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BIFURCATIONS AND EXACT SOLUTIONS FOR THE KUNDU EQUATION: DYNAMICAL APPROACH

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Abstract In this paper, we focus on the exact traveling wave solutions for the Kundu equation. By using the method of dynamical systems, we obtain bifurcations of the phase portraits of the corresponding planar dynamical system under different parameter conditions. Corresponding to different level curves, we derive all possible exact explicit parametric representations of the bounded solutions (including smooth periodic wave solutions, solitary solutions, kink wave solutions).

Keywords Solitary wave, periodic wave, kink wave, bifurcation, dynamical system, Kundu equation.

MSC(2010) 37L20, 34C37, 35C05, 37G10.

1. Introduction

Recently, Qiu and Zhang [7] considered the Riemann–Hilbert problem (RHP) for the Kundu equation, which reads as follows (see [4,7]):

$$q_t + q_{xx} - b|q|^2 q - ia(|q|^2 q)_x + \beta(4\beta - a)|q|^4 q + 4i\beta(|q|^2)_x q = 0, \tag{1.1}$$

where $a, b, \beta \in \mathbf{R}$. The authors of [7] stated that "This equation was firstly introduced by Kundu when he studied the gauge transformation for the nonlinear Schrödinger (NLS) type equations [4]. For special situations, the Kundu equation can be reduced to several famous soliton equations, such as the NLS equation (setting $a = 0, b = -2, \beta = 0$), the first-type derivative NLS equation (setting $a = -1, b = 0, \beta = 0$), the second-type derivative NLS equation (setting $a = -1, b = 0, \beta = -\frac{1}{4}$), the third-type derivative NLS equation (setting $a = 1, b = 0, \beta = \frac{1}{2}$), the Kundu–Eckhaus equation (setting $a = 0, b = -2$), the mixed NLS equation (setting $\beta = 0$), etc. The Kundu equation can also be viewed as an extension of cubic–quintic NLS equation, which models the evolution of few-cycle pulses in nonlinear meta-materials" (see [2,3,8,9]). By appealing to the research skills in [10–12], Qiu and Zhang solved the RHP for the Kundu equation with high-order poles and displayed the formula of N-th order bound-state soliton (BSS), which implies that the integral factor can be solved from the asymptotic behaviors of RHP.

In 2015, Qiu et al. [6] constructed an analytical and explicit representation of the Darboux transformation for the Kundu–Eckhaus (KE) equation. Furthermore, the

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formulae for the higher order rogue wave (RW) solutions of the KE equation were also obtained by using the Taylor expansion with the use of degenerate eigenvalues.

Different from the references [7] and [6], we use the method of dynamical systems to find the exact solutions for Eq. (1.1) in this paper. We consider the solutions of Eq. (1.1) with the following form:

$$q(x,t) = \phi(\xi) e^{i(\kappa x - \omega t)}, \quad \xi = x - vt, \tag{1.2}$$

where v, κ and ω are constant parameters. Substituting Eq. (1.2) into Eq. (1.1), decomposing into real and imaginary parts, we have

$$\phi'' = (\kappa^2 - \omega)\phi + (-a\kappa + b)\phi^3 + (a\beta - 4\beta^2)\phi^5 \tag{1.3}$$

and

$$(3a - 8\beta)\phi^2\phi' + (v - 2\kappa)\phi' = 0. \tag{1.4}$$

Eq. (1.4) follows from the parameters of Eq. (1.1) and Eq. (1.2), which must satisfies the relationship: $\beta = \frac{3}{8}a$ and $v = 2\kappa$. Write that $\alpha_0 = \kappa^2 - \omega, \alpha_2 = b - a\kappa, \alpha_4 = -\frac{3}{16}a^2 < 0$.

Thus, Eq. (1.3) is equivalent to the following planar Hamiltonian system:

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \phi\left(\alpha_0 + \alpha_2\phi^2 + \alpha_4\phi^4\right), \tag{1.5}$$

with the first integral

$$H(\phi, y) = \frac{1}{2}y^2 - \frac{1}{6}\alpha_4\phi^6 - \frac{1}{4}\alpha_2\phi^4 - \frac{1}{2}\alpha_0\phi^2 = h. \tag{1.6}$$

Clearly, since $H(-\phi, y) = H(\phi, y)$, the level curves defined by $H(\phi, y) = h$ are symmetry with respect to y-axis. So that, we can only discuss the phase portraits in the right half-phase plane for system (1.5).

Obviously, system (1.5) is a three-parameter system depending on parameter group $(\alpha_0, \alpha_2, \alpha_4)$. We next apply the method of dynamical systems to discuss the dynamical behavior of system (1.5) and the bifurcations of phase portraits (see [5]). Under different parameter conditions we shall find exact solutions of system (1.5) and calculate all possible exact explicit smooth periodic wave solutions, solitary solutions, kink wave solutions of system (1.5). The exact parametric representations of these solutions are presented.

The article is organized as follows. In section 2, we discuss the bifurcations of phase portraits for the systems (1.5). In section 3 and section 4, we derive exact explicit parametric representations for all bounded solutions of system (1.5). In section 5, we state the main results of this paper.

2. Bifurcations of phase portraits of system (1.5)

Obviously, system (1.5) has an equilibrium point at $O(0,0)$. Let $F(\psi) = \alpha_4\psi^2 + \alpha_2\psi + \alpha_0$, where $\psi = \phi^2$. Clearly, when $\Delta = \alpha_2^2 - 4\alpha_4\alpha_0 > 0, F(\psi)$ has two real zeros at $\psi = \psi_{1,2} = \frac{-\alpha_2 \mp \sqrt{\Delta}}{2\alpha_4}$. Let z_j is a positive real zero of function $F(\psi)$. Then, in the positive ϕ-axis of the phase plane, system (1.5) has an equilibrium point $E_j(\sqrt{z_j}, 0)$.

Because $\alpha_4 < 0$, it is easy to show that the following conclusions hold.

(i) When $\alpha_0 < 0, \alpha_2 > 0$ and $\Delta > 0$, then, $F(\psi)$ has two positive real zeros.

(ii) When $\alpha_0 < 0, \alpha_2 > 0$ and $\Delta = 0$, then, $F(\psi)$ has a double positive real zero.

(iii) When $\alpha_0 > 0, \alpha_2 > 0$, then $\Delta > 0$, and hence, $F(\psi)$ has one positive real zero at $\psi_1 = \frac{-\alpha_2 - \sqrt{\Delta}}{2\alpha_4}$.

(iv) When $\Delta < 0$, $F(\psi)$ has no real zero.

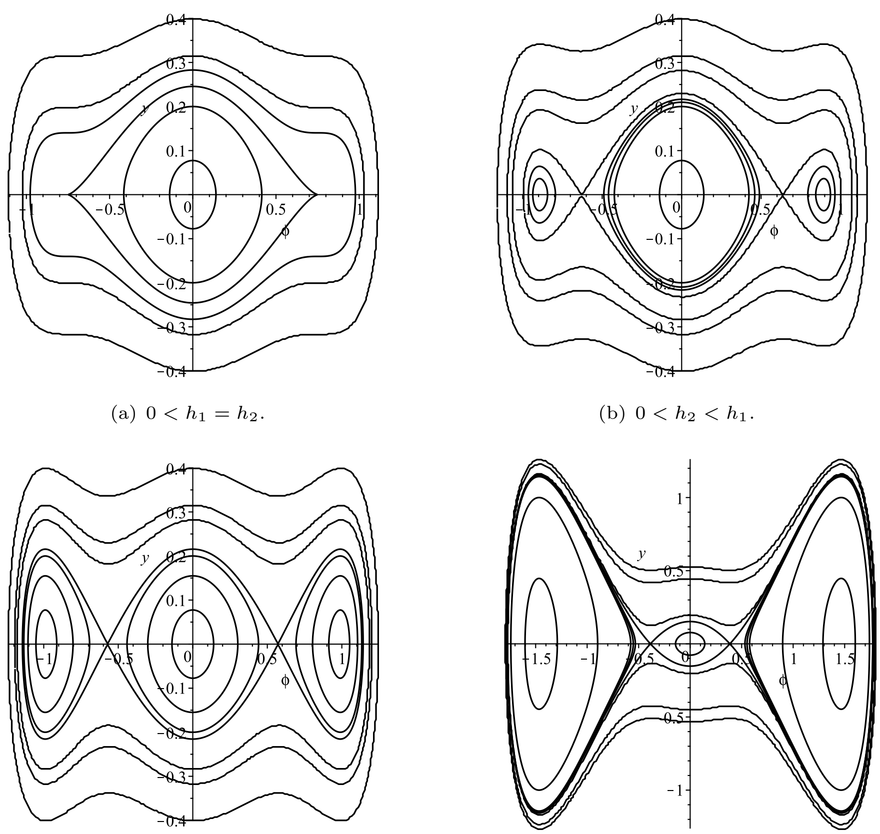
Let $M(\phi_j, 0)$ be the coefficient matrix of the linearized system of system (1.5) at an equilibrium point $E_j(\phi_j, 0), \phi_j = \sqrt{z_j}$. and $J(\phi_j, 0) = \det\mathrm{M}(\phi_j, 0)$. We have

$$J(0,0) = -\alpha_0, \quad J(\phi_j, 0) = -2\phi_j^2(\alpha_2 + 2\alpha_4\phi_j^2). \tag{2.1}$$

Write $h_0 = H(0,0), \quad h_j = H(\phi_j, 0)$, where H is given by (1.6).

By using the above information to do qualitative analysis, we have the following bifurcations of the phase portraits of system (1.5) shown in Fig.1-Fig.2.

1. The case that there exist two equilibrium points (including double points) of system (1.5) in the positive ϕ-axis.

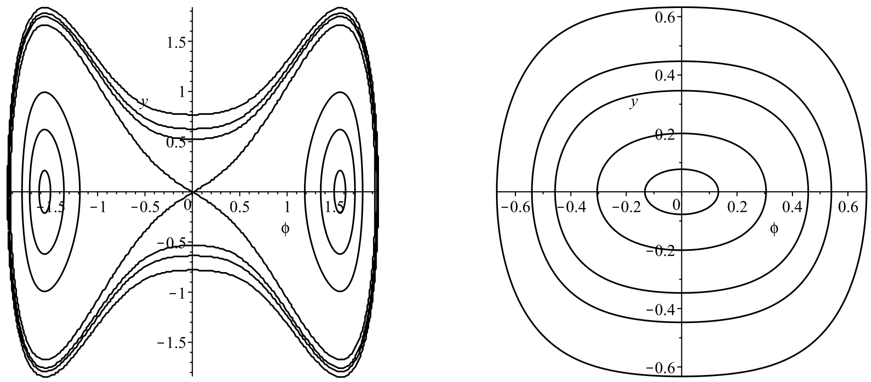


(a) $0 < h_1 = h_2$. (b) $0 < h_2 < h_1$.

(c) $0 = h2 < h_1$. (d) $h_2 < 0 < h_1$.

Figure 1. The bifurcations of phase portraits of system (1.5) when $\alpha_0 < 0, \alpha_2 > 0, \Delta > 0$.

2. The case that there exists one equilibrium point or no equilibrium point of system (1.5) in the positive ϕ-axis.



(a) $\alpha_0 > 0$. (b) $\alpha_0 < 0$.

Figure 2. Phase portraits of system (1.5) when $F(\psi)$ has one positive zero or has no zero.

3. Explicit exact parametric representations of solutions of system (1.5) in Fig.1

It is known that for a given real number h, the function $H(\phi, y) = h$ given by Eq. (1.6) defines level curves of system (1.5), which can have different branches. We see from Eq. (1.6) that $y^2 = 2h + \alpha_0\phi^2 + \frac{1}{2}\alpha_2\phi^4 + \frac{1}{3}\alpha_4\phi^6$. Hence, by using the first equation of system (1.5) we obtain

$$\begin{aligned} \xi &= \int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{|2h + \alpha_0\phi^2 + \frac{1}{2}\alpha_2\phi^4 + \frac{1}{3}\alpha_4\phi^6|}} \\ &= \int_{\psi_0}^{\psi} \frac{d\psi}{2\sqrt{\psi|2h + \alpha_0\psi + \frac{1}{2}\alpha_2\psi^2 + \frac{1}{3}\alpha_4\psi^3|}}, \end{aligned} \tag{3.1}$$

where $\psi = \phi^2$. By using Eq. (3.1), we can calculate the parametric representations of $\phi(\xi) = \sqrt{\psi(\xi)}$ of the orbits of system (1.5). Because $\phi = \sqrt{\psi}$, we only need to find the solutions of $\psi(\xi) > 0$.

3.1. The parametric representations of the periodic orbits and heteroclinic orbits given by Fig.1 (a).

(i) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (0, h_1), h_1 = h_2$, there exists a family of periodic orbits of system (1.5). Now, Eq. (3.1) can be written as

$$\sqrt{\frac{4|\alpha_4|}{3}} = \int_0^{\psi} \frac{d\psi}{\sqrt{(\psi_a - \psi)\psi[(\psi - \hat{b}_1)^2 + \hat{a}_1^2]}}.$$

Therefore, we obtain the parametric representation of the family of periodic solutions:

$$\phi(\xi) = \mp\left(\frac{\psi_a B_1(1 - \mathrm{cn}(\Omega_1\xi, k))}{(A_1 + B_1) + (A_1 - B_1)\mathrm{cn}(\Omega_0\xi, k)}\right)^{\frac{1}{2}}, \quad \xi \in (0, 2K(k)), \tag{3.2}$$

where $A_1^2 = (\psi_a - \hat{b}_1)^2 + \hat{a}_1^2, B_1^2 = \hat{b}_1^2 + \hat{a}_1^2, k^2 = \frac{\psi_a^2-(A_1-B_1)^2}{4A_1B_1}, \Omega_0 = \sqrt{\frac{4|\alpha_4|A_1B_1}{3}}$, $\mathrm{sn}(\cdot,k), \mathrm{cn}(\cdot,k), \mathrm{dn}(\cdot,k)$ are the Jacobin elliptic functions (see Byrd and Fridman [1]), $K(k)$ is the complete elliptic integral of the first kind.

(ii) Corresponding to the level curves defined by $H(\phi,y) = h_1 = h_2$, there exist two heteroclinic orbits connecting two cusp points $(\phi_1, 0)$ and $(-\phi_1, 0)$. Eq. (3.1) can be written as

$$\sqrt{\frac{4|\alpha_4|}{3}} = \int_0^{\psi} \frac{d\psi}{(\psi_1-\psi)\sqrt{(\psi_1-\psi)\psi}}.$$

Thus, we obtain the parametric representation of a kink wave and an anti-kink wave solutions of system (1.5) as follows:

$$\phi(\xi) = \pm\frac{\phi_1\omega_0\xi}{\sqrt{1+\omega_0^2\xi^2}}, \tag{3.3}$$

where $\omega_0 = \psi_1\sqrt{\frac{|\alpha_4|}{3}}$.

(iii) Corresponding to the level curves defined by $H(\phi,y) = h, h \in (h_1,\infty)$, there exists a global family of periodic orbits of system (1.5). It has the same parametric representation as Eq. (3.2).

3.2. The parametric representations of the periodic orbits, homoclinic and heteroclinic orbits given by Fig.1 (b).

(i) The level curves defined by $H(\phi,y) = h, h \in (0,h_2)$ are a family of periodic orbits enclosing the origin $O(0,0)$. These orbits have the same parametric representation as Eq. (3.2).

(ii) The level curves defined by $H(\phi,y) = h, h \in (h_2, 0)$ contain three families of periodic orbits enclosing the origin $O(0,0)$ and $(\mp\psi_2, 0)$, respectively. Now, for the right family of periodic orbits, Eq. (3.1) can be written as

$$\sqrt{\frac{4|\alpha_4|}{3}} = \int_{\psi_b}^{\psi} \frac{d\psi}{\sqrt{(\psi_a-\psi)(\psi-\psi_b)(\psi-\psi_c)\psi}}.$$

Therefore, we obtain the parametric representation of this family of periodic solutions:

$$\phi(\xi) = \left(\psi_c + \frac{\psi_b-\psi_c}{1-\tilde{\alpha}_1^2\mathrm{sn}^2(\Omega_1\xi,k)}\right)^{\frac{1}{2}}, \tag{3.4}$$

where $\tilde{\alpha}_1^2 = \frac{\psi_a-\psi_b}{\psi_a-\psi_c}, k^2 = \frac{\tilde{\alpha}_1^2\psi_c}{\psi_d}, \Omega_1 = \sqrt{\frac{|\alpha_4|(\psi_a-\psi_c)\psi_b}{3}}$.

For the mid family of periodic orbits, Eq. (3.1) can be written as

$$\sqrt{\frac{4|\alpha_4|}{3}} = \int_0^{\psi} \frac{d\psi}{\sqrt{(\psi_a-\psi)(\psi_b-\psi)(\psi_c-\psi)\psi}}.$$

Therefore, we obtain the parametric representation of this family of periodic solutions:

$$\phi(\xi) = \frac{\phi_a|\tilde{\alpha}_2|\mathrm{sn}(\Omega_1\xi,k)}{\sqrt{1-\tilde{\alpha}_2^2\mathrm{sn}^2(\Omega_1\xi,k)}}, \tag{3.5}$$

where $\tilde{\alpha}_2^2 = \frac{-\psi_c}{\psi_a - \psi_c}, k^2 = \frac{-\tilde{\alpha}_2^2(\psi_a - \psi_b)}{\psi_b}, \Omega_1 = \sqrt{\frac{|a_4|(\psi_a - \psi_c)\psi_b}{3}}$.

(iii) The level curves defined by $H(\phi, y) = h_1$ contain two heteroclinic orbits enclosing the origin $O(0,0)$ and two homoclinic orbits enclosing the singular points $(\mp\phi_2, 0)$, respectively. For the homoclinic orbit enclosing the equilibrium point $E_2(\phi_2, 0)$, Eq. (3.1) can be written as $\sqrt{\frac{4|\alpha_4|}{3}} = \int_{\psi}^{\psi_M} \frac{d\psi}{(\psi - \psi_1)\sqrt{(\psi_M - \psi)\psi}}$. It follows the parametric representation of a solitary wave solution of system (1.5):

$$\psi(\xi) = \left(\psi_1 + \frac{2(\psi_M - \psi_1)\psi_1}{\psi_M \cosh(\omega_1 \xi) + (2\psi_1 - \psi_M)}\right)^{\frac{1}{2}}, \tag{3.6}$$

where $\omega_1 = \sqrt{\frac{4}{3}|\alpha_4|\psi_1(\psi_M - \psi_1)}$.

For the upper heteroclinic orbit enclosing the equilibrium point $E_2(\phi_2, 0)$, Eq. (3.1) can be written as $\sqrt{\frac{4|\alpha_4|}{3}} = \int_0^{\psi} \frac{d\psi}{(\psi_1 - \psi)\sqrt{(\psi_M - \psi)\psi}}$. It follows the parametric representation of a kink wave solution of system (1.5):

$$\psi(\xi) = \mp\left(\psi_1 - \frac{2(\psi_M - \psi_1)\psi_1}{\psi_M \cosh(\omega_1 \xi) - (2\psi_1 - \psi_M)}\right)^{\frac{1}{2}}, \quad \omega_1 \xi \in (0, \infty), \tag{3.7}$$

where $\omega_1 = \sqrt{\frac{4}{3}|\alpha_4|\psi_1(\psi_M - \psi_1)}$.

(iv) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_1, \infty)$, there exists a global family of periodic orbits of system (1.5). It has the same parametric representation as Eq. (3.2).

Similarly, we can calculate the parametric representations for all bounded orbits in Fig.1 (c) and (d).

4. The parametric representations of the periodic and homoclinic orbits given by Fig.2

4.1. Explicit exact parametric representations of solutions of system (1.5) in Fig.2 (a)

(i) The level curves defined by $H(\phi, y) = h, h \in (h_1, 0)$ contain two families of periodic orbits enclosing two equilibrium points $(\mp\phi_1, 0)$, respectively. For the right family of periodic orbits, Eq. (3.1) can be written as

$$\sqrt{\frac{4|\alpha_4|}{3}} = \int_{\psi_b}^{\psi} \frac{d\psi}{\sqrt{(\psi_a - \psi)(\psi - \psi_b)\psi(\psi + \psi_d)}}.$$

Therefore, we obtain the parametric representation of this family of periodic solutions:

$$\phi(\xi) = \frac{\phi_b}{\sqrt{1 - \tilde{\alpha}_3^2 \mathrm{sn}^2(\Omega_3 \xi, k)}}, \tag{4.1}$$

where $\tilde{\alpha}_3^2 = \frac{\psi_a - \psi_b}{\psi_a}, k^2 = \frac{\tilde{\alpha}_3^2(-\psi_d)}{\psi_b + \psi_d}, \Omega_3 = \sqrt{\frac{|\alpha_4|\psi_a(\psi_b+\psi_d)}{3}}$.

(ii) The level curves defined by $H(\phi, y) = 0$ contain two homoclinic orbits enclosing two equilibrium points $(\mp\phi_1, 0)$, respectively. For the right homoclinic orbit, Eq. (3.1) can be written as $\sqrt{\frac{4|\alpha_4|}{3}} = \int_{\psi}^{\psi_M} \frac{d\psi}{\psi\sqrt{(\psi_M-\psi)(\psi+\psi_d)}}$. It gives rise to the following solitary wave solution:

$$\phi(\xi) = \frac{\sqrt{2}\phi_M\phi_d}{\sqrt{(\psi_M + \psi_d)\cosh(\omega_2\xi) - (\psi_M - \psi_d)}}, \tag{4.2}$$

where $\omega_2 = \sqrt{\frac{4}{3}|\alpha_4|\psi_M\psi_d}$.

(iii) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (0, \infty)$, there exists a global family of periodic orbits of system (1.5) enclosing three equilibrium points. It has the same parametric representation as Eq. (3.2).

4.2. Explicit exact parametric representations of solutions of system (1.5) in Fig.2 (b)

Corresponding to the level curves defined by $H(\phi, y) = h, h \in (0, \infty)$, there exists a family of periodic orbits of system (1.5) enclosing the origin $O(0, 0)$. It has the same parametric representation as Eq. (3.2).

5. Conclusion

By the above discussion, the following conclusion is established.

Theorem 5.1. i. *For the Kundu equation (1), if and only if* $\beta = \frac{3}{8}a,\ v = 2\kappa$, *it has the exact explicit solutions given by*

$$q(x, t) = \phi(\xi)e^{i(\kappa x - \omega t)}, \quad \xi = x - 2\kappa t, \tag{5.1}$$

where $\phi(\xi)$ *is a solution of planar dynamical system* (1.5).

ii. *Under different parameter conditions, system* (1.5) *has the bifurcations of phase portraits which are shown in Fig.1 and Fig.2.*

iii. $\phi(\xi)$ *has exact explicit solutions given by Eq.* (3.2)*-Eq.* (4.2).

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