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# A FINITE ELEMENT ITERATIVE ALGORITHM OF THE STEADY-STATE FLUID-FLUID INTERACTION PROBLEM*

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**Abstract** In this work, we propose a finite element iterative algorithm to solve the stationary fluid-fluid interaction model. First, we give the finite element discretization for the considered equations. Due that the finite element discretization system is nonlinear, then we design an iterative algorithm for solving the nonlinear equations, where error correction strategy is used to control iterative error at each iteration. Finally, some numerical tests are carried out to demonstrate theoretical results of the proposed algorithm.

**Keywords** Finite element method, error correction strategy, iterative algorithm.

**MSC(2010)** 65N30.

## 1. Introduction

Multi-domain and multi-physics coupling of two immiscible fluids often appears in many fields of production and life. Actually, fluid-fluid interaction model can describe this problem well and a great deal of effort has been devoted to the development of numerical methods for approximate solution of this model.

On one hand, for numerical work of the time-dependent fluid-fluid interaction model, Lions et al. [26] have given numerical analysis of the numerical method. Based on the operator-splitting and domain decomposition methods, Bresch and Koko [7] have shown a numerical simulation. In order to decouple multi-domain and multi-physics of the model, Connors et al. [10] have constructed two decoupled time stepping schemes: the first one is geometric averaging scheme, which is unconditionally stable; and the other one is implicit/exciplict scheme, which is conditionally stable [33]. Due to the unconditional stability of the geometric averaging scheme, it has undergone some evolution and been well further developed [1–4, 9, 17–22, 24, 27, 28]. Another unconditional stability scheme has been considered in [23].

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On other hand, for numerical work of the stationary case, a spectral discretization has been proposed in [5] where optimal error estimates were shown. A further work concerning standard Galerkin finite element approximation has been studied [6]. Moreover, two Uzawa-type domain decomposition algorithms have been established by Koko [16]. Besides, Li and Xu [25] have proposed a Schwarz domain decomposition algorithm iteratively, which allowed solving only single-physical model at each iteration. Zhang et al. [36] have considered a two-grid decoupled algorithm, which can save much computational time. By utilize Nitsche's interface conditions, Hussain et al. [15] have designed a stabilized finite element method to overcome numerical instability. In addition, Rebollo et al. [29] have presented and analyzed a finite element iterative scheme, which was mainly linear and a monotone nonlinearity being just kept at the interface between the fluids.

In this paper, we will construct and analyze a finite element iterative algorithm for the following governing equations of a steady-state fluid-fluid interaction model [34,36]. For $i, j = 1, 2$, and $i \neq j$, we consider

$$\begin{aligned}
&-\nu_i \Delta \mathbf{u}_i + \mathbf{u}_i \cdot \nabla \mathbf{u}_i + \nabla p_i = \mathbf{f}_i && \text{in } \Omega_i,\\
&-\nu_i \mathbf{n}_i \cdot \nabla \mathbf{u}_i \cdot \boldsymbol{\tau} = \kappa(\mathbf{u}_i - \mathbf{u}_j)\cdot \boldsymbol{\tau} && \text{on } I,\\
&\mathbf{u}_i \cdot \mathbf{n}_i = 0 && \text{on } I,\\
&\nabla \cdot \mathbf{u}_i = 0 && \text{in } \Omega_i,\\
&\mathbf{u}_i(0,x) = \mathbf{u}_{i,0}(x) && \text{in } \Omega_i,\\
&\mathbf{u}_i = 0 && \text{on } \Gamma_i := \partial\Omega_i \backslash I,
\end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain that consists of two sub-domains $\Omega_1$ and $\Omega_2$ coupled across their shared interface $I$. Additionally, in these equations, $\nu_i > 0$ is the kinematic viscosity, $\kappa > 0$ is the friction coefficient, $\mathbf{f}_i$ is the body force and $\mathbf{u}_{i,0}$ is the initial value. $\mathbf{u}_i$ represents the fluid velocity and $p_i$ is the pressure. The vector $\mathbf{n}_i$ is the unit outward normals on $I$, and $\boldsymbol{\tau}$ is any vector such that $\boldsymbol{\tau} \cdot \mathbf{n}_i = 0$.

The rest of the article is organized as follows: in Section 2, we introduce some basic notations and provide the corresponding variational form for the problem (1.1). In Section 3, the finite element method of the considered equations is shown and error estimates are deduced. Next, based on the previous finite element approximation, an iterative algorithm is proposed. Consequently, some numerical experiments are implemented to demonstrate the theoretical result of the presented iterative algorithm in the last section.

## 2. Preliminaries

For $i = 1, 2$, we use standard notations for the Lebesgue space $L^2(\Omega_i)$ and Sobolev space $H^m(\Omega_i) = W^{m,2}(\Omega_i)$, $1 \le m < \infty$. The $L^2(\Omega_i)$ norm is denoted by $\|\cdot\|_{L^2(\Omega_i)} = \|\cdot\|_0$ and $L^2$-scalar product $(\cdot,\cdot)$.

For the mathematical setting of the coupled fluid-fluid interaction model (1.1), we introduce the following function spaces:

$$\begin{aligned}
&\mathbf{X}_i = \{\mathbf{v}_i \in H^1(\Omega_i)^2 : \mathbf{v}_i|_{\Gamma_i} = 0; \quad \mathbf{v}_i \cdot \mathbf{n}_i = 0, \text{ on } I\},\\
&M_i = \{q_i \in L^2(\Omega_i) : (q_i, 1) = 0\}.
\end{aligned}$$

For $\mathbf{f}_i$ an element in the dual space of $\mathbf{X}_i$, denoted by $\mathbf{X}_i'$, its norm is defined by $\|\mathbf{f}_i\|_{-1} = \sup\limits_{\mathbf{v}_i\in\mathbf{X}_i} \frac{|(\mathbf{f}_i,\mathbf{v}_i)|}{\|\nabla\mathbf{v}_i\|_0}$. Besides, we denote norm $\||\mathbf{v}|\|_0^2 = \|\mathbf{v}_1\|_0^2 + \|\mathbf{v}_2\|_0^2$ for all $\mathbf{v}_i \in \mathbf{X}_i$.

Next, we define the bilinear forms

$$a(\mathbf{u}_i,\mathbf{v}_i) = \nu_i(\nabla\mathbf{u}_i,\nabla\mathbf{v}_i), \quad d(\mathbf{v}_i,q_i) = (\nabla\cdot\mathbf{v}_i,q_i), \quad \mathbf{u}_i,\mathbf{v}_i\in\mathbf{X}_i,\ q_i\in M_i$$

and the trilinear forms

$$\begin{aligned} b(\mathbf{u}_i,\mathbf{v}_i,\mathbf{w}_i) &= ((\mathbf{u}_i\cdot\nabla)\mathbf{v}_i,\mathbf{w}_i) + 0.5((\nabla\cdot\mathbf{u}_i)\mathbf{v}_i,\mathbf{w}_i) \\ &= 0.5((\mathbf{u}_i\cdot\nabla)\mathbf{v}_i,\mathbf{w}_i) - 0.5((\mathbf{u}_i\cdot\nabla)\mathbf{w}_i,\mathbf{v}_i), \quad \forall\mathbf{u}_i,\mathbf{v}_i,\mathbf{w}_i\in\mathbf{X}_i, \end{aligned}$$

with the following properties [30]:

$$\begin{aligned} &b(\mathbf{u}_i,\mathbf{v}_i,\mathbf{w}_i) = -b(\mathbf{u}_i,\mathbf{w}_i,\mathbf{v}_i), \\ &|b(\mathbf{u}_i,\mathbf{v}_i,\mathbf{w}_i)| \le N\|\nabla\mathbf{u}_i\|_0\|\nabla\mathbf{v}_i\|_0\|\nabla\mathbf{w}_i\|_0, \end{aligned} \tag{2.1}$$

where $N = \sup_{\mathbf{u}_i,\mathbf{v}_i,\mathbf{w}_i\in\mathbf{X}_i} \frac{|b(\mathbf{u}_i,\mathbf{v}_i,\mathbf{w}_i)|}{\|\nabla\mathbf{u}_i\|_0\|\nabla\mathbf{v}_i\|_0\|\nabla\mathbf{w}_i\|_0}$.

Now, we recall the trace inequality and the Poincaré inequality [8,30], which are useful in the following analysis. There exist some positive constants $C_p$ and $C_{tr}$, which depend on $\Omega_i$, such that

$$\|\mathbf{v}_i\|_0 \le C_p\|\nabla\mathbf{v}_i\|_0, \quad \|\mathbf{v}_i\|_{L^2(I)} \le C_{tr}\|\mathbf{v}_i\|_0^{\frac{1}{2}}\|\nabla\mathbf{v}_i\|_0^{\frac{1}{2}}. \tag{2.2}$$

Based on the above definitions, the corresponding variational formulation of the problem (1.1) is given as follows: find $(\mathbf{u}_i,p_i)\in\mathbf{X}_i\times M_i$ such that for $i,j=1,2$, $i\ne j$, all $(\mathbf{v}_i,q_i)\in\mathbf{X}_i\times M_i$,

$$a(\mathbf{u}_i,\mathbf{v}_i) - d(\mathbf{v}_i,p_i) + d(\mathbf{u}_i,q_i) + b(\mathbf{u}_i,\boldsymbol{u}_i,\mathbf{v}_i) + \int_I \kappa(\mathbf{u}_i-\mathbf{u}_j)\cdot\mathbf{v}_i\mathrm{ds} = (\mathbf{f}_i,\mathbf{v}_i). \tag{2.3}$$

Further, we show the following result concerning the variational formulation (2.3).

**Theorem 2.1.** *Assume that* $(\mathbf{u}_i,p_i)$ *is the solution of the variational formulation* (2.3). *If* $\nu_i$ *and* $\mathbf{f}_i$ *satisfy the uniqueness condition:*

$$0 < \sigma = \frac{N}{\nu^2}\sum_{i=1}^{2}\|\mathbf{f}_i\|_{-1} < 1, \tag{2.4}$$

*where* $\nu = \min\{\nu_i,\nu_j\}$, *then the solution pair* $(\mathbf{u}_i,p_i)$ *of* (2.3) *is unique. Furthermore, we have*

$$\nu\|\nabla\mathbf{u}_i\|_0 \le \sum_{i=1}^{2}\|\mathbf{f}_i\|_{-1}. \tag{2.5}$$

**Proof.** Set $(\mathbf{v}_i,q_i) = (\mathbf{u}_i,p_i)$ in (2.3) and use (2.1) to obtain

$$a(\mathbf{u}_i,\mathbf{u}_i) + \int_I \kappa(\mathbf{u}_i-\mathbf{u}_j)\cdot\mathbf{u}_i\mathrm{ds} = (\mathbf{f}_i,\mathbf{u}_i). \tag{2.6}$$

By summing (2.6) from $i = 1$ to 2, we have

$$\nu|||\nabla \mathbf{u}|||_0^2 + \int_I \kappa|\mathbf{u}_1 - \mathbf{u}_2|^2 \mathrm{ds} \le \sum_{i=1}^{2} \|\mathbf{f}_i\|_{-1}|||\nabla \mathbf{u}|||_0,$$

which applies (2.5).

Next, suppose that $\mathbf{u}_{i,1}, \mathbf{u}_{j,1}, p_{i,1}, p_{j,1}$ and $\mathbf{u}_{i,2}, \mathbf{u}_{j,2}, p_{i,2}, p_{j,2}$ are two solution pairs of (2.3), it follows that

$$\begin{aligned} &a(\mathbf{u}_{i,1}, \mathbf{v}_i) - d(\mathbf{v}_i, p_{i,1}) + d(\mathbf{u}_{i,1}, q_i) + b(\mathbf{u}_{i,1}, \mathbf{u}_{i,1}, \mathbf{v}_i) \\ &+ \int_I \kappa(\mathbf{u}_{i,1} - \mathbf{u}_{j,1}) \cdot \mathbf{v}_i \mathrm{ds} = (\mathbf{f}_i, \mathbf{v}_i), \end{aligned} \tag{2.7}$$

$$\begin{aligned} &a(\mathbf{u}_{i,2}, \mathbf{v}_i) - d(\mathbf{v}_i, p_{i,2}) + d(\mathbf{u}_{i,2}, q_i) + b(\mathbf{u}_{i,2}, \mathbf{u}_{i,2}, \mathbf{v}_i) \\ &+ \int_I \kappa(\mathbf{u}_{i,2} - \mathbf{u}_{j,2}) \cdot \mathbf{v}_i \mathrm{ds} = (\mathbf{f}_i, \mathbf{v}_i). \end{aligned} \tag{2.8}$$

Next, let $\mathbf{e}_i = \mathbf{u}_{i,1} - \mathbf{u}_{i,2}$, $\mathbf{e}_j = \mathbf{u}_{j,1} - \mathbf{u}_{j,2}$, $e_i = p_{i,1} - p_{i,2}$ and $e_j = p_{j,1} - p_{j,2}$. Then, subtracting (2.8) from (2.7), we have

$$\begin{aligned} &a(\mathbf{e}_i, \mathbf{v}_i) - d(\mathbf{v}_i, e_i) + d(\mathbf{e}_i, q_i) + b(\mathbf{e}_i, \mathbf{u}_{i,1}, \mathbf{v}_i) \\ &+ b(\mathbf{u}_{i,2}, \mathbf{e}_i, \mathbf{v}_i) + \int_I \kappa(\mathbf{e}_i - \mathbf{e}_j) \cdot \mathbf{v}_i \mathrm{ds} = 0. \end{aligned} \tag{2.9}$$

Moreover, taking $(\mathbf{v}_i, q_i) = (\mathbf{e}_i, e_i)$ in (2.9) and using (2.1), we give

$$\nu_i\|\nabla \mathbf{e}_i\|_0^2 + b(\mathbf{e}_i, \mathbf{u}_{i,1}, \mathbf{e}_i) + \int_I \kappa(\mathbf{e}_i - \mathbf{e}_j) \cdot \mathbf{e}_i \mathrm{d}s = 0. \tag{2.10}$$

Sum (2.10) from $i = 1$ to 2, and apply (2.1) again and (2.5).

$$\sum_{i=1}^{2} \nu_i\|\nabla \mathbf{e}_i\|_0^2 + \int_I \kappa|\mathbf{e}_1 - \mathbf{e}_2|^2 \mathrm{ds} \le \sum_{i=1}^{2} \nu^{-1} N\|\nabla \mathbf{e}_i\|_0^2(\|\mathbf{f}_i\|_{-1} + \|\mathbf{f}_j\|_{-1}).$$

Finally, rewrite the above inequality to get

$$\sum_{i=1}^{2} \left(\nu - N\nu^{-1}(\|\mathbf{f}_i\|_{-1} + \|\mathbf{f}_j\|_{-1})\right) \|\nabla \mathbf{e}_i\|_0^2 + \int_I \kappa|\mathbf{e}_1 - \mathbf{e}_2|^2 \mathrm{ds} \le 0,$$

which together with the uniqueness condition (2.4) finishes the proof. □

# 3. A finite element discretization

In this section, we will give a finite element method of the considered equations, and analyse error estimates.

Let $h > 0$ be a real positive parameter, and $K_h = \{K : \cup_{K\subset\Omega}\overline{K} = \overline{\Omega}\}$ a quasi-uniform partition of $\Omega$. Next, we consider the finite element subspace pair $\mathbf{X}_{i,h} \times M_{i,h} \subset \mathbf{X}_i \times M_i$, for $i = 1, 2$

$$\begin{aligned} \mathbf{X}_{i,h} &= \{\mathbf{v}_{i,h} \in \mathbf{X}_i \cap C^0(\overline{\Omega})^2 : \mathbf{v}_{i,h|K} \in P_2(K)^2, \forall K \in K_h\}, \\ M_{i,h} &= \{q_{i,h} \in M_i \cap C^0(\overline{\Omega}) : q_{i,h|K} \in P_1(K), \forall K \in K_h\}, \end{aligned}$$

where $P_l(K)$, $l = 1, 2$, is the set of all polynomials on $K$ of degree no more than $l$. Note that the considered finite element space pair $\mathbf{X}_{i,h} \times M_{i,h}$ satisfies the discrete inf-sup condition [8, 30]

$$\sup_{\mathbf{v}_{i,h}\in\mathbf{X}_{i,h},\mathbf{v}_{i,h}\neq 0} \frac{|d(\mathbf{v}_{i,h}, q_{i,h})|}{\|\nabla \mathbf{v}_{i,h}\|_0} \geq \beta_i \|q_{i,h}\|_0, \quad \forall q_{i,h} \in M_{i,h}, \tag{3.1}$$

where $\beta_i > 0$, $i = 1, 2$ is constant depending on $\Omega_i$.

Moreover, the finite element method of the fluid-fluid interaction model (1.1) is to find $(\mathbf{u}_{i,h}, p_{i,h}) \in \mathbf{X}_{i,h} \times M_{i,h}$ for all $(\mathbf{v}_{i,h}, q_{i,h}) \in \mathbf{X}_{i,h} \times M_{i,h}$, $i, j = 1, 2$ and $i \neq j$ such that

$$\begin{aligned} &a(\mathbf{u}_{i,h}, \mathbf{v}_{i,h}) - d(\mathbf{v}_{i,h}, p_{i,h}) + d(\mathbf{u}_{i,h}, q_{i,h}) + b(\mathbf{u}_{i,h}, \mathbf{u}_{i,h}, \mathbf{v}_{i,h}) \\ &+ \int_I \kappa(\mathbf{u}_{i,h} - \mathbf{u}_{j,h}) \cdot \mathbf{v}_{i,h} \mathrm{ds} = (\mathbf{f}_i, \mathbf{v}_{i,h}). \end{aligned} \tag{3.2}$$

In order to obtain the error estimates of the presented finite element method, we recall the Stokes-Stokes projection [11, 35]

$$(\mathbf{R}_{i,h}, T_{i,h}) = (\mathbf{R}_{i,h}(\mathbf{u}_i, p_i), T_{i,h}(\mathbf{u}_i, p_i)) : \mathbf{X}_i \times M_i \to \mathbf{X}_{i,h} \times M_{i,h}$$

by requiring

$$\begin{aligned} &a(\mathbf{u}_i - \mathbf{R}_{i,h}(\mathbf{u}_i, p_i), \mathbf{v}_{i,h}) - d(\mathbf{v}_{i,h}, p_i - T_{i,h}(\mathbf{u}_i, p_i)) \\ &+ d(\mathbf{u}_i - \mathbf{R}_{i,h}(\mathbf{u}_i, p_i), q_{i,h}) = 0, \end{aligned} \tag{3.3}$$

with the following approximate property, for all $(\mathbf{u}_i, p_i) \in (H^3(\Omega_i)^d \cap \mathbf{X}_i) \times (H^2(\Omega_i)^d \cap M_i)$

$$\|\nabla(\mathbf{u}_i - \mathbf{R}_{i,h}(\mathbf{u}_i, p_i))\|_0 + \|p_i - T_{i,h}(\mathbf{u}_i, p_i)\|_0 \leq c_s h^2, \tag{3.4}$$

where $c_s > 0$ is a constant independent on $h$.

Next, we denote errors of the velocities and pressures by $\mathbf{u}_i - \mathbf{u}_{i,h} = \mathbf{e}_i$, and $p_i - p_{i,h} = e_i$, then, we decompose them as

$$\begin{aligned} \mathbf{e}_i &= (\mathbf{u}_i - \mathbf{R}_{i,h}(\mathbf{u}_i, p_i)) + (\mathbf{R}_{i,h}(\mathbf{u}_i, p_i) - \mathbf{u}_{i,h}) =: \boldsymbol{\eta}_i + \boldsymbol{\phi}_i^h, \\ e_i &= (p_i - T_{i,h}(\mathbf{u}_i, p_i)) + (T_{i,h}(\mathbf{u}_i, p_i) - p_{i,h}) =: \xi_i + \psi_i^h. \end{aligned}$$

Now, for the finite element approximation (3.2), we present some results as follows.

**Theorem 3.1.** *Under the assumption of Theorem 2.1, the finite element solution* $(\mathbf{u}_{i,h}, p_{i,h})$ *of* (3.2) *satisfies*

$$\nu \|\nabla \mathbf{u}_{i,h}\|_0 \leq \sum_{i=1}^{2} \|\mathbf{f}_i\|_{-1}, \tag{3.5}$$

*and*

$$\|\nabla(\mathbf{u}_i - \mathbf{u}_{i,h})\|_0 + \|p_i - p_{i,h}\|_0 \leq Ch^2, \tag{3.6}$$

*where the positive constant* $C$ *is independent on* $h$.

**Proof.** By setting $(\mathbf{v}_{i,h}, q_{i,h}) = (\mathbf{u}_{i,h}, p_{i,h})$ in (3.2) and applying (2.1), it follows that

$$\nu_i \|\nabla \mathbf{u}_{i,h}\|_0^2 + \int_I \kappa(\mathbf{u}_{i,h} - \mathbf{u}_{j,h}) \cdot \mathbf{u}_{i,h} \mathrm{ds} = (\mathbf{f}_i, \mathbf{u}_{i,h}). \tag{3.7}$$

Then, summing (3.7) from $i = 1$ to 2, we have

$$\nu \||\nabla \mathbf{u}_h|\|_0^2 + \int_I \kappa |\mathbf{u}_{1,h} - \mathbf{u}_{2,h}|^2 \mathrm{ds} \le \sum_{i=1}^{2} \|\mathbf{f}_i\|_{-1} \||\nabla \mathbf{u}_h|\|_0,$$

which applies (3.5).

In the following part, we will establish error estimates for (3.2). Subtracting (3.2) from (2.3), we obtain the following error equation

$$\begin{aligned} &a(\mathbf{e}_i, \mathbf{v}_{i,h}) + b(\mathbf{e}_i, \mathbf{u}_i, \mathbf{v}_{i,h}) + b(\mathbf{u}_{i,h}, \mathbf{e}_i, \mathbf{v}_{i,h}) - d(\mathbf{v}_{i,h}, e_i) + d(\mathbf{e}_i, q_{i,h}) \\ &+ \int_I \kappa(\mathbf{u}_i - \mathbf{u}_j) \cdot \mathbf{v}_{i,h} \mathrm{ds} - \int_I \kappa(\mathbf{u}_{i,h} - \mathbf{u}_{j,h}) \cdot \mathbf{v}_{i,h} \mathrm{ds} = 0, \end{aligned} \tag{3.8}$$

which combines (3.3) to gain

$$\begin{aligned} &a(\boldsymbol{\phi}_i^h, \mathbf{v}_{i,h}) + b(\mathbf{e}_i, \mathbf{u}_i, \mathbf{v}_{i,h}) + b(\mathbf{u}_{i,h}, \mathbf{e}_i, \mathbf{v}_{i,h}) - d(\mathbf{v}_{i,h}, \psi_i^h) + d(\boldsymbol{\phi}_i^h, q_{i,h}) \\ &+ \int_I \kappa(\mathbf{u}_i - \mathbf{u}_j) \cdot \mathbf{v}_{i,h} \mathrm{ds} - \int_I \kappa(\mathbf{u}_{i,h} - \mathbf{u}_{j,h}) \cdot \mathbf{v}_{i,h} \mathrm{ds} = 0. \end{aligned} \tag{3.9}$$

Set $(\mathbf{v}_{i,h}, q_{i,h}) = (\boldsymbol{\phi}_i^h, \psi_i^h)$ in (3.9) and sum the ensuing equation up from 1 to 2.

$$\begin{aligned} &\sum_{i=1}^{2} \nu_i \|\nabla \boldsymbol{\phi}_i^h\|_0^2 + \underbrace{\sum_{i=1}^{2} (b(\mathbf{e}_i, \mathbf{u}_i, \boldsymbol{\phi}_i^h) + b(\mathbf{u}_{i,h}, \mathbf{e}_i, \boldsymbol{\phi}_i^h))}_{I_1} + \underbrace{\int_I \kappa(\mathbf{u}_1 - \mathbf{u}_2) \cdot (\boldsymbol{\phi}_1^h - \boldsymbol{\phi}_2^h) \mathrm{ds}}_{I_2} \\ &- \underbrace{\int_I \kappa(\mathbf{u}_{1,h} - \mathbf{u}_{2,h}) \cdot (\boldsymbol{\phi}_1^h - \boldsymbol{\phi}_2^h) \mathrm{ds}}_{I_3} = 0. \end{aligned} \tag{3.10}$$

For the trilinear terms in (3.10), by using (2.1), (2.5) and (3.5), we have following estimate

$$\begin{aligned} I_1 &\le \sum_{i=1}^{2} \big( N(\|\nabla \boldsymbol{\eta}_i\|_0 + \|\nabla \boldsymbol{\phi}_i^h\|_0) \|\nabla \mathbf{u}_i\|_0 \|\nabla \boldsymbol{\phi}_i^h\|_0 + N \|\nabla \boldsymbol{\eta}_i\|_0 \|\nabla \mathbf{u}_{i,h}\|_0 \|\nabla \boldsymbol{\phi}_i^h\|_0 \big) \\ &\le 2c_s N \nu^{-1} h^2 \sum_{i=1}^{2} \|\mathbf{f}_i\|_{-1} \sum_{i=1}^{2} \|\nabla \boldsymbol{\phi}_i^h\|_0 + \nu^{-1} N \sum_{i=1}^{2} \|\mathbf{f}_i\|_{-1} \sum_{i=1}^{2} \|\nabla \boldsymbol{\phi}_i^h\|_0^2, \end{aligned}$$

where we have used the approximate property (3.4). Besides, for the interaction terms in (3.10), we rewrite them as

$$I_2 - I_3 = \int_I \kappa(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2) \cdot (\boldsymbol{\phi}_1^h - \boldsymbol{\phi}_2^h) \mathrm{ds} + \int_I \kappa |\boldsymbol{\phi}_1^h - \boldsymbol{\phi}_2^h|^2 \mathrm{ds},$$

and utilizing (2.2) and (3.4) again, we arrive at

$$\int_I \kappa(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2)\cdot(\boldsymbol{\phi}_1^h - \boldsymbol{\phi}_2^h)\mathrm{ds} \le 2c_s h^2 \kappa C_{tr}^2 C_p \sum_{i=1}^{2} \|\nabla \boldsymbol{\phi}_i^h\|_0.$$

Moreover, from above estimates, we get the following bound from (3.10),

$$\nu(1-\sigma)\sum_{i=1}^{2}\|\nabla\boldsymbol{\phi}_i^h\|_0 \le Ch^2. \tag{3.11}$$

Further, select $(\mathbf{v}_{i,h}, q_{i,h}) = (\boldsymbol{\phi}_i^h, 0)$ in (3.9) to get

$$\begin{aligned} d(\boldsymbol{\phi}_i^h, \psi_i^h) =&\nu_i\|\nabla\boldsymbol{\phi}_i^h\|_0^2 + b(\boldsymbol{\eta}_i + \boldsymbol{\phi}_i^h, \mathbf{u}_i, \boldsymbol{\phi}_i^h) + b(\mathbf{u}_{i,h}, \boldsymbol{\eta}_i, \boldsymbol{\phi}_i^h) \\ &+ \int_I \kappa(\boldsymbol{\eta}_i + \boldsymbol{\phi}_i^h - \boldsymbol{\eta}_j - \boldsymbol{\phi}_j^h)\cdot\boldsymbol{\phi}_i^h \mathrm{ds}, \end{aligned}$$

which together with discrete inf-sup condition (3.1) and (2.2), (2.5) and (3.5) leads to

$$\begin{aligned} \beta_i\|\psi_i^h\|_0 \le& \left(\nu_i + \nu^{-1}N\sum_{i=1}^{2}\|\mathbf{f}_i\|_{-1} + \kappa C_{tr}^2 C_p\right)\|\nabla\boldsymbol{\phi}_i^h\|_0 + \kappa C_{tr}^2 C_p\|\nabla\boldsymbol{\phi}_j^h\|_0 \\ &+ \left(2\nu^{-1}N\sum_{i=1}^{2}\|\mathbf{f}_i\|_{-1} + \kappa C_{tr}^2 C_p\right)\|\nabla\boldsymbol{\eta}_i\|_0 + \kappa C_{tr}^2 C_p\|\nabla\boldsymbol{\eta}_j\|_0 \\ \le& Ch^2, \end{aligned} \tag{3.12}$$

where we have applied (3.11) and (3.4).

Finally, combine (3.12) with (3.11), (3.4) and the triangle inequality to finish the proof. □

Note that the finite element discretization (3.2) is nonlinear. Hence, in the rest of this section, as the work in [12–14] for nonlinear problems, we design an iterative algorithm for solving the considered equations. We also apply the error correction strategy [31, 32] to control the error at each iterative step for solving the nonlinear problem. Moreover, we deduce the bound of iterative solution.

Now, we give the iterative algorithm for solving the fluid-fluid interaction model.

**Algorithm 3.1.** *For $i, j = 1, 2$ and $i \neq j$, we run the following steps.*

**Step I**. *Let the initial guess $(\mathbf{u}_{i,h}^0, p_{i,h}^0) \in \mathbf{X}_{i,h} \times M_{i,h}$ be the solution by solving the following equations*

$$a(\mathbf{u}_{i,h}^0, \mathbf{v}_{i,h}) - d(\mathbf{v}_{i,h}, p_{i,h}^0) + d(\mathbf{u}_{i,h}^0, q_{i,h}) = (\mathbf{f}_i, \mathbf{v}_{i,h}), \quad \forall(\mathbf{v}_{i,h}, q_{i,h}) \in \mathbf{X}_{i,h}\times M_{i,h}.$$

*Set $\tilde{\mathbf{u}}_{i,h}^0 \equiv \mathbf{u}_{i,h}^0$ and $\tilde{p}_{i,h}^0 \equiv p_{i,h}^0$.*

**Step II**. *Find $(\tilde{\mathbf{u}}_{i,h}^n, \tilde{p}_{i,h}^n) \in \mathbf{X}_{i,h}\times M_{i,h}$ by solving the following iteration*

$$\begin{aligned} &a(\tilde{\mathbf{u}}_{i,h}^n, \mathbf{v}_{i,h}) - d(\mathbf{v}_{i,h}, \tilde{p}_{i,h}^n) + b(\mathbf{u}_{i,h}^{n-1}, \tilde{\mathbf{u}}_{i,h}^n, \mathbf{v}_{i,h}) \\ &+ \int_I \kappa(\tilde{\mathbf{u}}_{i,h}^n - \tilde{\mathbf{u}}_{j,h}^n)\cdot\mathbf{v}_{i,h}\mathrm{ds} = (\mathbf{f}_i, \mathbf{v}_{i,h}), \\ &d(\tilde{\mathbf{u}}_{i,h}^n, q_{i,h}) = 0, \qquad \forall(\mathbf{v}_{i,h}, q_{i,h}) \in \mathbf{X}_{i,h}\times M_{i,h}. \end{aligned} \tag{3.13}$$

**Step III**. *Find* $(\tilde{\boldsymbol{\epsilon}}^n_{i,h}, \tilde{\theta}^n_{i,h}) \in \mathbf{X}_{i,h} \times M_{i,h}$ *by solving the error correction equations*

$$\begin{aligned} & a(\tilde{\boldsymbol{\epsilon}}^n_{i,h}, \mathbf{v}^n_{i,h}) - d(\mathbf{v}^n_{i,n}, \tilde{\theta}^n_{i,h}) + b(\tilde{\boldsymbol{\epsilon}}^{n-1}_{i,h}, \tilde{\boldsymbol{\epsilon}}^n_{i,h}, \mathbf{v}_{i,h}) + b(\tilde{\mathbf{u}}^n_{i,h}, \tilde{\boldsymbol{\epsilon}}^n_{i,h}, \mathbf{v}_{i,h}) \\ & + b(\tilde{\boldsymbol{\epsilon}}^n_{i,h}, \tilde{\mathbf{u}}^n_{i,h}, \mathbf{v}_{i,h}) + b(\tilde{\mathbf{u}}^n_{i,h}, \tilde{\mathbf{u}}^n_{i,h}, \mathbf{v}_{i,h}) - b(\mathbf{u}^{n-1}_{i,h}, \tilde{\mathbf{u}}^n_{i,h}, \mathbf{v}_{i,h}) \\ & + \int_I \kappa(\tilde{\boldsymbol{\epsilon}}^n_{i,h} - \tilde{\boldsymbol{\epsilon}}^n_{j,h}) \cdot \mathbf{v}_{i,h} \mathrm{ds} = 0, \\ & d(\tilde{\boldsymbol{\epsilon}}^n_{i,h}, q_{i,h}) = 0, \qquad \forall (\mathbf{v}_{i,h}, q_{i,h}) \in \mathbf{X}_{i,h} \times M_{i,h}. \end{aligned} \tag{3.14}$$

**Step IV**. *Set* $\mathbf{u}^n_{i,h} \equiv \tilde{\mathbf{u}}^n_{i,h} + \tilde{\boldsymbol{\epsilon}}^n_{i,h}$ *and* $p^n_{i,h} \equiv \tilde{p}^n_{i,h} + \tilde{\theta}^n_{i,h}$. *If it satisfies the stopping criterion* $\|\mathbf{u}^n_{i,h} - \mathbf{u}^{n-1}_{i,h}\|_0 \le 10^{-6}$ *or the iteration step* $n \ge 1000$, *then stop; else set* $n = n+1$, *and return to* **Step II**.

Now, we will establish the bound of iterative solution for Algorithm 3.1.

**Theorem 3.2.** *Under the assumption of Theorem 3.1, if* $2\sigma < 1$, *then we obtain*

$$\nu \|\nabla \tilde{\mathbf{u}}^n_{i,h}\|_0 \le \sum_{i=1}^2 \|\mathbf{f}_i\|_{-1}, \tag{3.15}$$

$$\nu \|\nabla \mathbf{u}^n_{i,h}\|_0 \le \frac{\nu_1 + \nu_2 + \nu + 2\kappa C_{tr}^2 C_p}{\nu(1-2\sigma)} \sum_{i=1}^2 \|\mathbf{f}_i\|_{-1}. \tag{3.16}$$

**Proof.** Firstly, choose $(\mathbf{v}_{i,h}, q_{i,h}) = (\tilde{\mathbf{u}}^n_{i,h}, \tilde{p}^n_{i,h})$ in (3.13) and sum the ensuing the equation from $i = 1$ to 2. Then, use (2.1) to get

$$\nu |||\nabla \tilde{\mathbf{u}}^n_h|||_0 \le \sum_{i=1}^2 \|\mathbf{f}_i\|_{-1}. \tag{3.17}$$

Secondly, taking $(\mathbf{v}_{i,h}, q_{i,h}) = (\tilde{\boldsymbol{\epsilon}}^n_{i,h}, \tilde{\theta}^n_{i,h})$ in (3.14) gives

$$\begin{aligned} & \nu_i \|\nabla \tilde{\boldsymbol{\epsilon}}^n_{i,h}\|_0^2 + b(\tilde{\boldsymbol{\epsilon}}^n_{i,h}, \tilde{\mathbf{u}}^n_{i,h}, \tilde{\boldsymbol{\epsilon}}^n_{i,h}) + b(\tilde{\mathbf{u}}^n_{i,h}, \tilde{\mathbf{u}}^n_{i,h}, \tilde{\boldsymbol{\epsilon}}^n_{i,h}) - b(\mathbf{u}^{n-1}_{i,h}, \tilde{\mathbf{u}}^n_{i,h}, \tilde{\boldsymbol{\epsilon}}^n_{i,h}) \\ & + \int_I \kappa(\tilde{\boldsymbol{\epsilon}}^n_{i,h} - \tilde{\boldsymbol{\epsilon}}^n_{j,h}) \cdot \tilde{\boldsymbol{\epsilon}}^n_{i,h} \mathrm{ds} = 0. \end{aligned} \tag{3.18}$$

Next, set $\mathbf{v}_{i,h} = \tilde{\boldsymbol{\epsilon}}^n_{i,h}$ in (3.13) and notice that $d(\tilde{\boldsymbol{\epsilon}}^n_{i,h}, q_{i,h}) = 0$ for all $q_{i,h} \in M_{i,h}$. Then, adding the ensuing equation with (3.18), we have

$$\begin{aligned} & a(\tilde{\mathbf{u}}^n_{i,h}, \tilde{\boldsymbol{\epsilon}}^n_{i,h}) + \nu_i \|\nabla \tilde{\boldsymbol{\epsilon}}^n_{i,h}\|_0^2 + b(\tilde{\boldsymbol{\epsilon}}^n_{i,h}, \tilde{\mathbf{u}}^n_{i,h}, \tilde{\boldsymbol{\epsilon}}^n_{i,h}) + b(\tilde{\mathbf{u}}^n_{i,h}, \tilde{\mathbf{u}}^n_{i,h}, \tilde{\boldsymbol{\epsilon}}^n_{i,h}) \\ & + \int_I \kappa(\tilde{\mathbf{u}}^n_{i,h} - \tilde{\mathbf{u}}^n_{j,h}) \cdot \tilde{\boldsymbol{\epsilon}}^n_{i,h} \mathrm{ds} + \int_I \kappa(\tilde{\boldsymbol{\epsilon}}^n_{i,h} - \tilde{\boldsymbol{\epsilon}}^n_{j,h}) \cdot \tilde{\boldsymbol{\epsilon}}^n_{i,h} \mathrm{ds} = 0. \end{aligned}$$

Now, sum the above equation from $i = 1$ to 2, and apply (2.1) and (2.2) to the ensuing equation.

$$\begin{aligned} & \sum_{i=1}^2 \nu_i \|\nabla \tilde{\boldsymbol{\epsilon}}^n_{i,h}\|_0^2 + \int_I \kappa |\tilde{\boldsymbol{\epsilon}}^n_{1,h} - \tilde{\boldsymbol{\epsilon}}^n_{2,h}|^2 \mathrm{ds} \\ \le & \sum_{i=1}^2 \nu_i \|\nabla \tilde{\mathbf{u}}^n_{i,h}\|_0 \sum_{i=1}^2 \|\nabla \tilde{\boldsymbol{\epsilon}}^n_{i,h}\|_0 + \sum_{i=1}^2 N \|\nabla \tilde{\boldsymbol{\epsilon}}^n_{i,h}\|_0^2 \sum_{i=1}^2 \|\nabla \tilde{\mathbf{u}}^n_{i,h}\|_0 \\ & + \sum_{i=1}^2 N \|\nabla \tilde{\mathbf{u}}^n_{i,h}\|_0^2 \sum_{i=1}^2 \|\nabla \tilde{\boldsymbol{\epsilon}}^n_{i,h}\|_0 + \kappa C_{tr}^2 C_p \sum_{i=1}^2 \|\nabla \tilde{\mathbf{u}}^n_{i,h}\|_0 \sum_{i=1}^2 \|\nabla \tilde{\boldsymbol{\epsilon}}^n_{i,h}\|_0. \end{aligned} \tag{3.19}$$

Rewrite (3.19) and apply (3.15) to gain

$$\begin{aligned}\nu(1-2\sigma)\sum_{i=1}^{2}\|\nabla\tilde{\boldsymbol{\epsilon}}_{i,h}^{n}\|_0 &\leq \sum_{i=1}^{2}\nu_i\|\nabla\tilde{\mathbf{u}}_{i,h}^{n}\|_0+\sum_{i=1}^{2}N\|\nabla\tilde{\mathbf{u}}_{i,h}^{n}\|_0^2+\kappa C_{tr}^2C_p\sum_{i=1}^{2}\|\nabla\tilde{\mathbf{u}}_{i,h}^{n}\|_0\\ &\leq \frac{\nu_1+\nu_2+2\sigma\nu+2\kappa C_{tr}^2C_p}{\nu}\sum_{i=1}^{2}\|\mathbf{f}_i\|_{-1},\end{aligned}\tag{3.20}$$

which leads to

$$\nu\|\nabla\tilde{\boldsymbol{\epsilon}}_{i,h}^{n}\|_0 \leq \frac{\nu_1+\nu_2+2\sigma\nu+2\kappa C_{tr}^2C_p}{\nu(1-2\sigma)}\sum_{i=1}^{2}\|\mathbf{f}_i\|_{-1}.\tag{3.21}$$

Finally, combining (3.15) with (3.17) and the definitions of $\mathbf{u}_h^n$ in **Step** IV, and applying the triangle inequality, we finish the proof. □

## 4. Numerical experiments

In this section, we will test the performance and effectiveness of the proposed iterative algorithm.

In the first experiment, we access to the convergence performance of Algorithm 3.1. Consider the problem (1.1) on the domain $\Omega=\Omega_1\cup\Omega_2$, where $\Omega_1=[0,1]\times[0,1]$ and $\Omega_2=[0,1]\times[-1,0]$. Suppose the right-hand sides of functions $\mathbf{f}_i(x,y), i=1,2$ are

$$\begin{aligned}f_{1,1}&=\exp(x^2+y^2)\cos(x)\sin(y), & f_{1,2}&=\exp(x^2+y^2)\sin(x+y)(x^2+y^2),\\ f_{2,1}&=\exp(x^2+y^2)\sin(x)\cos(y), & f_{2,2}&=\exp(x^2+y^2)(y^3+x^2).\end{aligned}$$

For $i=1,2$, $u_{i,\frac{h}{2}}$ and $p_{i,\frac{h}{2}}$ are the numerical solutions of the fluid velocity and pressure when the mesh size is $\frac{h}{2}$. Then we display the errors and convergence orders for velocities and pressure in Table 1 and 2 with different values of the parameters. From these tables, we can see that as $h$ decreases, it is easy to see that the convergence rates of the velocity approximate to 3 and the pressure to 2.

**Table 1.** Errors and convergence rates with respect to $h$ with $\nu_1=0.05, \nu_2=0.1, \kappa=0.1$.

| $h$ | $\|\mathbf{u}_{1,h}-\mathbf{u}_{1,\frac{h}{2}}\|_0$ | Rate | $\|\mathbf{u}_{2,h}-\mathbf{u}_{2,\frac{h}{2}}\|_0$ | Rate | $\|p_{1,h}-p_{1,\frac{h}{2}}\|_0$ | Rate | $\|p_{2,h}-p_{2,\frac{h}{2}}\|_0$ | Rate |
|---|---|---|---|---|---|---|---|---|
| $\frac{1}{8}$ | 1.58E−3 | — | 8.60E−4 | — | 8.61E−3 | — | 4.99E−3 | — |
| $\frac{1}{16}$ | 1.84E−4 | 3.10 | 1.08E−4 | 2.99 | 2.12E−3 | 2.02 | 1.22E−3 | 2.03 |
| $\frac{1}{32}$ | 2.15E−5 | 3.10 | 1.28E−5 | 3.07 | 4.64E−4 | 2.19 | 2.75E−4 | 2.15 |
| $\frac{1}{64}$ | 2.36E−6 | 3.19 | 1.42E−6 | 3.18 | 1.04E−4 | 2.16 | 6.13E−5 | 2.17 |
| $\frac{1}{128}$ | 2.92E−7 | 3.01 | 1.75E−7 | 3.02 | 2.83E−5 | 1.87 | 1.60E−5 | 1.94 |

In the second experiment, motivated by previous study in [7], we consider a simple ocean/atmosphere model on rectangular subdomains $\Omega_1=[0,5]\times[0,1]$ and $\Omega_2=[0,5]\times[0,-1]$. Obviously, the interface $I=[0,5]\times\{0\}$. Set same boundary conditions, initial velocities, physical parameter and inflow profile as in [7]. Then, we perform simulation by using Algorithm 3.1 with mesh size $h=\frac{1}{10}$.

Figure 1 shows the viscous drag force exerted by the velocity difference at the interface induces circular motion in $\Omega_2$.

**Table 2.** Errors and convergence rates with respect to $h$ with $\nu_1 = 0.005, \nu_2 = 0.01, \kappa = 0.01$.

| $h$ | $\|\mathbf{u}_{1,h} - \mathbf{u}_{1,\frac{h}{2}}\|_0$ | Rate | $\|\mathbf{u}_{2,h} - \mathbf{u}_{2,\frac{h}{2}}\|_0$ | Rate | $\|p_{1,h} - p_{1,\frac{h}{2}}\|_0$ | Rate | $\|p_{2,h} - p_{2,\frac{h}{2}}\|_0$ | Rate |
|---|---|---|---|---|---|---|---|---|
| $\frac{1}{8}$ | 5.76E−2 | — | 2.35E−2 | — | 4.25E−2 | — | 2.03E−2 | — |
| $\frac{1}{16}$ | 3.91E−3 | 3.88 | 2.05E−3 | 3.52 | 3.84E−3 | 3.47 | 3.29E−3 | 2.62 |
| $\frac{1}{32}$ | 3.89E−4 | 3.33 | 2.22E−4 | 3.21 | 8.28E−4 | 2.21 | 7.37E−4 | 2.16 |
| $\frac{1}{64}$ | 4.12E−5 | 3.24 | 2.39E−5 | 3.21 | 1.80E−4 | 2.20 | 1.61E−4 | 2.20 |
| $\frac{1}{128}$ | 5.10E−6 | 3.01 | 2.84E−6 | 3.07 | 4.62E−5 | 1.95 | 4.06E−5 | 1.98 |

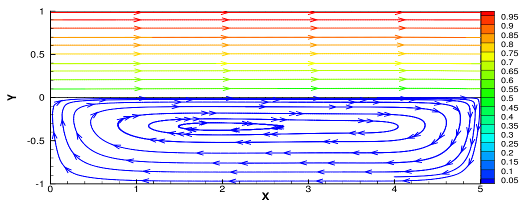



**Figure 1.** The numerical velocity streamlines with $\nu_1 = 5.0E-4, \nu_2 = 5.0E-3, \kappa = 2.45E-3$.

In the third experiment, we test Algorithm 3.1 with a practical problem, submarine mountain problem. This problem describes the fluid that flows in a domain including the submarine mountain. Set $\Omega_1 = [0, 1] \times [0.05, 0.15]$ and $\Omega_2 = \{(x, y) : \frac{7}{40}(\sin(\frac{7}{2}) - (2x - 1)\sin(7x - \frac{7}{2})) \le y \le 0.05, 0 \le x \le 1\}$. The body forces $\mathbf{f}_1$ and $\mathbf{f}_2$ are chosen to ensure that

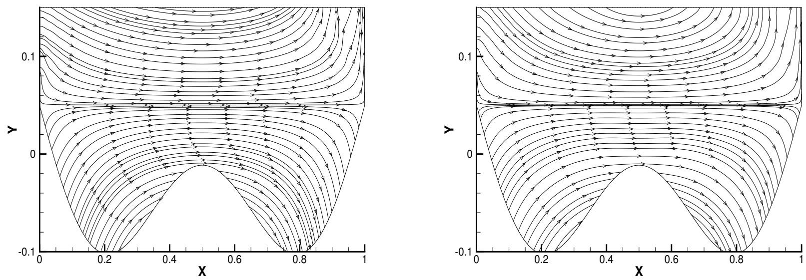
$$\begin{aligned}
u_{1,1}(x, y) &= x^2(1 - x)^2(0.1 - y),\\
u_{1,2}(x, y) &= x(y - 0.05)(-0.2 + y + 0.6x - 3xy - 0.4x^2 + 2x^2y),\\
u_{2,1}(x, y) &= x^2(1 - x)^2(0.1 + y),\\
u_{2,2}(x, y) &= x(y - 0.05)(-0.2 - y + 0.6x + 3xy - 0.4x^2 - 2x^2y),\\
p_1(x, y) &= p_2(x, y) = \cos(\pi x)\sin(\pi y),
\end{aligned}$$

Besides, the boundary and initial condition are chosen by the above exact solutions. We take $\nu_1 = 0.1, \nu_2 = 0.5, \kappa = 1.0, h = \frac{1}{64}$. Then, we apply Algorithm 3.1 to get numerical results.

In Figure 2, we present the velocity streamlines. From the figure, we can see that the numerical result obtained by the proposed iterative algorithm are in good agreement with the exact solution. Hence, the iterative algorithm is effective for the submarine mountain problem.

## 5. Conclusion

In this work, we introduce a iterative algorithm to solve the stationary fluid-fluid interaction model. The new algorithm is consist of the Oseen scheme and the error correction which can control the error of the iterative step arising for solving the nonlinear problem. We give the stability analysis in this study. The numerical tests show that the proposed algorithm is effective for solving the stationary fluid-fluid interaction problem.



(a) (b)

**Figure 2.** Velocity streamlines: Exact solution(a), numerical solution(b).

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