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BOUNDED AND BLOW-UP SOLUTIONS OF K-HESSIAN SYSTEM WITH AUGMENTED TERMS*

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Abstract The radial solutions of the k-Hessian system with augmented terms are considered. We not only prove the existence of entire bounded radial solutions, but also provide a necessary and sufficient condition for the existence of blow-up radial solutions under some suitable growth conditions of nonlinearity by using the monotone iterative method. Two concrete examples are presented to show an application of the main results.

Keywords k-Hessian system, blow-up solutions, monotone iterative method.

MSC(2010) 37G05, 37G10, 34K18.

1. Introduction

In this paper, we consider the existence of radial solutions for the following k-Hessian system with augmented terms

$$\begin{cases} S_k(\lambda(D^2u+\mu|\nabla u|I)) = p(|x|)\varphi(u,v), & \text{in } \mathbb{R}^n, \\ S_k(\lambda(D^2v+\nu|\nabla v|I)) = q(|x|)\psi(u,v), & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where μ and ν are nonnegative constants, $p(\cdot)$ and $q(\cdot)$ are positive weight functions, the nonlinear terms $\varphi,\psi \in C([0,\infty)\times[0,\infty),[0,\infty))$ are increasing in each variables, $S_k(\lambda(D^2u))$ $(k=1,2,\cdots,n)$ is the k-Hessian operator of u. In general, the k-Hessian operator is defined by

$$S_k(\lambda(D^2u)) = P_k(\Lambda) = \sum_{1\le j_1<\cdots<j_k\le n} \lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_k}, \quad k=1,2,\cdots,n,$$

where $P_k(\Lambda)$ denotes the k-th elementary symmetric function of Λ, $\Lambda=(\lambda_1,\lambda_2,\cdots,\lambda_n)$ are the eigenvalues of Hessian matrix D^2u. In addition, the k-Hessian operator can also be written in the divergence form

$$S_k(\lambda(D^2u)) = \frac{1}{k}\sum_{i,j=1}^{N}(S_k^{ij}u_i)_j,$$

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where $S_k^{ij}=\frac{\partial S_k(\lambda(D^2u))}{\partial u_{ij}}$, see details in [2, 20] and the references cited therein. Noticed that the k-Hessian operator is a generalization of both Laplace operator and Monge-Ampère operator, that is, when $k=1$, k-Hessian operator reduces to Laplace operator $S_1(\lambda(D^2u))=\sum_{i=1}^{n}\lambda_i=\Delta u$; when $k=n$, k-Hessian operator is Monge-Ampère operator $S_n(\lambda(D^2u))=\prod_{i=1}^{n}\lambda_i=\det(D^2u)$. About Laplace problem and Monge-Ampère problem, there are a lot of brilliant papers, we refer the readers to [1, 6, 10, 19, 31–33].

In recent years, the k-Hessian equation has attracted the attention of a large number of scholars due to its wide range of applications in many fields, including fluid mechanics, geometric analysis and other disciplines. The scope of the study includes singular solutions, regular solutions and the corresponding asymptotic behavior [3, 7, 9, 13–18, 22]. Many scholars have also investigated the existence of blow-up solutions. In 2018, Zhang and Feng [28] studied the boundary blow-up solutions for the k-Hessian equation

$$S_k(\lambda(D^2u))=H(x)u^p \quad \text{in } \ \Omega, \quad u=+\infty \ \text{ on } \ \partial\Omega,$$

where Ω is a smooth, bounded, strictly convex domain in $\mathbb{R}^n (n\geq 2)$, and $H(x)$ is smooth positive function. The author obtained the existence, nonexistence, uniqueness results, global estimates and estimates near the boundary for the solutions by constructing suitable sub- and super- solutions. Furthermore, they considered the existence and asymptotic behavior of k-convex solution to the boundary blow-up k-Hessian problem

$$S_k(\lambda(D^2u))=H(x)\varphi(u), \quad \text{in } \Omega,$$

where φ is a smooth positive function that satisfies the Keller-Osserman condition. Zhang [29] considered the existence of the entire radial large solution for the following modified quasilinear Schrödinger elliptic system

$$\begin{cases} \triangle u+\triangle(|u|^{2\gamma})|u|^{2\gamma-2}=p(|x|)\varphi(v)\chi_\gamma(u),\\ \triangle v+\triangle(|v|^{2\delta})|v|^{2\delta-2}=q(|x|)\psi(v)\chi_\delta(v),\\ \lim\limits_{|x|\to\infty}u(x)=\lim\limits_{|x|\to\infty}v(x)=\infty \ (\text{i.e. } u,v \text{ are large}),\end{cases}$$

where $x\in\mathbb{R}^n(n\geq 3)$, $\gamma,\delta>\frac{1}{2}$, $\chi_i(s)=\sqrt{1+2i|s|^{2(2i-1)}}$, $i>\frac{1}{2}$, and the nonnegative functions p and q are continuous on $\mathbb{R}^n$, φ and ψ are also required to be increasing. For the study of blow-up solutions, we can see [24–27].

In addition, not only the study of blow-up solutions , but also the entire k-convex radial solutions have attracted the interests of many scholars. For $\mu=\nu=0$, there are several works that deals with the existence of radial solutions of (1.1), such as, Zhang and Zhou [34] who obtained the existence of entire positive k-convex radial solutions of the following k-Hessian by using the monotone iterative method

$$\begin{cases} S_k(\lambda(D^2u))=p(|x|)\varphi(v), & \text{in } \mathbb{R}^n,\\ S_k(\lambda(D^2v))=q(|x|)\psi(u), & \text{in } \mathbb{R}^n.\end{cases}$$

In 2018, Covei [4] established a necessary condition and a sufficient condition of the existence of positive radial solution. It is notable that the monotonicity of φ and

ψ are required in order to get the existence of radial solutions. In 2019, Feng [8] applied a new fixed point theorem to investigate the existence and multiplicity of nontrivial radial solutions of k-Hessian system. One point to note compared to the previous two papers is that monotonicity of the nonlinear term is not required anymore.

For $\mu \neq 0$ or $\nu \neq 0$, Cui [5] considered for the first time the existence of entire k-convex radial solutions for a system of the form $\varphi(u,v) = \varphi_1(u)\varphi_2(v)$. The author obtained the existence and nonexistence of entire k-convex radial solutions. In 2022, Ji [11] studied the single k-Hessian equation

$$S_k^{\frac{1}{k}}(\lambda(D^2u + \mu|Du|I)) = \varphi(u), \text{ in } \mathbb{R}^n,$$

the author obtained a necessary and sufficient condition of φ on the existence and nonexistence of entire admissible subsolutions under the generalized Keller-Osserman condition. Zhang [30] analyzes the existence of entire subsolutions to the p-k-Hessian equation

$$S_k^{\frac{1}{k}}(\lambda((D_i(|Du|^{p-2}D_ju)) + \alpha|Du|^{\beta(p-1)}I)) = f(u),\ i,j \in \{1,2,\cdots,n\},\ u \in \mathbb{R}^n,$$

where $\beta = 0$ or 1, $p \geq 2$, α is a nonnegative constant. The necessary and sufficient conditions on f for the existence of entire subsolutions are established. In 2023, Yang [21] deals with the k-Hessian type system with the gradients

$$\begin{cases} S_k(\lambda(D^2u_i + \alpha|\nabla u_i|I)) = \varphi_i(|x|, -u_1, -u_2, \cdots, -u_n), & \text{in } \Omega, \\ u_i = 0, \text{on } \ \partial\Omega, i = 1, 2, \ldots, n, \end{cases}$$

where $\alpha \geq 0, n \geq 2, 1 \leq k \leq N$ is a positive integer, I is the identity matrix and Ω stands for the open unit ball in $\mathbb{R}^N (N \geq 2)$. Based on appropriate assumptions about $\varphi_i (i = 1, 2, \ldots, n)$, some results regarding existence of negative k-convex radial solution are established. In the same year, Zhang [23] applies the sub-super-solution method to investigate the multiplicity of radial k-convex solutions of an augmented Hessian problem

$$\begin{cases} S_k(\lambda(D^2u - \mu I)) = b(|x|)g(u), x \in \Omega, \\ u = +\infty, x \in \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N (N \geq 2)$ is a ball, $k \in \{1, 2, \cdots, N\}$, μ is a positive constant.

However, few papers studied the k-Hessian system with the form (1.1), for which it is worthwhile to consider what kind of radial solutions properties of system (1.1) exist. Inspired by the above works, in this paper we try to generalize and improve some known results about k-Hessian equation, such as, [34] and [4], from three aspects: 1) the k-Hessian operator with augmentation item is considered, that is, $S_k(\lambda(D^2u + \mu|\nabla u|I))$, here μ maybe not equal to zero; 2)both the existence of blow-up solution and the existence of entire bounded radial solution of the system (1.1) are studied; 3)the form of nonlinear term in (1.1) is more general. By applying the monotone iterative method, the sufficient conditions of the existence of entire radial solutions and a necessary and sufficient conditions of the existence of blow-up radial solutions of the system (1.1) is obtained, respectively.In addition, the nonexistence of blow-up radial solutions is also discussed as a direct result of the main theorem.

The rest of this paper is organized as follows. In Section 2, we provide some preliminary results, which are useful in the following proofs. In Section 3, the existence of entire bounded solutions of (1.1) is discussed by using the monotone iterative method. In Section 4, a necessary and sufficient conditions of the existence of blow-up radial solutions of the system (1.1) is given. Finally, two examples are given to verify our results in Section 5.

2. Preliminaries

Throughout the paper, we assume

(H1) $\varphi, \psi \in C([0,\infty)\times[0,\infty),[0,\infty))$ are increasing in each variables and $\varphi(u,v)>0, \psi(u,v)>0$ for all u and v satisfying $u^2+v^2>0$.

(H2) $p, q : [0,\infty)\to[0,\infty)$ are continuous functions.

Lemma 2.1. [12] *Assume $z(\cdot)\in C^2[0,\infty)$ with $z'(0)=0$. Then for $u(x)=z(r)$, we have that $u(x)\in C^2(\mathbb{R}^n)$ and the eigenvalues of the operator $D^2u+\mu|\nabla u|I$ are*

$$\lambda(D^2u+\mu|\nabla u|I)=\begin{cases}\left(z''(r)+\mu z'(r),\dfrac{1+\mu r}{r}z'(r),\cdots,\dfrac{1+\mu r}{r}z'(r)\right), & r>0,\\ \left(z''(0),z''(0),\cdots,z''(0)\right), & r=0,\end{cases}$$

where $\mu>0$.

Lemma 2.2. *If $(z(x),w(x))=(u(r),v(r))$ is a radial solution of the k-Hessian system(1.1), then, for any initial values $u(0)=\gamma$, $v(0)=\delta$, the radial solution $(u(r),v(r))$ can be expressed by the integral form*

$$\begin{cases}u(r)=\gamma+\displaystyle\int_0^r\left(\frac{t^{k-n}}{e^{n\mu t}}\int_0^t\frac{n}{C_n^k}\frac{e^{n\mu s}s^{n-1}}{(1+\mu s)^{k-1}}p(s)\varphi(u(s),v(s))ds\right)^{\frac{1}{k}}dt,\\ v(r)=\delta+\displaystyle\int_0^r\left(\frac{t^{k-n}}{e^{n\nu t}}\int_0^t\frac{n}{C_n^k}\frac{e^{n\nu s}s^{n-1}}{(1+\nu s)^{k-1}}q(s)\psi(u(s),v(s))ds\right)^{\frac{1}{k}}dt.\end{cases}$$

Proof. By Lemma 2.1 and the definition of S_k, we can get the expression of k-Hessian operator S_k

$$\begin{aligned}&S_k(\lambda(D^2u+\mu|\nabla u|I))\\ =&C_{n-1}^{k-1}\left(u''(r)+\mu u'(r)\right)\left(\frac{1+\mu r}{r}u'(r)\right)^{k-1}+C_{n-1}^{k}\left(\frac{1+\mu r}{r}u'(r)\right)^{k}\\ =&\frac{C_n^k}{n}\left(\frac{1+\mu r}{r}\right)^{k-1}\left[ku''(r)(u'(r))^{k-1}+\left(n\mu+\frac{n-k}{r}\right)(u'(r))^k\right].\end{aligned}$$

Then, the first equation in (1.1) can be written as

$$\frac{C_n^k}{n}\left(\frac{1+\mu r}{r}\right)^{k-1}\left[ku''(r)(u'(r))^{k-1}+\left(n\mu+\frac{n-k}{r}\right)(u'(r))^k\right]=p(r)\varphi(u(r),v(r)),$$

which means

$$ku''(r)(u'(r))^{k-1}+\left(n\mu+\frac{n-k}{r}\right)(u'(r))^k=\frac{n}{C_n^k}\left(\frac{r}{1+\mu r}\right)^{k-1}p(r)\varphi(u(r),v(r)). \tag{2.1}$$

Let $\sigma(r) = n\mu r + (n-k)\ln r$. Multiplying both sides of (2.1) by $e^{\sigma(r)}$, we get

$$ku''(r)(u'(r))^{k-1}e^{\sigma(r)} + \left(n\mu + \frac{n-k}{r}\right)(u'(r))^k e^{\sigma(r)}$$
$$= \frac{n}{C_n^k}\left(\frac{r}{1+\mu r}\right)^{k-1} p(r)\varphi(u(r),v(r))e^{\sigma(r)},$$

that is,

$$\left\{\frac{e^{n\mu r}}{r^{k-n}}(u'(r))^k\right\}' = \frac{n}{C_n^k}\frac{e^{n\mu r}r^{n-1}}{(1+\mu r)^{k-1}}p(r)\varphi(u(r),v(r)),$$

$$u'(r) = \left(\frac{r^{k-n}}{e^{n\mu r}}\int_0^r \frac{n}{C_n^k}\frac{e^{n\mu s}s^{n-1}}{(1+\mu s)^{k-1}}p(s)\varphi(u(s),v(s))ds\right)^{\frac{1}{k}}. \tag{2.2}$$

Similarly, from the second equation of (1.1),we can deduce the expression of $v'(r)$

$$v'(r) = \left(\frac{r^{k-n}}{e^{n\nu r}}\int_0^r \frac{n}{C_n^k}\frac{e^{n\nu s}s^{n-1}}{(1+\nu s)^{k-1}}q(s)\psi(u(s),v(s))ds\right)^{\frac{1}{k}}. \tag{2.3}$$

For any given initial value $u(0) = \gamma, v(0) = \delta$, we can transform (1.1) into the integral equations

$$\begin{cases} u(r) = \gamma + \displaystyle\int_0^r \left(\frac{t^{k-n}}{e^{n\mu t}}\int_0^t \frac{n}{C_n^k}\frac{e^{n\mu s}s^{n-1}}{(1+\mu s)^{k-1}}p(s)\varphi(u(s),v(s))ds\right)^{\frac{1}{k}} dt, \\ v(r) = \delta + \displaystyle\int_0^r \left(\frac{t^{k-n}}{e^{n\nu t}}\int_0^t \frac{n}{C_n^k}\frac{e^{n\nu s}s^{n-1}}{(1+\nu s)^{k-1}}q(s)\psi(u(s),v(s))ds\right)^{\frac{1}{k}} dt, \end{cases}$$

by integrating both sides of equation (2.2) and (2.3) from zero to r, respectively. This completes the proof. □

3. Existence of bounded radial solutions

In order to make the form of our main results concise, we first give some notations. Denote

$$A(r) = \int_{\gamma+\delta}^r \frac{dt}{\varphi(t,t)+\psi(t,t)+1}, \quad r \geq \gamma+\delta > 0, \quad A(\infty) := \lim_{r\to\infty} A(r), \tag{3.1}$$

$$P(r) = \int_0^r \left(\frac{t^{k-n}}{e^{n\mu t}}\int_0^t \frac{n}{C_n^k}\frac{e^{n\mu s}s^{n-1}}{(1+\mu s)^{k-1}}p(s)ds\right)^{\frac{1}{k}} dt, \quad r \geq 0, \quad P(\infty) := \lim_{r\to\infty} P(r), \tag{3.2}$$

$$Q(r) = \int_0^r \left(\frac{t^{k-n}}{e^{n\nu t}}\int_0^t \frac{n}{C_n^k}\frac{e^{n\nu s}s^{n-1}}{(1+\nu s)^{k-1}}q(s)ds\right)^{\frac{1}{k}} dt, \quad r \geq 0, \quad Q(\infty) := \lim_{r\to\infty} Q(r). \tag{3.3}$$

Theorem 3.1. *Assume (H1) and (H2) hold, $A(\infty) = \infty$, $P(\infty) < \infty$ and $Q(\infty) < \infty$. For every given initial value, system (1.1) exists positive bounded radial solutions in space $C^2[0,\infty) \times C^2[0,\infty)$.*

Proof. For any $r \geq 0$, we construct sequences $\{u_m(r)\}_{m\geq 0}$ and $\{v_m(r)\}_{m\geq 0}$ in the following format

$$\begin{cases} u_m(r) = \gamma + \int_0^r \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s)\varphi(u_{m-1}(s), v_{m-1}(s)) ds \right)^{\frac{1}{k}} dt, \\ v_m(r) = \delta + \int_0^r \left(\frac{t^{k-n}}{e^{n\nu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\nu s} s^{n-1}}{(1+\nu s)^{k-1}} q(s)\psi(u_{m-1}(s), v_{m-1}(s)) ds \right)^{\frac{1}{k}} dt, \\ u_0(r) \equiv \gamma, \quad v_0(r) \equiv \delta. \end{cases} \tag{3.4}$$

Firstly, we can conclude that $\{u_m(r)\}_{m\geq 0}$ and $\{v_m(r)\}_{m\geq 0}$ are non-decreasing sequences, respectively. To this end, we introduce a semi-ordering $\preceq$ in $C[0,\infty) \times C[0,\infty)$ by

$$(u_m, v_m) \preceq (u_{m+1}, v_{m+1}) \Leftrightarrow u_m(r) \leq u_{m+1}(r), v_m(r) \leq v_{m+1}(r), r \geq 0, \quad m \in \mathbb{N}.$$

From (H1),(H2) and (3.4), it is easy to see that $u_0(r) \leq u_1(r)$, $v_0(r) \leq v_1(r)$ for any $r \geq 0$, that is, $(u_0, v_0) \preceq (u_1, v_1)$. Assume that $(u_{l-1}, v_{l-1}) \preceq (u_l, v_l)$ hold, then $u_{l-1}(r) \leq u_{l-1}(r)$ and $v_{l-1}(r) \leq v_{l-1}(r)$ for any $r \geq 0$. From the monotonicity of φ and the first equation of (3.4), we have

$$\begin{aligned} u_l(r) &= \gamma + \int_0^r \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s)\varphi(u_{l-1}(s), v_{l-1}(s)) ds \right)^{\frac{1}{k}} dt \\ &\leq \gamma + \int_0^r \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s)\varphi(u_l(s), v_l(s)) ds \right)^{\frac{1}{k}} dt \\ &= u_{l+1}(r), \quad r \geq 0. \end{aligned}$$

Similarly, the monotonicity of ψ and the second equation of (3.4) indicate $v_l(r) \leq v_{l+1}(r)$ for any $r \geq 0$. thus, $(u_l, v_l) \preceq (u_{l+1}, v_{l+1})$. By mathematical induction we can obtain

$$(u_m, v_m) \preceq (u_{m+1}, v_{m+1}), \quad m \in \mathbb{N},$$

which means that $\{u_m(r)\}_{m\geq 0}$ and $\{v_m(r)\}_{m\geq 0}$ are non-decreasing sequences, respectively.

Secondly, the sequences $\{u_m(r)\}_{m\geq 0}$ and $\{v_m(r)\}_{m\geq 0}$ are convergent.

i) The sequences $\{u_m(r)\}_{m\geq 0}$ and $\{v_m(r)\}_{m\geq 0}$ are uniformly bounded. In fact,

$$\begin{aligned} u_m'(r) &= \left(\frac{r^{k-n}}{e^{n\mu r}} \int_0^r \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s)\varphi(u_{m-1}(s), v_{m-1}(s)) ds \right)^{\frac{1}{k}} \\ &\leq \left(\frac{r^{k-n}}{e^{n\mu r}} \int_0^r \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s)\varphi(u_m(s), v_m(s)) ds \right)^{\frac{1}{k}} \\ &\leq \left(\frac{r^{k-n}}{e^{n\mu r}} \int_0^r \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s)\Big(\Phi(r,r)+1\Big) ds \right)^{\frac{1}{k}} \\ &\leq \Big(\Phi(r,r)+1\Big) \left(\frac{r^{k-n}}{e^{n\mu r}} \int_0^r \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s) ds \right)^{\frac{1}{k}}, \end{aligned} \tag{3.5}$$

by (H1) and (H2), where $\Phi(r,r) = \varphi(u_m(r)+v_m(r), u_m(r)+v_m(r))$. Similarly,

$$v_m'(r) \leq \Big(\Psi(r,r)+1\Big) \left(\frac{r^{k-n}}{e^{n\nu r}} \int_0^r \frac{n}{C_n^k} \frac{e^{n\nu s} s^{n-1}}{(1+\nu s)^{k-1}} q(s) ds \right)^{\frac{1}{k}}, \tag{3.6}$$

where $\Psi(r,r) = \psi(u_m(r)+v_m(r), u_m(r)+v_m(r))$. Adding (3.5) and (3.6), we have

$$\begin{aligned} & u_m'(r)+v_m'(r) \\ \leq & \Big(\Phi(r,r)+\Psi(r,r)+1\Big) \\ & \times \left\{ \left(\frac{r^{k-n}}{e^{n\mu r}} \int_0^r \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s) ds \right)^{\frac{1}{k}} \right. \\ & \left. + \left(\frac{r^{k-n}}{e^{n\nu r}} \int_0^r \frac{n}{C_n^k} \frac{e^{n\nu s} s^{n-1}}{(1+\nu s)^{k-1}} q(s) ds \right)^{\frac{1}{k}} \right\}, \end{aligned}$$

furthermore,

$$\begin{aligned} \frac{u_m'(r)+v_m'(r)}{\Phi(r,r)+\Psi(r,r)+1} \leq & \left(\frac{r^{k-n}}{e^{n\mu r}} \int_0^r \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s) ds \right)^{\frac{1}{k}} \\ & + \left(\frac{r^{k-n}}{e^{n\nu r}} \int_0^r \frac{n}{C_n^k} \frac{e^{n\nu s} s^{n-1}}{(1+\nu s)^{k-1}} q(s) ds \right)^{\frac{1}{k}}. \end{aligned}$$

Integrating the above inequality from zero to $r (r \geq 0)$, we can get

$$\int_0^r \frac{u_m'(t)+v_m'(t)}{\Phi(t,t)+\Psi(t,t)+1} dt \leq P(r)+Q(r),$$

where P and Q are defined by (3.2) and (3.3),respectively. Furthermore,

$$\int_{\gamma+\delta}^{u_m(r)+v_m(r)} \frac{dt}{\varphi(t,t)+\psi(t,t)+1} \leq P(r)+Q(r),$$

that is,

$$A(u_m(r)+v_m(r)) \leq P(r)+Q(r),$$

where A is defined by (3.1). Due to the continuity and monotonicity of φ and ψ, A is bijective and the inverse mapping A^{-1} is strictly increasing on $[0, A(\infty))$, thus, from the above inequality, we get

$$u_m(r)+v_m(r) \leq A^{-1}(P(r)+Q(r)), \quad r \in [0,\infty).$$

Since $A(\infty) = \infty$ and A is bijective, we can know $A^{-1}(\infty) = \infty$. Therefore the sequences $\{u_m(r)\}_{m\geq 0}$ and $\{v_m(r)\}_{m\geq 0}$ are uniformly bounded by the conditions $P(\infty) < \infty$ and $Q(\infty) < \infty$.

ii) The sequences $\{u_m(r)\}_{m\geq 0}$ and $\{v_m(r)\}_{m\geq 0}$ are equicontinuous.

Since the sequences $\{u_m(r)\}_{m\geq 0}$ and $\{v_m(r)\}_{m\geq 0}$ are uniformly bounded, there exist positive constants M_{11} and M_{12}, such that

$$u_m(r) \leq M_{11}, \quad v_m(r) \leq M_{12}, \quad r \geq 0, \quad m \in \mathbb{N},$$

then, we have, by the monotonicity of φ,

$$\varphi(u_m(r), v_m(r)) \leq \varphi(M_{11}, M_{12}) \triangleq M_1.$$

For any $\varepsilon > 0$, choosing $\delta = \frac{\varepsilon}{\sqrt[k]{M_1 M_2}}$, such that for $t_1, t_2 \in [0, c_0]$ and $|t_1 - t_2| < \delta$, we have

$$\begin{aligned}
&|u_m(t_1) - u_m(t_2)| \\
=&\left| \int_{t_1}^{t_2} \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s)\varphi(u_{m-1}(s), v_{m-1}(s))ds \right)^{\frac{1}{k}} dt \right| \\
\leq& \sqrt[k]{M_1} \left| \int_{t_1}^{t_2} \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s)ds \right)^{\frac{1}{k}} dt \right| \\
\leq& \sqrt[k]{M_1 M_2} |t_2 - t_1| \\
<& \varepsilon,
\end{aligned}$$

where

$$M_2 = \max_{t \in [0, c_0]} \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s)ds \right).$$

Therefore, for any $c_0 > 0$, the sequences $\{u_m(r)\}_{m \geq 0}$ is equicontinuous on $[0, c_0]$. There are similar results for the sequences $\{v_m(r)\}_{m \geq 0}$.

Combining i) and ii), $\{u_m(r)\}_{m \geq 0}$ and $\{v_m(r)\}_{m \geq 0}$ contain convergent subsequences, without loss of generality, still denoted by $\{u_m(r)\}_{m \geq 0}$ and $\{v_m(r)\}_{m \geq 0}$, which convergent uniformly to $u(r)$ and $v(r)$ on $[0, c_0]$ by using Arzela-Ascoli theorem, respectively.

It follows from the arbitrariness of c_0 that $(u(r), v(r))$ is a positive radial solution of the system (1.1). Then, according to the arbitrariness of the initial values $\gamma, \delta \in (0, \infty)$, we can know that the system (1.1) has many positive radial solutions.

Moreover, it follows from $P(\infty) < \infty$, $Q(\infty) < \infty$ that

$$u_m(r) + v_m(r) \leq A^{-1}(P(r) + Q(r)) \leq A^{-1}(P(\infty) + Q(\infty)),$$

which means that the radial solutions of (1.1) are bounded.

Finally, we show that $u, v \in C^2[0, \infty)$. It is easy to see that $u, v \in C^2(0, \infty)$. We only need to prove that $u'(r)$, $v'(r)$, $u''(r)$ and $v''(r)$ are continuous at $r = 0$. In fact, we have, by the definition of derivative

$$\begin{aligned}
u'(0) =& \lim_{r \to 0} \frac{u(r) - u(0)}{r} \\
=& \lim_{r \to 0} \frac{\int_0^r \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s)\varphi(u(s), v(s))ds \right)^{\frac{1}{k}} dt}{r} \\
=& \lim_{r \to 0} \left(\frac{r^{k-n}}{e^{n\mu r}} \int_0^r \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s)\varphi(u(s), v(s))ds \right)^{\frac{1}{k}} \\
=& 0
\end{aligned}$$

and

$$\lim_{r \to 0} u'(r) = \lim_{r \to 0} \left(\frac{r^{k-n}}{e^{n\mu r}} \int_0^r \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s)\varphi(u(s), v(s))ds \right)^{\frac{1}{k}} = 0.$$

Therefore, $\lim_{r\to 0} u'(r) = 0 = u'(0)$, which indicates $u'(r)$ is continuous at $r = 0$. Furthermore,

$$\begin{aligned} u''(r) =& \frac{1}{k}\left(\frac{n}{C_n^k}\right)^{\frac{1}{k}} \Bigg[\left(\frac{k-n}{r} - n\mu\right)\left(\frac{r^{k-n}}{e^{n\mu r}}\int_0^r \frac{e^{n\mu s}s^{n-1}}{(1+\mu s)^{k-1}} p(s)\varphi(u(s),v(s))ds\right)^{\frac{1}{k}} \\ &+ \left(\frac{r}{1+\mu r}\right)^{k-1} p(r)\varphi(u(r),v(r)) \\ &\times \left(\frac{r^{k-n}}{e^{n\mu r}}\int_0^r \frac{e^{n\mu s}s^{n-1}}{(1+\mu s)^{k-1}} p(s)\varphi(u(s),v(s))ds\right)^{\frac{1}{k}-1} \Bigg]. \end{aligned} \tag{3.7}$$

On the one hand, taking limit of (3.7) as $r \to 0$

$$\begin{aligned} &\lim_{r\to 0} u''(r) \\ =& \lim_{r\to 0}\frac{1}{k}\left(\frac{n}{C_n^k}\right)^{\frac{1}{k}} \Bigg[\left(\frac{k-n}{r} - n\mu\right)\left(\frac{r^{k-n}}{e^{n\mu r}}\int_0^r \frac{e^{n\mu s}s^{n-1}}{(1+\mu s)^{k-1}} p(s)\varphi(u(s),v(s))ds\right)^{\frac{1}{k}} \\ &+ \left(\frac{r}{1+\mu r}\right)^{k-1} p(r)\varphi(u(r),v(r)) \\ &\times \left(\frac{r^{k-n}}{e^{n\mu r}}\int_0^r \frac{e^{n\mu s}s^{n-1}}{(1+\mu s)^{k-1}} p(s)\varphi(u(s),v(s))ds\right)^{\frac{1}{k}-1} \Bigg] \\ =& \lim_{r\to 0}\frac{k-n}{k}\left(\frac{n}{C_n^k}\right)^{\frac{1}{k}} \left(\frac{\int_0^r \frac{e^{n\mu s}s^{n-1}}{(1+\mu s)^{k-1}} p(s)\varphi(u(s),v(s))ds}{r^n}\right)^{\frac{1}{k}} \\ &+ \lim_{r\to 0}\frac{1}{k}\left(\frac{n}{C_n^k}\right)^{\frac{1}{k}} p(r)\varphi(u(r),v(r)) \left(\frac{\int_0^r \frac{e^{n\mu s}s^{n-1}}{(1+\mu s)^{k-1}} p(s)\varphi(u(s),v(s))ds}{r^n}\right)^{\frac{1}{k}-1} \\ =& \frac{k-n}{k}\left(\frac{n}{C_n^k}\right)^{\frac{1}{k}} \left(\frac{p(0)\varphi(u(0),v(0))}{n}\right)^{\frac{1}{k}} \\ &+ \frac{n}{k}\left(\frac{n}{C_n^k}\right)^{\frac{1}{k}} p(0)\varphi(u(0),v(0)) \left(\frac{p(0)\varphi(u(0),v(0))}{n}\right)^{\frac{1}{k}-1} \\ =& \frac{(p(0)\varphi(u(0),v(0)))^{\frac{1}{k}}}{(C_n^k)^{\frac{1}{k}}}. \end{aligned}$$

On the other hand, by the definition of the second derivative, we have

$$\begin{aligned} u''(0) &= \lim_{r\to 0}\frac{u'(r)-u'(0)}{r} \\ &= \lim_{r\to 0}\frac{\left(\frac{r^{k-n}}{e^{n\mu r}}\int_0^r \frac{n}{C_n^k}\frac{e^{n\mu s}s^{n-1}}{(1+\mu s)^{k-1}} p(s)\varphi(u(s),v(s))ds\right)^{\frac{1}{k}}}{r} \\ &= \left(\frac{n}{C_n^k}\right)^{\frac{1}{k}} \left(\lim_{r\to 0}\frac{\int_0^r e^{n\mu s}\frac{s^{n-1}}{(1+\mu s)^{k-1}} p(s)\varphi(u(s),v(s))ds}{e^{n\mu r}r^n}\right)^{\frac{1}{k}} \end{aligned}$$

$$= \left(\frac{n}{C_n^k}\right)^{\frac{1}{k}} \left(\lim_{r\to 0} \frac{p(r)\varphi(u(r),v(r))}{(n+n\mu r)(1+\mu r)^{k-1}}\right)^{\frac{1}{k}}$$
$$= \left(\frac{n}{C_n^k}\right)^{\frac{1}{k}} \frac{(p(0)\varphi(u(0),v(0)))^{\frac{1}{k}}}{n^{\frac{1}{k}}}$$
$$= \frac{(p(0)\varphi(u(0),v(0)))^{\frac{1}{k}}}{(C_n^k)^{\frac{1}{k}}}.$$

Thus $\lim\limits_{r\to 0} u''(r) = u''(0)$, which shows that u'' is continuous at $r = 0$. thus, $u(r) \in C^2[0,\infty)$. Similarly, we can prove that $v(r)$ are in $C^2[0,\infty)$.

Based on the above discussion, we can conclude that the k-Hessian system (1.1) has many positive bounded radial solutions (u,v) in space $C^2[0,\infty) \times C^2[0,\infty)$. □

4. Existence of blow-up radial solutions

In order to obtain the blow-up solutions of (1.1), we need to make further assumptions about functions φ and ψ.

(H3) there exists a constant $\alpha \in (0, \frac{1}{2})$ such that

$$\varphi(c_1u, c_2v) \geq (c_1c_2)^\alpha \varphi(u,v), \quad \psi(c_1u, c_2v) \geq (c_1c_2)^\alpha \psi(u,v), \quad c_1, c_2 \in (0,1].$$

Remark 4.1. We easily obtain the equivalent condition of (H3), that is, there exists a constant $\alpha \in (0, \frac{1}{2})$ such that

$$\varphi(c_1u, c_2v) \leq (c_1c_2)^\alpha \varphi(u,v), \quad \psi(c_1u, c_2v) \leq (c_1c_2)^\alpha \psi(u,v), \quad c_1, c_2 \geq 1.$$

Theorem 4.1. *Assume (H1) ,(H2) and (H3) hold. For every given initial value, system (1.1) exists positive blow-up radial solutions in space* $C^2[0,\infty) \times C^2[0,\infty)$ *if and only if* $P(\infty) = \infty$ *and* $Q(\infty) = \infty$ *hold.*

Proof. Firstly, we prove the sufficiency of the Theorem 4.1. Define the same iterative sequence $\{u_m(r)\}_{m\geq 0}$ and $\{v_m(r)\}_{m\geq 0}$ as in Theorem 3.1. For fixed $R > 0, r \in [0,R]$, we can get

$$u_m(r) \leq \gamma + \int_0^R \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s)\varphi(u_{m-1}(s), v_{m-1}(s))ds\right)^{\frac{1}{k}} dt < \infty, \tag{4.1}$$

and

$$v_m(r) \leq \delta + \int_0^R \left(\frac{t^{k-n}}{e^{n\nu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\nu s} s^{n-1}}{(1+\nu s)^{k-1}} q(s)\psi(u_{m-1}(s), v_{m-1}(s))ds\right)^{\frac{1}{k}} dt < \infty, \tag{4.2}$$

then, the iterative sequence $\{u_m(r)\}_{m\geq 0}$ and $\{v_m(r)\}_{m\geq 0}$ bounded on interval $[0,R]$, according to (H1) and (H3), we know

$$u_m(R) = \gamma + \int_0^R \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s)\varphi(u_{m-1}(s), v_{m-1}(s))ds\right)^{\frac{1}{k}} dt$$
$$\leq \gamma + \int_0^R \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s)\varphi(u_m(R), v_m(R))ds\right)^{\frac{1}{k}} dt$$

$$\leq \gamma + \int_0^R \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} M_R p(s) \varphi(1,1) ds \right)^{\frac{1}{k}} dt$$

$$\leq \gamma + \int_0^R \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} M_R p(s) \Big(\varphi(1,1) + \psi(1,1) \Big) ds \right)^{\frac{1}{k}} dt$$

$$\leq \gamma + \Big[M_R \Big(\varphi(1,1) + \psi(1,1) \Big) \Big]^{\frac{1}{k}} \int_0^R \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s) ds \right)^{\frac{1}{k}} dt,$$

where $M_R = (u_m(R)+1)^\alpha (v_m(R)+1)^\alpha$. Similarly, we can obtain

$$\begin{aligned} v_m(R) \leq & \delta + \Big[M_R \Big(\varphi(1,1) + \psi(1,1) \Big) \Big]^{\frac{1}{k}} \\ & \times \int_0^R \left(\frac{t^{k-n}}{e^{n\nu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\nu s} s^{n-1}}{(1+\nu s)^{k-1}} q(s) ds \right)^{\frac{1}{k}} dt. \end{aligned}$$

Let

$$M_1(R) := \lim_{m\to\infty} u_m(R), \quad M_2(R) := \lim_{m\to\infty} v_m(R),$$

according to (4.1) and (4.2), we can get $M_1(R) < \infty$, $M_2(R) < \infty$. Otherwise, then there is equation $M_1(R) + M_2(R) = \infty$, it's impossible for $M_1(R), M_2(R)$ not to exist here, thus, for any $0 < \alpha < \frac{1}{2}$, we have

$$\begin{aligned} u_m(R) + v_m(R) \leq & \gamma + \delta + \Big[M_R \Big(\varphi(1,1) + \psi(1,1) \Big) \Big]^{\frac{1}{k}} \\ & \times \Bigg[\int_0^R \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s) ds \right)^{\frac{1}{k}} dt \\ & + \int_0^R \left(\frac{t^{k-n}}{e^{n\nu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\nu s} s^{n-1}}{(1+\nu s)^{k-1}} q(s) ds \right)^{\frac{1}{k}} dt \Bigg]. \end{aligned}$$

Furthermore,

$$\begin{aligned} 1 \leq & \frac{\gamma + \delta + \Big[M_R \Big(\varphi(1,1) + \psi(1,1) \Big) \Big]^{\frac{1}{k}}}{u_m(R) + v_m(R)} \\ & \times \Bigg[\int_0^R \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s) ds \right)^{\frac{1}{k}} dt \\ & + \int_0^R \left(\frac{t^{k-n}}{e^{n\nu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\nu s} s^{n-1}}{(1+\nu s)^{k-1}} q(s) ds \right)^{\frac{1}{k}} dt \Bigg] \\ \leq & \frac{\gamma + \delta + \Big[(u_m(R) + v_m(R) + 2)^{2\alpha} \Big(\varphi(1,1) + \psi(1,1) \Big) \Big]^{\frac{1}{k}}}{u_m(R) + v_m(R)} \\ & \times \Bigg[\int_0^R \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s) ds \right)^{\frac{1}{k}} dt \\ & + \int_0^R \left(\frac{t^{k-n}}{e^{n\nu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\nu s} s^{n-1}}{(1+\nu s)^{k-1}} q(s) ds \right)^{\frac{1}{k}} dt \Bigg], \end{aligned} \tag{4.3}$$

the right side of the inequality (4.3) tends to zero as $m \to \infty$. This is a contradiction, which means $M_1(R)$ and $M_2(R)$ are not finite. It is worthy that $M_i : (0, \infty) \to (0, \infty), (i = 1, 2)$ is a increasing mapping due to the increasing of the sequence $\{u_m(r)\}_{m\geq 0}$ and $\{v_m(r)\}_{m\geq 0}$. Moreover, for any $r \in [0, R]$ and $m \geq 0$, we have

$$u_m(r) \leq u_m(R) \leq M_1(R), \quad v_m(r) \leq v_m(R) \leq M_2(R),$$

it is implied that $\{u_m(r)\}_{m\geq 0}$ and $\{v_m(r)\}_{m\geq 0}$ bounded on interval $[0, R]$.

Let

$$u(r) = \lim_{m\to\infty} u_m(r), \quad v(r) = \lim_{m\to\infty} v_m(r), \quad r \geq 0,$$

combining conditions (H1)and(H2) , we can obtain $(u(r), v(r))$ is a positive radial solution.

Next, we prove $(u(r), v(r))$ is a blow-up solution

$$\begin{aligned}
u(r) &= \gamma + \int_0^r \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s)\varphi(u(s), v(s)) ds \right)^{\frac{1}{k}} dt \\
&\geq \gamma + \int_0^r \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s)\varphi(\gamma, \delta) ds \right)^{\frac{1}{k}} dt \\
&\geq \gamma + \varphi(\gamma, \delta)^{\frac{1}{k}} \int_0^r \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s) ds \right)^{\frac{1}{k}} dt,
\end{aligned}$$

it easy to know $u(r) \to \infty$ as $r \to \infty$ by the condition $P(\infty) = \infty$. Similarly, we have

$$v(r) \geq \delta + \psi(\gamma, \delta)^{\frac{1}{k}} \int_0^r \left(\frac{t^{k-n}}{e^{n\nu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\nu s} s^{n-1}}{(1+\nu s)^{k-1}} q(s) ds \right)^{\frac{1}{k}} dt$$

when $r \to \infty$, we get $v(r) \to \infty$. it is implied that $(u(r), v(r))$ is blow-up solution of k-Hessian system (1.1). For the arbitrariness of $\gamma, \delta > 0$, the k-Hessian system (1.1) has many positive blow-up radial solutions.

In the following, we will prove the necessity of Theorem 4.1. Assume that (1.1) has many blow-up radical solutions, we will prove that $P(\infty) = \infty$ and $Q(\infty) = \infty$. On the contrary, if $P(\infty) < \infty$ and $Q(\infty) < \infty$, that is

$$\int_0^\infty \left(\frac{t^{k-n}}{e^{n\mu r}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s) ds \right)^{\frac{1}{k}} dt < \infty$$

and

$$\int_0^\infty \left(\frac{t^{k-n}}{e^{n\nu r}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\nu s} s^{n-1}}{(1+\nu s)^{k-1}} q(s) ds \right)^{\frac{1}{k}} dt < \infty,$$

thus $\lim\limits_{r\to\infty} u(r) = \infty$ and $\lim\limits_{r\to\infty} v(r) = \infty$. We know from the expression of $u'(r)$ and $v'(r)$ that $u(r)$ and $v(r)$ are increasing function, so we can get that

$$\begin{aligned}
u(r) + v(r) \leq& \gamma + \delta + \left[(u(r)+1)^\alpha (v(r)+1)^\alpha \Big(\varphi(1,1) + \psi(1,1) \Big) \right]^{\frac{1}{k}} \\
&\times \left[\int_0^r \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s) ds \right)^{\frac{1}{k}} dt \right. \\
&\left. + \int_0^r \left(\frac{t^{k-n}}{e^{n\nu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\nu s} s^{n-1}}{(1+\nu s)^{k-1}} q(s) ds \right)^{\frac{1}{k}} dt \right].
\end{aligned}$$

Furthermore, we have

$$
\begin{aligned}
1 \leq & \frac{\gamma+\delta+\left[(u(r)+1)^{\alpha}(v(r)+1)^{\alpha}\Big(\varphi(1,1)+\psi(1,1)\Big)\right]^{\frac{1}{k}}}{u(r)+v(r)} \\
& \times \left[\int_0^r \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s) ds\right)^{\frac{1}{k}} dt\right. \\
& \left. + \int_0^r \left(\frac{t^{k-n}}{e^{n\nu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\nu s} s^{n-1}}{(1+\nu s)^{k-1}} q(s) ds\right)^{\frac{1}{k}} dt\right] \\
\leq & \frac{\gamma+\delta+\left[(u(r)+v(r)+2)^{2\alpha}\Big(\varphi(1,1)+\psi(1,1)\Big)\right]^{\frac{1}{k}}}{u(r)+v(r)} \\
& \times \left[\int_0^r \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s) ds\right)^{\frac{1}{k}} dt\right. \\
& \left. + \int_0^r \left(\frac{t^{k-n}}{e^{n\nu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\nu s} s^{n-1}}{(1+\nu s)^{k-1}} q(s) ds\right)^{\frac{1}{k}} dt\right],
\end{aligned}
\tag{4.4}
$$

the right hand of inequality (4.4) tends to zero as $r \to \infty$, which is a contradiction. This completes the proof. $\square$

Remark 4.2. Assume (H1), (H2) and (H3) hold. Then the k-Hessian system (1.1) has no blow-up radial solutions (u, v) in space $C^2[0,\infty) \times C^2[0,\infty)$ if $P(\infty) < \infty$ and $Q(\infty) < \infty$.

The argument is analogous to that in Theorem 4.1, so it is omitted.

5. Examples

Example 5.1. Consider the following 3-Hessian system

$$
\begin{cases}
S_3(\lambda(D^2u+2|\nabla u|I)) = \dfrac{(8|x|^3-5|x|^2+8|x|+1)(1+2|x|^2)}{|x|^3(1+|x|^2)^4}\left(u^2+v^2\right)^{\frac{1}{3}}, x \in \mathbb{R}^4, \\
S_3(\lambda(D^2v+3|\nabla v|I)) = \dfrac{(12|x|^3-5|x|^2+12|x|+1)(1+3|x|)^2}{|x|^3(1+|x|^2)^4}\left(u^{\frac{1}{3}}+v^{\frac{1}{3}}\right), x \in \mathbb{R}^4,
\end{cases}
\tag{5.1}
$$

where $\varphi(u,v) = \left(u^2+v^2\right)^{\frac{1}{3}}$, $\psi(u,v) = u^{\frac{1}{3}} + v^{\frac{1}{3}}$ and

$$
p(r) = \frac{(8r^3-5r^2+8r+1)(1+2r)^2}{r^3(1+r^2)^4}, \quad q(r) = \frac{(12r^3-5r^2+12r+1)(1+3r)^2}{r^3\left(1+r^2\right)^4}.
$$

It is not difficult to check that φ and ψ are increasing for each variables.

Now we check $A(\infty)$, $P(\infty)$ and $Q(\infty)$ satisfying the conditions of Theorem 3.1.

$$
\begin{aligned}
A(\infty) &= \int_{\gamma+\delta}^{\infty} \frac{dt}{\varphi(t,t)+\psi(t,t)+1} \\
&= \int_{\gamma+\delta}^{\infty} \frac{dt}{(t^2+t^2)^{\frac{1}{3}}+t^{\frac{1}{3}}+t^{\frac{1}{3}}+1}
\end{aligned}
$$

$$
\begin{aligned}
&\geq \int_{\gamma+\delta}^{\infty} \frac{1}{4t+1} dt \\
&= \infty, \\
P(\infty) &= \int_0^{\infty} \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s) ds \right)^{\frac{1}{k}} dt \\
&= \int_0^{\infty} \left(\frac{t^{3-4}}{e^{8t}} \int_0^t \frac{4}{C_4^3} \frac{e^{8s} s^3}{(1+2s)^2} \frac{(8s^3 - 5s^2 + 8s + 1)(1+2s)^4}{s^3(1+s^2)^2} ds \right)^{\frac{1}{3}} dt \\
&= \int_0^{\infty} \frac{1}{1+t^2} dt \\
&< \infty, \\
Q(\infty) &= \int_0^{\infty} \left(\frac{t^{k-n}}{e^{n\nu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\nu s} s^{n-1}}{(1+\nu s)^{k-1}} q(s) ds \right)^{\frac{1}{k}} dt \\
&= \int_0^{\infty} \left(\frac{t^{3-4}}{e^{12t}} \int_0^t \frac{4}{C_4^3} \frac{e^{12s} s^3}{(1+3s)^2} \frac{(12s^3 - 5s^2 + 12s + 1)(1+3s)^2}{(1+s^2)^4 s^3} ds \right)^{\frac{1}{3}} dt \\
&= \int_0^{\infty} \frac{1}{1+t^2} dt \\
&< \infty.
\end{aligned}
$$

Therefore, all the conditions of Theorem 3.1 are satisfied, the conclusion follows, that is, the system (5.1) has many bounded radial solutions.

Example 5.2. Consider the following 3-Hessian system

$$
\begin{cases}
S_3(\lambda(D^2 u + 2|\nabla u| I)) = (1 + 2|x|^2) \left(u^{\frac{1}{3}} + v^{\frac{1}{3}} \right), & x \text{ in } \mathbb{R}^4, \\
S_3(\lambda(D^2 v + 3|\nabla v| I)) = (1 + 3|x|)^2 \left(u^{\frac{2}{5}} + v^{\frac{2}{5}} \right), & x \text{ in } \mathbb{R}^4,
\end{cases}
\tag{5.2}
$$

where $\varphi(u,v) = u^{\frac{1}{3}} + v^{\frac{1}{3}}$, $\psi(u,v) = u^{\frac{2}{5}} + v^{\frac{2}{5}}$ and

$$
p(r) = (1+2r)^2, \quad q(r) = (1+3r)^2.
$$

When $\alpha = \frac{1}{4}$, it is not difficult to check that (H3) hold and φ, ψ are increasing in each variables which satisfied (H1). Now we check $P(\infty)$ and $Q(\infty)$satisfying the conditions of Theorem 4.1.

$$
\begin{aligned}
P(\infty) &= \int_0^{\infty} \left(\frac{t^{k-n}}{e^{n\mu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\mu s} s^{n-1}}{(1+\mu s)^{k-1}} p(s) ds \right)^{\frac{1}{k}} dt \\
&= \int_0^{\infty} \left(\frac{t^{3-4}}{e^{8t}} \int_0^t \frac{4}{C_4^3} e^{8s} s^3 ds \right)^{\frac{1}{3}} dt \\
&= \infty, \\
Q(\infty) &= \int_0^{\infty} \left(\frac{t^{k-n}}{e^{n\nu t}} \int_0^t \frac{n}{C_n^k} \frac{e^{n\nu s} s^{n-1}}{(1+\nu s)^{k-1}} q(s) ds \right)^{\frac{1}{k}} dt \\
&= \int_0^{\infty} \left(\frac{t^{3-4}}{e^{12t}} \int_0^t \frac{4}{C_4^3} e^{12s} s^3 ds \right)^{\frac{1}{3}} dt \\
&= \infty.
\end{aligned}
$$

Therefore, all the conditions of Theorem 4.1 are satisfied, therefore, the system (5.2) has many positive blow-up radial solutions.

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