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ESTIMATES FOR BILINEAR Θ-TYPE CALDERÓN-ZYGMUND OPERATORS AND THEIR COMMUTATORS ON NON-HOMOGENEOUS GENERALIZED WEIGHTED MORREY SPACES*

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Abstract Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space satisfying geometrically doubling and upper doubling conditions. Under assumption that a dominating function λ satisfies ε-weak reverse doubling condition, the authors prove that a bilinear θ-type Calderón-Zygmund operator $\widetilde{T}_\theta$ is bounded from product of generalized weighted Morrey spaces $\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu) \times \mathcal{L}^{p_2,\Phi,\varrho}_{\omega_2}(\mu)$ into weak generalized weighted Morrey spaces $W\mathcal{L}^{p,\Phi,\varrho}_{\nu_{\vec{\omega}}}(\mu)$, and also show that the commutator $\widetilde{T}_{\theta,b_1,b_2}$ generated by $b_1, b_2 \in \widetilde{\mathrm{RBMO}}(\mu)$ and $\widetilde{T}_\theta$ are bounded from product of spaces $\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu) \times \mathcal{L}^{p_2,\Phi,\varrho}_{\omega_2}(\mu)$ into spaces $W\mathcal{L}^{p,\Phi,\varrho}_{\nu_{\vec{\omega}}}(\mu)$, where $\Phi : (0,\infty) \to (0,\infty)$ is a Lebesgue measurable function, $\varrho \in (1,\infty)$, $\vec{p} = (p_1, p_2)$, $\vec{\omega} = (\omega_1, \omega_2) \in A^{\tau}_{\vec{p}}(\mu)$, $\nu_{\vec{\omega}} \in RH_r(\mu)$ for $r \in (1,\infty)$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ with $1 < p_1, p_2 < \infty$. Furthermore, the strong and weak type results for the $\widetilde{T}_\theta$ and $\widetilde{T}_{\theta,b_1,b_2}$ on the product of spaces $\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu) \times \mathcal{L}^{p_2,\Phi,\varrho}_{\omega_2}(\mu)$ are established.

Keywords Non-homogeneous metric measure space, bilinear θ-type Calder ón-Zygmund operator, commutator, space $\widetilde{\mathrm{RBMO}}(\mu)$, generalized weighted Morrey space.

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1. Introduction

It is well known that the researches on the boundedness of operators is not only a hot topic in modern harmonic analysis, but also their use is best justified by the variety of applications in which they appear; for example, see [3, 4, 8]. To investigate the local behaviour of solutions for the second order elliptic partial differential equations, C.B. Morrey [35] introduced the classical Morrey space. On the basis of this, B. Muckenhoupt and R. Wheeden [36] established the weighted norm inequalities for the Hardy maximal functions; in 1994, E. Nakai [37] introduced a generalized Morrey space $L^{p,\omega}(\mathbb{R}^n)$, and also obtained the boundedness of the Hardy-

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Littlewood maximal operator M, the singular integral operator T and the Riesz potential I_α on spaces $L^{p,\omega}(\mathbb{R}^n)$. In 2009, T.Y. Komori and S. Shirai [18] introduced a weighted Morrey space $L^{p,\kappa}_\omega(\mathbb{R}^n)$, and proved that the Hardy-Littlewood maximal operator M, the Calderón-Zygmund operator T and the fractional integral operator I_α are bounded on spaces $L^{p,\kappa}_\omega(\mathbb{R}^n)$. In recent years, many papers focus the various Morrey spaces on different kinds of underlying spaces. For example, in 2021, I. Ekincioglu et al. [10] introduced a generalized variable exponent Morrey space $M^{p(\cdot),\varphi}(\mathbb{R}^n)$, and showed that the multilinear commutators $T_{\mathbf{b}}$ generated by Calderón-Zygmund operators T and $\mathbf{b}=(b_1,\cdots,b_m)\in\big(\mathrm{BMO}(\mathbb{R}^n)\big)^m$ are bounded on spaces $M^{p(\cdot),\varphi}(\mathbb{R}^n)$. In 2022, Wei [47] obtained the definition of a generalized mixed Morrey space $M^u_{\vec{p}}(\mathbb{R}^n)$ and its dual space, and then established the boundedness of Calderón-Zygmund singular integral operators T on spaces $M^u_{\vec{p}}(\mathbb{R}^n)$ for $\vec{p}=(p_1,\cdots,p_n)\in(1,\infty)^n$. In 2023, F. Deringoz [9] obtained the definition of a generalized weighted Orlicz-Morrey space $M^{\Phi,\varphi}_\omega(\mathbb{R}^n)$, and proved that the Calderón-Zygmund operators T and their commutators $[b,T]$ associated with BMO functions are bounded on spaces $M^{\Phi,\varphi}_\omega(\mathbb{R}^n)$. Recently, Lu et al. [32] obtain the definition of a generalized Morrey space over RD-spaces satisfying the doubling conditions in the sense of Coifman and Weiss in [6,7] and the reverse doubling conditions, and show that the bilinear generalized fractional integral operator $\widetilde{T}_\alpha$ and its commutator $\widetilde{T}_{\alpha,b_1,b_2}$ which is formed by $b_1,b_2\in\mathrm{BMO}(X)$ are bounded on product of spaces $\mathcal{L}^{\varphi_1,p_1}(X)\times\mathcal{L}^{\varphi_2,p_2}(X)$. More development on the various generalized Morrey spaces can be seen in [5,19,20,23,29,30,39].

Regarding two important class of function spaces in harmonic analysis, i.e., spaces of homogeneous type in the sense of Coifman and Weiss [6,7] and non-doubling measure spaces whose measures satisfy the polynomial growth conditions (see [38, 42,43,46]), many results from real analysis and harmonic analysis on spaces $\mathbb{R}^n$ are proved still valid on these two spaces. But, generally, some results hold on spaces of homogeneous type many not be correct on spaces without doubling measures. To unify the two class of spaces, in 2010, T. Hytönen [15] introduced a new class of metric measure spaces satisfying so-called geometrically doubling and upper doubling conditions, which are now called *non-homogeneous metric measure spaces* and simply denoted by $(\mathcal{X},d,\mu)$. Since then, many papers focus on the various properties of function spaces and integral operators over $(\mathcal{X},d,\mu)$. For example, in 2021, Lu [25] showed that an θ-type Calderón-Zygmund operator T_θ and its commutator $[b,T_\theta]$ generated by $b\in\mathrm{RBMO}(\mu)$ and T_θ are bounded on weighted weak Lebesgue spaces $WL^p(\omega)$ and weighted weak Morrey spaces $WL^{p,\kappa,\rho}(\omega)$. At the same year, Zhao et al. in [50] obtained some weak-type multiple weighted estimates for the iterated commutator $T_{\prod\vec{b}}$ formed by $\vec{b}=(b_1,\cdots,b_m)\in[\widetilde{\mathrm{RBMO}}(\mu)]^m$ and a multilinear Calderón-Zygmund operator T. In 2022, Lu [26] proved that fractional type Marcinkiewicz integrals $\mathcal{M}_{\iota,\rho,m}$ and their commutators $\mathcal{M}_{\iota,\rho,m,b}$ generated by $b\in\widetilde{\mathrm{RBMO}}(\mu)$ and the $\mathcal{M}_{\iota,\rho,m,b}$ are bounded on generalized Morrey spaces $L^{p,\phi}(\mu)$ and on Morrey spaces $M^q_p(\mu)$, where ϕ is a Lebesgue measurable function defined on $(0,\infty)$ and $1<p\le q<\infty$. Recently, Lu et al. [33] show that the bilinear strongly generalized fractional integrals $\widetilde{T}_\alpha$ and their commutator $\widetilde{T}_{\alpha,b_1,b_2}$ formed by $b_1,b_2\in\widetilde{\mathrm{RBMO}}(\mu)$ and $\widetilde{T}_\alpha$ on product of Lebesgue spaces $L^{p_1}(\mu)\times L^{p_2}(\mu)$, product of Morrey spaces $M^{p_1}_{q_1}(\mu)\times M^{p_2}_{q_2}(\mu)$ and product of generalized Morrey spaces $\mathcal{L}^{p_1,u_1}(\mu)\times\mathcal{L}^{p_2,u_2}(\mu)$. More researches about the integral operators and function spaces on $(\mathcal{X},d,\mu)$ can be seen in [13,16,24,27,28,34,41,44,45,48,49].

It is position to state the organizations of this paper as follows: in section 2, we mainly recall some necessary notation and notions. In section 3, the authors showed that $\widetilde{T}_\theta$ is bounded form the product of generalized weighted Morrey spaces $\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu)\times\mathcal{L}^{p_2,\Phi,\varrho}_{\omega_2}(\mu)$ into weak generalized weighted Morrey spaces $W\mathcal{L}^{p,\Phi,\varrho}_{\nu_{\vec{\omega}}}(\mu)$, where Φ is a non-negative Lebesgue measurable function defined on $(0,\infty)$, $\vec{\omega}=(\omega_1,\omega_2)\in A^\tau_{\vec{p}}(\mu)$, $\vec{p}=(p_1,p_2)$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ for $p_1,p_2\in[1,\infty)$, and $\nu_{\vec{\omega}}=\prod\limits_{j=1}^{2}\omega_j^{\frac{p}{p_j}}\in RH_r(\mu)$ with $r\in(1,\infty)$. In section 4, the authors prove that the $\widetilde{T}_{\theta,b_1,b_2}$ formed by $b_1,b_2\in\widetilde{\mathrm{RBMO}}(\mu)$ and the $\widetilde{T}_\theta$ are bounded from product of spaces $\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu)\times\mathcal{L}^{p_2,\Phi,\varrho}_{\omega_2}(\mu)$ into spaces $W\mathcal{L}^{p,\Phi,\varrho}_{\nu_{\vec{\omega}}}(\mu)$. The strong (weak) type boundedness for the $\widetilde{T}_\theta$ and $\widetilde{T}_{\theta,b_1,b_2}$ on product of spaces $\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu)\times\mathcal{L}^{p_2,\Phi,\varrho}_{\omega_2}(\mu)$ are obtained in section 5.

Finally, we make some conventions on notation. Throughout this paper, we always denote by C a positive constant being independent of the main parameters, but it may vary from line to line. Given any $p\in[1,\infty)$, we denote p' as its conjugate index, that is, $1/p+1/p'=1$. For any measurable set E, χ_E denotes its characteristic function,

$$\nu_{\vec{\omega}}(E)=\int_E\nu_{\vec{\omega}}(x)\mathrm{d}\mu(x)$$

with $\vec{\omega}\in A^\tau_{\vec{p}}(\mu)$ and

$$m_E(f)=\frac{1}{\mu(E)}\int_E f(x)\mathrm{d}\mu(x)$$

represents the average of the function f on E.

2. Preliminaries

In this section, we recall some necessary notions and notation, including the dominating function, the discrete coefficient $\widetilde{K}^{(\rho)}_{B,S}$, the spaces $\widetilde{\mathrm{RBMO}}(\mu)$, the bilinear θ-type Calderón-Zygmund operator and generalized weighted Morrey spaces $\mathcal{L}^{p,\Phi,\varrho}_{\omega}(\mu)$. The following definitions of upper doubling is from [15].

Definition 2.1. A metric measure space $(\mathcal{X},d,\mu)$ is said to be upper doubling if μ is a Borel measure on $\mathcal{X}$ and there exist a dominating function $\lambda:\mathcal{X}\times(0,\infty)\to(0,\infty)$ and a positive constant $C_{(\lambda)}$, only depending on λ, such that, for each $x\in\mathcal{X}$, $r\to\lambda(x,r)$ is non-decreasing and, for all $x\in\mathcal{X}$ and $r\in(0,\infty)$.

$$\mu(B(x,r))\leq\lambda(x,r)\leq C_{(\lambda)}\lambda(x,r/2).\tag{2.1}$$

Remark 2.1. T. Hytönen [16] showed that there exists another dominating function $\widetilde{\lambda}$ such that $\widetilde{\lambda}\leq\lambda$, $C_{(\widetilde{\lambda})}\leq C_{(\lambda)}$ and, for all $x,y\in\mathcal{X}$ with $d(x,y)\leq r$,

$$\widetilde{\lambda}(x,r)\leq C_{(\lambda)}\widetilde{\lambda}(y,r).\tag{2.2}$$

Hence, in this paper, we also assume that the λ defined as in (2.1) satisfies (2.2).

The following notion of the geometrically doubling is well known in analysis on metric measure spaces, which can be found in [6].

Definition 2.2. A metric space $(\mathcal{X}, d)$ is said to be geometrically doubling if there exists some $N_0 \in \mathbb{N}$ such that, for any ball $B(x, r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$, there exists a finite ball covering $\{B(x_i, \frac{r}{2})\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0, here $i = 1, 2, \cdots, N_0$.

Remark 2.2. Let $(\mathcal{X}, d)$ be a metric measure. T. Hytönen [15] showed that the geometrically doubling is equivalent to the following statement: for every $\epsilon \in (0, 1)$, any ball $B(x, r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$ contains at most $N_0 \epsilon^{-n_0}$ centers of disjoint balls $\{B(x_i, \epsilon r)\}_i (i = 1, 2, \cdots)$, here and in what follows, $n_0 = \log_2 N_0$ and N_0 is as in Definition 2.2.

For any ball $B \subset \mathcal{X}$, we respectively denote its center and radius by c_B and r_B and, moreover, for any $\zeta \in (0, \infty)$, we denote the ball $B(c_B, \zeta r_B)$ by ζB. The following definition of discrete coefficients $\widetilde{K}^{(\rho)}_{B,S}$, which is more close to the quantity $K_{B,S}$ introduced by X. Tolsa in [42], is from [1].

Definition 2.3. For any $\rho \in (1, \infty)$ and any two balls B, S with $B \subset S$, define

$$\widetilde{K}^{(\rho)}_{B,S} = 1 + \sum_{k=-\lfloor \log_\rho 2 \rfloor}^{N^{(\rho)}_{B,S}} \frac{\mu(\rho^k B)}{\lambda(c_B, \rho^k r_B)}. \tag{2.3}$$

Here and hereafter, for any $a \in \mathbb{R}$, $\lfloor a \rfloor$ represents the largest integer smaller than or equal to a, and $N^{(\rho)}_{B,S}$ is the smallest integer satisfying $\rho^{N^{(\rho)}_{B,S}} r_B \geq r_S$. Moreover, more properties on the coefficients $\widetilde{K}^{(\rho)}_{B,S}$ can be seen Remark 2.8 in [22].

In [15], Hytönen introduced a (α, β)-doubling ball, i.e., let $\alpha, \beta \in (1, \infty)$, a ball $B \subset \mathcal{X}$ is said to be (α, β)-doubling if $\mu(\alpha B) \leq \beta \mu(B)$. The other properties on the (α, β)-doubling ball can be seen Lemmas 3.2 and 3.3 in [15]. In what follows, let $\nu = \log_2 C_{(\lambda)}$ and $n_0 = \log_2 N_0$. Throughout this article, for any $\alpha \in (1, \infty)$ and ball B, the smallest (α, β_α)-doubling ball of the form $\alpha^j B$ with $j \in \mathbb{N}$ is denoted by $\widetilde{B}^\alpha$, where

$$\beta_\alpha = \max\{\alpha^{n_0}, \alpha^\nu\} + 30^{n_0} + 30^\nu. \tag{2.4}$$

In addition, if there is no special explanation in this paper, we always set $\alpha = 6$ in (2.4) and simply denote $\widetilde{B}^6$ by $\widetilde{B}$.

The following definition of the spaces RBMO with discrete coefficient is from [11].

Definition 2.4. Let $\rho \in (1, \infty)$ and $\gamma \in [1, \infty)$. A real-valued function $f \in L^1_{\text{loc}}(\mu)$ is said to belong to the *space* $\widetilde{\text{RBMO}}_{\rho,\gamma}(\mu)$ if there exist a positive constant C such that, for any ball $B \subset \mathcal{X}$ and a number f_B,

$$\frac{1}{\mu(\rho B)} \int_B |f(x) - f_B| \mathrm{d}\mu(x) \leq C \tag{2.5}$$

and, for any two balls B and S such that $B \subset S$,

$$|f_B - f_S| \leq C \big[\widetilde{K}^{(\rho)}_{B,S}\big]^\gamma, \tag{2.6}$$

where f_B represents the mean value of functions f over ball B, that is,

$$f_B = \frac{1}{\mu(B)} \int_B f(y) \mathrm{d}\mu(y).$$

The infimum of the positive constants C satisfying (2.5) and (2.6) is defined to be the $\widetilde{\mathrm{RBMO}}_{\rho,\gamma}(\mu)$ norm of f and simply denoted by $\|f\|_{\widetilde{\mathrm{RBMO}}_{\rho,\gamma}(\mu)}$. Furthermore, Fu et al. [11] showed that the space $\widetilde{\mathrm{RBMO}}_{\rho,\gamma}(\mu)$ is independent of choices of $\rho \in (1,\infty)$ and $\gamma \in [1,\infty)$. Hence, in this paper, the space $\widetilde{\mathrm{RBMO}}_{\rho,\gamma}(\mu)$ is simply denoted by $\widetilde{\mathrm{RBMO}}(\mu)$.

Now we recall the definition of a bilinear θ-type Calderón-Zygmund operator introduced in [48].

Definition 2.5. Let θ be a non-negative and non-decreasing function defined on $(0,\infty)$ and satisfy

$$\int_0^1 \frac{\theta(t)}{t} \log\left(\frac{1}{t}\right) \mathrm{d}t < \infty. \tag{2.7}$$

A kernel $K(\cdot,\cdot,\cdot) \in L^1_{\mathrm{loc}}\big(\mathcal{X}^3 \backslash \{(x,x,x) : x \in \mathcal{X}\}\big)$ is called a bilinear θ-type Calderón-Zygmund kernel if it satisfies the following conditions:

(i) For all $(x,y_1,y_2) \in \mathcal{X} \times \mathcal{X} \times \mathcal{X}$ with $x \neq y_j$, $j = 1,2$,

$$|K(x,y_1,y_2)| \leq C\left[\sum_{j=1}^{2} \lambda(x,d(x,y_j))\right]^{-2}; \tag{2.8}$$

(ii) There exists a constant $c \in (0,\infty)$ such that, for all x,x',y_1,y_2 with satisfying $cd(y_1,y_1') \leq \max\limits_{1\leq j\leq 2} d(x,y_j)$,

$$|K(x,y_1,y_2) - K(x',y_1,y_2)| \leq C\theta\left(\frac{d(x,x')}{d(x,y_1)+d(x,y_2)}\right)\left[\sum_{j=1}^{2} \lambda\big(x,d(x,y_j)\big)\right]^{-2}; \tag{2.9}$$

(iii) There exists a constant $c \in (0,\infty)$ such that, for all x,y_1,y_1',y_2 with satisfying $cd(y_1,y_1') \leq \max\limits_{1\leq j\leq 2} d(x,y_j)$,

$$|K(x,y_1,y_2) - K(x,y_1',y_2)| \leq C\theta\left(\frac{d(y_1,y_1')}{d(x,y_1)+d(x,y_2)}\right)\left[\sum_{j=1}^{2} \lambda\big(x,d(x,y_j)\big)\right]^{-2}. \tag{2.10}$$

Let $L^\infty_b(\mu)$ be the spaces of all $L^\infty(\mu)$ functions with bounded support. A bilinear operator $\widetilde{T}_\theta$ is called a bilinear θ-type Calderón-Zygmund operator with kernels K satisfying (2.8), (2.9) and (2.10) if for all $f_1,f_2 \in L^\infty_b(\mu)$ and $x \in \mathcal{X} \setminus \big(\mathrm{supp}(f_1) \bigcap \mathrm{supp}(f_2)\big)$,

$$\widetilde{T}_\theta(f_1,f_2)(x) = \int_{\mathcal{X}^2} K(x,y_1,y_2) f_1(y_1) f_2(y_2) \mathrm{d}\mu(y_1)\mathrm{d}\mu(y_2). \tag{2.11}$$

Given $b_1,b_2 \in \widetilde{\mathrm{RBMO}}(\mu)$, the commutator $\widetilde{T}_{\theta,b_1,b_2}$ generated by b_1,b_2 and the $\widetilde{T}_\theta$ is defined by

$$\widetilde{T}_{\theta,b_1,b_2}(f_1,f_2)(x) = b_1(x)b_2(x)\widetilde{T}_\theta(f_1,f_2)(x) - b_1(x)\widetilde{T}_\theta(f_1,b_2(\cdot)f_2)(x)$$

$$- b_2(x)\widetilde{T}_\theta(b_1(\cdot)f_1, f_2)(x) + \widetilde{T}_\theta(b_1(\cdot)f_1, b_2(\cdot)f_2)(x). \tag{2.12}$$

Equivalently, the $\widetilde{T}_{\theta,b_1,b_2}(f_1, f_2)(x)$ can be formally written as

$$\int_{\mathcal{X}^2} K(x, y_1, y_2)\big(b_1(x) - b_1(y_1)\big)\big(b_2(x) - b_2(y_2)\big) f_1(y_1) f_2(y_2)\mathrm{d}\mu(y_1)\mathrm{d}\mu(y_2).$$

Also, the commutators $\widetilde{T}_{\theta,b_1}$ and $\widetilde{T}_{\theta,b_2}$ are respectively defined by

$$\widetilde{T}_{\theta,b_1}(f_1, f_2)(x) = b_1(x)\widetilde{T}_\theta(f_1, f_2)(x) - \widetilde{T}_\theta\big(b_1(\cdot)f_1, f_2\big)(x) \tag{2.13}$$

and

$$\widetilde{T}_{\theta,b_2}(f_1, f_2)(x) = b_2(x)\widetilde{T}_\theta(f_1, f_2)(x) - \widetilde{T}_\theta\big(f_1, b_2(\cdot)f_2\big)(x). \tag{2.14}$$

The following definition of a multiple $A^\tau_{\vec{p}}(\mu)$ weight is from [50].

Definition 2.6. Let $\tau \in [1, \infty)$, $\vec{p} = (p_1, p_2)$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ with $p_1, p_2 \in [1, \infty)$. A multiple-weight $\vec{\omega}$ with ω_1, ω_2 being non-negative μ-measurable functions is called an $A^\tau_{\vec{p}}(\mu)$ weight if there exists a positive constant C such that, for any ball $B \subset \mathcal{X}$,

$$\frac{1}{\mu(\tau B)} \int_B \nu_{\vec{\omega}}(x)\mathrm{d}\mu(x) \prod_{j=1}^{2} \left[\frac{1}{\mu(\tau B)} \int_B \omega_j^{1-p'_j}\mathrm{d}\mu(x)\right]^{\frac{p}{p'_j}} \le C, \tag{2.15}$$

where

$$\nu_{\vec{\omega}}(x) = \prod_{j=1}^{2} [\omega_j(x)]^{p/p_j}$$

and, when $p_j = 1$,

$$\left[\frac{1}{\mu(\tau B)} \int_B \omega_j^{1-p'_j}(x)\mathrm{d}\mu(x)\right]^{\frac{1}{p'_j}}$$

is understood as $(\inf_B \omega_j)^{-1}$ for $j \in \{1, 2\}$.

Remark 2.3. (i) If we take $(\mathcal{X}, d, \mu) = (\mathbb{R}^n, |\cdot|, \mathrm{d}x)$ and $\tau = 1$ in Definition 2.6, then the $A^1_{\vec{p}}(\mu)$ weight reduces to the multiple weight introduced by Lerner et al. [21].

(ii) From the Hölder inequality, it follows that, $\nu_{\vec{\omega}} \in A^\tau_{2p}$ if $\vec{\omega} \in A^\tau_{\vec{p}}$ for $\vec{p} = (p_1, p_2)$.

(iii) If we take $j = 1$ in Definition 2.6, then the multiple weight $A^\tau_{\vec{p}}(\mu)$ is just the $A^\tau_p(\mu)$ weight introduced by Hu et al. in [14]. Namely, let $\tau \in [1, \infty)$ and $p \in (1, \infty)$. A non-negative μ-measure function ω is called an $A^\tau_p(\mu)$ weight if there exists some positive constant C such that, for all balls $B \subset \mathcal{X}$,

$$\left(\frac{1}{\mu(\tau B)} \int_B \omega(x)\mathrm{d}\mu(x)\right)\left\{\frac{1}{\mu(\tau B)} \int_B [\omega(x)]^{1-p'}\mathrm{d}\mu(x)\right\}^{p-1} \le C. \tag{2.16}$$

And a weight ω is called an $A^\tau_1(\mu)$ weight if there exists some positive constant C such that, for all balls $B \subset \mathcal{X}$,

$$\frac{1}{\mu(\tau B)} \int_B \omega(x)\mathrm{d}\mu(x) \le C \inf_{y\in B} \omega(y).$$

As in the classical setting, let $A^\tau_\infty(\mu) = \bigcup_{p=1}^{\infty} A^\tau_p(\mu)$.

The following definition of a reverse Hölder class is from [17].

Definition 2.7. A weight ω is said to belong to the reverse Hölder class $RH_r(\mu)$ with $r \in (1, \infty)$ if there exists a positive constant C such that, for any ball $B \subset \mathcal{X}$,

$$\left\{\frac{1}{\mu(B)} \int_B [\omega(x)]^r \mathrm{d}\mu(x)\right\}^{\frac{1}{r}} \leq C\left(\frac{1}{\mu(B)} \int_B \omega(x) \mathrm{d}\mu(x)\right). \tag{2.17}$$

Next, we recall the definition of a generalized weighted Morrey space introduced in [28].

Definition 2.8. Let $\varrho \in (1, \infty)$, $p \in [1, \infty)$ and ω be a weight. Suppose that $\Phi : (0, \infty) \to (0, \infty)$ is an increasing function. Then the generalized weighted Morrey space $\mathcal{L}^{p,\Phi,\varrho}_\omega(\mu)$ is defined by

$$\mathcal{L}^{p,\Phi,\varrho}_\omega(\mu) = \left\{f \in L^p_{\mathrm{loc}}(\omega, \mu) : \|f\|_{\mathcal{L}^{p,\Phi,\varrho}_\omega(\mu)} < \infty\right\},$$

where

$$\|f\|_{\mathcal{L}^{p,\Phi,\varrho}_\omega(\mu)} = \sup_B [\Phi(\omega(\varrho B))]^{-\frac{1}{p}} \left(\int_B |f(x)|^p \omega(x) \mathrm{d}\mu(x)\right)^{\frac{1}{p}}. \tag{2.18}$$

Also, we denote by $W\mathcal{L}^{p,\Phi,\varrho}_\omega(\mu)$ the weak generalized weighted Morrey space of all locally integrable functions satisfying

$$\|f\|_{W\mathcal{L}^{p,\Phi,\varrho}_\omega(\mu)} = \sup_B \sup_{t>0} [\Phi(\omega(\varrho B))]^{-\frac{1}{p}} t\omega\big(\{x \in B : |f(x)| > t\}\big)^{\frac{1}{p}}. \tag{2.19}$$

Moreover, Lu [28] showed that the norms $\|\cdot\|_{\mathcal{L}^{p,\Phi,\varrho}_\omega(\mu)}$ and $\|\cdot\|_{W\mathcal{L}^{p,\Phi,\varrho}_\omega(\mu)}$ are independent of the choice of $\varrho > 1$.

Remark 2.4. (i) If we take $\omega(\cdot) \equiv 1$ in (2.18) and (2.19), then the generalized weighted Morrey space $\mathcal{L}^{p,\Phi,\varrho}_\omega(\mu)$ and the weak generalized weighted Morrey space $W\mathcal{L}^{p,\Phi,\varrho}_\omega(\mu)$ are just the generalized Morrey space $\mathcal{L}^{p,\Phi,\varrho}(\mu)$ and the weak generalized Morrey space $W\mathcal{L}^{p,\Phi,\varrho}(\mu)$ introduced by Lu and Tao in [31].

(ii) If we take $(\mathcal{X}, d, \mu) = (\mathbb{R}^n, |\cdot|, \mathrm{d}x)$ and $\omega \equiv 1$ in Definition 2.8, then the generalized weighted Morrey space $\mathcal{L}^{p,\Phi,\varrho}_\omega(\mu)$ and the weak generalized weighted Morrey space $W\mathcal{L}^{p,\Phi,\varrho}_\omega(\mu)$ are just the generalized Morrey space $\mathcal{L}^{p,\Phi,\varrho}(\mu)$ and the weak generalized Morrey space $W\mathcal{L}^{p,\Phi,\varrho}(\mu)$ introduced in [38].

(iii) If we take $\Phi(t) = t^{1-\frac{p}{q}}$ with $t > 0$ and $1 < p \leq q < \infty$ in (2.18) and (2.19), then the spaces $\mathcal{L}^{p,\Phi,\varrho}_\omega(\mu)$ and $W\mathcal{L}^{p,\Phi,\varrho}_\omega(\mu)$ are just the weighted Morrey spaces $L^{p,\kappa,\rho}(\omega)$ and the weighted weak Morrey spaces $WL^{p,\kappa,\rho}(\omega)$ introduced in [49]. Furthermore, if we take $\omega(\cdot) \equiv 1$, then the spaces $\mathcal{L}^{p,\Phi,\varrho}_\omega(\mu)$ and $W\mathcal{L}^{p,\Phi,\varrho}_\omega(\mu)$ are just the Morrey spaces $M^q_p(\mu)$ and the weak Morrey spaces $WM^q_p(\mu)$ in [2].

(iv) When $\Phi(\cdot) \equiv 1$, then $\mathcal{L}^{p,\Phi,\varrho}_\omega(\mu) = L^p_\omega(\mu)$ and $\mathcal{L}^{p,\Phi,\varrho}_\omega(\mu) = L^{p,\infty}_\omega(\mu)$.

The following definition of an ε-weak reverse doubling condition is from [31], also see [12].

Definition 2.9. Let $\varepsilon \in (0, \infty)$. A dominating function λ is said to satisfy ε-weak reverse doubling condition if, for all $r \in (0, 2\mathrm{diam}(\mathcal{X}))$ and $a \in (1, 2\mathrm{diam}(\mathcal{X})/r)$, there exists some number $C(a) \in [1,\infty)$, depending only on a and $\mathcal{X}$, such that, for all $x \in \mathcal{X}$,

$$\lambda(x, ar) \geq C(a)\lambda(x,r)$$

and, moreover,

$$\sum_{k=1}^{\infty} \frac{1}{[C(a^k)]^{\varepsilon}} < \infty. \tag{2.20}$$

3. Estimate for $\widetilde{T}_\theta$ on spaces $\mathcal{L}^{p,\phi,\varphi}_{\omega}(\mu)$

The main theorem of this section is stated as follows:

Theorem 3.1. *Let $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ for $p_1, p_2 \in [1,\infty)$, $\vec{p} = (p_1, p_2)$, $\vec{\omega} = (\omega_1, \omega_2) \in A^{\tau}_{\vec{p}}(\mu)$, $\tau \in [1,\infty)$, $\nu_{\vec{\omega}} \in RH_r(\mu)$ with $r \in [1,\infty)$, and $\Phi : (0,\infty) \to (0,\infty)$ be an increasing function satisfying*

$$\int_r^{\infty} \frac{\Phi(t)}{t}\frac{\mathrm{d}t}{t} \leq C\frac{\Phi(r)}{r}, \quad \text{for any } r \in (0,\infty). \tag{3.1}$$

Moreover, the mapping $t \mapsto \frac{\Phi(t)}{t}$ is almost decreasing: there is some positive constant C such that

$$\frac{\Phi(t)}{t} \leq C\frac{\Phi(s)}{s} \tag{3.2}$$

holds for all $s \leq t$. Suppose that $\widetilde{T}_\theta$ defined as in (2.11) is bounded from product of spaces $L^1(\mu) \times L^1(\mu)$ into spaces $L^{\frac{1}{2},\infty}(\mu)$. Then there exists some positive constant C such that, for any $f \in \mathcal{L}^{p_i,\Phi,\varrho}_{\omega_i}(\mu)$, $i = 1,2$,

$$\|\widetilde{T}_\theta(f_1,f_2)\|_{W\mathcal{L}^{p,\Phi,\varrho}_{\nu_{\vec{\omega}}}(\mu)} \leq C\|f_1\|_{\mathcal{L}^{p_1,\Phi,\varrho}_{\omega}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\Phi,\varrho}_{\omega}(\mu)}.$$

To prove Theorem 3.1, we need to recall the following results.

Lemma 3.1 (Lemma 2.7(ii), [14]). *Let $\varrho, p \in [1,\infty)$, $\omega \in A^{\tau}_p(\mu)$, and $\tau \in [5\varrho,\infty)$. Then there exists a constant $C_1 \in [1,\infty)$ such that, for any $(6,\beta_6)$-doubling ball B and any μ-measurable set $E \subset B$,*

$$C_1^{-1}\left[\frac{\mu(E)}{\mu(B)}\right]^p \leq \frac{\omega(E)}{\omega(B)}. \tag{3.3}$$

Lemma 3.2 (Lemma 2, [37]). *Suppose that $\psi : (0,\infty) \to (0,\infty)$ is a function and satisfies*

$$\int_s^{\infty} \psi(t)\frac{\mathrm{d}t}{t} \leq C\psi(s), \quad \text{for all } s > 0.$$

Then there exists a positive constant ϵ such that, for all $s > 0$, the following equation

$$\int_s^{\infty} \psi(t)t^{\epsilon}\frac{\mathrm{d}t}{t} \leq C\psi(s)s^{\epsilon}$$

holds. In particular, for every $\xi \in (-\infty,1]$, there exists a positive constant C such that, for all $s > 0$,

$$\int_s^{\infty} [\psi(t)]^{\xi}\frac{\mathrm{d}t}{t} \leq C[\psi(s)]^{\xi}.$$

Lemma 3.3 ([40]). *A weight $\omega \in RH_r(\mu)$ for some $r \in (1,\infty)$ if and only if there exist two positive constants C_2 and $\kappa \in (0,1)$ such that, for any ball B and any μ-measurable set $E \subset B$,*

$$\frac{\omega(E)}{\omega(B)} \le C_2\left[\frac{\mu(E)}{\mu(B)}\right]^{\kappa}. \tag{3.4}$$

Also, we need the following lemma on the operator $\widetilde{T}_\theta$.

Lemma 3.4 (Lemma 3.1, [50]). *Let $\tau \in [1,\infty)$, $\vec{p} = (p_1,p_2)$, $\vec{\omega} = (\omega_1,\omega_2) \in A^{\tau}_{\vec{p}}(\mu)$, $\nu_{\vec{\omega}} \in RH_r(\mu)$ with $r \in [1,\infty)$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ for $p_1,p_2 \in [1,\infty)$. Suppose that $\widetilde{T}_\theta$ defined as in (2.11) is bounded from product of spaces $L^1(\mu)\times L^1(\mu)$ into spaces $L^{\frac{1}{2},\infty}(\mu)$. Then there exists some positive constant C such that, for all $f_i \in L^{p_i}_{\omega_i}(\mu)$, $i = 1,2$,*

$$\|\widetilde{T}_\theta(f_1,f_2)\|_{L^{p,\infty}_{\nu_{\vec{\omega}}}(\mu)} \le C\|f_1\|_{L^{p_1}_{\omega_1}(\mu)}\|f_2\|_{L^{p_2}_{\omega_2}(\mu)}.$$

Lemma 3.5. *Let $p \in [1,\infty)$, $\omega \in A_p(\mu)$ and $\Phi : (0,\infty) \to (0,\infty)$ be an increasing function satisfying (3.1). Assume that the mapping $t \mapsto \Phi(t)/t$ satisfies (3.2). Then there exists a positive constant C such that, for any ball $B \subset \mathcal{X}$,*

$$\sum_{\ell=1}^{\infty}\left[\frac{\Phi(\omega(6^\ell B))}{\omega(6^\ell B)}\right]^{\frac{1}{p}} \le C\left[\frac{\Phi(\omega(B))}{\omega(B)}\right]^{\frac{1}{p}}.$$

Remark 3.1. By applying Lemma 3.2 and a way similar to that used in the Lemma 2.8 in [5], it is easy to show that Lemma 3.5 holds. Hence, to avoid the repeatability, we do not state the process of proof.

It is now position to state the proof of Theorem 3.1 as follows:
Proof. Without loss of generality, we may assume that $\varrho = 6$ in (2.18) and (2.19). And let $B = B(c_B, r_B)$ be a fixed doubling ball centered at $c_B \in \mathcal{X}$ with its radius $r_B > 0$. Represent functions $f_i (i = 1,2)$ as

$$f_i = f_i^1 + f_i^\infty = f_i\chi_{6B} + f_i\chi_{\mathcal{X}\setminus 6B}. \tag{3.5}$$

Then, by the property of the distribution function, write

$$\begin{aligned}
&\|\widetilde{T}_\theta(f_1,f_2)\|_{W\mathcal{L}^{p,\Phi,\varrho}_{\nu_{\vec{\omega}}}(\mu)}\\
&= \sup_B \sup_{t>0}[\Phi(\nu_{\vec{\omega}}(6B))]^{-\frac{1}{p}} t\nu_{\vec{\omega}}\big(\{x \in B : |\widetilde{T}_\theta(f_1,f_2)(x)| > t\}\big)^{-\frac{1}{p}}\\
&\le \sup_B \sup_{t>0}[\Phi(\nu_{\vec{\omega}}(6B))]^{-\frac{1}{p}} t\nu_{\vec{\omega}}\big(\{x \in B : |\widetilde{T}_\theta(f_1^1,f_2^1)(x)| > t/4\}\big)^{\frac{1}{p}}\\
&\quad + \sup_B \sup_{t>0}[\Phi(\nu_{\vec{\omega}}(6B))]^{-\frac{1}{p}} t\nu_{\vec{\omega}}\big(\{x \in B : |\widetilde{T}_\theta(f_1^1,f_2^\infty)(x)| > t/4\}\big)^{\frac{1}{p}}\\
&\quad + \sup_B \sup_{t>0}[\Phi(\nu_{\vec{\omega}}(6B))]^{-\frac{1}{p}} t\nu_{\vec{\omega}}\big(\{x \in B : |\widetilde{T}_\theta(f_1^\infty,f_2^1)(x)| > t/4\}\big)^{\frac{1}{p}}\\
&\quad + \sup_B \sup_{t>0}[\Phi(\nu_{\vec{\omega}}(6B))]^{-\frac{1}{p}} t\nu_{\vec{\omega}}\big(\{x \in B : |\widetilde{T}_\theta(f_1^\infty,f_2^\infty)(x)| > t/4\}\big)^{\frac{1}{p}}\\
&= \mathrm{D}_1 + \mathrm{D}_2 + \mathrm{D}_3 + \mathrm{D}_4.
\end{aligned}$$

From (2.15), Remark 2.3 (ii), (2.18), (3.2) and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, it then follows that

$$\begin{aligned}
\mathrm{D}_1 &\le C\sup_B[\Phi(\nu_{\vec{\omega}}(6B))]^{-\frac{1}{p}}\|f_1\chi_{6B}\|_{L^{p_1}_{\omega_1}(\mu)}\|f_2\chi_{6B}\|_{L^{p_2}_{\omega_2}(\mu)}\\
&\le C\|f_1\|_{\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\Phi,\varrho}_{\omega_2}(\mu)}\sup_B[\Phi(\nu_{\vec{\omega}}(6B))]^{-\frac{1}{p}}[\Phi(\omega_1(6B))]^{\frac{1}{p_1}}[\Phi(\omega_2(6B))]^{\frac{1}{p_2}}\\
&\le C\|f_1\|_{\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\Phi,\varrho}_{\omega_2}(\mu)}\sup_B\left[\frac{\Phi(\omega_1(6B))}{\Phi(\omega_1^{\frac{p}{p_1}}(6B)\omega_2^{\frac{p}{p_2}}(6B))}\right]^{\frac{1}{p_1}}\\
&\quad\times\left[\frac{\Phi(\omega_2(6B))}{\Phi(\omega_1^{\frac{p}{p_1}}(6B)\omega_2^{\frac{p}{p_2}}(6B))}\right]^{\frac{1}{p_2}}\\
&\le C\|f_1\|_{\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\Phi,\varrho}_{\omega_2}(\mu)}\sup_B\left[\frac{\omega_1(6B)}{\omega_1^{\frac{p}{p_1}}(6B)\omega_2^{\frac{p}{p_2}}(6B)}\right]^{\frac{1}{p_1}}\\
&\quad\times\left[\frac{\omega_2(6B)}{\omega_1^{\frac{p}{p_1}}(6B)\omega_2^{\frac{p}{p_2}}(6B)}\right]^{\frac{1}{p_2}}\\
&\le C\|f_1\|_{\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\Phi,\varrho}_{\omega_2}(\mu)}.
\end{aligned}$$

To estimate D_2, we first consider $|\widetilde{T}_\theta(f_1^1,f_2^\infty)(x)|$ for $x\in B$. By applying (2.1), (2.8), (2.15), (2.18), (2.20), the Hölder inequality and Lemma 3.5, we have

$$\begin{aligned}
&|\widetilde{T}_\theta(f_1^1,f_2^\infty)(x)|\\
&\le C\int_{6B}|f_1(y_1)|\mathrm{d}\mu(y_1)\left\{\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus(6^kB)}\frac{|f_2(y_2)|}{[\lambda(c_B,d(c_B,y_2))]^2}\mathrm{d}\mu(y_2)\right\}\\
&\le C\left(\int_{6B}|f_1(y_1)|^{p_1}\omega_1(y_1)\mathrm{d}\mu(y_1)\right)^{\frac{1}{p_1}}\left(\int_{6B}[\omega_1(y_1)]^{1-p_1'}\mathrm{d}\mu(y_1)\right)^{\frac{p_1-1}{p_1}}\\
&\quad\times\left\{\sum_{k=1}^{\infty}\frac{1}{[\lambda(c_B,6^kr_B)]^2}\int_{6^{k+1}B}|f_2(y_2)|[\omega_2(y_2)]^{\frac{1}{p_2}-\frac{1}{p_2}}\mathrm{d}\mu(y_2)\right\}\\
&\le C\|f_1\|_{\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}[\Phi(\omega_1(6\times6B))]^{\frac{1}{p_1}}\mu(2\times6B)[\omega_1(6B)]^{-\frac{1}{p_1}}\left\{\sum_{k=1}^{\infty}\frac{1}{[\lambda(c_B,6^kr_B)]^2}\right.\\
&\quad\times\left.\left(\int_{6^{k+1}B}|f_2(y_2)|^{p_2}\omega_2(y_2)\mathrm{d}\mu(y_2)\right)^{\frac{1}{p_2}}\left(\int_{6^{k+1}B}[\omega_2(y_2)]^{1-p_2'}\mathrm{d}\mu(y_2)\right)^{\frac{p_2-1}{p_2}}\right\}\\
&\le C\|f_1\|_{\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\Phi,\varrho}_{\omega_2}(\mu)}[\Phi(\omega_1(6\times6B))]^{\frac{1}{p_1}}\mu(2\times6B)[\omega_1(6B)]^{-\frac{1}{p_1}}\\
&\quad\times\left\{\sum_{k=1}^{\infty}\frac{\mu(2\times6^{k+1}B)}{[\lambda(c_B,6^kr_B)]^2}[\Phi(\omega_2(6\times6^{k+1}B))]^{\frac{1}{p_2}}[\omega_2(6^{k+1}B)]^{-\frac{1}{p_2}}\right\}\\
&\le C\|f_1\|_{\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\Phi,\varrho}_{\omega_2}(\mu)}\left[\frac{\Phi(\omega_1(6^2B))}{\omega_1(6^2B)}\right]^{\frac{1}{p_1}}\left[\frac{\omega_1(6^2B)}{\omega_1(6B)}\right]^{\frac{1}{p_1}}\\
&\quad\times\left\{\sum_{k=1}^{\infty}\frac{\mu(2\times6^{k+1}B)}{[\lambda(c_B,6^kr_B)]^2}\left[\frac{\Phi(\omega_2(6^{k+2}B))}{\omega_2(6^{k+2}B)}\right]^{\frac{1}{p_2}}\left[\frac{\omega_2(6^{k+2}B)}{\omega_2(6^{k+1}B)}\right]^{\frac{1}{p_2}}\right\}
\end{aligned}$$

$$
\begin{aligned}
&\le C\|f_1\|_{\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\Phi,\varrho}_{\omega_2}(\mu)}\left[\frac{\Phi(\omega_1(6^2B))}{\omega_1(6^2B)}\right]^{\frac{1}{p_1}}\\
&\quad\times\left\{\sum_{k=1}^{\infty}\frac{\mu(2\times 6^{k+1}B)}{[\lambda(c_B,6^kr_B)]^2}\left[\frac{\Phi(\omega_2(6^{k+2}B))}{\omega_2(6^{k+2}B)}\right]^{\frac{1}{p_2}}\frac{\mu(6^{k+2}B)}{\mu(6^{k+1}B)}\right\}\\
&\le C\|f_1\|_{\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\Phi,\varrho}_{\omega_2}(\mu)}\left[\frac{\Phi(\omega_1(6^2B))}{\omega_1(6^2B)}\right]^{\frac{1}{p_1}}\left\{\sum_{k=1}^{\infty}\left[\frac{\Phi(\omega_2(6^{k+2}B))}{\omega_2(6^{k+2}B)}\right]^{\frac{1}{p_2}}\right\}\\
&\le C\|f_1\|_{\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\Phi,\varrho}_{\omega_2}(\mu)}\left[\frac{\Phi(\omega_1(6B))}{\omega_1(6B)}\right]^{\frac{1}{p_1}}\left[\frac{\Phi(\omega_2(6B))}{\omega_2(6B)}\right]^{\frac{1}{p_2}},
\end{aligned}
$$

further, from (3.2), (3.4), $\nu_{\vec{\omega}}=\prod_{j=1}^{2}\omega_j^{\frac{p}{p_j}}$ and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, it follows that

$$
\begin{aligned}
&\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\omega}}(6B))]^{-\frac{1}{p}}t\nu_{\vec{\omega}}\big(\{x\in B:|\widetilde{T}_\theta(f_1^1,f_2^\infty)(x)|>t/4\}\big)^{\frac{1}{p}}\\
\le&C\|f_1\|_{\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\Phi,\varrho}_{\omega_2}(\mu)}\sup_B\left[\frac{\nu_{\vec{\omega}}(B)}{\Phi(\nu_{\vec{\omega}}(6B))}\right]^{\frac{1}{p}}\left[\frac{\Phi(\omega_1(6B))}{\omega_1(6B)}\right]^{\frac{1}{p_1}}\left[\frac{\Phi(\omega_2(6B))}{\omega_2(6B)}\right]^{\frac{1}{p_2}}\\
\le&C\|f_1\|_{\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\Phi,\varrho}_{\omega_2}(\mu)}\sup_B\left[\frac{\Phi(\omega_1(6B))}{\Phi(\nu_{\vec{\omega}}(6B))}\right]^{\frac{1}{p_1}}\left[\frac{\Phi(\omega_2(6B))}{\Phi(\nu_{\vec{\omega}}(6B))}\right]^{\frac{1}{p_2}}\\
\le&C\|f_1\|_{\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\Phi,\varrho}_{\omega_2}(\mu)}\sup_B\left[\frac{\omega_1(6B)}{\nu_{\vec{\omega}}(6B)}\right]^{\frac{1}{p_1}}\left[\frac{\omega_2(6B)}{\nu_{\vec{\omega}}(6B)}\right]^{\frac{1}{p_2}}\\
\le&C\|f_1\|_{\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\Phi,\varrho}_{\omega_2}(\mu)}.
\end{aligned}
$$

With an argument similar to that used in the estimate for D_2, it is easy to get

$$\mathrm{D}_3\le C\|f_1\|_{\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\Phi,\varrho}_{\omega_2}(\mu)}.$$

Now we turn D_4. For any $x\in B$, applying (2.1), (2.8), (2.16), (2.18), the Hölder inequality, (2.20), (3.3) and Lemma 3.5, we obtain

$$
\begin{aligned}
&|\widetilde{T}_\theta(f_1^\infty,f_2^\infty)(x)|\\
&\le C\int_{\mathcal{X}^2}\frac{|f_1^\infty(y_1)||f_2^\infty(y_2)|}{[\lambda(x,d(x,y_1))+\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_1)\mathrm{d}\mu(y_2)\\
&\le C\left(\sum_{k=1}^{\infty}\frac{1}{\lambda(c_B,6^kr_B)}\int_{6^{k+1}B}|f_1(y_1)|\mathrm{d}\mu(y_1)\right)\\
&\quad\times\left(\sum_{i=1}^{\infty}\frac{1}{\lambda(c_B,6^ir_B)}\int_{6^{i+1}B}|f_2(y_2)|\mathrm{d}\mu(y_2)\right)\\
&\le C\left\{\sum_{k=1}^{\infty}\frac{1}{\lambda(c_B,6^kr_B)}\left(\int_{6^{k+1}B}|f_1(y_1)|^{p_1}\omega_1(y_1)\mathrm{d}\mu(y_1)\right)^{\frac{1}{p_1}}\right.\\
&\quad\left.\times\left(\int_{6^{k+1}B}[\omega_1(y_1)]^{1-p_1'}\mathrm{d}\mu(y_1)\right)^{\frac{p_1-1}{p_1}}\right\}\\
&\quad\times\left\{\sum_{i=1}^{\infty}\frac{1}{\lambda(c_B,6^ir_B)}\left(\int_{6^{i+1}B}|f_2(y_2)|^{p_2}\omega_1(y_2)\mathrm{d}\mu(y_2)\right)^{\frac{1}{p_2}}\right.
\end{aligned}
$$

$$\times\left(\int_{6^{i+1}B}[\omega_2(y_2)]^{1-p_2'}\mathrm{d}\mu(y_2)\right)^{\frac{p_2-1}{p_2}}\Bigg\}$$

$$\le C\|f_1\|_{L^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}\|f_2\|_{L^{p_2,\Phi,\varrho}_{\omega_2}(\mu)}\left\{\sum_{k=1}^{\infty}\frac{\mu(6^{k+1}B)}{\lambda(c_B,6^k r_B)}\frac{[\Phi(\omega_1(6\times 6^{k+1}B))]^{\frac{1}{p_1}}}{[\omega_1(2\times 6^{k+1}B)]^{\frac{1}{p_1}}}\right\}$$

$$\times\left\{\sum_{i=1}^{\infty}\frac{\mu(6^{i+1}B)}{\lambda(c_B,6^i r_B)}\frac{[\Phi(\omega_1(6\times 6^{i+1}B))]^{\frac{1}{p_2}}}{[\omega_1(2\times 6^{i+1}B)]^{\frac{1}{p_2}}}\right\}$$

$$\le C\|f_1\|_{L^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}\|f_2\|_{L^{p_2,\Phi,\varrho}_{\omega_2}(\mu)}\left\{\sum_{k=1}^{\infty}\left[\frac{\Phi(\omega_1(6^{k+2}B))}{\omega_1(6^{k+2}B)}\right]^{\frac{1}{p_1}}\left[\frac{\omega_1(6^{k+2}B)}{\omega_1(2\times 6^{k+1}B)}\right]^{\frac{1}{p_1}}\right\}$$

$$\times\left\{\sum_{i=1}^{\infty}\left[\frac{\Phi(\omega_1(6^{i+2}B))}{\omega_2(6^{i+2}B)}\right]^{\frac{1}{p_2}}\left[\frac{\omega_1(6^{i+2}B)}{\omega_1(2\times 6^{i+1}B)}\right]^{\frac{1}{p_2}}\right\}$$

$$\le C\|f_1\|_{L^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}\|f_2\|_{L^{p_2,\Phi,\varrho}_{\omega_2}(\mu)}\left[\frac{\Phi(\omega_1(6B))}{\omega_1(6B)}\right]^{\frac{1}{p_1}}\left[\frac{\Phi(\omega_1(6B))}{\omega_2(6B)}\right]^{\frac{1}{p_2}},$$

further, by applying (3.2), (3.4), $\nu_{\vec{\omega}}=\prod_{j=1}^{2}\omega_j^{\frac{p}{p_j}}$ and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, we deduce

$$\begin{aligned}\mathrm{D}_4&=\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\omega}}(6B))]^{-\frac{1}{p}}t\nu_{\vec{\omega}}\big(\{x\in B:|\widetilde{T}_\theta(f_1^\infty,f_2^\infty)(x)|>t/4\}\big)^{\frac{1}{p}}\\&\le C\|f_1\|_{L^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}\|f_2\|_{L^{p_2,\Phi,\varrho}_{\omega_2}(\mu)}\sup_B\left[\frac{\Phi(\omega_1(6B))}{\Phi(\nu_{\vec{\omega}}(6B))}\right]^{\frac{1}{p_1}}\left[\frac{\Phi(\omega_2(6B))}{\Phi(\nu_{\vec{\omega}}(6B))}\right]^{\frac{1}{p_2}}\\&\le C\|f_1\|_{L^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}\|f_2\|_{L^{p_2,\Phi,\varrho}_{\omega_2}(\mu)}\sup_B\left[\frac{\omega_1(6B)}{\nu_{\vec{\omega}}(6B)}\right]^{\frac{1}{p_1}}\left[\frac{\omega_2(6B)}{\nu_{\vec{\omega}}(6B)}\right]^{\frac{1}{p_2}}\\&\le C\|f_1\|_{L^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}\|f_2\|_{L^{p_2,\Phi,\varrho}_{\omega_2}(\mu)}.\end{aligned}$$

Which, combining the estimates for D_1, D_2 and D_3, yields the desired results. Hence, the proof of Theorem 3.1 is finished. □

4. Estimate for $\widetilde{T}_{\theta,b_1,b_2}$ on spaces $\mathcal{L}^{p,\Phi,\varrho}_{\omega}(\mu)$

The main theorem of this section is stated as follows:

Theorem 4.1. *Let $b_1,b_2\in\widetilde{\mathrm{RBMO}}(\mu)$, $\tau\in[1,\infty)$, $\vec{p}=(p_1,p_2)$, $\vec{\omega}=(\omega_1,\omega_2)\in A^{\tau}_{\vec{p}}(\mu)$, $\nu_{\vec{\omega}}\in RH_r(\mu)$ with $r\in(1,\infty)$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ for $p_1,p_2\in[1,\infty)$, and $\Phi:(0,\infty)\to(0,\infty)$ be an increasing function satisfying (3.1) and (3.2). Suppose that $\widetilde{T}_\theta$ defined as in (2.11) is bounded from product of spaces $L^1(\mu)\times L^1(\mu)$ into space $L^{\frac{1}{2},\infty}(\mu)$. Then there exists some positive constant C such that, for all $f\in\mathcal{L}^{p_i,\Phi,\varrho}_{\omega_i}(\mu)$, $i=1,2$,*

$$\begin{aligned}&\|\widetilde{T}_{\theta,b_1,b_2}(f_1,f_2)\|_{W\mathcal{L}^{p,\Phi,\varrho}_{\nu_{\vec{\omega}}}(\mu)}\\ \le& C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\Phi,\varrho}_{\omega_2}(\mu)}.\end{aligned}$$

To prove the above theorem, we need to recall the following results on the maximal operators N and $M_{s,\zeta}$.

Lemma 4.1 (Lemma 2.5, [12])**.** (i) *Let $p \in (1,\infty)$, $s \in (1,p)$ and $\zeta \in [5,\infty)$. The following maximal operators defined, respectively, by setting, for all $f \in L^1_{\mathrm{loc}}(\mu)$ and $x \in \mathcal{X}$,*

$$M_{s,\zeta} f(x) = \sup_{B \ni x} \left(\frac{1}{\mu(\zeta B)} \int_B |f(y)|^s \mathrm{d}\mu(y) \right)^{\frac{1}{s}}, \tag{4.1}$$

$$Nf(x) = \sup_{B \ni x,\ B \text{ doubling}} \frac{1}{\mu(B)} \int_B |f(y)| \mathrm{d}\mu(y)$$

and

$$M_\zeta f(x) = \sup_{B \ni x} \frac{1}{\mu(\zeta B)} \int_B |f(y)| \mathrm{d}\mu(y), \tag{4.2}$$

are bounded on $L^p(\mu)$ and also bounded from spaces $L^1(\mu)$ into spaces $L^{1,\infty}(\mu)$.

(ii) *For all $f \in L^1_{\mathrm{loc}}(\mu)$, it holds true that $|f(x)| \le Nf(x)$ for μ-a.e. $x \in \mathcal{X}$.*

Lemma 4.2 (Lemma 3.1, [14])**.** *Let $\varrho \in [1,\infty)$, $\zeta \in [5\varrho,\infty)$, $s \in (1,\infty)$ and $M_{s,\zeta}$ be defined as in (4.1). For $\vec{p} = (p_1,p_2)$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ with $p_1,p_2 \in [1,\infty)$, $\vec{\omega} = (\omega_1, \omega_2) \in A^{\varrho}_{\vec{p}}(\mu)$, $\nu_{\vec{\omega}} \in RH_r(\mu)$, the operators $M_{s,\zeta}$ is bounded from product of spaces $L^{p_1}_{\omega_1}(\mu) \times L^{p_2}_{\omega_2}(\mu)$ into spaces $L^{p,\infty}_{\nu_{\vec{\omega}}}(\mu)$.*

The following lemma on the operators $\widetilde{T}_{\theta,b_1,b_2}$ is sightly modified from [24,48,50].

Lemma 4.3. *Let $b_1,\ b_2 \in \widetilde{\mathrm{RBMO}}(\mu)$, $1 < s < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ for $1 \le p_1,p_2 < \infty$ and $5 < \varsigma, \varsigma_1 < \infty$ with $\varsigma_1 < \varsigma$. Assume that $\widetilde{T}_\theta$ defined as in (2.11) is bounded from product of spaces $L^1(\mu) \times L^1(\mu)$ to spaces $L^{\frac{1}{2},\infty}(\mu)$, and λ satisfies the ϵ-weak reverse doubling condition. Then there exists some positive constant C such that, for any $\delta \in (0,\frac{1}{2})$, $\gamma \in (\delta,\frac{1}{2})$, $x \in \mathcal{X}$, $f_i \in L^{p_i}(\mu)$, $i = 1,2$,*

$$\begin{aligned}
&M^{\sharp}_{\varsigma,\delta}\big(\widetilde{T}_{\theta,b_1,b_2}(f_1,f_2)\big)(x)\\
\le& C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)} M_{\varsigma,\gamma}(T_\theta(f_1,f_2))(x)\\
&+ C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)} M_{\varsigma,\gamma}\big(T_{\theta,b_2}(f_1,f_2)\big)(x)\\
&+ C\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)} M_{\varsigma,\gamma}\big(T_{\theta,b_1}(f_1,f_2)\big)(x)\\
&+ C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)} M_{L(\log L),\rho_1}(f_1,f_2)(x),\\
&M^{\sharp}_{\varsigma,\delta}\big(T_{\theta,b_1}(f_1,f_2)\big)(x)\\
\le& C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)} \Big[M_{\varsigma,\gamma}(T_\theta(f_1,f_2))(x) + M_{L(\log L),\rho_1}(f_1,f_2)(x)\Big]
\end{aligned}$$

and

$$\begin{aligned}
&M^{\sharp}_{\varsigma,\delta}\big(T_{\theta,b_2}(f_1,f_2)\big)(x)\\
\le& C\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)} \Big[M_{\varsigma,\gamma}(T_\theta(f_1,f_2))(x) + M_{L(\log L),\rho_1}(f_1,f_2)(x)\Big],
\end{aligned}$$

where the sharp maximal function $M^{\sharp}(f)$ is defined by

$$M^{\sharp}_{\rho}(f)(x)$$

$$= \sup_{B \ni x} \frac{1}{\mu(\rho B)} \int_B |f(y) - m_{\widetilde{B}^{(\rho)}}(f)| \mathrm{d}\mu(y) + \sup_{\substack{x \in B \subset S \\ B, S \ (\rho,\beta_\rho)-\text{doubling}}} \frac{|m_B(f) - m_S(f)|}{\widetilde{K}^{(\rho)}_{B,S}},$$

$M^{\sharp}_{\rho,\delta}(f)(x) = [M^{\sharp}_{\rho}(|f|^{\delta})(x)]^{\frac{1}{\delta}}$ *for any* $\delta \in (0, \infty)$, *and*

$$M_{L(\log L),\rho}(f_1, f_2)(x) = \sup_{B \ni x} \prod_{i=1}^{2} \|f_i\|_{L(\log L),\rho,B}.$$

Lemma 4.4 (Lemma 5.5, [50])**.** *Let* $\delta \in (0, \frac{1}{2})$, $\varrho \in [1, \infty)$ *and* $\zeta \in [5\varrho, \infty)$. *Then, for any* $p \in [1, \infty)$ *and* $\omega \in A^{\varrho}_{2p}(\mu)$, *there exits some positive constant* C, *depending only on* δ, *such that, for any suitable function* f *and* $t \in (0, \infty)$,

$$\omega(\{x \in \mathcal{X} : M_{\zeta,\delta}(f)(x) > t\}) \le Ct^{-p} \sup_{\varsigma \ge Ct} \varsigma^p \omega(\{x \in \mathcal{X} : |f(x)| > t\}). \tag{4.3}$$

Also, we need to establish the following lemma modified from [27].

Lemma 4.5. *Let* $\varrho \in [1, \infty)$, $\delta \in (0, 1)$, ω *be a weight, and* $f \in L^1_{\rm loc}(\omega)$ *satisfy* $\int_{\mathcal{X}} f(x)\omega(x)\mathrm{d}\mu(x) = 0$ *when* $\|\mu\| = \mu(\mathcal{X}) < \infty$. *Assume that* $\inf\{1, N_\delta\} \in W\mathcal{L}^{p,\Phi,\varrho}_{\omega}(\mu)$ *for some* p *satisfying* $1 < p < \infty$. *Then there exists some positive constant* C *being independent of* f, *such that,*

$$\|N_\delta(f)\|_{W\mathcal{L}^{p,\Phi,\varrho}_{\omega}(\mu)} \le C\|M^{\sharp}_{\rho,\delta}(f)\|_{W\mathcal{L}^{p,\Phi,\varrho}_{\omega}(\mu)}, \tag{4.4}$$

where $N_\delta(f)(x) = [N(|f|^{\delta})(x)]^{\frac{1}{\delta}}$.

The proof of Theorem 4.1 is stated as follows:

Proof. By applying (3.2), $\nu_{\vec{\omega}} = \prod_{j=1}^{2} \omega_j^{\frac{p}{p_j}}$ and Lemmas 4.2, 4.3, 4.4 and 4.5, we get

$$\begin{aligned}
&\|T_{\theta,b_1,b_2}(f_1, f_2)\|_{W\mathcal{L}^{p,\phi,\varrho}_{\omega}(\mu)} \\
\le&\|N_\delta\big(T_{\theta,b_1,b_2}(f_1, f_2)\big)\|_{W\mathcal{L}^{p,\phi,\varrho}_{\omega}(\mu)} \\
\le&C\|M^{\sharp}_{\rho,\delta}\big(T_{\theta,b_1,b_2}(f_1, f_2)\big)\|_{W\mathcal{L}^{p,\phi,\varrho}_{\omega}(\mu)} \\
\le&C \sup_{B} \sup_{t>0} [\Phi(\nu_{\vec{\omega}}(6B))]^{\frac{1}{p}} t \nu_{\vec{\omega}}\big(\{x \in B : |M^{\sharp}_{\rho,\delta}\big(T_{\theta,b_1,b_2}(f_1, f_2)(x)| > t\}\big)^{\frac{1}{p}} \\
\le&C \sup_{B} \sup_{t>0} [\Phi(\nu_{\vec{\omega}}(6B))]^{-\frac{1}{p}} t \\
&\times \nu_{\vec{\omega}}\big(\{x \in B : C\|b_1\|_{\widetilde{\rm RBMO}(\mu)}\|b_2\|_{\widetilde{\rm RBMO}(\mu)} M_{\varsigma,\gamma}(T_\theta(f_1, f_2))(x) > t/4\}\big)^{\frac{1}{p}} \\
&+ C \sup_{B} \sup_{t>0} [\Phi(\nu_{\vec{\omega}}(6B))]^{\frac{1}{p}} t \\
&\times \nu_{\vec{\omega}}\big(\{x \in B : C\|b_1\|_{\widetilde{\rm RBMO}(\mu)} M_{\varsigma,\gamma}\big(T_{\theta,b_2}(f_1, f_2)\big)(x) > t/4\}\big)^{\frac{1}{p}} \\
&+ C \sup_{B} \sup_{t>0} [\Phi(\nu_{\vec{\omega}}(6B))]^{\frac{1}{p}} t \\
&\times \nu_{\vec{\omega}}\big(\{x \in B : C\|b_2\|_{\widetilde{\rm RBMO}(\mu)} M_{\varsigma,\gamma}\big(T_{\theta,b_1}(f_1, f_2)\big)(x) > t/4\}\big)^{\frac{1}{p}} \\
&+ C \sup_{B} \sup_{t>0} [\Phi(\nu_{\vec{\omega}}(6B))]^{\frac{1}{p}} t \\
&\times \nu_{\vec{\omega}}\big(\{x \in B : C\|b_1\|_{\widetilde{\rm RBMO}(\mu)}\|b_2\|_{\widetilde{\rm RBMO}(\mu)} M_{L(\log L),\rho_1}(f_1, f_2)(x) > t/4\}\big)^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
\leq & C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\varpi}}(6B))]^{-\frac{1}{p}}t\sup_{\iota>Ct}t^{-1}\iota\\
&\times\nu_{\vec{\varpi}}\big(\{x\in B:|T_\theta(f_1,f_2)(x)|>\iota\}\big)^{\frac{1}{p}}\\
&+C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\varpi}}(6B))]^{\frac{1}{p}}t\sup_{\iota>Ct}t^{-1}\iota\\
&\times\nu_{\vec{\varpi}}\big(\{x\in B:|T_{\theta,b_2}(f_1,f_2)(x)|>\iota\}\big)^{\frac{1}{p}}\\
&+C\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\varpi}}(6B))]^{\frac{1}{p}}t\sup_{\iota>Ct}t^{-1}\iota\\
&\times\nu_{\vec{\varpi}}\big(\{x\in B:|T_{\theta,b_1}(f_1,f_2)(x)|>\iota\}\big)^{\frac{1}{p}}\\
&+C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\varpi}}(6B))]^{\frac{1}{p}}t\\
&\times\nu_{\vec{\varpi}}\big(\{x\in B:M_{L(\log L),\rho_1}(f_1,f_2)(x)>t\}\big)^{\frac{1}{p}}\\
\leq & C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\varpi}}(6B))]^{-\frac{1}{p}}t\sup_{\iota>Ct}t^{-1}\iota\\
&\times\nu_{\vec{\varpi}}\big(\{x\in B:|T_\theta(f_1,f_2)(x)|>\iota\}\big)^{\frac{1}{p}}\\
&+C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\varpi}}(6B))]^{\frac{1}{p}}t\sup_{\iota>Ct}t^{-1}\iota\\
&\times\nu_{\vec{\varpi}}\big(\{x\in B:|M^\sharp_{\zeta,\delta}\big(T_{\theta,b_2}(f_1,f_2)\big)(x)|>\iota\}\big)^{\frac{1}{p}}\\
&+C\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\varpi}}(6B))]^{\frac{1}{p}}t\sup_{\iota>Ct}t^{-1}\iota\\
&\times\nu_{\vec{\varpi}}\big(\{x\in B:|M^\sharp_{\zeta,\delta}\big(T_{\theta,b_1}(f_1,f_2)\big)(x)|>\iota\}\big)^{\frac{1}{p}}\\
&+C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\varpi}}(6B))]^{\frac{1}{p}}t\\
&\times\nu_{\vec{\varpi}}\big(\{x\in B:M_{L(\log L),\rho_1}(f_1,f_2)(x)>t\}\big)^{\frac{1}{p}}\\
\leq & C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\varpi}}(6B))]^{-\frac{1}{p}}t\sup_{\iota>Ct}t^{-1}\iota\\
&\times\nu_{\vec{\varpi}}\big(\{x\in B:|T_\theta(f_1,f_2)(x)|>\iota\}\big)^{\frac{1}{p}}\\
&+C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\varpi}}(6B))]^{\frac{1}{p}}t\sup_{\iota>Ct}t^{-1}\iota\\
&\times\nu_{\vec{\varpi}}\big(\{x\in B:C\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}M_{\varsigma,\gamma}(T_\theta(f_1,f_2))(x)>\iota/2\}\big)^{\frac{1}{p}}\\
&+C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\varpi}}(6B))]^{\frac{1}{p}}t\sup_{\iota>Ct}t^{-1}\iota\nu_{\vec{\varpi}}\\
&\times\big(\{x\in B:C\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}M_{L(\log L),\rho_1}(f_1,f_2)(x)>\iota/2\}\big)^{\frac{1}{p}}\\
&+C\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\varpi}}(6B))]^{\frac{1}{p}}t\sup_{\iota>Ct}t^{-1}\iota\\
&\times\nu_{\vec{\varpi}}\big(\{x\in B:C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}M_{\varsigma,\gamma}(T_\theta(f_1,f_2))(x)>\iota/2\}\big)^{\frac{1}{p}}\\
&+C\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\varpi}}(6B))]^{\frac{1}{p}}t\sup_{\iota>Ct}t^{-1}\iota\\
&\times\nu_{\vec{\varpi}}\big(\{x\in B:C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}M_{L(\log L),\rho_1}(f_1,f_2)(x)>\iota/2\}\big)^{\frac{1}{p}}
\end{aligned}
$$

$$
\begin{aligned}
&+C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\omega}}(6B))]^{\frac{1}{p}}t\\
&\times\nu_{\vec{\omega}}\big(\{x\in B:M_{L(\log L),\rho_1}(f_1,f_2)(x)>t\}\big)^{\frac{1}{p}}\\
\leq&C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\omega}}(6B))]^{-\frac{1}{p}}t\sup_{\iota>Ct}t^{-1}\iota\\
&\times\nu_{\vec{\omega}}\big(\{x\in B:|T_\theta(f_1,f_2)(x)|>\iota\}\big)^{\frac{1}{p}}\\
&+C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\omega}}(6B))]^{\frac{1}{p}}t\sup_{\iota>Ct}t^{-1}\iota\sup_{\varpi>C\iota}\iota^{-1}\varpi\\
&\times\nu_{\vec{\omega}}\big(\{x\in B:|T_\theta(f_1,f_2))(x)|>\varpi\}\big)^{\frac{1}{p}}\\
&+C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\omega}}(6B))]^{\frac{1}{p}}t\sup_{\iota>Ct}t^{-1}\iota\\
&\times\nu_{\vec{\omega}}\big(\{x\in B:M_{L(\log L),\rho_1}(f_1,f_2)(x)>\iota\}\big)^{\frac{1}{p}}\\
&+C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\omega}}(6B))]^{\frac{1}{p}}t\sup_{\iota>Ct}t^{-1}\iota\sup_{\varpi>C\iota}\iota^{-1}\varpi\\
&\times\nu_{\vec{\omega}}\big(\{x\in B:|T_\theta(f_1,f_2))(x)|>\varpi\}\big)^{\frac{1}{p}}\\
&+C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\omega}}(6B))]^{\frac{1}{p}}t\sup_{\iota>Ct}t^{-1}\iota\\
&\times\nu_{\vec{\omega}}\big(\{x\in B:M_{L(\log L),\rho_1}(f_1,f_2)(x)>\iota\}\big)^{\frac{1}{p}}\\
&+C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\omega}}(6B))]^{\frac{1}{p}}t\\
&\times\nu_{\vec{\omega}}\big(\{x\in B:M_{L(\log L),\rho_1}(f_1,f_2)(x)>t\}\big)^{\frac{1}{p}}\\
\leq&C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\omega}}(6B))]^{-\frac{1}{p}}t\sup_{\iota>Ct}t^{-1}\iota\\
&\times\nu_{\vec{\omega}}\big(\{x\in B:|T_\theta(f_1,f_2)(x)|>\iota\}\big)^{\frac{1}{p}}\\
&+C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\omega}}(6B))]^{\frac{1}{p}}t\sup_{\iota>Ct}t^{-1}\iota\sup_{\varpi>C\iota}\iota^{-1}\varpi\\
&\times\nu_{\vec{\omega}}\big(\{x\in B:|T_\theta(f_1,f_2))(x)|>\varpi\}\big)^{\frac{1}{p}}\\
&+C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\omega}}(6B))]^{\frac{1}{p}}t\sup_{\iota>Ct}t^{-1}\iota\\
&\times\nu_{\vec{\omega}}\big(\{x\in B:M_{L(\log L),\rho_1}(f_1,f_2)(x)>\iota\}\big)^{\frac{1}{p}}\\
&+C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\omega}}(6B))]^{\frac{1}{p}}t\sup_{\iota>Ct}t^{-1}\iota\sup_{\varpi>C\iota}\iota^{-1}\varpi\\
&\times\nu_{\vec{\omega}}\big(\{x\in B:|T_\theta(f_1,f_2))(x)|>\varpi\}\big)^{\frac{1}{p}}\\
&+C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\omega}}(6B))]^{\frac{1}{p}}t\sup_{\iota>Ct}t^{-1}\iota\\
&\times\nu_{\vec{\omega}}\big(\{x\in B:M_{L(\log L),\rho_1}(f_1,f_2)(x)>\iota\}\big)^{\frac{1}{p}}\\
&+C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\omega}}(6B))]^{\frac{1}{p}}t\\
&\times\nu_{\vec{\omega}}\big(\{x\in B:M_{L(\log L),\rho_1}(f_1,f_2)(x)>t\}\big)^{\frac{1}{p}}\\
\leq&C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\Phi,\varrho}_{\omega_2}(\mu)}
\end{aligned}
$$

$$
\begin{aligned}
&+C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\omega}}(6B))]^{\frac{1}{p}}t\sup_{\iota>Ct}t^{-1}\iota\\
&\times\nu_{\vec{\omega}}\big(\{x\in B:CM_{s,\zeta_1}(f_1,f_2(x))>\iota\}\big)^{\frac{1}{p}}\\
&+C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sup_B\sup_{t>0}[\Phi(\nu_{\vec{\omega}}(6B))]^{\frac{1}{p}}t\\
&\times\nu_{\vec{\omega}}\big(\{x\in B:CM_{s,\zeta_1}(f_1,f_2(x))>t\}\big)^{\frac{1}{p}}\\
\le&C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\Phi,\varrho}_{\omega_1}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\Phi,\varrho}_{\omega_2}(\mu)},
\end{aligned}
$$

where we use the following fact introduced in [24]

$$M_{L(\log L),\rho_1}(f_1,f_2)(x)\le CM_{s,\zeta_1}(f_1,f_2(x)).$$

Which is our desired result. Hence, we complete the proof of Theorem 4.1. □

5. Estimate for $\widetilde{T}_\theta$ and $\widetilde{T}_{\theta,b_1,b_2}$ on spaces $\mathcal{L}^{p,\Phi,\varrho}(\mu)$

The main results of this section are stated as follows:

Theorem 5.1. *Let $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ for $p_1,p_2\in[1,\infty)$, $\Phi:(0,\infty)\to(0,\infty)$ be an increasing function satisfying (3.1), and the mapping $t\mapsto\Phi(t)/t$ satisfy (3.2). Suppose that $\widetilde{T}_\theta$ defined as in (2.11) is bounded from product of spaces $L^1(\mu)\times L^1(\mu)$ into spaces $L^{\frac{1}{2},\infty}(\mu)$. Then there exists some positive constant C such that, for all $f_i\in\mathcal{L}^{p_i,\Phi,\varrho}(\mu)$, $i=1,2$,*

$$\|\widetilde{T}_\theta(f_1,f_2)\|_{W\mathcal{L}^{p,\Phi,\varrho}(\mu)}\le C\|f_1\|_{\mathcal{L}^{p_1,\Phi,\varrho}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\Phi,\varrho}(\mu)}.$$

Theorem 5.2. *Let $b_1,b_2\in\widetilde{\mathrm{RBMO}}(\mu)$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ with $p_1,p_2\in[1,\infty)$, $\Phi:(0,\infty)\to(0,\infty)$ be an increasing function satisfying (3.1), and the mapping $t\mapsto\Phi(t)/t$ satisfy (3.2). Suppose that $\widetilde{T}_\theta$ defined as in (2.11) is bounded from the product of spaces $L^1(\mu)\times L^1(\mu)$ into spaces $L^{\frac{1}{2},\infty}(\mu)$. Then there exists some positive constant C such that, for all $f_i\in\mathcal{L}^{p_i,\Phi,\varrho}(\mu)$, $i=1,2$,*

$$\|\widetilde{T}_{\theta,b_1,b_2}(f_1,f_2)\|_{W\mathcal{L}^{p,\Phi,\varrho}(\mu)}\le C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\Phi,\varrho}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\Phi,\varrho}(\mu)}.$$

Remark 5.1. By applying Definition 2.8 and Lemmas 3.3 and 3.4 in [34], it is easy to show that Theorems 5.1 and 5.2 hold. Thus, in this paper, we omit the process of proofs.

Also, with a way similar to that used in the estimates for Theorems 1.1 and 1.4 in [34], it is easy to obtain the strong type results for the $\widetilde{T}_\theta$ and $\widetilde{T}_{\theta,b_1,b_2}$ on product of spaces $\mathcal{L}^{p_1,\Phi,\varrho}(\mu)\times\mathcal{L}^{p_2,\Phi,\varrho}(\mu)$ for $p_1,p_2\in(1,\infty)$.

Theorem 5.3. *Let $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ for $p_1,p_2\in[1,\infty)$, $\Phi:(0,\infty)\to(0,\infty)$ be an increasing function satisfying (3.1), and the mapping $t\mapsto\Phi(t)/t$ satisfy (3.2). Suppose that $\widetilde{T}_\theta$ defined as in (2.11) is bounded from the product of spaces $L^1(\mu)\times L^1(\mu)$ into spaces $L^{\frac{1}{2},\infty}(\mu)$. Then there exits some positive constant C such that, for all $f_i\in\mathcal{L}^{p_i,\Phi,\varrho}(\mu)$, $i=1,2$,*

$$\|\widetilde{T}_\theta(f_1,f_2)\|_{\mathcal{L}^{p,\Phi,\varrho}(\mu)}\le C\|f_1\|_{\mathcal{L}^{p_1,\Phi,\varrho}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\Phi,\varrho}(\mu)}.$$

Theorem 5.4. *Let $b_1, b_2 \in \widetilde{\mathrm{RBMO}}(\mu)$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ with $p_1, p_2 \in [1, \infty)$, $\Phi : (0, \infty) \to (0, \infty)$ be an increasing function satisfying* (3.1), *and the mapping $t \mapsto \Phi(t)/t$ satisfy* (3.2). *Suppose that $\widetilde{T}_\theta$ defined as in* (2.11) *is bounded from the product of spaces $L^1(\mu) \times L^1(\mu)$ into spaces $L^{\frac{1}{2},\infty}(\mu)$. Then there exists some positive constant C such that, for all $f_i \in \mathcal{L}^{p_i,\Phi,\varrho}(\mu)$, $i = 1, 2$,*

$$\|\widetilde{T}_{\theta,b_1,b_2}(f_1, f_2)\|_{\mathcal{L}^{p,\Phi,\varrho}(\mu)} \leq C\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|b_2\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\Phi,\varrho}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\Phi,\varrho}(\mu)}.$$

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