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ON THE EXISTENCE AND STABILITY OF SOLUTIONS OF A TYPE-III THERMOELASTIC TRUNCATED TIMOSHENKO SYSTEM

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Abstract This paper is concerned with the well-posedness and stability of a one-dimensional thermoelastic truncated Timoshenko system of Type III. In order to establish the well-posedness, we first solve an auxiliary problem and give the proof in details, using the semigroup theory and some non traditional operators. Then, we use this result to solve our original problem. After that, we prove that the presence of the thermal effect in one equation only is strong enough to drive the system exponentially to rest, irrespective to any relation between the coefficients. By the end of the work, we present some numerical tests to illustrate our theoretical findings.

Keywords Exponential decay, Timoshenko, thermoelasticity type III, well-posedness, truncated.

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1. Introduction

The issue of modelling the motion of thick beams was a concern of many engineers and scientists during the last two centuries. In 1921, the following system

$$\begin{cases} \rho_1\phi_{tt} - \kappa\left(\phi_x + \psi\right)_x = 0, & \text{in } \ (0,L)\times(0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa\left(\phi_x + \psi\right) = 0, & \text{in } \ (0,L)\times(0,+\infty), \end{cases} \tag{1.1}$$

was introduced by Timoshenko as answer to such a concern. Here, ϕ is the transverse displacement, ψ is the rotational angle of the filament of the beam and ρ_1, ρ_2, b and κ are fixed positive physical constants.

This system, together with several types of boundary conditions, is conservative. So, to stabilize it, various kinds of dissipations have been added to one of the equations or both. As a product, many results concerning the well-posedness and

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long-time behavior of the system have been established. We can see, for example, [1–3, 5, 10, 12, 14, 15, 20, 21, 23, 25, 26, 30].

In fact, it is well-known that the exponential stability of (1.1) can be obtained, without imposing any condition on the parameters, by the presence of linear dampings in both equations. However, if only one linear damping is acting on the system, then the exponential stability is achieved if and only if the equal speeds of wave propagation holds; that is,

$$\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}. \tag{1.2}$$

The interested reader is advised to see the above references, among others, for detailed analysis on the well-posedness and the stability of Timoshenko systems. In particular, the Timoshenko system of thermoelasticity of type-III of the form

$$\begin{cases} \rho_1\varphi_{tt} - \kappa\left(\varphi_x + \psi\right)_x = 0, & \text{in } \ (0,L)\times(0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa\left(\varphi_x + \psi\right) + \delta\theta_x = 0, & \text{in } \ (0,L)\times(0,+\infty), \\ \rho_3\theta_{tt} - \beta\theta_{xx} + \gamma\psi_{ttx} - \delta\theta_{txx} = 0, & \text{in } \ (0,L)\times(0,+\infty), \end{cases} \tag{1.3}$$

was derived, taking into account Green and Naghdi's theory, where φ, ψ and θ model, respectively, the transverse displacement of the beam, the rotation angle of the filament, and the temperature displacement.

This system, with initial and boundary conditions, has been studied by many authors. In this regard, we mention Messaoudi and Said-Houari [27] who established an exponential decay result for the system (1.3) when (1.2) holds. The case of non-equal speeds was studied later by Messaoudi and Fareh [24] and they established a polynomial decay result. For more results, see [6, 13].

From the physics point of view, the original Timoshenko system (1.1) is characterized by two natural frequencies which lead to a paradox, known as the second spectrum, which was not discovered in Timoshenko's original work. This imposed (mathematically) a relation on the system coefficients, called the equal-speed propagation. However, this is an unrealistic requirement. To get around this paradox and to eliminate the anomaly of the second spectrum, Elishakoff [7] proposed in 2009 the following truncated form

$$\begin{cases} \rho_1\varphi_{tt} - \kappa\left(\varphi_x + \psi\right)_x = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \kappa\left(\varphi_x + \psi\right) = 0, \end{cases} \tag{1.4}$$

in $(0, L) \times \mathbb{R}^+$. The system (1.4) has been studied by a number of researchers and results concerning the stability have been established. For example, Almeida Jùnior et al. [16] considered the following truncated dissipative shear beam model

$$\begin{cases} \rho_1\varphi_{tt} - \kappa\left(\varphi_x + \psi\right)_x + \mu\varphi_t = 0, \\ -b\psi_{xx} + \kappa\left(\varphi_x + \psi\right) = 0, \end{cases} \tag{1.5}$$

and established the exponential decay of the system without imposing any relationship between the coefficients. Also, Feng et al. [11] looked into the following truncated system in the presence of time-varying delay

$$\begin{cases} \rho_1\varphi_{tt} - \kappa\left(\varphi_x + \psi\right)_x = 0, \\ -\rho_2\varphi_{xtt} - b\psi_{xx} + \kappa\left(\varphi_x + \psi\right) + \xi_1\psi_t + \xi_2\psi_t(x, t-\tau) = 0, \end{cases} \tag{1.6}$$

and proved an exponential stability result. In particular, for the stabilization via heat dissipation, not much work has been done. For instance, Apalara et al. [4] considered the following system

$$\begin{cases} \rho_1\varphi_{tt} - \kappa\left(\varphi_x + \psi\right)_x = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \kappa\left(\varphi_x + \psi\right) + \gamma\theta_x = 0, \\ \rho_3\theta_t - \beta\theta_{xx} + \gamma\psi_{xtt} = 0, \end{cases} \tag{1.7}$$

where θ is the temperature difference and $\rho_3, \beta > 0, \gamma \neq 0$ are the capacity, diffusivity and adhesive stiffness, respectively. They discussed briefly the well-posedness and proved an exponential decay result irrespective of the coefficients of the system. Also, Keddi et al. [22] discussed the following second-sound type thermoelastic truncated Timoshenko system

$$\begin{cases} \rho_1\varphi_{tt} - \kappa\left(\varphi_x + \psi\right)_x = 0, & \text{in } \ (0,1)\times(0,+\infty), \\ \rho_2\psi_{ttx} - b\psi_{xx} + \kappa\left(\varphi_x + \psi\right) + \delta\theta_x = 0, & \text{in } \ (0,1)\times(0,+\infty), \\ c\theta_t + q_x + \delta\psi_{xt} = 0, & \text{in } \ (0,1)\times(0,+\infty), \\ \tau q_t + \beta q + \theta_x = 0, & \text{in } \ (0,1)\times(0,+\infty), \end{cases} \tag{1.8}$$

established the well-posedness in details, using the semigroup theory and introducing some non classical operators, and showed that the system is exponentially stable irrespective of the coefficients. For more results in this direction, we refer the reader to visit [8, 9, 17–19, 28, 29].

In this paper, we are concerned with the following Type-III thermoelastic Timoshenko system in the light of the second spectrum of frequency

$$\begin{cases} \rho_1\varphi_{tt} - \kappa\left(\varphi_x + \psi\right)_x + \mu\theta_x = 0, \\ -b\psi_{xx} + \kappa\left(\varphi_x + \psi\right) = 0, \\ c\theta_{tt} - \kappa\theta_{xx} - \delta\theta_{xxt} + \mu\varphi_{xtt} = 0, \end{cases} \tag{1.9}$$

together with the initial and boundary conditions

$$\begin{cases} \varphi\left(x,0\right) = \varphi_0(x),\ \varphi_t\left(x,0\right) = \varphi_1(x),\ \theta\left(x,0\right) = \theta_0(x),\ \theta_t\left(x,0\right) = \theta_1(x), \\ \varphi\left(0,t\right) = \varphi\left(1,t\right) = \psi_x\left(0,t\right) = \psi_x\left(1,t\right) = \theta_x\left(1,t\right) = \theta_x\left(0,t\right) = 0. \end{cases} \tag{1.10}$$

By differentiating the first and second equations of (1.9) with respect to t and introducing the new variables $\phi = \varphi_t$ and $\omega = \psi_t$, the problem (1.9),(1.10) becomes

$$\begin{cases} \rho_1\phi_{tt} - k\left(\phi_x + \omega\right)_x + \mu\theta_{xt} = 0, \\ -b\omega_{xx} + k\left(\phi_x + \omega\right) = 0, \\ c\theta_{tt} - \kappa\theta_{xx} - \delta\theta_{xxt} + \mu\phi_{xt} = 0, \end{cases} \tag{1.11}$$

with the following boundary conditions

$$\phi(0,t)=\phi(1,t)=\omega_x(0,t)=\omega_x(1,t)=\theta_x(1,t)=\theta_x(0,t)=0,\ \forall t\geq 0, \tag{1.12}$$

and the initial data

$$\begin{aligned}&\phi(x,0)=\phi_0(x)=\varphi_1(x),\ \phi_t(x,0)=\phi_1(x)=\tfrac{k}{\rho_1}(\varphi_{0x}+\psi_0)_x(x)+\mu\theta_{0x},\\&\theta(x,0)=\theta_0(x),\ \theta_t(x,0)=\theta_1(x),\qquad \forall x\in[0,1].\end{aligned} \tag{1.13}$$

Since the boundary conditions on ω and θ are of Neumann type. We are unable to apply Poincaré's inequality. However, from the second and the third equation of (1.11) and the boundary conditions (1.12), we can deduce

$$\int_0^1 \omega(x,t)dx=0,\quad \frac{d^2}{dt^2}\int_0^1 \theta(x,t)dx=0, \tag{1.14}$$

which entails

$$\int_0^1 \theta(x,t)dx=\left[\int_0^1 \theta_1(x)dx\right]t+\int_0^1 \theta_0(x)dx. \tag{1.15}$$

Therefore, if we set

$$\overline{\theta}(x,t)=\theta(x,t)-\left[\int_0^1 \theta_1(x)dx\right]t-\int_0^1 \theta_0(x)dx, \tag{1.16}$$

we get

$$\int_0^1 \overline{\theta}(x,t)dx=0, \tag{1.17}$$

which allows the application of Poincaré's inequality. In the sequel, we work with $(\phi,\omega,\overline{\theta})$, but for convenience, we write (ϕ,ω,θ).

The remaining of the paper is organized as follows. In Section 2, we present the existence and uniqueness in details, using the semigroup theory and some unusual operators. The stability result is given in Section 3. In Section 4, we check the stability of the discrete system and illustrate our theoretical results by performing some numerical tests.

2. Well posedness

In this section, we prove the existence and uniqueness of the solution for the problem (1.11)-(1.13).

From the second equation of (1.11), we get

$$\omega=-k(k-b\partial_{xx})^{-1}\phi_x$$

and by substituting this latter equation into the first equation of (1.11), we obtain

$$\begin{cases}\rho_1\phi_{tt}-k\phi_{xx}+k^2(k-b\partial_{xx})^{-1}\phi_{xx}+\mu\theta_{xt}=0,\\ c\theta_{tt}-\kappa\theta_{xx}-\delta\theta_{xxt}+\mu\phi_{xt}=0.\end{cases}$$

Consequently, our problem reduces to

$$\begin{cases} \rho_1\phi_{tt} - BS^{-1}\phi_{xx} + \mu\theta_{xt} = 0, \\ c\theta_{tt} - \kappa\theta_{xx} - \delta\theta_{xxt} + \mu\phi_{xt} = 0, \end{cases} \tag{2.1}$$

and

$$\begin{cases} \phi\left(0,t\right) = \phi\left(1,t\right) = \theta_x\left(1,t\right) = \theta_x\left(0,t\right) = 0, & \forall t \geq 0, \\ \phi\left(x,0\right) = \phi_0\left(x\right) = \varphi_1(x), & \\ \phi_t\left(x,0\right) = \phi_1\left(x\right) = \dfrac{k}{\rho_1}\left(\varphi_{0x} + \psi_0\right)_x\left(x\right) + \mu\theta_{0x}, & \forall x \in (0,1), \\ \theta\left(x,0\right) = \theta_0, \quad \theta_t\left(x,0\right) = \theta_1(x), & \forall x \in (0,1), \end{cases} \tag{2.2}$$

where $B,\, S : L^2\left(0,1\right) \to L^2\left(0,1\right)$ are positive self-adjoint operators defined by

$$\begin{aligned} B &= -bk\partial_{xx}, \\ S &= k - b\partial_{xx}, \end{aligned}$$

with domains

$$D\left(B\right) = D\left(S\right) = H^2\left(0,1\right) \cap H_0^1(0,1).$$

Next, we define the Hilbert space

$$H = H_0^1\left(0,1\right) \times L^2\left(0,1\right) \times H_*^1\left(0,1\right) \times L_*^2(0,1),$$

equipped with the inner product

$$\begin{aligned} &\left\langle (\phi, v, \theta, \vartheta)^T, (\phi^*, v^*, \theta^*, \vartheta^*)^T \right\rangle \\ =&\rho_1\left\langle v, v^*\right\rangle + \left\langle B^{1/2}S^{-1/2}\phi_x, B^{1/2}S^{-1/2}\phi_x^* \right\rangle + c\left\langle \vartheta, \vartheta^*\right\rangle + \kappa\langle\theta_x, \theta_x^*\rangle, \end{aligned}$$

where

$$L_*^2\left(0,1\right) = \left\{ \phi \in L^2\left(0,1\right) / \int_0^1 \phi(x)dx = 0 \right\} \text{ and } H_*^1\left(0,1\right) = H^1\left(0,1\right) \cap L_*^2(0,1).$$

Now, we introduce the new variables $\phi_t = v$ and $\theta_t = \vartheta$ and the problem (2.1)-(2.2) becomes

$$\begin{cases} \Phi_t + A\Phi = 0, \\ \Phi\left(0\right) = \Phi_0 = (\phi_0, \phi_1, \theta_0, \theta_1)^T, \end{cases}$$

where $\Phi = (\phi, v, \theta, \vartheta)$ and the operator $A : D\left(A\right) \subset H \to H$ is defined by

$$A\Phi = \begin{pmatrix} -v \\ -\frac{1}{\rho_1}BS^{-1}\phi_{xx} + \frac{\mu}{\rho_1}\vartheta_x \\ -\vartheta \\ -\frac{\kappa}{c}\theta_{xx} - \frac{\delta}{c}\vartheta_{xx} + \frac{\mu}{c}v_x \end{pmatrix},$$

with domain

$$D(A)=\left\{\begin{array}{c}\Phi\in H:\phi\in H^2(0,1),\ v\in H_0^1(0,1),\ \vartheta\in H_*^1(0,1),\\ \kappa\theta+\delta\vartheta\in H_*^2(0,1)\end{array}\right\},$$

where

$$H_*^2(0,1)=\Big\{\phi\in H^2(0,1):\phi_x(0)=\phi_x(1)=0\Big\}.$$

Theorem 2.1. *Let $\Phi_0\in D(A)$. Then there exists a unique solution of the problem (2.1), $\Phi\in C(\mathbb{R}^+;D(A))\cap C^1(\mathbb{R}^+;H)$.*

Proof. We use the semigroup method. According to the Hille-Yosida theorem, it suffices to prove that A is a maximal monotone operator.

First, we prove that A is monotone. Let $\Phi\in D(A)$, then the inner product, the properties of the operators B and S and integration by parts lead to

$$\begin{aligned}\langle A\Phi,\Phi\rangle_H&=-\left\langle BS^{-1}\phi_{xx}-\mu\vartheta_x,v\right\rangle-\left\langle B^{1/2}S^{-1/2}v_x,B^{1/2}S^{-1/2}\phi_x\right\rangle\\&\quad-\langle\kappa\theta_{xx}+\delta\vartheta_{xx}-\mu v_x,\vartheta\rangle-\kappa\langle\vartheta_x,\theta_x\rangle\\&=\left\langle B^{1/2}S^{-1/2}\phi_x,B^{1/2}S^{-1/2}v_x\right\rangle+\mu\langle\vartheta_x,v\rangle\\&\quad-\left\langle B^{1/2}S^{-1/2}v_x,B^{1/2}S^{-1/2}\phi_x\right\rangle+\kappa\langle\theta_x,\vartheta_x\rangle\\&\quad+\delta\langle\vartheta_x,\vartheta_x\rangle-\mu\langle v,\vartheta_x\rangle-\kappa\langle\vartheta_x,\theta_x\rangle\\&=\delta\langle\vartheta_x,\vartheta_x\rangle\\&\geq 0.\end{aligned}$$

Thus, A is monotone.

Next, let $F=(f_1,f_2,f_3,f_4)^T\in H$, we seek $\Phi\in D(A)$ such that $(I-A)\Phi=F$, that is

$$\begin{cases}\phi-v=f_1,\\ \rho_1 v-BS^{-1}\phi_{xx}+\mu\vartheta_x=\rho_1 f_2,\\ \theta-\vartheta=f_3,\\ c\vartheta-\kappa\theta_{xx}-\delta\vartheta_{xx}+\mu v_x=cf_4.\end{cases}\tag{2.3}$$

From the first and third equations of (2.3), we get

$$v=\phi-f_1,\ \vartheta=\theta-f_3\tag{2.4}$$

and by substituting (2.4) into the second and fourth equations of (2.3), we obtain

$$\begin{cases}\rho_1\phi-BS^{-1}\phi_{xx}+\mu\theta_x=g_1,\\ c\theta-\kappa\theta_{xx}-\delta\theta_{xx}+\mu\phi_x=g_2,\end{cases}\tag{2.5}$$

where

$$g_1=\rho_1 f_1+\rho_1 f_2+\mu f_x^3\in L^2(0,1),\ g_2=\mu f_{1x}-\delta f_{3xx}-cf_3+cf_4\in H^{-1}(0,1).$$

Let's define the following variational formulation

$$B\left((\phi,\theta),(\widetilde{\phi},\widetilde{\theta})\right)=L\Big(\widetilde{\phi},\widetilde{\theta}\Big),\ \forall\left(\widetilde{\phi},\widetilde{\theta}\right)\in\mathcal{W}, \tag{2.6}$$

where

$$\begin{aligned}B\left((\phi,\theta),(\widetilde{\phi},\widetilde{\theta})\right)=&\rho_1\int_0^1\phi\widetilde{\phi}dx+\int_0^1 BS^{-1}\phi_x\widetilde{\phi}_xdx+\mu\int_0^1\theta_x\widetilde{\phi}dx+c\int_0^1\theta\widetilde{\theta}dx\\&+(\kappa+\delta)\int_0^1\theta_x\widetilde{\theta}_xdx+\mu\int_0^1\phi_x\widetilde{\theta}dx\end{aligned}$$

is the bilinear form over $\mathcal{W}=H_0^1(0,1)\times H_*^1(0,1)$ and

$$L\left(\widetilde{\phi},\widetilde{\theta}\right)=\int_0^1 g_1\widetilde{\phi}dx+\mu\int_0^1 f_{1x}\widetilde{\theta}dx+\delta\int_0^1 f_{3x}\widetilde{\theta}_xdx-c\int_0^1 f_3\widetilde{\theta}dx+c\int_0^1 f_4\widetilde{\theta}dx$$

is a linear form. It is easy to check that B and L are bounded. Moreover,

$$\begin{aligned}&B\left((\phi,\theta),(\phi,\theta)\right)\\&=\rho_1\langle\phi,\phi\rangle+\left\langle BS^{-1}\phi_x,\phi_x\right\rangle+c\langle\theta,\theta\rangle+(\kappa+\delta)\langle\theta_x,\theta_x\rangle\\&=\rho_1\langle\phi,\phi\rangle+k\langle\phi_x,\phi_x\rangle+k^2\left\langle S^{-1}\phi_x,\phi_x\right\rangle+c\langle\theta,\theta\rangle+(\kappa+\delta)\langle\theta_x,\theta_x\rangle\\&=\rho_1\langle\phi,\phi\rangle+k\langle\phi_x,\phi_x\rangle+k^2\left\langle S^{-1/2}\phi_x,S^{-1/2}\phi_x\right\rangle+c\langle\theta,\theta\rangle+(\kappa+\delta)\langle\theta_x,\theta_x\rangle\\&\geq C\|(\phi,\theta)\|_{\mathcal{W}}^2,\end{aligned}$$

for some $C>0$; which shows that B is coercive. Thus, the Lax-Milgram theorem guarantees the existence of a unique solution $(\phi,\theta)\in\mathcal{W}$ to the problem (2.6).

Next, we take $\left(\widetilde{\phi},\widetilde{\theta}\right)=\left(\widetilde{\phi},0\right)$ in (2.6) to arrive at

$$\int_0^1 BS^{-1}\phi_x\widetilde{\phi}_xdx=\int_0^1\left[g_1-\rho_1\phi-\mu\theta_x\right]\widetilde{\phi}dx,\ \forall\widetilde{\phi}\in H_0^1(0,1).$$

Then the elliptic regularity theory gives

$$BS^{-1}\phi\in H^2(0,1),$$

with

$$BS^{-1}\phi_{xx}=\rho_1\phi+\mu\theta_x-\rho_1 f_1-\rho_1 f_2-\mu f_x^3\in L^2(0,1).$$

Since $BS^{-1}:L^2(0,1)\to L^2(0,1)$ is a bijection, we conclude that $\phi_{xx}\in L^2(0,1)$; consequently

$$\phi\in H_0^1(0,1)\cap H^2(0,1). \tag{2.7}$$

Further, by using (2.4), we get

$$\rho_1 v-BS^{-1}\phi_{xx}+\mu\vartheta_x=\rho_1 f_2.$$

Similarly, taking $\left(\widetilde{\phi},\widetilde{\theta}\right)=\left(0,\widetilde{\theta}\right)$, we infer that

$$\int_0^1\left[\kappa\theta_x+\delta\vartheta_x\right]\widetilde{\theta}_xdx=\int_0^1\left[\mu f_{1x}-cf_3+cf_4-\mu\phi_x-c\theta\right]\widetilde{\theta}dx,\ \forall\widetilde{\theta}\in H_*^1(0,1).$$

To be able to apply the regularity results, let $\widetilde{\Theta} \in H_0^1(0,1)$ and set $\widetilde{\theta} = \widetilde{\Theta} - \int_0^1 \widetilde{\Theta}(x)\,dx$. Clearly $\widetilde{\theta} \in H_*^1(0,1)$ and we have

$$\int_0^1 [\kappa\theta_x + \delta\vartheta_x]\, \widetilde{\Theta}_x dx = \int_0^1 [\mu f_{1x} - cf_3 + cf_4 - \mu\phi_x - c\theta]\, \widetilde{\Theta} dx,\ \forall \widetilde{\Theta} \in H_0^1(0,1).$$

Therefore,

$$\kappa\theta + \delta\vartheta \in H^2(0,1)$$

and

$$\kappa\theta_{xx} + \delta\vartheta_{xx} = \mu\phi_x + c\theta - \mu f_{1x} + cf_3 - cf_4 \in L^2(0,1).$$

Then, use of (2.4) yields

$$c\vartheta - \kappa\theta_{xx} - \delta\vartheta_{xx} + \mu v_x = cf_4.$$

On the other hand, since $\kappa\theta_{xx} + \delta\vartheta_{xx} = R \in L^2(0,1)$, then

$$-\int_0^1 [\kappa\theta_{xx} + \delta\vartheta_{xx}]\, \Psi dx = \int_0^1 R\Psi dx,\ \forall \Psi \in H^1(0,1). \tag{2.8}$$

Integration by parts gives

$$\int_0^1 [\kappa\theta_x + \delta\vartheta_x]\, \Psi_x dx - [(\kappa\theta_x + \delta\vartheta_x)\, \Psi]_0^1 = \int_0^1 R\Psi dx,\ \forall \Psi \in H^1(0,1),$$

consequently, we have

$$\int_0^1 [\kappa\theta_x + \delta\vartheta_x]\, \Psi_x dx - [(\kappa\theta_x + \delta\vartheta_x)\, \Psi]_0^1 = \int_0^1 R\Psi dx,\ \forall \Psi \in H_*^1(0,1).$$

We use (2.8) to get

$$[\kappa\theta_x(1) + \delta\vartheta_x(1)]\, \Psi(1) - [\kappa\theta_x(0) + \delta\vartheta_x(0)]\, \Psi(0) = 0.$$

Since $\Psi \in H_*^1(0,1)$ is arbitrary, then

$$\kappa\theta_x(1) + \delta\vartheta_x(1) = \kappa\theta_x(0) + \delta\vartheta_x(0) = 0,$$

which means that

$$\kappa\theta + \delta\vartheta \in H_*^2(0,1). \tag{2.9}$$

Substituting (2.7) and (2.9) into (2.4), we arrive at

$$v \in H_0^1(0,1) \text{ and } \vartheta \in H_*^1(0,1). \tag{2.10}$$

Hence, the solution Φ belongs to $D(A)$. This shows that A is maximal. Thus, the Hille-Yosida theorem guarantees the existence of a unique solution to the problem (2.1),(2.2). This completes the proof of Theorem 2.1. $\square$

Now, we turn to our problem (1.11)-(1.13). For this purpose, we define the following set

$$\mathcal{D} = \left\{ \begin{array}{l} U = (\phi, v, \omega, \theta, \vartheta) \in H_0^1(0,1) \times L^2(0,1) \times H^1(0,1) \times H_*^1(0,1) \times L_*^2(0,1) : \\ \phi \in H^2(0,1),\ v \in H_0^1(0,1),\ \vartheta \in H_*^1(0,1),\ \kappa\theta + \delta\vartheta \in H_*^2(0,1),\ \omega \in H_*^2(0,1) \end{array} \right\}.$$

Our well-posedness result reads as follows:

Theorem 2.2. *For any* $(\phi_0, \phi_1, \theta_0, \theta_1) \in D(A)$*, problem (1.11)-(1.13) has a unique solution,* $(\phi, \phi_t, \omega, \theta, \theta_t) \in C(\mathbb{R}^+; \mathcal{D})$ *and* $(\phi, \phi_t, \theta, \theta_t) \in C^1(\mathbb{R}^+; H)$*.*

Proof. By using Theorem 2.1, we find $(\phi, \phi_t, \theta, \theta_t) \in C(\mathbb{R}^+; D(A)) \cap C^1(\mathbb{R}^+; H)$.

To fined ω, we use the second equation of (1.11).

Namely, we consider the following problem

$$\begin{cases} -b\omega_{xx} + k\omega = -k\phi_x, \\ \omega_x(0,t) = \omega_x(1,t) = 0. \end{cases}$$

A simple application of Lax-Milgram theorem, we obtain $\omega \in C\left(\mathbb{R}^+; H_*^2(0,1)\right)$, since $\phi_x \in C\left(\mathbb{R}^+; H^1(0,1)\right)$. Therefore, $(\phi, \phi_t, \omega, \theta, \theta_t) \in C(\mathbb{R}^+; \mathcal{D})$.

By recalling (2.1), we easily check that (ϕ, ω, θ) is the desired solution. □

3. Exponential stability

In this section, we present and prove our exponential decay result. For this, we need the following essential lemmas.

Lemma 3.1. *The energy functional of the system (1.11)-(1.13), given by*

$$E(t) = \frac{1}{2}\int_0^1 \left(\rho_1\phi_t^2 + k(\phi_x + \omega)^2 + b\omega_x^2 + c\theta_t^2 + \kappa\theta_x^2\right) dx, \tag{3.1}$$

satisfies

$$E'(t) = -\delta \int_0^1 \theta_{xt}^2 dx \le 0. \tag{3.2}$$

Proof. Taking the L^2-product of equation $(1.11)_1$ with ϕ_t, equation $(1.11)_2$ with ω_t, and equation $(1.11)_3$ with θ_t, and applying integration by parts, we get

$$\begin{aligned} &\frac{\rho_1}{2}\frac{d}{dt}\int_0^1 \phi_t^2 dx + k\int_0^1 (\phi_x + \omega)\phi_{xt} dx - \mu\int_0^1 \theta_t \phi_{xt} dx = 0, \\ &\frac{b}{2}\frac{d}{dt}\int_0^1 \omega_x^2 dx + k\int_0^1 (\phi_x + \omega)\omega_t dx = 0 \end{aligned}$$

and

$$\frac{c}{2}\frac{d}{dt}\int_0^1 \theta_t^2 dx + \frac{\kappa}{2}\frac{d}{dt}\int_0^1 \theta_x^2 dx + \delta\int_0^1 \theta_{xt}^2 dx + \mu\int_0^1 \phi_{xt}\theta_t dx = 0.$$

Thus, summing up, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\rho_1\int_0^1 \phi_t^2 dx + k\int_0^1 (\phi_x + \omega)^2 dx + b\int_0^1 \omega_x^2 dx + c\int_0^1 \theta_t^2 dx + \kappa\int_0^1 \theta_x^2 dx\right] \\ &= -\delta\int_0^1 \theta_{xt}^2 dx. \end{aligned} \tag{3.3}$$

□

Lemma 3.2. *The functional*

$$F_1(t) = c\int_0^1 \theta_t\theta dx + \frac{\delta}{2}\int_0^1 \theta_x^2 dx + \mu\int_0^1 \phi_x\theta dx$$

satisfies, along the solution of (1.11)-(1.13) and for any $\varepsilon_1 > 0$, *the estimate*

$$F_1^{'}(t) \leq -\kappa\int_0^1 \theta_x^2 dx + \left(c + \frac{\mu^2}{4\varepsilon_1}\right)c_p\int_0^1 \theta_{xt}^2 dx + 2\varepsilon_1\int_0^1 (\phi_x+\omega)^2 dx + 2c_p\varepsilon_1\int_0^1 \omega_x^2 dx, \tag{3.4}$$

where c_p *is the Poincaré constant.*

Proof. Differentiating F_1, using (1.11), integration by parts, and the boundary conditions, we obtain

$$\begin{aligned} F_1^{'}(t) &= \int_0^1 (\kappa\theta_{xx} + \delta\theta_{xxt} - \mu\phi_{xt})\theta dx + c\int_0^1 \theta_t^2 dx + \delta\int_0^1 \theta_x\theta_{xt} dx + \mu\int_0^1 \phi_{xt}\theta dx \\ &\quad + \mu\int_0^1 \phi_x\theta_t dx \\ &= -\kappa\int_0^1 \theta_x^2 dx - \delta\int_0^1 \theta_{xt}\theta_x dx + \delta\int_0^1 \theta_x\theta_{xt} dx + c\int_0^1 \theta_t^2 dx + \mu\int_0^1 \phi_x\theta_t dx \\ &= -\kappa\int_0^1 \theta_x^2 dx + c\int_0^1 \theta_t^2 dx + \mu\int_0^1 \phi_x\theta_t dx. \end{aligned}$$

Using Young's and Poincaré's inequalities, we get, for any $\varepsilon_1 > 0$,

$$F_1^{'}(t) \leq -\kappa\int_0^1 \theta_x^2 dx + \left(c + \frac{\mu^2}{4\varepsilon_1}\right)c_p\int_0^1 \theta_{xt}^2 dx + \varepsilon_1\int_0^1 \phi_x^2 dx.$$

The fact that $\phi_x^2 \leq 2(\phi_x+\omega)^2 + 2\omega$ and Poincaré's inequality lead to (3.4). □

Lemma 3.3. *The functional*

$$F_2(t) = \rho_1\int_0^1 \phi_t\phi dx$$

satisfies, along the solution of (1.11)-(1.13), the estimate

$$F_2^{'}(t) \leq \rho_1\int_0^1 \phi_t^2 dx - \frac{k}{2}\int_0^1 (\phi_x+\omega)^2 dx - \frac{b}{2}\int_0^1 \omega_x^2 dx + M\int_0^1 \theta_{tx}^2 dx, \tag{3.5}$$

where $M > 0$ *is a constant.*

Proof. The differentiation of F_2, using the first equation of (1.11) and integration by parts, gives

$$\begin{aligned} F_2^{'}(t) &= \rho_1\int_0^1 \phi_t^2 dx + \rho_1\int_0^1 \phi_{tt}\phi dx \\ &= \rho_1\int_0^1 \phi_t^2 dx + k\int_0^1 \phi\left[(\phi_x+\omega)_x - \mu\theta_{xt}\right] dx \end{aligned}$$

$$= \rho_1 \int_0^1 \phi_t^2 dx - k \int_0^1 \phi_x (\phi_x + \omega) dx - \mu \int_0^1 \phi \theta_{xt} dx$$
$$= \rho_1 \int_0^1 \phi_t^2 dx - k \int_0^1 (\phi_x + \omega)^2 dx + k \int_0^1 (\phi_x + \omega) \omega dx + \mu \int_0^1 \phi \theta_{xt} dx. \tag{3.6}$$

From the second equation of (1.11) and (1.12), we have

$$k \int_0^1 (\phi_x + \omega) \omega dx = -b \int_0^1 \omega_x^2 dx. \tag{3.7}$$

Plugging (3.7) into (3.6), we arrive at

$$F_2'(t) = \rho_1 \int_0^1 \phi_t^2 dx - k \int_0^1 (\phi_x + \omega)^2 dx - b \int_0^1 \omega_x^2 dx + \mu \int_0^1 \phi \theta_{xt} dx$$

and using Young's and Poincaré's inequalities, and again the fact $\phi_x^2 \le 2(\phi_x + \omega)^2 + 2\omega$, we obtain, $\forall \varepsilon_2 > 0$,

$$F_2'(t) = \rho_1 \int_0^1 \phi_t^2 dx - k \int_0^1 (\phi_x + \omega)^2 dx - b \int_0^1 \omega_x^2 dx + 2\varepsilon_2 \int_0^1 (\phi_x + \omega)^2 dx$$
$$+ 2\varepsilon_2 c_p \int_0^1 \omega_x^2 dx + \frac{\mu^2 c_p}{4\varepsilon_2} \int_0^1 \theta_{xt}^2 dx.$$

Choosing $\varepsilon_2 \le \min\left(\frac{k}{4}, \frac{b}{4c_p}\right)$, the estimate (3.5) follows immediately. □

Lemma 3.4. *The functional*

$$F_3(t) = \rho_1 c \int_0^1 \phi_t \int_0^x \theta_t(y) dy dx$$

satisfies, along the solution of (1.11)-(1.13) and for all $\varepsilon_3 > 0$, the estimate

$$F_3'(t) \le -\frac{\rho_1}{2} \int_0^1 \phi_t^2 dx + \varepsilon_3 \int_0^1 (\phi_x + \omega)^2 dx + \frac{\kappa^2 \rho_1}{\mu^2} \int_0^1 \theta_x^2 dx$$
$$+ \left[\frac{c^2 k^2 c_p}{4\mu^2 \varepsilon_3} + c c_p + \frac{\delta^2 \rho_1}{\mu^2}\right] \int_0^1 \theta_{xt}^2 dx, \tag{3.8}$$

where c_p is the Poincaré constant.

Proof. Direct differentiation of F_3, using the first equation of (1.11) and integration by parts, yields

$$F_3'(t) = \frac{c}{\mu} \int_0^1 [k(\phi_x + \omega)_x - \mu \theta_{xt}] \Big(\int_0^x \theta_t(y) dy \Big) dx$$
$$+ \frac{\rho_1}{\mu} \int_0^1 \phi_t \Big(\int_0^x (\kappa \theta_{xx} + \delta \theta_{xxt} - \mu \phi_{xt})(y) dy \Big) dx$$
$$= -\frac{c}{\mu} \int_0^1 [k(\phi_x + \omega) - \mu \theta_t] \theta_t dx + \frac{\rho_1}{\mu} \int_0^1 \phi_t (\kappa \theta_x + \delta \theta_{xt} - \mu \phi_t) dx$$
$$= -\frac{ck}{\mu} \int_0^1 (\phi_x + \omega) \theta_t dx + c \int_0^1 \theta_t^2 dx + \frac{\kappa \rho_1}{\mu} \int_0^1 \theta_x \phi_t dx$$

$$+\frac{\delta\rho_1}{\mu}\int_0^1 \theta_{xt}\phi_t dx - \rho_1 \int_0^1 \phi_t^2 dx.$$

Exploiting Young's and Poincaré's inequalities, we arrive at (3.8). □

Theorem 3.1. *Let (ϕ, ω, θ) be the solution of problem (1.11)-(1.13). Then the energy functional (3.1) satisfies*

$$E(t) \le \sigma e^{-\varpi t}, \forall t \ge 0,$$

where σ and ϖ are two positive constants.

Proof. First, we define the Lyapunov functional by

$$L(t) = NE(t) + N_1 F_1(t) + F_2(t) + 3F_3(t),$$

for N and N_1, positive constants to be determined appropriately.

Clearly, we have

$$\begin{aligned} & |L(t) - NE(t)| \\ \le & C\left[\left|\int_0^1 \theta_t \theta dx\right| + \left|\int_0^1 \theta_x^2 dx\right| + \left|\int_0^1 \phi_x \theta dx\right| + \left|\int_0^1 \phi_t \phi dx\right| + \left|\int_0^1 \phi_t \int_0^x \theta_t(y)\, dy dx\right|\right].\end{aligned}$$

Cauchy–Schwarz, Young's and Poincaré's inequalities give, for some $\lambda > 0$,

$$|L(t) - NE(t)| \le \lambda E(t).$$

Consequently, we obtain

$$(N - \lambda) E(t) \le L(t) \le (\lambda + N) E(t), \quad \forall t > 0. \tag{3.9}$$

Next, by differentiating L and using (3.2), (3.4), (3.5) and (3.8), we get

$$\begin{aligned} L'(t) \le & -\left[\delta N - \left(c + \frac{\mu^2}{4\varepsilon_1}\right) c_p N_1 - M - 3\left[\frac{c^2 k^2 c_p}{4\mu^2 \varepsilon_3} + c c_p + \frac{\delta^2 \rho_1}{\mu^2}\right]\right] \int_0^1 \theta_{xt}^2 dx \\ & - \left[\kappa N_1 - 3\frac{\kappa^2 \rho_1}{\mu^2}\right] \int_0^1 \theta_x^2 dx - \left[\frac{b}{2} - 2c_p \varepsilon_1 N_1\right] \int_0^1 \omega_x^2 dx \\ & - \left[\frac{k}{2} - 2\varepsilon_1 N_1 - 3\varepsilon_3\right] \int_0^1 (\phi_x + \omega)^2 dx - \frac{\rho_1}{2} \int_0^1 \phi_t^2 dx. \end{aligned}$$

At this point, we choose $N_1 > 3\frac{\kappa\rho_1}{\mu^2}$ and select ε_1 such that

$$\frac{b}{2} - 2c_p \varepsilon_1 N_1 > 0 \quad \text{and} \quad \frac{k}{2} - 2\varepsilon_1 N_1 = C_1 > 0.$$

After that, we pick ε_3 small enough such that

$$C_2 = C_1 - 3\varepsilon_3 > 0.$$

Finally, we choose N very large so that

$$\delta N - \left(c + \frac{\mu^2}{4\varepsilon_1}\right) c_p N_1 - M - 3\left[\frac{c^2 k^2 c_p}{4\mu^2 \varepsilon_3} + c c_p + \frac{\delta^2 \rho_1}{\mu^2}\right] > 0, \quad N - \lambda > 0.$$

Thus, there exist $\zeta, \xi, m_1, m_2 > 0$ such that

$$L^{'}(t) \leq -\zeta \left[\int_0^1 \theta_x^2 dx + \int_0^1 \omega_x^2 dx + \int_0^1 (\phi_x + \omega)^2 dx + \int_0^1 \phi_t^2 dx\right] - \xi \int_0^1 \theta_{xt}^2 dx$$

and

$$m_1 E(t) \leq L(t) \leq m_2 E(t), \quad \forall t \geq 0. \tag{3.10}$$

Using Poincaré's inequality, we infer that there exists $\gamma, \varpi > 0$ such that

$$L^{'}(t) \leq -\gamma E(t) \leq -\varpi L(t), \quad \forall t \geq 0.$$

Therefore, by integration we obtain

$$L(t) \leq L(0) e^{-\varpi t}, \quad \forall t \geq 0.$$

Again, using (3.10), we get, for some $\sigma > 0$,

$$E(t) \leq \sigma e^{-\varpi t}, \quad \forall t \geq 0.$$

This completes the proof of Theorem 3.1. □

4. Numerical experiments

In this section, we propose a finite element approximation to system (1.11) with the initial conditions (1.13) and under the boundary conditions (1.12). Moreover, we prove that the discrete energy decays, from which we derive a discrete stability property. Multiplying (1.11) by test functions $\xi, \eta, \zeta \in H_0^1(0,1)$, we get the following weak form:

$$\begin{cases} \rho_1(\phi_{tt}, \xi) + k(\phi_x + \omega, \xi_x) - \mu(\theta_t, \xi_x) = 0, \\ b(\omega_x, \eta_x) + k(\phi_x + \omega, \eta) = 0, \\ c(\theta_{tt}, \zeta) + \kappa(\theta_x, \zeta_x) + \delta(\theta_{tx}, \zeta_x) - \mu(\phi_t, \zeta_x) = 0. \end{cases} \tag{4.1}$$

Let us partition the interval $(0,1)$ into subintervals $I_i = (x_{i-1}, x_i)$ of length $h = 1/N_h$ with $0 = x_0 < x_1 < \cdots < x_{N_h} = 1$ and define

$$P_h^1 = \left\{u \in H_0^1(0,1) \cap C(0,1), u|_{I_i} \text{ is a linear function}\right\}.$$

For a given final time T and a positive integer N_t, let $\Delta t = T/N_t$ be the time step and $t_n = n\Delta t$, $n = 0, \ldots, N_t$. The finite element method for (4.1) is to find ϕ_h^n, ω_h^n, $\theta_h^n \in P_h^1$, such that, for all ξ_h, η_h, $\zeta_h \in P_h^1$,

$$\begin{cases} \frac{\rho_1}{\Delta t}(\phi_{ht}^n - \phi_{ht}^{n-1}, \xi_h) + k(\phi_{hx}^n + \omega_h^n, \xi_{hx}) - \mu(\theta_{ht}^n, \xi_{hx}) = 0, \\ b(\omega_{hx}^n, \eta_{hx}) + k(\phi_{hx}^n + \omega_h^n, \eta_h) = 0, \\ \frac{c}{\Delta t}(\theta_{ht}^n - \theta_{ht}^{n-1}, \zeta_h) + \kappa(\theta_{hx}^n, \zeta_{hx}) + \delta(\theta_{htx}^n, \zeta_{hx}) - \mu(\phi_{ht}^n, \zeta_{hx}) = 0, \end{cases} \tag{4.2}$$

where $\phi_h^n = \phi_h^{n-1} + \Delta t \phi_{ht}^n$ and $\theta_h^n = \theta_h^{n-1} + \Delta t \theta_{ht}^n$. The notations ϕ_h^0, ϕ_{ht}^0, ω_h^0, θ_h^0, θ_{ht}^0 are adequate approximations to ϕ_0, ϕ_1, ω_0, θ_0 and θ_1, respectively. Then, the discrete energy is given by

$$E_h^n = \frac{1}{2}\left(\rho_1 \|\phi_{ht}^n\|_2^2 + k\|\phi_{hx}^n + \omega_h^n\|_2^2 + b\|\omega_{hx}^n\|_2^2 + c\|\theta_{ht}^n\|_2^2 + \kappa\|\theta_{hx}^n\|_2^2\right). \tag{4.3}$$

The next result is a discrete version of the energy decay property satisfied by the solution of system (1.11).

Theorem 4.1. *For $n = 1, 2, \ldots, N_t$, the discrete energy satisfies the following decay property*

$$\frac{1}{\Delta t}\left(E_h^n - E_h^{n-1}\right) \leq 0. \tag{4.4}$$

Proof. Choosing $\xi_h = \phi_{ht}^n$, $\eta_h = \omega_h^n$, and $\zeta_h = \theta_{ht}^n$ in (4.2) and thanks to the following equality

$$(a-b, a) = \frac{1}{2}\left(\|a-b\|_2^2 + \|a\|_2^2 - \|b\|_2^2\right),$$

we obtain

$$\begin{cases} \dfrac{\rho_1}{2\Delta t}(\|\phi_{ht}^n - \phi_{ht}^{n-1}\|_2^2 + \|\phi_{ht}^n\|_2^2 - \|\phi_{ht}^{n-1}\|_2^2) + k(\phi_{hx}^n + \omega_h^n, \phi_{htx}^n) - \mu(\theta_{ht}^n, \phi_{htx}^n) = 0, \\ b\|\omega_{hx}^n\|_2^2 + k(\phi_{hx}^n + \omega_h^n, \omega_h^n) = 0, \\ \dfrac{c}{2\Delta t}(\|\theta_{ht}^n - \theta_{ht}^{n-1}\|_2^2 + \|\theta_{ht}^n\|_2^2 - \|\theta_{ht}^{n-1}\|_2^2) + \kappa(\theta_{hx}^n, \theta_{htx}^n) \\ +\delta\|\theta_{htx}^n\|_2^2 - \mu(\phi_{ht}^n, \theta_{htx}^n) = 0. \end{cases}$$

Adding the latter equations, we find that

$$\begin{aligned} &\frac{\rho_1}{2\Delta t}(\|\phi_{ht}^n - \phi_{ht}^{n-1}\|_2^2 + \|\phi_{ht}^n\|_2^2 - \|\phi_{ht}^{n-1}\|_2^2) + k(\phi_{hx}^n + \omega_h^n, \phi_{htx}^n + \omega_h^n) + b\|\omega_{hx}^n\|_2^2 \\ &+ \frac{c}{2\Delta t}(\|\theta_{ht}^n - \theta_{ht}^{n-1}\|_2^2 + \|\theta_{ht}^n\|_2^2 - \|\theta_{ht}^{n-1}\|_2^2) + \kappa(\theta_{hx}^n, \theta_{htx}^n) + \delta\|\theta_{htx}^n\|_2^2 = 0. \end{aligned}$$

We note that

$$\begin{aligned} &k(\phi_{hx}^n + \omega_h^n, \phi_{htx}^n + \omega_h^n) \\ =&\frac{k}{\Delta t}(\phi_{hx}^n + \omega_h^n, \phi_{hx}^n + \omega_h^n - (\phi_{hx}^{n-1} + \omega_h^{n-1})) \\ &+ \frac{k}{\Delta t}(\phi_{hx}^n + \omega_h^n, -\omega_h^n + \omega_h^{n-1}) + k(\phi_{hx}^n + \omega_h^n, \omega_h^n) \\ =&\frac{k}{2\Delta t}(\|\phi_{hx}^n + \omega_h^n - (\phi_{hx}^{n-1} + \omega_h^{n-1})\|_2^2 + \|\phi_{hx}^n + \omega_h^n\|_2^2 - \|\phi_{hx}^{n-1} + \omega_h^{n-1}\|_2^2) \\ &- \frac{b}{\Delta t}(\omega_{hx}^n, \omega_{hx}^{n-1}) + (\frac{b}{\Delta t} - b)\|\omega_{hx}^n\|_2^2 \\ \geq&\frac{k}{2\Delta t}(\|\phi_{hx}^n + \omega_h^n - (\phi_{hx}^{n-1} + \omega_h^{n-1})\|_2^2 + \|\phi_{hx}^n + \omega_h^n\|_2^2 - \|\phi_{hx}^{n-1} + \omega_h^{n-1}\|_2^2) \\ &- \frac{b}{2\Delta t}(\|\omega_{hx}^n\|_2^2 + \|\omega_{hx}^{n-1}\|_2^2) \end{aligned}$$

and

$$\begin{aligned} \kappa(\theta_{hx}^n, \theta_{htx}^n) =&\frac{\kappa}{\Delta t}(\theta_{hx}^n, \theta_{hx}^n - \theta_{hx}^{n-1}) \\ =&\frac{\kappa}{2\Delta t}\left(\|\theta_{hx}^n - \theta_{hx}^{n-1}\|_2^2 + \|\theta_{hx}^n\|_2^2 - \|\theta_{hx}^{n-1}\|_2^2\right). \end{aligned}$$

These results together, yield

$$\frac{1}{\Delta t}\Big[\frac{1}{2}\left(\rho_1\|\phi_{ht}^n\|_2^2 + k\|\phi_{hx}^n + \omega_h^n\|_2^2 + b\|\omega_{hx}^n\|_2^2 + c\|\theta_{ht}^n\|_2^2 + \kappa\|\theta_{hx}^n\|_2^2\right)$$

$$- \frac{1}{2} \left(\rho_1 \|\phi_{ht}^{n-1}\|_2^2 + k\|\phi_{hx}^{n-1} + \omega_h^{n-1}\|_2^2 + b\|\omega_{hx}^{n-1}\|_2^2 + c\|\theta_{ht}^{n-1}\|_2^2 + \kappa\|\theta_{hx}^{n-1}\|_2^2 \right) \Big] \leq 0$$

and the theorem is proved using the definition of the discrete energy (4.3). □

In order to illustrate the theoretical results, some numerical experiments have been performed using the numerical method analysed in the previous section. We divide the spatial interval $[0, 1]$ into $N_h = 50$ subintervals, where the spatial step size $h = 0.02$. The temporal interval $[0, T] = [0, 150]$ with a time step size $\Delta t = 10^{-2}$.

We run our code for N_t time steps ($N_t = T/\Delta t$) using the following initial conditions:

$$\phi_h^0(x) = \frac{1}{2}\sin(\pi x), \quad \phi_{ht}^0(x) = -\frac{k\pi}{\rho_1}(\pi \sin(\pi x) + \sin(\pi x)) + \frac{\mu\pi}{\rho_1}\sin(\pi x),$$
$$\omega_h^0(x) = 0, \quad \theta_h^0(x) = \cos(\pi x), \quad \theta_{ht}^0(x) = 2\cos(\pi x).$$

The numerical tests are done for different entries as follows:

- Test 1:
$$\rho_1 = k = \mu = b = \kappa = \delta = c = 1.$$
- Test 2:
$$\rho_1 = k = \mu = b = \kappa = c = 1 \text{ and } \delta = 0.1.$$
- Test 3:
$$\rho_1 = k = b = \kappa = \delta = c = 1 \text{ and } \mu = 2.$$

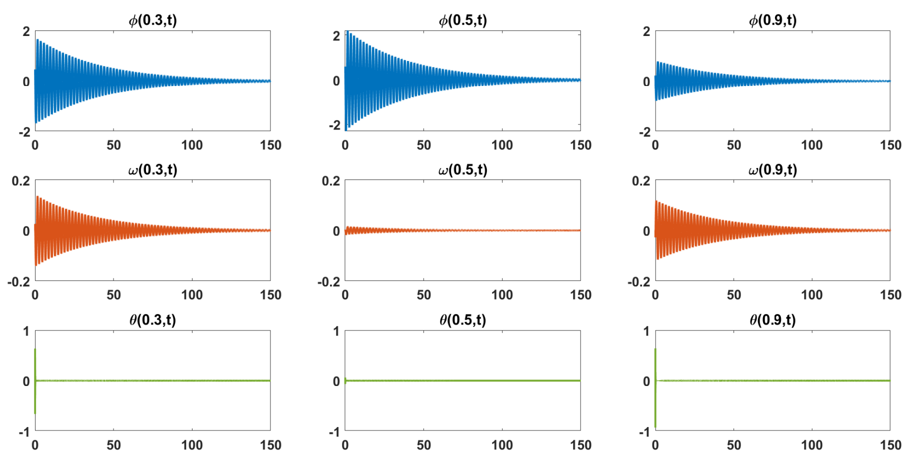

Figure 1. Test 1: Damping cross section waves.

For each numerical test, we plot the cross section cuts for the approximate solution (ϕ, ω, θ) at different points ($x = 0.3$, $x = 0.5$ and $x = 0.9$). The numerical results are shown in Figure 1, Figure 2 and Figure 3, successively. Next, we present the energy curves for the three cases in Figure 4, for times between $t = 5$ and $t = T = 150$, to show the difference between the energy decays according to the parameters chosen in each test.

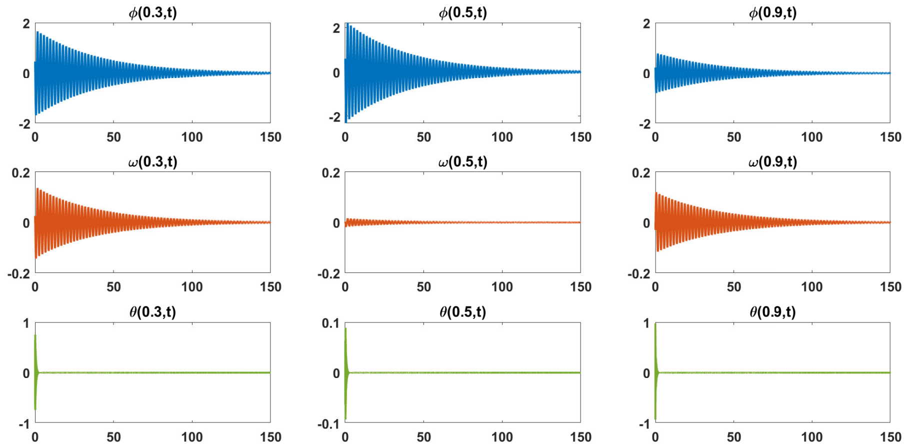

Figure 2. Test 2: Damping cross section waves.

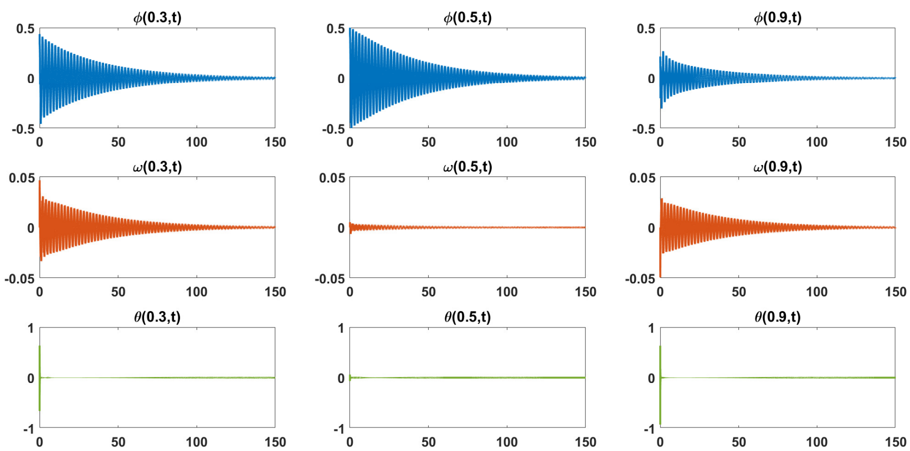

Figure 3. Test 3: Damping cross section waves.

As a conclusion, for all tests, we observed that the numerical solution converges to zero and an exponential decay with different rates (see log scales of the energy on the right of Figure 4) seems to be reached which is compatible with the theoretical results.

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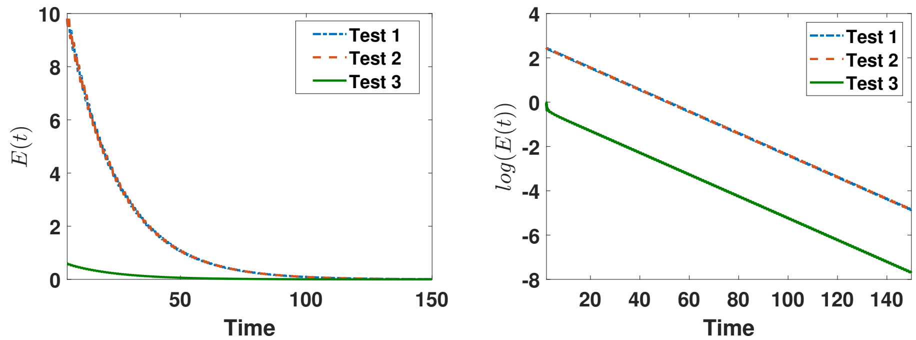

Figure 4. Energy decay in natural and log scales.

Disclosure statement

The authors have no conflict of interest.

Data availability statement

No data are involved in this research.

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