Journal of Applied Analysis and Computation
Volume 14, Number 6, December 2024, 3450–3457
Website:http://www.jaac-online.com
DOI:10.11948/20240039

LEVINSON'S CONJECTURE TO NEWTONIAN SYSTEMS WITH JUMPING NONLINEARITY

Yingying Li[1,†]

Abstract This paper concerns the jumping nonlinear Newtonian systems with friction. We show the existence of periodic solutions by using Lyapunov's methods and the modular degree theory. Furthermore, we apply our main result to find periodic solutions in a classical suspension bridge model.

Keywords Jumping nonlinearity, periodic solutions, Newtonian systems with friction, Lyapunov's methods.

MSC(2010) 34C25.

1. Introduction

At the end of the 19th century, Poincaré and Lyapunov established the qualitative theory of differential equations, in which the study of periodic solutions is fundamental, see [2, 5–7, 10, 13, 15, 16, 18–20, 22] for some developments. Fixed point theorems play an significant role in the study of periodic solutions, and remarkable progress has been done in this field. For example, Furumochi and Naito [10] discussed the autonomous difference equations and showed the existence of periodic solutions by the Schauder fixed point theorem. Wang [20] established a second-order non-autonomous singular dynamic systems and proved the existence of positive periodic solutions by the Krasnoselskii fixed point theorem. For a periodic system, Fink [8] showed that if systems are exponential uniform asymptotic stable, then a periodic solution exists. Li et al. [16] studied the existence of affine-periodic solutions for Newtonian systems with friction, and proved Levinson's conjecture by Lyapunov's methods.

In physical applications, differential systems with piecewise linearity may characterize various vibration processes, such as engineering [1, 14], neural networks ones [21], and in particular in mechanics [17], among others. Fonda et al. [9] proved the existence of large-scale subharmonic solutions by means of a method known as the Poincaré-Birkhoff theorem. Humphreys and Mackenna [12] showed multiple periodic solutions by Leray-Schauder degree theory. Aravinth et al. [3] considered the time periodic piecewise systems and obtained the criterion for the stability of the system by constructing a Lyapunov function. To the best of our knowledge, the bridge will appear large vibration in a large storm. Thereby, the stability analysis for the nonlinear suspension bridge model will be explored later as an application of the main result of this paper. The primary goal of this paper is to study the

[†]The corresponding author.
[1]School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China
Email: yingyingli0306@nenu.edu.cn(Y. Li)

periodic solutions under jumping nonlinearity for Newtonian systems with friction. Based on the theoretical underpinning of the Lyapunov function approach and the modular degree theorem [16, 23], we are now in a position to prove the theorem associated with periodic solutions.

This article is organized as follows. In section 2, we state the main result and give its proof. Finally, we apply our main result to the suspension bridge model.

2. Newtonian systems with jump

2.1. Preliminaries

In this section, we study the following Newtonian equation

$$\ddot{x} + D(t, x, \dot{x})\dot{x} + \nabla V(x) = p(t), \tag{2.1}$$

where $D : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $p : \mathbb{R} \to \mathbb{R}$ are continuous and $V \in C^1(\mathbb{R}, \mathbb{R})$.

When

$$V(x) = \begin{cases} \dfrac{1}{2}bx^2, & x \geq A, \\ \dfrac{1}{2}ax^2, & x < -A, \end{cases}$$

where $a, b > 0$ are constant, $|x| \geq A > 0$, then (2.1) becomes

$$\ddot{x} + D(t, x, \dot{x})\dot{x} + bx^+ - ax^- = p(t), \tag{2.2}$$

where $x^+ = max\{x, 0\}$, $x^- = max\{-x, 0\}$.

If $a \neq b$, such an equation is usually called the jumping nonlinear Newtonian one. The above equation is equivalent to

$$\ddot{x} + D(t, x, \dot{x})\dot{x} + bx = p(t), \qquad x \geq A, \tag{2.3}$$

$$\ddot{x} + D(t, x, \dot{x})\dot{x} + ax = p(t), \qquad x < -A. \tag{2.4}$$

2.2. Main result

Theorem 2.1. *Suppose D and p are continuous and satisfy the following T-periodicity:*

$$D(t + T, x, \dot{x}) = D(t, x, \dot{x}),$$
$$p(t + T) = p(t),$$

and $D \geq \sigma_0 > 0$. Also assume the solution of (2.2) *with respect to initial value is unique. Then system* (2.2) *has T-periodic solution.*

Proof. Equation (2.3) is equivalent to

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -Dy - bx + p(t). \end{cases}$$

Let

$$U(x, y) = \lambda y^2 + (x + y)^2, \quad \lambda \gg 1,$$

then

$$\begin{aligned}
\dot{U}_{(2.3)} =& 2\lambda y(-Dy-bx+p)+2(x+y)(y-Dy-bx+p)\\
=& -2\lambda Dy^2-2\lambda bxy+2\lambda py+2xy-2Dxy-2bx^2+2px+2y^2\\
& -2Dy^2-2bxy+2py\\
=& -(2\lambda+2)Dy^2-2bx^2+2px+(-2\lambda bxy+2\lambda py+2xy-2Dxy\\
& +2y^2-2bxy+2py)\\
=& -(\lambda+2)Dy^2-bx^2-b(x^2-2\frac{p}{b}x)-\lambda\Big[(\sqrt{D}y)^2+2(\sqrt{D}y)\frac{bx}{\sqrt{D}}\\
& -2(\sqrt{D}y)\frac{p}{\sqrt{D}}-2(\sqrt{D}y)\frac{x}{\lambda\sqrt{D}}+2(\sqrt{D}y)\frac{\sqrt{D}x}{\lambda}-2(\sqrt{D}y)\frac{y}{\lambda\sqrt{D}}\\
& +2(\sqrt{D}y)\frac{bx}{\lambda\sqrt{D}}-2(\sqrt{D}y)\frac{p}{\lambda\sqrt{D}}\Big]\\
=& -(\lambda+2)Dy^2-bx^2-b\big(x-\frac{p}{b}\big)^2+b\big(\frac{p}{b}\big)^2-\lambda\Big[\sqrt{D}y-\Big(-\frac{b}{\sqrt{D}}x\\
& +\frac{p}{\sqrt{D}}+\frac{x}{\lambda\sqrt{D}}-\frac{\sqrt{D}x}{\lambda}+\frac{y}{\lambda\sqrt{D}}-\frac{bx}{\lambda\sqrt{D}}+\frac{p}{\lambda\sqrt{D}}\Big)\Big]^2\\
& +\lambda\Big[-\frac{bx}{\sqrt{D}}+\frac{p}{\sqrt{D}}+\frac{x}{\lambda\sqrt{D}}-\frac{\sqrt{D}x}{\lambda}+\frac{y}{\lambda\sqrt{D}}-\frac{bx}{\lambda\sqrt{D}}+\frac{p}{\lambda\sqrt{D}}\Big]^2\\
\le& -(\lambda+2)Dy^2-bx^2+\frac{p^2}{b}+\lambda\Big[-\frac{bx}{\sqrt{D}}+\frac{p}{\sqrt{D}}+\frac{x}{\lambda\sqrt{D}}-\frac{\sqrt{D}x}{\lambda}\\
& +\frac{y}{\lambda\sqrt{D}}-\frac{bx}{\lambda\sqrt{D}}+\frac{p}{\lambda\sqrt{D}}\Big]^2\\
=& -(\lambda+2)Dy^2-bx^2+\frac{p^2}{b}+\lambda\Big[\frac{y}{\lambda\sqrt{D}}+\Big(-\frac{bx}{\sqrt{D}}+\frac{1}{\lambda\sqrt{D}}-\frac{\sqrt{D}}{\lambda}\\
& -\frac{b}{\lambda\sqrt{D}}\Big)x+\Big(\frac{1}{\sqrt{D}}+\frac{1}{\lambda\sqrt{D}}\Big)p\Big]^2\\
\le& -(\lambda+2)Dy^2-bx^2+\frac{p^2}{b}+\frac{W}{\lambda}y^2+\frac{W}{\lambda}(\lambda^2b^2+1+D+b^2)x^2\\
& +\frac{W}{\lambda}(1+\lambda^2)p^2\\
=& \Big(-(\lambda+2)D+\frac{W}{\lambda}\Big)y^2+\Big(-b+\frac{W}{\lambda}(\lambda^2b^2+1+D+b^2)\Big)x^2\\
& +\Big(\frac{1}{b}+\lambda W+\frac{W}{\lambda}\Big)p^2,
\end{aligned}$$

for some $W>0$. Thus, for sufficiently large λ and sufficiently small $\xi(>0)$ and some $H>0$,

$$\dot{U}\le -\xi(y^2+x^2)+H, \tag{2.5}$$

where $-(\lambda+2)D+\frac{W}{\lambda}\le-\xi$, $-b+\frac{W}{\lambda}(1+\lambda^2b^2+D+b^2)\le-\xi$ and $H=\Big(\frac{1}{b}+\lambda W+\frac{W}{\lambda}\Big)p^2$. Obviously, there exists $1\ge\eta>0$ such that

$$U\ge\eta(y^2+x^2). \tag{2.6}$$

For (2.4), $\dot{U}_{(2.4)}$ is also similar. In fact,

$$\begin{aligned}
\dot{U}_{(2.4)} =& 2\lambda y(-Dy-ax+p)+2(x+y)(y-Dy-ax+p)\\
=&-(2\lambda+2)Dy^2-2ax^2+2px+(-2\lambda axy+2\lambda py+2xy-2Dxy\\
&+2y^2-2axy+2py)\\
=&-(\lambda+2)Dy^2-ax^2-a\big(x^2-2x\frac{p}{a}\big)-\lambda\Big[(\sqrt{D}y)^2+2(\sqrt{D}y)\frac{ax}{\sqrt{D}}\\
&-2(\sqrt{D}y)\frac{p}{\sqrt{D}}-2(\sqrt{D}y)\frac{x}{\lambda\sqrt{D}}+2(\sqrt{D}y)\frac{\sqrt{C}x}{\lambda}-2(\sqrt{D}y)\frac{y}{\lambda\sqrt{D}}\\
&+2(\sqrt{D}y)\frac{ax}{\lambda\sqrt{D}}-2(\sqrt{D}y)\frac{p}{\lambda\sqrt{D}}\Big]\\
=&-(\lambda+2)Dy^2-ax^2-a\big(x-\frac{p}{a}\big)^2+a\big(\frac{p}{a}\big)^2-\lambda\Big[\sqrt{D}y-(-\frac{a}{\sqrt{D}}x\\
&+\frac{p}{\sqrt{D}}+\frac{x}{\lambda\sqrt{D}}-\frac{\sqrt{D}x}{\lambda}+\frac{y}{\lambda\sqrt{D}}-\frac{ax}{\lambda\sqrt{D}}+\frac{p}{\lambda\sqrt{D}})\Big]^2\\
&+\lambda\Big[-\frac{a}{\sqrt{D}}x+\frac{p}{\sqrt{D}}+\frac{x}{\lambda\sqrt{D}}-\frac{\sqrt{D}x}{\lambda}+\frac{y}{\lambda\sqrt{D}}-\frac{ax}{\lambda\sqrt{D}}+\frac{p}{\lambda\sqrt{D}}\Big]^2\\
\leq&-(\lambda+2)Dy^2-ax^2+\frac{p^2}{a}+\lambda\Big[-\frac{a}{\sqrt{D}}x+\frac{p}{\sqrt{C}}+\frac{x}{\lambda\sqrt{D}}-\frac{\sqrt{D}x}{\lambda}\\
&+\frac{y}{\lambda\sqrt{D}}-\frac{ax}{\lambda\sqrt{D}}+\frac{p}{\lambda\sqrt{D}}\Big]^2\\
\leq&-(\lambda+2)Dy^2-ax^2+\frac{p^2}{a}+\lambda\Big[\frac{y}{\lambda\sqrt{D}}+\Big(-\frac{a}{\sqrt{D}}+\frac{1}{\lambda\sqrt{D}}\\
&-\frac{\sqrt{D}}{\lambda}-\frac{a}{\lambda\sqrt{D}}\Big)x+\Big(\frac{1}{\lambda\sqrt{D}}+\frac{1}{\sqrt{D}}\Big)p\Big]^2\\
\leq&-(\lambda+2)Dy^2-ax^2+\frac{p^2}{a}+\frac{W}{\lambda}y^2+\frac{W}{\lambda}(\lambda^2a^2+1+D\\
&+a^2)x^2+\frac{W}{\lambda}(1+\lambda^2)p^2\\
=&\Big(-(\lambda+2)D+\frac{W}{\lambda}\Big)y^2+\Big(-a+\frac{W}{\lambda}(\lambda^2a^2+1+D\\
&+a^2)\Big)x^2+\Big(\frac{1}{a}+\lambda W+\frac{W}{\lambda}\Big)p^2
\end{aligned}$$

for some $W>0$.

Applying Gronwall-Bellman inequality to (2.5), we get

$$U(x(t),y(t))\leq U(x_0,y_0)e^{\int_0^t[-\xi(x^2(t)+y^2(t))+H]ds},$$

for any solution $\omega(t,\omega_0)=(x(t,x_0,y_0),y(t,x_0,y_0))$ of (2.2) with the initial value $\omega(0)=(x(0),y(0))=(x_0,y_0)$, which together with (2.6) implies

$$\eta|\omega(t,\omega_0)|^2\leq U(\omega(t,\omega_0))\leq U(\omega_0)e^{\int_0^t(-\xi|\omega(s,\omega_0)|^2+H)ds}, \tag{2.7}$$

that is to say, for any ω_0, the solution $\omega(t,\omega_0)$ exists on $\mathbb{R}^1_+$. That is,

$$|\omega(t,\omega_0)|^2\geq 0,$$

$$
\begin{gathered}
-\xi|\omega(t,\omega_0)|^2 + H \le H, \\
e^{\int_0^t(-\xi|\omega(s,\omega_0)|^2+H)ds} \le e^{Ht}.
\end{gathered}
$$

Take

$$
M = \max\left\{U(x,y)||x|^2+|y|^2 \le \frac{H+1}{\xi}\right\}, \tag{2.8}
$$

$$
\bar{M} = \max\left\{U(x,y)||x|^2+|y|^2 \le R^2\right\}, \quad R^2 \ge \frac{H+1}{\xi}. \tag{2.9}
$$

Then

$$
\begin{gathered}
-\xi|\omega(t,\omega_0)|^2 + H \ge -1, \\
e^{\int_0^t(-\xi|\omega(s,\omega_0)|^2+H)ds} \ge e^{-t}.
\end{gathered}
$$

According to the above,

$$
e^{-t} \le e^{\int_0^t(-\xi|\omega(s,\omega_0)|^2+H)ds} \le e^{Ht},
$$

and according to (2.7) and (2.8), $\exists N(R) \in \mathbb{N}$, we have

$$
\eta|\omega(t,\omega_0)|^2 \le Me^{HN(R)T}, \quad \forall t \ge N(R)T,
$$

so

$$
|\omega(t,\omega_0)|^2 \le \frac{M}{\eta}e^{HN(R)T},
$$

i.e.

$$
|\omega(t,\omega_0)|^2 \le B, \quad \forall t \ge N(R)T, \tag{2.10}
$$

where $B^2 = \dfrac{M}{\eta}e^{HN(R)T}$. Set

$$
\Omega = \{\omega_0 \in \mathbb{R}^2 ||\omega_0| < B+1\}. \tag{2.11}
$$

And define Poincaré map P, we have

$$
P(\omega_0) = \omega(T,\omega_0), \tag{2.12}
$$

then

$$
P^i(\omega_0) = \omega(iT,\omega_0) \quad \forall i \ge 1.
$$

By (2.10), we get

$$
|P^i(\omega_0)| = |\omega(iT,\omega_0)| \le B \quad \forall i = N(B+1), N(B+1)+1, \quad \forall \omega_0 \in \partial\Omega. \tag{2.13}
$$

By (2.11) and (2.13), we get

$$
P^i(\omega_0) \in \Omega, \quad i = N, N+1. \tag{2.14}
$$

We set N to be a prime number. By (2.11) – (2.14) and the Rothe theorem, we get

$$
\deg(\mathrm{id} - P^N, \Omega, 0) = 1,
$$

according to the modular degree theorem [16, 23], we have

$$\deg(\mathrm{id} - P, \Omega, 0) = 1 \neq 0,$$

so

$$P(\omega^*) = \omega^*, \quad \omega^* \in \Omega.$$

According to (2.12),

$$\omega^* = \omega(T, \omega^*).$$

For any t, $\omega(t + T, \omega^*) = \omega(t, \omega^*)$. Hence, $\omega(t, \omega^*)$ is a T-periodic solution of equation (2.2). □

3. An application to a suspension bridge model

Due to its remarkable flexibility, the suspension bridge is extensively employed in both practical and engineering applications. However, in the face of adverse weather conditions like thunderstorms or storms, the suspension bridge may undergo significant oscillations with large amplitudes, potentially resulting in detrimental consequences, such as the catastrophic failure witnessed in the Tacoma Narrows suspension bridge [1, 11]. Therefore, we consider the steady state of a one-dimensional suspended bridge in references [4, 11] as follows:

$$m\ddot{z} + \delta\dot{z} + c(\pi/L)^4 z + dz^+ = mg + h(t), \tag{3.1}$$

where m is the mass per unit of length, δ is a small viscous damping coefficient, c and L represent the flexibility and length of the bridge respectively, d is the stiffness of nonlinear springs and m, δ, c, L, $d > 0$, moreover, h is continuous T-periodic.

Let $\alpha = \frac{\delta}{m}$, $\mu = \frac{c}{m}\left(\frac{\pi}{L}\right)^4 + \frac{d}{m}$, $\nu = \frac{c}{m}\left(\frac{\pi}{L}\right)^4$, $f(t) = \frac{1}{m}(mg + h(t))$, then system (3.1) becomes

$$\ddot{z} + \alpha\dot{z} + \mu z^+ - \nu z^- = f(t), \tag{3.2}$$

that is

$$\ddot{z} + \alpha\dot{z} + \mu z = f(t), \quad x \geq 0, \tag{3.3}$$

$$\ddot{z} + \alpha\dot{z} + \nu z = f(t), \quad x < 0. \tag{3.4}$$

We can obtain the following conclusion:

Theorem 3.1. *System* (2.1) *has a T-periodic solution if f is continuous and satisfies T-periodicity with respect to t.*

Proof. We can apply the main result of the part 2, so the details are omitted. □

Acknowledgements

The author is grateful to the anonymous referees for their useful comments and suggestions.

References

[1] O. H. Amann, T. V. Kármán and G. B. Woodruff, *The Failure of the Tacoma Narrows Bridge*, Federal Works Agency, 1941.

[2] M. E. Anacleto, J. Llibre, C. Valls and C. Vidal, *Limit cycles of discontinuous piecewise differential systems formed by linear centers in $\mathbb{R}^2$ and separated by two circles*, Nonlinear Anal. Real World Appl., 2021. DOI: 10.1016/j.nonrwa.2020.103281.

[3] N. Aravinth, T. Satheesh, R. Sakthivel, G. Ran and A. Mohammadzadeh, *Input-output finite-time stabilization of periodic piecewise systems with multiple disturbances*, Appl. Math. Comput., 2023. DOI: 10.1016/j.amc.2023.128080.

[4] A. Buică, J. Llibre and O. Makarenkov, *Asymptotic stability of periodic solutions for nonsmooth differential equations with application to the nonsmooth Van Der Pol oscillator*, SIAM J. Math. Anal., 2009, 40(6), 2478–2495.

[5] T. A. Burton and B. Zhang, *Uniform ultimate boundedness and periodicity in functional differential equations*, Tohoku Math. J., 1990, 42(1), 93–100.

[6] T. A. Burton and S. Zhang, *Unified boundedness, periodicity, and stability in ordinary and functional-differential equations*, Ann. Mat. Pura Appl., 1986. DOI: 10.1007/BF01790540.

[7] V. Carmona, F. F. Sánchez and D. D. Novaes, *Uniqueness and stability of limit cycles in planar piecewise linear differential systems without sliding region*, Commun. Nonlinear Sci. Numer. Simul., 2023. DOI: 10.1016/j.cnsns.2023.107257.

[8] A. M. Fink, *Convergence and almost periodicity of solutions of forced lienard equation*, SIAM. J. Appl. Math., 1974, 26(1), 26–34.

[9] A. Fonda, Z. Schneider and F. Zanolin, *Periodic oscillations for a nonlinear suspension bridge model*, J. Comput. Appl. Math., 1994, 52(1–3), 113–140.

[10] T. Furumochi and T. Naito, *Periodic solutions of difference equations*, Nonlinear Anal-Theor., 2009, 71(12), e2217–e2222.

[11] J. Glover, A. C. Lazer and P. J. McKenna, *Existence and stability of large scale nonlinear oscillations in suspension bridges*, Z. Angew. Math. Phys., 1989, 40(2), 172–200.

[12] L. D. Humphreys and P. J. Mckenna, *Multiple periodic solutions for a nonlinear suspension bridge equation*, IMA J. Appl. Math., 1999, 63(1), 37–49.

[13] T. Küpper, Y. Li and B. Zhang, *Periodic solutions for dissipative-repulsive systems*, Tohoku Math. J., 2000, 52(3), 321–329.

[14] A. C. Lazer and P. J. Mckenna, *Large scale oscillatory behaviour in loaded asymmetric systems*, Ann. Inst. Henri Poincaré, 1987, 4(3), 243–274.

[15] Y. Li and F. Huang, *Levinson's problem on affine-periodic solutions*, Adv. Nonlinear Stud., 2015, 15(1), 241–252.

[16] Y. Li, H. Wang and X. Yang, *Fink type conjecture on affine-periodic solutions and Levinson's conjecture to Newtonian systems*, Discrete Contin. Dyn. Syst. Ser. B, 2018, 23(6), 2607–2623.

[17] A. Macrina, L. A. Mengütürk and M. C. Mengütürk, *Captive jump processes for bounded random systems with discontinuous dynamics*, Commun. Nonlinear Sci. Numer. Simul., 2024. DOI: 10.1016/j.cnsns.2023.107646.

[18] T. T. Nguyen and T. T. Nguyen, *The inviscid limit of Navier-stokes equations for vortex-wave data on* $\mathbb{R}^2$, SIAM J. Math. Anal., 2019, 51(3), 2575–2598.

[19] C. Wang, X. Yang and Y. Li, *Affine-periodic solutions for nonlinear differential equations*, Rocky Mt. J. Math., 2016, 46(5), 1717–1737.

[20] H. Wang, *Positive periodic solutions of singular systems*, J. Diff. Eqs., 2010, 249(12), 2986–3002.

[21] T. Wu, J. D. Cao, L. L. Xiong, H. Y. Zhang and J. L. Shu, *Sampled-data synchronization criteria for Markovian jumping neural networks with additive time-varying delays using new techniques*, Appl. Math. Comput., 2022. DOI: 10.1016/j.amc.2021.126604.

[22] J. M. Xing, X. Yang and Y. Li, *Affine-periodic solutions by averaging methods*, Sci. China Math., 2018. DOI: 10.1007/s11425-016-0455-1.

[23] P. P. Zabreiko and M. A. Krasnosel'skii, *Iterations of operators, and fixed points*, Dokl. Akad. Nauk SSSR, 1971, 196(5), 1006–1009.