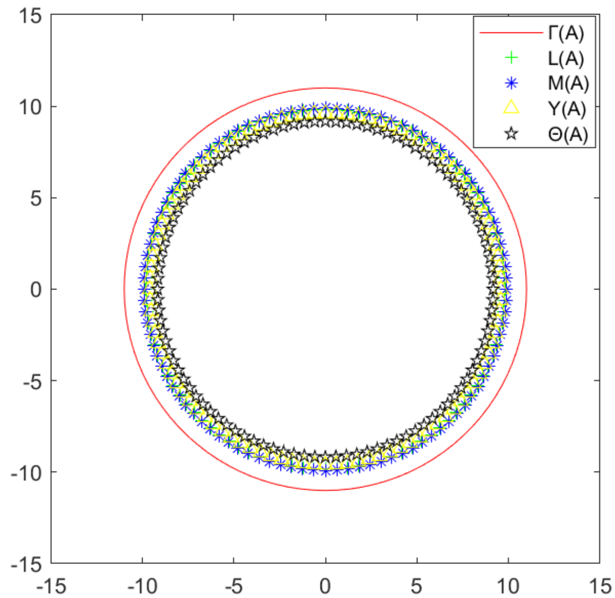

Figure 2. Comparison of inclusion sets of Example 2.2.

Theorem 3.1. [25] *Let* $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ *be a partially symmetric tensor and* $\mathcal{F}_{\mathcal{M}}$ *be an M-identity tensor. For any* $\alpha = (\alpha_1, ..., \alpha_m)^{\mathrm{T}} \in \mathbb{R}^m$*, then*

$$\sigma(\mathcal{A}) \subseteq \mathfrak{X}(\mathcal{A}, \alpha) = \bigcup_{i\in[m]} \mathfrak{X}_i(\mathcal{A}, \alpha),$$

where

$$\mathfrak{X}_i(\mathcal{A}, \alpha) = \{\lambda \in \mathbb{R} : |\lambda - \alpha_i| \leq R_i(\mathcal{A}, \alpha_i)\},$$
$$R_i(\mathcal{A}, \alpha_i) = \sum_{k\in[m],\ j,l\in[n]} |a_{ijkl} - \alpha_i(\mathcal{F}_{\mathcal{M}})_{ijkl}|.$$

Further,

$$\sigma(\mathcal{A}) \subseteq \bigcap_{\alpha\in\mathbb{R}^m} \mathfrak{X}(\mathcal{A}, \alpha).$$

Theorem 3.2. [25] *Let* $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ *be a partially symmetric tensor and* $\mathcal{F}_{\mathcal{M}}$ *be an M-identity tensor. For any* $\alpha = (\alpha_1, ..., \alpha_m)^{\mathrm{T}} \in \mathbb{R}^m$*, then*

$$\sigma(\mathcal{A}) \subseteq \mathfrak{K}(\mathcal{A}, \alpha) = \bigcup_{i\in[m]} \Big(\bigcap_{k\neq i, k\in[m]} \mathfrak{K}_{i,k}(\mathcal{A}, \alpha)\Big),$$

where

$$\begin{aligned}\mathfrak{K}_{i,k}(\mathcal{A}, \alpha) = \{\lambda \in \mathbb{R} : &[|\lambda - \alpha_i| - (R_i(\mathcal{A}, \alpha_i) - R_i^k(\mathcal{A}, \alpha_i))]|\lambda - \alpha_k| \\ &\leq R_i^k(\mathcal{A}, \alpha_i) R_k(\mathcal{A}, \alpha_k)\},\\ R_i^k(\mathcal{A}, \alpha_i) = &\sum_{j,l\in[n]} |a_{ijkl} - \alpha_i(\mathcal{F}_{\mathcal{M}})_{ijkl}|.\end{aligned}$$

Further,

$$\sigma(\mathcal{A}) \subseteq \bigcap_{\alpha\in\mathbb{R}^m} \mathfrak{K}(\mathcal{A},\alpha).$$

Theorem 3.3. [25] *Let $\mathcal{A}=(a_{ijkl})\in\mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor and $\mathcal{F}_{\mathcal{M}}$ be an M-identity tensor. For any $\alpha=(\alpha_1,...,\alpha_m)^{\mathrm{T}}\in\mathbb{R}^m$, then*

$$\sigma(\mathcal{A}) \subseteq \mathfrak{K}(\mathcal{A},\alpha) \subseteq \mathfrak{X}(\mathcal{A},\alpha).$$

Now, we give two new M-eigenvalue inclusion theorems and establish the corresponding inclusion relationships.

Theorem 3.4. *Let $\mathcal{A}=(a_{ijkl})\in\mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor and $\mathcal{F}_{\mathcal{M}}$ be an M-identity tensor. For any $\alpha=(\alpha_1,...,\alpha_m)^{\mathrm{T}}\in\mathbb{R}^m$, then*

$$\sigma(\mathcal{A}) \subseteq \mathfrak{N}(\mathcal{A},\alpha) = \bigcup_{i,k\in[m],\ k\neq i} \mathfrak{N}_{i,k}(\mathcal{A},\alpha),$$

where

$$\begin{aligned}\mathfrak{N}_{i,k}(\mathcal{A},\alpha) = \{\lambda\in\mathbb{R} : {} & [|\lambda-\alpha_i| - (R_i^i(\mathcal{A},\alpha_i))]|\lambda-\alpha_k| \\ & \leq [R_i(\mathcal{A},\alpha_i) - R_i^i(\mathcal{A},\alpha_i)]R_k(\mathcal{A},\alpha_k)\}, \\ R_i^i(\mathcal{A},\alpha_i) = {} & \sum_{j,l\in[n]} |a_{ijil} - \alpha_i(\mathcal{F}_{\mathcal{M}})_{ijil}|.\end{aligned}$$

Further,

$$\sigma(\mathcal{A}) \subseteq \bigcap_{\alpha\in\mathbb{R}^m} \mathfrak{N}(\mathcal{A},\alpha).$$

Proof. Assume that λ is an M-eigenvalue of $\mathcal{A}$, $x=(x_1,x_2,...,x_m)^T\in\mathbb{R}^m\backslash\{0\}$ and $y=(y_1,y_2,...,y_n)^T\in\mathbb{R}^n\backslash\{0\}$ are the corresponding left and right M-eigenvectors, and $\mathcal{F}_{\mathcal{M}}$ is an M-identity tensor, then

$$\mathcal{A}\cdot yxy = \lambda x = \lambda\mathcal{F}_{\mathcal{M}}\cdot yxy,\ \ x^{\mathrm{T}}x=1 \ \text{ and } \ y^{\mathrm{T}}y=1.$$

Let

$$|x_t| \geq |x_s| = \max_{i\in[m],i\neq t}|x_i|, \quad 0<|x_t|\leq 1.$$

From $\mathcal{A}\cdot yxy=\lambda x=\lambda\mathcal{F}_{\mathcal{M}}\cdot yxy$, it holds that

$$\sum_{k\in[m],\ j,l\in[n]} \lambda(\mathcal{F}_{\mathcal{M}tjkl})y_jx_ky_l = \sum_{k\in[m],\ j,l\in[n]} a_{tjkl}y_jx_ky_l.$$

Then, for any real number α_t, it follows that

$$\begin{aligned}(\lambda-\alpha_t)x_t &= \sum_{k\in[m],\ j,l\in[n]} (\lambda-\alpha_t)(\mathcal{F}_{\mathcal{M}})_{tjkl}y_jx_ky_l \\ &= \sum_{k\neq t,k\in[m],\ j,l\in[n]} (a_{tjkl}-\alpha_t(\mathcal{F}_{\mathcal{M}})_{tjkl})y_jx_ky_l \\ &\quad + \sum_{j,l\in[n]} (a_{tjtl}-\alpha_t(\mathcal{F}_{\mathcal{M}})_{tjtl})y_jx_ty_l.\end{aligned}$$

Taking modulus in the above equation and using the triangle inequality leads to

$$\begin{aligned}|\lambda-\alpha_t||x_t| \le & \sum_{k\in[m],k\neq t,\ j,l\in[n]} |a_{tjkl}-\alpha_t(\mathcal{F}_{\mathcal{M}})_{tjkl}||y_j||x_k||y_l| \\ & + \sum_{j,l\in[n]} |a_{tjtl}-\alpha_t(\mathcal{F}_{\mathcal{M}})_{tjtl}||y_j||x_t||y_l| \\ & \le (R_t(\mathcal{A},\alpha_t)-R_t^t(\mathcal{A},\alpha_t))|x_s| + R_t^t(\mathcal{A},\alpha_t)|x_t|.\end{aligned}$$

Therefore,

$$|\lambda-\alpha_t| - R_t^t(\mathcal{A},\alpha_t) \le (R_t(\mathcal{A},\alpha_t)-R_t^t(\mathcal{A},\alpha_t))\frac{|x_s|}{|x_t|}. \tag{3.1}$$

(1) If $|x_s|=0$, then $|\lambda-\alpha_t|-R_t^t(\mathcal{A},\alpha_t)\le 0$, which implies $\lambda\in\mathfrak{N}_{t,s}(\mathcal{A},\alpha)\subseteq\mathfrak{N}(\mathcal{A},\alpha)$.
(2) If $|x_s|>0$, we have

$$(\lambda-\alpha_s)x_s = \sum_{k\in[m],\ j,l\in[n]} (a_{sjkl}-\alpha_s(\mathcal{F}_{\mathcal{M}})_{sjkl})y_jx_ky_l.$$

Taking modulus in the above equation, we have

$$\begin{aligned}|\lambda-\alpha_s||x_s| \le & \sum_{k\in[m],\ j,l\in[n]} |a_{sjkl}-\alpha_s(\mathcal{F}_{\mathcal{M}})_{sjkl}||y_j||x_k||y_l| \\ & \le R_s(\mathcal{A},\alpha_s)|x_t|.\end{aligned}$$

Therefore,

$$|\lambda-\alpha_s| \le R_s(\mathcal{A},\alpha_s)\frac{|x_t|}{|x_s|}. \tag{3.2}$$

(i) If $|\lambda-\alpha_t|-R_t^t(\mathcal{A},\alpha_t)\ge 0$, multiplying (10) with (11) yields

$$[|\lambda-\alpha_t|-R_t^t(\mathcal{A},\alpha_t)]|\lambda-\alpha_s| \le [R_t(\mathcal{A},\alpha_t)-R_t^t(\mathcal{A},\alpha_t)]R_s(\mathcal{A},\alpha_s).$$

That is

$$\lambda\in\mathfrak{N}_{t,s}(\mathcal{A},\alpha)\subseteq\mathfrak{N}(\mathcal{A},\alpha).$$

(ii) If $|\lambda-\alpha_t|-R_t^t(\mathcal{A},\alpha_t)<0$, then $\lambda\in\mathfrak{N}_{t,s}(\mathcal{A},\alpha)\subseteq\mathfrak{N}(\mathcal{A},\alpha)$. Thus $\sigma(\mathcal{A})\subseteq\mathfrak{N}(\mathcal{A},\alpha)$. □

On the basis of Theorem 3.1 and Theorem 3.4, we can establish the following inclusion relationship between $\mathfrak{X}(\mathcal{A},\alpha)$ and $\mathfrak{N}(\mathcal{A},\alpha)$.

Corollary 3.1. *Let $\mathcal{A}=(a_{ijkl})\in\mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor and $\mathcal{F}_{\mathcal{M}}$ be an M-identity tensor. For any $\alpha=(\alpha_1,...,\alpha_m)^{\mathrm{T}}\in\mathbb{R}^m$, then*

$$\sigma(\mathcal{A})\subseteq\mathfrak{N}(\mathcal{A},\alpha)\subseteq\mathfrak{X}(\mathcal{A},\alpha).$$

Proof. For any $\lambda\in\mathfrak{N}(\mathcal{A},\alpha)$, without loss of generality, there exists $t\in[m]$ such that $\lambda\in\mathfrak{N}_{t,k}(\mathcal{A},\alpha)$, for all $t\neq k$. Thus,

$$[|\lambda-\alpha_t|-R_t^t(\mathcal{A},\alpha_t)]|\lambda-\alpha_k| \le [R_t(\mathcal{A},\alpha_t)-R_t^t(\mathcal{A},\alpha_t)]R_k(\mathcal{A},\alpha_k).$$

We now break up the argument into two cases.

Case 1. If $[R_t(\mathcal{A},\alpha_t)-R_t^t(\mathcal{A},\alpha_t)]R_k(\mathcal{A},\alpha_k)=0$, then

$$|\lambda-\alpha_t|-R_t^t(\mathcal{A},\alpha_t)\le 0 \text{ or } \lambda=\alpha_k.$$

Hence,

$$|\lambda-\alpha_t|\le R_t^t(\mathcal{A},\alpha_t)\le R_t(\mathcal{A},\alpha_t) \text{ or } \lambda=\alpha_k.$$

Therefore, $\lambda\in\mathfrak{X}_t(\mathcal{A},\alpha)\bigcup\mathfrak{X}_k(\mathcal{A},\alpha)\subseteq\mathfrak{X}(\mathcal{A},\alpha)$.

Case 2. If $[R_t(\mathcal{A},\alpha_t)-R_t^t(\mathcal{A},\alpha_t)]R_k(\mathcal{A},\alpha_k)>0$, then

$$\frac{|\lambda-\alpha_t|-R_t^t(\mathcal{A},\alpha_t)}{R_t(\mathcal{A},\alpha_t)-R_t^t(\mathcal{A},\alpha_t)}\cdot\frac{|\lambda-\alpha_k|}{R_k(\mathcal{A},\alpha_k)}\le 1.$$

That is

$$\frac{|\lambda-\alpha_t|-R_t^t(\mathcal{A},\alpha_t)}{R_t(\mathcal{A},\alpha_t)-R_t^t(\mathcal{A},\alpha_t)}\le 1 \text{ or } \frac{|\lambda-\alpha_k|}{R_k(\mathcal{A},\alpha_k)}\le 1.$$

Therefore,

$$|\lambda-\alpha_t|\le R_t(\mathcal{A},\alpha_t) \text{ or } |\lambda-\alpha_k|\le R_k(\mathcal{A},\alpha_k),$$

which implies $\lambda\in\mathfrak{X}_t(\mathcal{A},\alpha)\bigcup\mathfrak{X}_k(\mathcal{A},\alpha)\subseteq\mathfrak{X}(\mathcal{A},\alpha)$. Thus $\mathfrak{N}(\mathcal{A},\alpha)\subseteq\mathfrak{X}(\mathcal{A},\alpha)$. $\square$

Theorem 3.5. *Let $\mathcal{A}=(a_{ijkl})\in\mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor and $\mathcal{F}_{\mathcal{M}}$ be an M-identity tensor. For any $\alpha=(\alpha_1,...,\alpha_m)^{\mathrm{T}}\in\mathbb{R}^m$, then*

$$\sigma(\mathcal{A})\subseteq\mathfrak{M}(\mathcal{A},\alpha)=\bigcup_{i,k\in[m],\ k\neq i}\left(\mathfrak{M}_{i,k}(\mathcal{A},\alpha)\bigcup\mathfrak{H}_{i,k}(\mathcal{A},\alpha)\right),$$

where

$$\begin{aligned}\mathfrak{M}_{i,k}(\mathcal{A},\alpha)=\{\lambda\in\mathbb{R}:&[|\lambda-\alpha_i|-(R_i(\mathcal{A},\alpha_i)-R_i^k(\mathcal{A},\alpha_i))][|\lambda-\alpha_k|-R_k^k(\mathcal{A},\alpha_k)]\\ &\le R_i^k(\mathcal{A},\alpha_i)[R_k(\mathcal{A},\alpha_k)-R_k^k(\mathcal{A},\alpha_k)]\},\end{aligned}$$

and

$$\mathfrak{H}_{i,k}(\mathcal{A},\alpha)=\{\lambda\in\mathbb{R}:|\lambda-\alpha_i|-(R_i(\mathcal{A},\alpha_i)-R_i^k(\mathcal{A},\alpha_i))\le 0,\ |\lambda-\alpha_k|-R_k^k(\mathcal{A},\alpha_k)<0\}.$$

Further,

$$\sigma(\mathcal{A})\subseteq\bigcap_{\alpha\in\mathbb{R}^m}\mathfrak{M}(\mathcal{A},\alpha).$$

Proof. Assume that λ is an M-eigenvalue of $\mathcal{A}$, $x=(x_1,x_2,...,x_m)^T\in\mathbb{R}^m\backslash\{0\}$ and $y=(y_1,y_2,...,y_n)^T\in\mathbb{R}^n\backslash\{0\}$ are the corresponding left and right M-eigenvectors, and $\mathcal{F}_{\mathcal{M}}$ is an M-identity tensor, then

$$\mathcal{A}\cdot yxy=\lambda x=\lambda\mathcal{F}_{\mathcal{M}}\cdot yxy,\ x^{\mathrm{T}}x=1 \text{ and } y^{\mathrm{T}}y=1.$$

Let

$$|x_t|\ge|x_s|=\max_{i\in[m],i\neq t}|x_i|,\quad 0<|x_t|\le 1.$$

From $\mathcal{A}\cdot yxy=\lambda x=\lambda\mathcal{F}_{\mathcal{M}}\cdot yxy$, it holds that

$$\sum_{k\in[m],\ j,l\in[n]}\lambda(\mathcal{F}_{\mathcal{M}tjkl})y_jx_ky_l=\sum_{k\in[m],\ j,l\in[n]}a_{tjkl}y_jx_ky_l.$$

Then, for any real number α_t, it follows that

$$\begin{aligned}(\lambda-\alpha_t)x_t &= \sum_{k\in[m],\ j,l\in[n]} (\lambda-\alpha_t)(\mathcal{F}_{\mathcal{M}})_{tjkl}y_jx_ky_l \\ &= \sum_{k\neq s,k\in[m],\ j,l\in[n]} (a_{tjkl}-\alpha_t(\mathcal{F}_{\mathcal{M}})_{tjkl})y_jx_ky_l \\ &\quad+\sum_{j,l\in[n]}(a_{tjsl}-\alpha_t(\mathcal{F}_{\mathcal{M}})_{tjsl})y_jx_sy_l.\end{aligned}$$

Taking modulus in the above equation and using the triangle inequality gives

$$\begin{aligned}|\lambda-\alpha_t||x_t| &\leq \sum_{k\in[m],k\neq s,\ j,l\in[n]} |a_{tjkl}-\alpha_t(\mathcal{F}_{\mathcal{M}})_{tjkl}||y_j||x_k||y_l| \\ &\quad+\sum_{j,l\in[n]}|a_{tjsl}-\alpha_t(\mathcal{F}_{\mathcal{M}})_{tjsl}||y_j||x_s||y_l| \\ &\leq (R_t(\mathcal{A},\alpha_t)-R_t^s(\mathcal{A},\alpha_t))|x_t|+R_t^s(\mathcal{A},\alpha_t)|x_s|.\end{aligned}$$

Therefore,

$$|\lambda-\alpha_t|-(R_t(\mathcal{A},\alpha_t)-R_t^s(\mathcal{A},\alpha_t))\leq R_t^s(\mathcal{A},\alpha_t)\frac{|x_s|}{|x_t|}. \tag{3.3}$$

(1) If $|x_s|=0$, then $|\lambda-\alpha_t|-(R_t(\mathcal{A},\alpha_t)-R_t^s(\mathcal{A},\alpha_t))\leq 0$.
 (i) If $|\lambda-\alpha_s|-R_s^s(\mathcal{A},\alpha_s)\geq 0$, then $\lambda\in\mathfrak{M}_{t,s}(\mathcal{A},\alpha)\subseteq\mathfrak{M}(\mathcal{A},\alpha)$.
 (ii) If $|\lambda-\alpha_s|-R_s^s(\mathcal{A},\alpha_s)<0$, then $\lambda\in\mathfrak{H}_{t,s}(\mathcal{A},\alpha)\subseteq\mathfrak{M}(\mathcal{A},\alpha)$.
(2) If $|x_s|>0$, we have

$$\begin{aligned}|\lambda-\alpha_s||x_s| &\leq \sum_{k\in[m],k\neq s,\ j,l\in[n]} |a_{sjkl}-\alpha_s(\mathcal{F}_{\mathcal{M}})_{sjkl}||y_j||x_k||y_l| \\ &\quad+\sum_{j,l\in[n]}|a_{sjsl}-\alpha_s(\mathcal{F}_{\mathcal{M}})_{sjsl}||y_j||x_s||y_l| \\ &\leq (R_s(\mathcal{A},\alpha_s)-R_s^s(\mathcal{A},\alpha_s))|x_t|+R_s^s(\mathcal{A},\alpha_s)|x_s|.\end{aligned}$$

Therefore,

$$|\lambda-\alpha_s|-R_s^s(\mathcal{A},\alpha_s)\leq (R_s(\mathcal{A},\alpha_s)-R_s^s(\mathcal{A},\alpha_s))\frac{|x_t|}{|x_s|}. \tag{3.4}$$

(i) If $|\lambda-\alpha_t|-(R_t(\mathcal{A},\alpha_t)-R_t^s(\mathcal{A},\alpha_t))\geq 0$ *or* $|\lambda-\alpha_s|-R_s^s(\mathcal{A},\alpha_s)\geq 0$, multiplying (12) with (13) yields

$$\begin{aligned}&[|\lambda-\alpha_t|-(R_t(\mathcal{A},\alpha_t)-R_t^s(\mathcal{A},\alpha_t))][|\lambda-\alpha_s|-R_s^s(\mathcal{A},\alpha_s)] \\ &\leq R_t^s(\mathcal{A},\alpha_t)(R_s(\mathcal{A},\alpha_s)-R_s^s(\mathcal{A},\alpha_s)).\end{aligned}$$

That is

$$\lambda\in\mathfrak{M}_{t,s}(\mathcal{A},\alpha)\subseteq\mathfrak{M}(\mathcal{A},\alpha).$$

(ii) If $|\lambda-\alpha_t|-(R_t(\mathcal{A},\alpha_t)-R_t^s(\mathcal{A},\alpha_t)))<0$ *and* $|\lambda-\alpha_s|-R_s^s(\mathcal{A},\alpha_s)<0$, then $\lambda\in\mathfrak{H}_{t,s}(\mathcal{A},\alpha)\subseteq\mathfrak{M}(\mathcal{A},\alpha)$. This shows that $\sigma(\mathcal{A})\subseteq\mathfrak{M}(\mathcal{A},\alpha)$. □

On the basis of Theorem 3.1 and Theorem 3.5, we can establish the following inclusion relationship between $\mathfrak{X}(\mathcal{A},\alpha)$ and $\mathfrak{M}(\mathcal{A},\alpha)$.

Corollary 3.2. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor and $\mathcal{F}_{\mathcal{M}}$ be an M-identity tensor. For any $\alpha = (\alpha_1, ..., \alpha_m)^{\mathrm{T}} \in \mathbb{R}^m$, then*

$$\sigma(\mathcal{A}) \subseteq \mathfrak{M}(\mathcal{A}, \alpha) \subseteq \mathfrak{X}(\mathcal{A}, \alpha).$$

Proof. For any $\lambda \in \mathfrak{M}(\mathcal{A}, \alpha)$, without loss of generality, there exists $t \in [m]$ such that $\lambda \in \mathfrak{M}_{t,k}(\mathcal{A}, \alpha)$, for all $t \neq k$. We break the proof into two cases.

Case 1. If $\lambda \in \mathfrak{H}_{t,k}(\mathcal{A}, \alpha)$, then

$$|\lambda - \alpha_t| - (R_t(\mathcal{A}, \alpha_t) - R_t^k(\mathcal{A}, \alpha_t)) \leq 0 \text{ and } |\lambda - \alpha_k| - R_k^k(\mathcal{A}, \alpha_k) < 0.$$

Therefore,

$$|\lambda - \alpha_t| \leq R_t(\mathcal{A}, \alpha_t) \text{ and } |\lambda - \alpha_k| \leq R_k(\mathcal{A}, \alpha_k),$$

which implies $\lambda \in \mathfrak{X}_t(\mathcal{A}, \alpha) \bigcup \mathfrak{X}_k(\mathcal{A}, \alpha) \subseteq \mathfrak{X}(\mathcal{A}, \alpha)$.

Case 2. If $\lambda \in \mathfrak{M}_{t,k}(\mathcal{A}, \alpha)$, then

$$\begin{aligned}&[|\lambda - \alpha_t| - (R_t(\mathcal{A}, \alpha_t) - R_t^k(\mathcal{A}, \alpha)_t)][|\lambda - \alpha_k| - R_k^k(\mathcal{A}, \alpha_k)] \\ &\leq R_t^k(\mathcal{A}, \alpha_t)[R_k(\mathcal{A}, \alpha_k) - R_k^k(\mathcal{A}, \alpha_k)].\end{aligned}$$

(i) If $R_t^k(\mathcal{A}, \alpha_t)[R_k(\mathcal{A}, \alpha_k) - R_k^k(\mathcal{A}, \alpha_k)] = 0$, then

$$[|\lambda - \alpha_t| - (R_t(\mathcal{A}, \alpha_t) - R_t^k(\mathcal{A}, \alpha)_t)][|\lambda - \alpha_k| - R_k^k(\mathcal{A}, \alpha_k)] \leq 0.$$

Therefore,

$$|\lambda - \alpha_t| - (R_t(\mathcal{A}, \alpha_t) - R_t^k(\mathcal{A}, \alpha)_t) \leq 0 \text{ or } [|\lambda - \alpha_k| - R_k^k(\mathcal{A}, \alpha_k)] \leq 0.$$

This is

$$|\lambda - \alpha_t| \leq R_t(\mathcal{A}, \alpha_t) \text{ or } |\lambda - \alpha_k| \leq R_k(\mathcal{A}, \alpha_k),$$

which implies $\lambda \in \mathfrak{X}_t(\mathcal{A}, \alpha) \bigcup \mathfrak{X}_k(\mathcal{A}, \alpha) \subseteq \mathfrak{X}(\mathcal{A}, \alpha)$.

(ii) If $R_t^k(\mathcal{A}, \alpha_t)[R_k(\mathcal{A}, \alpha_k) - R_k^k(\mathcal{A}, \alpha_k)] > 0$, then

$$\frac{|\lambda - \alpha_t| - (R_t(\mathcal{A}, \alpha_t) - R_t^k(\mathcal{A}, \alpha_t))}{R_t^k(\mathcal{A}, \alpha_t)} \cdot \frac{|\lambda - \alpha_k| - R_k^k(\mathcal{A}, \alpha_k)}{R_k(\mathcal{A}, \alpha_k) - R_k^k(\mathcal{A}, \alpha_k)} \leq 1.$$

That is

$$\frac{|\lambda - \alpha_t| - (R_t(\mathcal{A}, \alpha_t) - R_t^k(\mathcal{A}, \alpha_t))}{R_t^k(\mathcal{A}, \alpha_t)} \leq 1 \text{ or } \frac{|\lambda - \alpha_k| - R_k^k(\mathcal{A}, \alpha_k)}{R_k(\mathcal{A}, \alpha_k) - R_k^k(\mathcal{A}, \alpha_k)} \leq 1.$$

Therefore,

$$|\lambda - \alpha_t| \leq R_t(\mathcal{A}, \alpha_t) \text{ or } |\lambda - \alpha_k| \leq R_k(\mathcal{A}, \alpha_k),$$

which implies $\lambda \in \mathfrak{X}_t(\mathcal{A}, \alpha) \bigcup \mathfrak{X}_k(\mathcal{A}, \alpha) \subseteq \mathfrak{X}(\mathcal{A}, \alpha)$. Thus $\mathfrak{M}(\mathcal{A}, \alpha) \subseteq \mathfrak{X}(\mathcal{A}, \alpha)$. □

Example 3.1. Consider the partially symmetric tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[2]\times[2]\times[2]\times[2]}$ with

$$a_{ijkl} = \begin{cases} a_{1111} = 2, a_{1211} = a_{1112} = 3, a_{1121} = 6, a_{1212} = 2, \\ a_{1222} = 10, a_{2111} = 6, a_{2212} = 10, a_{2222} = 5, \\ a_{ijkl} = 0, \ \text{otherwise}. \end{cases}$$

Here, we set $\alpha = (2, 5)^{\mathrm{T}}$ (This optimal parameter is obtained by traversal). The bounds via different inclusion theorems are shown in Table 1.

Table 1. Comparison of the inclusion intervals of Example 3.1.

Theorem	Inclusion interval
Theorem 2.1 [2]	$\Gamma(\mathcal{A}) = [-26, 26]$
Theorem 2.2 [2]	$\mathcal{L}(\mathcal{A}) = [-24, 24]$
Theorem 2.3 [2]	$\mathcal{M}(\mathcal{A}) = [-23.6941, 23.6941]$
Theorem 2.4 [2]	$\mathcal{N}(\mathcal{A}) = [-24, 24]$
Theorem 2.5 Ours	$\Upsilon(\mathcal{A}) = [-23.6941, 23.6941]$
Theorem 2.6 Ours	$\Theta(\mathcal{A}) = [-23.6941, 23.6941]$
Theorem 3.1 [25]	$\mathfrak{X}(\mathcal{A}, (2, 5)) = [-22, 24]$
Theorem 3.2 [25]	$\mathfrak{K}(\mathcal{A}, (2, 5)) = [-16.1208, 22.5702]$
Theorem 3.4 Ours	$\mathfrak{N}(\mathcal{A}, (2, 5)) = [-16.1208, 22.5702]$
Theorem 3.5 Ours	$\mathfrak{M}(\mathcal{A}, (2, 5)) = [-16.1208, 22.5702]$

Example 3.2. Consider the partially symmetric tensor with

$$a_{ijkl} = \begin{cases} a_{1111} = 20, a_{1122} = a_{1221} = 1, a_{1212} = 8, \\ a_{2222} = 10, a_{2112} = a_{2211} = 1, a_{2121} = 7, \\ a_{ijkl} = 0, \text{ otherwise.} \end{cases}$$

Here, we set $\alpha = (14, 8.5)^{\mathrm{T}}$ (This optimal parameter is obtained by traversal [25]). The bounds via different inclusion theorems are shown in Table 2.

Table 2. Comparison of the inclusion interval of Example 3.2.

References	Inclusion interval
Theorem 2.1 [2]	$\Gamma(\mathcal{A}) = [-30, 30]$
Theorem 2.2 [2]	$\mathcal{L}(\mathcal{A}) = [-29.2971, 29.2971]$
Theorem 2.3 [2]	$\mathcal{M}(\mathcal{A}) = [-28.3523, 28.3523]$
Theorem 2.4 [2]	$\mathcal{N}(\mathcal{A}) = [-29.2971, 29.2971]$
Theorem 2.5 Ours	$\Upsilon(\mathcal{A}) = [-28.3523, 28.3523]$
Theorem 2.6 Ours	$\Theta(\mathcal{A}) = [-28.3523, 28.3523]$
Theorem 3.1 [25]	$\mathfrak{X}(\mathcal{A}, (14, 8.5)) = [0, 28]$
Theorem 3.2 [25]	$\mathfrak{K}(\mathcal{A}, (14, 8.5)) = [0.7154, 26.5539]$
Theorem 3.4 Ours	$\mathfrak{N}(\mathcal{A}, (14, 8.5)) = [0.7154, 26.5539]$
Theorem 3.5 Ours	$\mathfrak{M}(\mathcal{A}, (14, 8.5)) = [1.0925, 26.2708]$

Example 3.1 and Example 3.2 give the comparison between the M-eigenvalue inclusion intervals. From Table 1 and Table 2, we can see that the inclusion intervals obtained in Section 3 are significantly smaller than Section 2. When $m = n = 2$, $\mathfrak{N}(\mathcal{A}, \alpha) = \mathfrak{K}(\mathcal{A}, \alpha)$. From Table 1, $\mathfrak{N}(\mathcal{A}, \alpha)$ and $\mathfrak{M}(\mathcal{A}, \alpha)$ are more accurate than $\mathfrak{X}(\mathcal{A}, \alpha)$ and $\mathcal{L}(\mathcal{A})$. From Table 2, it can be seen that $\mathfrak{M}(\mathcal{A}, \alpha)$ is more accurate than $\mathfrak{X}(\mathcal{A}, \alpha)$ and $\mathfrak{K}(\mathcal{A}, \alpha)$. This shows that our inclusion intervals are better than the existing results in some cases. Moreover, our inclusion intervals can be positioned on the non-negative axis.

4. Application to WQZ-algorithm

In this section, we first present new upper bounds of the fourth-order partially symmetric tensors using the results derived in Section 2. Then, as an application, taking these new upper bounds as a parameter in WQZ-algorithm, can make the generated sequence more rapidly converge to a good approximation of the M-spectral radius. The WQZ-algorithm for solving the largest M-eigenvalue is summarized as follows.

Algorithm 1 WQZ-Algorithm [27]

1: Initial Step: Input $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ and unfold it into a matrix $A = (A_{st}) \in \mathbb{R}^{[mn]\times[mn]}$ by mapping $A_{st} = a_{ijkl}$ with $s = n(i-1)+j, \quad t = n(k-1)+l$.

2: Substep 1: Take $\tau = \sum\limits_{1\le s\le t\le mn} |A_{st}|$ and $\overline{\mathcal{A}} = \tau\mathcal{I} + \mathcal{A}$, where $\mathcal{I} = (\delta_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ with $\delta_{ijkl} = 1$ if $i = k$ and $j = l$, otherwise, $\delta_{ijkl} = 0$. Then unfold $\overline{\mathcal{A}} = (\overline{a}_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ into a matrix $\overline{A} = (\overline{A}_{st}) \in \mathbb{R}^{[mn]\times[mn]}$

3: Substep 2: Compute the unit eigenvalue $w = (w_i)_{i=1}^{mn} \in \mathbb{R}^{mn}$ of matrix $\overline{A}$ associated with its largest eigenvalue, and fold vector w into the matrix $W = (W_{ij}) \in \mathbb{R}^{[m]\times[n]}$, $W_{ij} = w_k$, where $i = \lceil k/n \rceil$, $j = (k-1)modn + 1, \quad \forall k = 1, 2, ..., mn$.

4: Substep 3: Compute the singular vectors u_1 and v_1 corresponding to the largest singular value σ_1 of the matrix W. Specifically, the singular value decomposition of W is $W = U^{\mathrm{T}}\Sigma V = \sum\limits_{i=1}^{r} \sigma_i u_i v_i^{\mathrm{T}}$, where $\sigma_1 \ge \sigma_2 \ge ... \ge \sigma_r$ and r is the rank of W.

5: Substep 4: Take $x_0 = u_1, y_0 = v_1$, and let $k = 0$.

6: Iterative Step: Execute the following procedures alternatively until certain convergence criterion is satisfied and output x^*, y^*:

$$\overline{x}_{k+1} = \overline{\mathcal{A}} \cdot y_k x_k y_k, \quad x_{k+1} = \frac{\overline{x}_{k+1}}{\|\overline{x}_{k+1}\|},$$

$$\overline{y}_{k+1} = \overline{\mathcal{A}} x_{k+1} y_k x_{k+1}, \quad y_{k+1} = \frac{\overline{y}_{k+1}}{\|\overline{y}_{k+1}\|},$$

$$k = k+1.$$

7: Final Step: Output the largest M-eigenvalue of the tensor $\mathcal{A}$: $\lambda_{\max}(\mathcal{A}) = f(x^*, y^*) - \tau$, where $f(x^*, y^*) = \sum\limits_{i,k=1}^{m} \sum\limits_{j,l=1}^{n} \overline{a}_{ijkl} x_i^* y_j^* x_k^* y_l^*$ and the associated M-eigenvectors: x^*, y^*.

We recall some existing upper bounds for M-eigenvalues of the fourth-order partially symmetric tensor in [2].

Theorem 4.1. [2] *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric tensor. Then*

$$\rho(\mathcal{A}) \le \tau_1 = \max_{i\in[m]} R_i(\mathcal{A}).$$

Theorem 4.2. [2] *Let* $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ *be a partially symmetric tensor. Then*

$$\begin{aligned}\rho(\mathcal{A}) \leq& \tau_2 \\ =&\max_{i\in[m]} \min_{k\in[m],\ k\neq i} \tfrac{1}{2}\left\{R_i(\mathcal{A}) - R_i^k(\mathcal{A}) + \sqrt{(R_i(\mathcal{A}) - R_i^k(\mathcal{A}))^2 + 4R_i^k(\mathcal{A})R_k(\mathcal{A})}\right\}.\end{aligned}$$

Theorem 4.3. [2] *Let* $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ *be a partially symmetric tensor. Then*

$$\begin{aligned}\rho(\mathcal{A}) &\leq \tau_3 \\ &= \max_{i,k\in[m],\ k\neq i} \left\{\frac{1}{2}(R_i(\mathcal{A}) - R_i^k(\mathcal{A}) + R_k^k(\mathcal{A}) + \delta_i^k), R_i(\mathcal{A}) - R_i^k(\mathcal{A}), R_k^k(\mathcal{A})\right\},\end{aligned}$$

where

$$\begin{aligned}\delta_i^k(\mathcal{A}) = ((R_i(\mathcal{A}) &- R_i^k(\mathcal{A}) + R_k^k(\mathcal{A}))^2 - 4[(R_i(\mathcal{A}) - R_i^k(\mathcal{A}))R_k^k(\mathcal{A}) \\ &-R_i^k(\mathcal{A})(R_k(\mathcal{A}) - R_k^k(\mathcal{A}))])^{1/2}.\end{aligned}$$

Theorem 4.4. [2] *Let* $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ *be a partially symmetric tensor. Then*

$$\rho(\mathcal{A}) \leq \tau_4 = \max_{i,k\in[m],\ k\neq i} \left\{\frac{1}{2}(R_i^i(\mathcal{A}) + \sqrt{R_i^i(\mathcal{A})^2 + 4((R_i(\mathcal{A}) - R_i^i(\mathcal{A}))R_k(\mathcal{A})))}\right\}.$$

By Theorem 2.5 and Theorem 2.6, we obtain the following result.

Theorem 4.5. *Let* $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ *be a partially symmetric tensor. Then*

$$\begin{aligned}\rho(\mathcal{A}) \leq \tau_5 = \max_{i,k\in[m],\ k\neq i} &\left\{\frac{1}{2}\left([(R_i(\mathcal{A}) - R_i^k(\mathcal{A})) + (R_k(\mathcal{A}) - R_k^i(\mathcal{A}))] + \delta_i^k(\mathcal{A})\right),\right. \\ &\left. R_i(\mathcal{A}) - R_i^k(\mathcal{A}), R_k(\mathcal{A}) - R_k^i(\mathcal{A})\right\},\end{aligned}$$

where

$$\begin{aligned}\delta_i^k(\mathcal{A}) = ([(R_i(\mathcal{A}) &- R_i^k(\mathcal{A})) + (R_k(\mathcal{A}) - R_k^i(\mathcal{A}))]^2 - 4[(R_i(\mathcal{A}) - R_i^k(\mathcal{A})) \\ &\times(R_k(\mathcal{A}) - R_k^i(\mathcal{A})) - R_i^k(\mathcal{A})R_k^i(\mathcal{A})])^{1/2}.\end{aligned}$$

Proof. Suppose $\rho(\mathcal{A})$ is the largest M-eigenvalue of $\mathcal{A}$. We complete the proof by two cases.

Case 1. There exist $i, k \in [m], i \neq k$ such that $\rho(\mathcal{A}) \in \widetilde{\gamma}_{i,k}(\mathcal{A})$. In this case, we have

$$(\rho(\mathcal{A}) - R_i(\mathcal{A}) + R_i^k(\mathcal{A}))(\rho(\mathcal{A}) - R_k(\mathcal{A}) + R_k^i(\mathcal{A})) \leq R_i^k(\mathcal{A})R_k^i(\mathcal{A}),$$

which yields that

$$\begin{aligned}\rho(\mathcal{A}) &\leq \frac{1}{2}\left([(R_i(\mathcal{A}) - R_i^k(\mathcal{A})) + (R_k(\mathcal{A}) - R_k^i(\mathcal{A}))] + \delta_i^k(\mathcal{A})\right) \\ &\leq \max_{i,k\in[m],\ k\neq i} \frac{1}{2}\left([(R_i(\mathcal{A}) - R_i^k(\mathcal{A})) + (R_k(\mathcal{A}) - R_k^i(\mathcal{A}))] + \delta_i^k(\mathcal{A})\right).\end{aligned}$$

Case 2. There exist $i, k \in [m], i \neq k$ such that $\rho(\mathcal{A}) \in \widehat{\gamma}_{i,k}(\mathcal{A})$. In this case, we get

$$\rho(\mathcal{A}) \leq R_i(\mathcal{A}) - R_i^k(\mathcal{A}),$$

and

$$\rho(\mathcal{A}) \leq R_k(\mathcal{A}) - R_k^i(\mathcal{A}).$$

Thus, we complete the proof. □

Similar to the proof of Theorem 4.5, the following conclusion is true.

Theorem 4.6. *Let* $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ *be a partially symmetric tensor. Then*

$$\rho(\mathcal{A}) \leq \tau_6 = \max_{i,k\in[m],\ k\neq i} \left\{ \frac{1}{2}(R_i^i(\mathcal{A}) + R_k^k(\mathcal{A}) + \delta_i^k(\mathcal{A})), R_i^i(\mathcal{A}) \right\},$$

where

$$\begin{aligned}&\delta_i^k(\mathcal{A})\\ =&\sqrt{(R_i^i(\mathcal{A}) + R_k^k(\mathcal{A})) - 4(R_i^i(\mathcal{A})R_k^k(\mathcal{A}) - ((R_i(\mathcal{A}) - R_i^k(\mathcal{A}))(R_k(\mathcal{A}) - R_k^i(\mathcal{A}))}.\end{aligned}$$

Viewing Theorem 4.1 to Theorem 4.6, τ_1 to τ_6 are upper bounds for the M-spectral radius of a fourth-order partially symmetric tensor, hence they can be taken as the parameter τ in WQZ-algorithm. Li et al. [18] illustrated that the selection for the parameter τ in the WQZ-algorithm has a significant impact on the convergence rate. The comparison is illustrated by the following example, refer to [27].

Example 4.1. [27] Consider the tensor $\mathcal{A}_2$ with

$$\mathcal{A}_2(:,:,1,1) = \begin{bmatrix} -0.9727 & 0.3169 & -0.3437 \\ -0.6332 & -0.7866 & 0.4257 \\ -0.3350 & -0.9896 & -0.4323 \end{bmatrix},$$

$$\mathcal{A}_2(:,:,2,1) = \begin{bmatrix} -0.6332 & -0.7866 & 0.4257 \\ 0.7387 & 0.6873 & -0.3248 \\ -0.7986 & -0.5988 & -0.9485 \end{bmatrix},$$

$$\mathcal{A}_2(:,:,3,1) = \begin{bmatrix} -0.3350 & -0.9896 & -0.4323 \\ -0.7986 & -0.5988 & -0.9485 \\ 0.5853 & 0.5921 & 0.6301 \end{bmatrix},$$

$$\mathcal{A}_2(:,:,1,2) = \begin{bmatrix} 0.3169 & 0.6158 & -0.0184 \\ -0.7866 & 0.0160 & 0.0085 \\ -0.9896 & -0.6663 & 0.2559 \end{bmatrix},$$

$$\mathcal{A}_2(:,:,2,2)=\begin{bmatrix}-0.7866 & 0.0160 & 0.0085\\ 0.6873 & 0.5160 & -0.0216\\ -0.5988 & 0.0411 & 0.9857\end{bmatrix},$$

$$\mathcal{A}_2(:,:,3,2)=\begin{bmatrix}-0.9896 & -0.6663 & 0.2559\\ -0.5988 & 0.0411 & 0.9857\\ 0.5921 & -0.2907 & -0.3881\end{bmatrix},$$

$$\mathcal{A}_2(:,:,1,3)=\begin{bmatrix}-0.3437 & -0.0184 & 0.5649\\ 0.4257 & 0.0085 & -0.1439\\ -0.4323 & 0.2559 & 0.6162\end{bmatrix},$$

$$\mathcal{A}_2(:,:,2,3)=\begin{bmatrix}0.4257 & 0.0085 & -0.1439\\ -0.3248 & -0.0216 & -0.0037\\ -0.9485 & 0.9857 & -0.7734\end{bmatrix},$$

$$\mathcal{A}_2(:,:,3,3)=\begin{bmatrix}-0.4323 & 0.2559 & 0.6162\\ -0.9485 & 0.9857 & -0.7734\\ 0.6301 & -0.3881 & -0.8526\end{bmatrix}.$$

By calculation, we can get $\tau = 23.3503$. The values of $\tau_1, ..., \tau_6$ are as follows.

τ_1	τ_2	τ_3	τ_4	τ_5	τ_6
16.6014	15.4102	15.1288	14.9160	15.4044	15.1393

Taking $\tau_1, ..., \tau_6$ to τ in the WQZ-algorithm. The numerical result is given in Figure 3.

From Figure 3, it can be seen that, when taking $\tau = \tau_5, \tau_6$, the WQZ-algorithm needs fewer iterations and converges more rapidly to the largest M-eigenvalue $\lambda_{\max}(\mathcal{A})$ than τ_1, τ_2. This shows that our upper bounds are more tighter than the existing results in some cases.

5. Application to strong ellipticity conditions

In this section, using the bounds derived in Section 3, we first propose some new sufficient conditions for the positive definiteness of fourth-order partially symmetric tensors. Subsequently, as an application, the strong ellipticity conditions of elastic materials are obtained through the new sufficient conditions. The following lemma and some existing sufficient conditions for the positive definiteness are required.

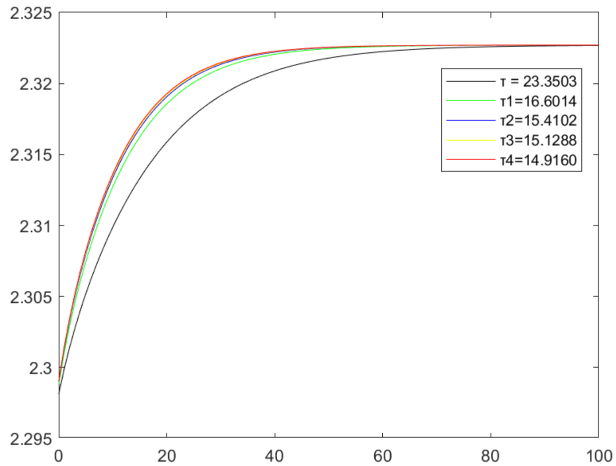

Figure 3. Numerical results of Example 4.1.

Lemma 5.1. [10] *Let* $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ *be a partially symmetric tensor. The strong ellipticity condition holds. i.e.,*

$$f(x, y) = \mathcal{A}xyxy = \sum_{i,k=1}^{m} \sum_{j,l=1}^{n} a_{ijkl} x_i y_j x_k y_l > 0,$$

for all nonzero vectors $x,\ y\ \in \mathbb{R}^n$ *if and only if the smallest M-eigenvalue of* $\mathcal{A}$ *is positive.*

Theorem 5.1. [25] *Let* $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ *be a partially symmetric nonnegative tensor and* $\mathcal{F}_{\mathcal{M}}$ *be an M-identity tensor. For* $i \in [m]$, *if there exists positive real vector* $\alpha = (\alpha_1, ..., \alpha_m)^{\mathrm{T}} \in \mathbb{R}^m$ *such that*

$$\alpha_i > R_i(\mathcal{A}, \alpha_i),$$

then $\mathcal{A}$ *is positive definite.*

Theorem 5.2. [25] *Let* $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ *be a partially symmetric nonnegative tensor and* $\mathcal{F}_{\mathcal{M}}$ *be an M-identity tensor. For* $i \in [m]$, *if there exists positive real vector* $\alpha = (\alpha_1, ..., \alpha_m)^{\mathrm{T}} \in \mathbb{R}^m$ *and* $k \neq i$ *such that*

$$(\alpha_i - (R_i(\mathcal{A}, \alpha_i) - R_i^k(\mathcal{A}, \alpha_i)))\alpha_k > R_i^k(\mathcal{A}, \alpha_i) R_k(\mathcal{A}, \alpha_k),$$

then $\mathcal{A}$ *is positive definite.*

Theorem 5.3. *Let* $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ *be a partially symmetric nonnegative tensor and* $\mathcal{F}_{\mathcal{M}}$ *be an M-identity tensor. For* $i \in [m]$, *if there exists positive real vector* $\alpha = (\alpha_1, ..., \alpha_m)^{\mathrm{T}} \in \mathbb{R}^m$ *and* $k \neq i$ *such that*

$$(\alpha_i - R_i^i(\mathcal{A}, \alpha_i))\alpha_k > [R_i(\mathcal{A}, \alpha_i) - R_i^i(\mathcal{A}, \alpha_i)] R_k(\mathcal{A}, \alpha_k), \tag{5.1}$$

then $\mathcal{A}$ *is positive definite. That is, the strong ellipticity condition holds.*

Proof. We complete the proof by contradiction. Suppose $\lambda \leq 0$. From Theorem 3.4, there exists $i_0 \in [m]$ such that $\alpha \in \mathfrak{N}_{i_0,p}(\mathcal{A}, \alpha)$, then

$$[|\lambda - \alpha_{i_0}| - R_{i_0}^{i_0}(\mathcal{A}, \alpha_{i_0})]|\lambda - \alpha_p| \leq [R_{i_0}(\mathcal{A}, \alpha_{i_0}) - R_{i_0}^{i_0}(\mathcal{A}, \alpha_{i_0})] R_p(\mathcal{A}, \alpha_p),\ \forall p \neq i_0.$$

Further, it follows from $\alpha_{i_0}, \alpha_p > 0$ and $\lambda \leq 0$ that

$$\begin{aligned}[\alpha_{i_0} - R_{i_0}^{i_0}(\mathcal{A}, \alpha_{i_0})]\alpha_p &\leq [|\lambda - \alpha_{i_0}| - R_{i_0}^{i_0}(\mathcal{A}, \alpha_{i_0})]|\lambda - \alpha_p| \\ &\leq [R_{i_0}(\mathcal{A}, \alpha_{i_0}) - R_{i_0}^{i_0}(\mathcal{A}, \alpha_{i_0})]R_p(\mathcal{A}, \alpha_p),\end{aligned}$$

which contradicts (14). Hence, $\lambda > 0$. Since $\mathcal{A}$ is partially symmetric and all M-eigenvalues are positive, then $\mathcal{A}$ is positive definite. That is, the strong ellipticity condition of the elastic material is established. □

Theorem 5.4. *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[m]\times[n]\times[m]\times[n]}$ be a partially symmetric nonnegative tensor and $\mathcal{F}_{\mathcal{M}}$ be an M-identity tensor. For $i \in [m]$, if there exists positive real vector $\alpha = (\alpha_1, ..., \alpha_m)^{\mathrm{T}} \in \mathbb{R}^m$ and $k \neq i$ such that*

$$\begin{aligned}&[\alpha_i - (R_i(\mathcal{A}, \alpha_i) - R_i^k(\mathcal{A}, \alpha_i))][\alpha_k - R_k^k(\mathcal{A}, \alpha_k)] \\ > \ &R_i^k(\mathcal{A}, \alpha_i)[R_k(\mathcal{A}, \alpha_k) - R_k^k(\mathcal{A}, \alpha_k)],\end{aligned} \tag{5.2}$$

or

$$\alpha_i - (R_i(\mathcal{A}, \alpha_i) - R_i^k(\mathcal{A}, \alpha_i)) > 0 \ \ \textit{and} \ \ \alpha_k - R_k^k > 0, \tag{5.3}$$

then $\mathcal{A}$ is positive definite. That is, the strong ellipticity condition holds.

Proof. We complete the proof by contradiction. Suppose $\lambda \leq 0$. From Theorem 3.5, we consider two cases.

Case 1. There exists $i_0 \in [m]$ such that $\alpha \in \mathfrak{M}_{i_0,p}(\mathcal{A}, \alpha)$, then for $\forall p \neq i_0$,

$$\begin{aligned}&[|\lambda - \alpha_{i_0}| - (R_{i_0}(\mathcal{A}, \alpha_{i_0}) - R_{i_0}^p(\mathcal{A}, \alpha_{i_0}))][|\lambda - \alpha_p| - R_p^p(\mathcal{A}, \alpha_p)] \\ \leq &R_{i_0}^p(\mathcal{A}, \alpha_{i_0})[R_p(\mathcal{A}, \alpha_p) - R_p^p(\mathcal{A}, \alpha_p)].\end{aligned}$$

Further, it follows from $\alpha_{i_0}, \alpha_p > 0$ and $\lambda \leq 0$ that

$$\begin{aligned}&[\alpha_{i_0} - (R_{i_0}(\mathcal{A}, \alpha_{i_0}) - R_{i_0}^p(\mathcal{A}, \alpha_{i_0})][\alpha_p - R_p^p(\mathcal{A}, \alpha_p)] \\ \leq &[|\lambda - \alpha_{i_0}| - (R_{i_0}(\mathcal{A}, \alpha_{i_0}) - R_{i_0}^p(\mathcal{A}, \alpha_{i_0}))][|\lambda - \alpha_p| - R_p^p(\mathcal{A}, \alpha_p)] \\ \leq &R_{i_0}^p(\mathcal{A}, \alpha_{i_0})[R_p(\mathcal{A}, \alpha_p) - R_p^p(\mathcal{A}, \alpha_p)],\end{aligned}$$

which contradicts with (15). Hence, $\lambda > 0$.

Case 2. There exists $i_0 \in [m]$ such that $\alpha \in \mathfrak{H}_{i_0,p}(\mathcal{A}, \alpha)$, then

$$|\lambda - \alpha_{i_0}| - (R_{i_0}(\mathcal{A}, \alpha_{i_0}) - R_{i_0}^p(\mathcal{A}, \alpha_{i_0})) \leq 0 \ \ \textit{and} \ \ |\lambda - \alpha_p| - R_p^p \leq 0.$$

Further, it follows from $\alpha_{i_0}, \alpha_p > 0$ and $\lambda \leq 0$ that

$$\alpha_{i_0} - (R_{i_0}(\mathcal{A}, \alpha_{i_0}) - R_{i_0}^p(\mathcal{A}, \alpha_{i_0})) \leq |\lambda - \alpha_{i_0}| - (R_{i_0}(\mathcal{A}, \alpha_{i_0}) - R_{i_0}^p(\mathcal{A}, \alpha_{i_0})) \leq 0,$$

and

$$\alpha_p - R_p^p \leq |\lambda - \alpha_p| - R_p^p \leq 0,$$

which contradicts with (16). Hence, $\lambda > 0$. □

In summary, $\mathcal{A}$ is partially symmetric and all M-eigenvalue are positive, $\mathcal{A}$ is positive definite. Thus, Theorem 5.3 and Theorem 5.4 are sufficient conditions for the strong ellipticity of elastic materials. Moreover, we offer corresponding numerical examples to verify the validity of the obtained results below.

Example 5.1. Consider the partially symmetric tensor

$$\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[2]\times[2]\times[2]\times[2]}$$

with

$$a_{ijkl} = \begin{cases} a_{1111} = 10, a_{1122} = a_{1221} = -0.5, a_{1212} = 4, \\ a_{2222} = 3, a_{2112} = a_{2211} = -0.5, a_{2121} = 5, \\ a_{ijkl} = 0, \text{ otherwise.} \end{cases}$$

By Theorem 7 of [23], we obtain that the minimum M-eigenvalue and corresponding with left and right M-eigenvectors are

$$(\lambda, x, y) = (3, (0, 1), (0, 1)).$$

Hence, $\mathcal{A}$ is positive definite. That is, the strong ellipticity condition holds.

Here, we set $\alpha = (8, 4)^{\mathrm{T}}$ (This optimal parameter is obtained by traversal). According to Theorem 5.3, we have

$$(\alpha_1 - R_1^1(\mathcal{A}, \alpha_1))\alpha_2 = 6 > [R_1(\mathcal{A}, \alpha_1) - R_1^1(\mathcal{A}, \alpha_1)]R_2(\mathcal{A}, \alpha_2) = 3,$$
$$(\alpha_2 - R_2^2(\mathcal{A}, \alpha_2))\alpha_1 = 14 > [R_2(\mathcal{A}, \alpha_2) - R_2^2(\mathcal{A}, \alpha_2)]R_1(\mathcal{A}, \alpha_1) = 7.$$

Hence, $\mathcal{A}$ satisfies the condition of Theorem 5.3, which implies that $\mathcal{A}$ is positive definite. That is, the strong ellipticity condition holds.

According to Theorem 5.4, we have

$$\begin{aligned} &[\alpha_1 - (R_1(\mathcal{A}, \alpha_1) - R_1^2(\mathcal{A}, \alpha_1))][\alpha_2 - R_2^2(\mathcal{A}, \alpha_2)] \\ &= 2 \\ &> R_1^2(\mathcal{A}, \alpha_1)[R_2(\mathcal{A}, \alpha_2) - R_2^2(\mathcal{A}, \alpha_2)] \\ &= 1, \\ &[\alpha_2 - (R_2(\mathcal{A}, \alpha_2) - R_2^1(\mathcal{A}, \alpha_2))][\alpha_1 - R_1^1(\mathcal{A}, \alpha_1)] \\ &= 4 \\ &> R_2^1(\mathcal{A}, \alpha_2)[R_1(\mathcal{A}, \alpha_1) - R_1^1(\mathcal{A}, \alpha_1)] \\ &= 1. \end{aligned}$$

Hence, $\mathcal{A}$ satisfies the condition of Theorem 5.4, which implies that $\mathcal{A}$ is positive definite. That is, the strong ellipticity condition holds.

Example 5.2. Consider the partially symmetric tensor

$$\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[2]\times[2]\times[2]\times[2]}$$

with

$$a_{ijkl} = \begin{cases} a_{1111} = 10, a_{1212} = 8, a_{1122} = a_{1221} = 0.5, \\ a_{1222} = -1.5, a_{1112} = a_{1211} = -0.1, a_{1121} = 1.5, \\ a_{2222} = 3, a_{2121} = 5, a_{2112} = a_{2211} = 0.5, \\ a_{2212} = -1.5, a_{2221} = a_{2122} = -0.1, a_{2111} = 1.5. \end{cases}$$

By Theorem 7 of [23], we obtain that the minimum M-eigenvalue and corresponding with left and right M-eigenvectors are

$$(\lambda, x, y) = (2.5774, (0.2724, 0.9622), (-0.0452, 0.9990)).$$

Hence, $\mathcal{A}$ is positive definite.

Here, we set $\alpha = (8,4)^{\mathrm{T}}$ (This optimal parameter is obtained by traversal). According to Theorem 5.3, we have

$$(\alpha_1 - R_1^1(\mathcal{A},\alpha_1))\alpha_2 = 23.2 < [R_1(\mathcal{A},\alpha_1) - R_1^1(\mathcal{A},\alpha_1)]R_2(\mathcal{A},\alpha_2) = 24.8,$$
$$(\alpha_2 - R_2^2(\mathcal{A},\alpha_2))\alpha_1 = 14.4 > [R_2(\mathcal{A},\alpha_2) - R_2^2(\mathcal{A},\alpha_2)]R_1(\mathcal{A},\alpha_1) = 24.8,$$

which implies that the condition of Theorem 5.3 is not satisfied. Thus, Theorem 5.3 is not suitable in this case. However, from Theorem 5.4, we have

$$\alpha_1 - (R_1(\mathcal{A},\alpha_1) - R_1^2(\mathcal{A},\alpha_1)) = 5.8 > 0 \ \ \textit{and} \ \ \alpha_2 - R_2^2(\mathcal{A},\alpha_2) = 1.8 > 0,$$
$$\alpha_2 - (R_2(\mathcal{A},\alpha_2) - R_2^1(\mathcal{A},\alpha_2)) = 1.8 > 0 \ \ \textit{and} \ \ \alpha_1 - R_1^1(\mathcal{A},\alpha_1)] = 5.8 > 0.$$

Hence, $\mathcal{A}$ satisfies the condition of Theorem 5.4, which implies that $\mathcal{A}$ is positive definite. That is, the strong ellipticity of the elastic material can be checked.

6. Conclusion

In this paper, we have proposed some new M-eigenvalue inclusion theorems for fourth-order partially symmetric tensors, which are more accurate than some existing theorems. As applications, we have applied the upper bound to the WQZ-algorithm to solve the largest M-eigenvalue. Numerical experiments have shown that using the obtained upper bound as a parameter can make the sequence generated by the WQZ-algorithm rapidly converge to a good approximation of the M-spectral radius of the fourth-order partially symmetric tensor. Moreover, the judgment theorem about the sufficient condition of the strong ellipticity of elastic material has been obtained. Through numerical examples, we have verified that the sufficient conditions for the strong ellipticity condition holds of the elastic materials.

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