DYNAMICAL INVESTIGATION AND NUMERICAL MODELING OF A FRACTIONAL MIXED NONLINEAR PARTIAL INTEGRO-DIFFERENTIAL PROBLEM IN TIME AND SPACE

D. Sh. Mohamed[1], M. A. Abdou[2] and A. M. S. Mahdy[1,3,†]

Abstract In the current study, a novel and effective method for solving the nonlinear fractional mixed partial integro-differential equation (NfrPIo-DE) based on a continuous kernel is presented and discussed. The NfrPIo-DE is transformed into the nonlinear Fredholm integral equation (NFIE) through the utilization of the separation of variables. The NFIE reduction was then transformed into a system of nonlinear algebraic equations (SNAE) with the application of Chebyshev polynomials of the sixth type (CP6K). By utilizing the Banach fixed point theorem, we can describe the existence of the solution of NfrPIo-DE as well as its uniqueness. Furthermore, the convergence and the stability of the reduced error have been described. Finally, a numerical example is presented to illustrate the theoretical results. The Maple 18 software is responsible for getting all of the computational outcomes.

Keywords NfrPIo-DE, Caputo fractional derivative, existence and uniqueness, convergence and stability, separation of variables, CP6K.

MSC(2010) 65R20, 45M10, 35R09.

1. Introduction

There are types of math equations called nonlinear integral equations that use both integrals and nonlinear functions [13]. In many areas of science and engineering, these equations are very important for explaining things where the link between variables is not linear [20, 24]. Over the past few years, much of the author's attention has been drawn to the mixed integral equations (MIE) in time and position. Mahdy et al. in [18], presented computational methods for solving mixed-difference integro-differential. Using the Lerch polynomial approach, the technique for responding to MIE was presented by Alhazmi and colleagues [8], and it employed a

[†]The corresponding author.

[1]Department of Mathematics, Faculty of Science, Zagazig University, P.O. Box 44519, Zagazig, Egypt

[2]Department of Mathematics, Faculty of Education, Alexandria University, P.O. Box 21526, Alexandria, Egypt

[3]Department of Mathematics & Statistics, College of Science, Taif University, P.O. Box 11099, Taif, 21944, Saudi Arabia

Email: doaashokry203@yahoo.com(D. Sh. Mohamed),
abdella_777@yahoo.com(M. A. Abdou), amr_mahdy85@yahoo.com,
amr_mahdy85@zu.edu.eg and amattaya@tu.edu.sa(A. M. S. Mahdy)

solitary kernel that was essentially symmetric. A numerical algorithm for generating MIEs with singular kernels was presented by Mahdy et al. in the publication [19]. Abdou and Sharifah in [7], used the Toeplitz matrix method for solving NfrMIo-DE. The Toeplitz matrix approach was applied by Jan in [16] to determine the MIE. To discuss quadratic nonlinear MIE, Basseem, and Alalyani presented the Toeplitz and Nystrom algorithms in [11]. Abdou and Youssef [4] proposed analytical and numerical approaches to the resolution of three MIE problems. In [3], Abdou and Basseem used the separation of variables and the Toeplitz matrix algorithm for solving nonlinear MIEs.

Most of the researchers focused their attention on the first, second, third, and fourth types of CPs, but they did not address the sixth type of CPs, as an illustration, Youssri et al. suggested a numerical to telegraph hyperbolic model by applying shifted CP6K [10]. The fractional partial differential equations were solved by Sadri and Aminikhah using CP6K, as described in [25]. In [27] Yaghoubi et al. introduce shifted CP6K to obtain the solution of frIo-DEs.

Several benefits of the suggested method include superior accuracy in approximations and reduced computation time achieved by selecting CP6K as basic functions and extracting a few terms from the retained modes. Furthermore, the resulting inaccuracies are negligible.

Fractional calculus has been a popular topic for academics in a variety of scientific and technical domains in recent years. The ability of fractional derivatives to offer a better method of characterizing the memory and genetic characteristics of different materials and processes is one of the main benefits of fractional calculus. For instance, there are many benefits for employing fractional derivatives: Fractional derivatives can be used to model systems with memory. Fractional order differential equations readily relate to systems with memory, which are present in most physical problems, models in extended thermoelasticity theory, and biological systems (FDEs). FDEs show the true biphasic decline behavior of diseases and elastic media, albeit more slowly. FDEs fit better than integer-order models when it comes to modeling complex systems, like biological systems, extended thermoelasticity theory models, and physical challenges.

It is difficult to find efficient numerical solutions for most of these equations since it is not possible to find exact solutions for them. Different approaches to solve these kinds of equations have been discussed and studied, for example, Alsulaiman et al. [9] used the Bernoulli matrix approach for solving NfrIo-DE. Alharbi et al. in [6], introduced the Haar-Wavelet method for displaying FrODEs. In [15], Hamdan et al. used the Haar Wavelet method for solving LFrIEs. In [12], Bekkouche and colleagues conducted research on the computational approach to the fractional boundary value question. Khalouta in [17], discussed the solution of FrODEs by utilizing a new general integral transform. Euler wavelets were introduced by Mohammad et al. in [22] to solve the fractional diffusion wave equation. Panda and Mohapatra in [23] used semi-analytical methods for solving FrPDEs. Youssri and Atta in [28] presented Fejér-quadrature collocation algorithm for solving frIo-DEs. Also in [29], Youssri and Hafez introduced the Chebyshev collocation treatment to get the solution of VFIE. While Youssri and Hafez in [14] introduced spectral Legendre-Chebyshev treatment for solving 2D nonlinear MVFIE. While dealing with FrMIEs are considered modern for this purpose. To solve the continuous kernels mixed nonlinear partial integro-differential equations, Al-Bugami et al. introduced a sixth type of Chebyshev and Bernoulli polynomial numerical approaches [5].

This paper is devoted to investigate a new technique to demonstrate the computational approach for dealing with the study of the numerical solution of NfrMIo-DE based on a continuous kernel.

Consider the following NfrPIo-DE:

$$\frac{\partial^\alpha}{\partial t^\alpha}[\frac{\Phi(x,t)-f(x,t)}{\delta(G(t),\Phi(x,t))}] = F(t)\int_{-1}^{1}k(x,y)\gamma(\Phi(y,t))dy, \quad 0<\alpha<1. \tag{1.1}$$

Under the condition

$$\Phi(x,0)=u(x), \tag{1.2}$$

where these continuous functions are known: $k(x,t), f(x,t), G(t), F(t)$ and $\Phi(x,t)$ is unknown.

This type of equation, in this case, or its special cases, can appear widely in genetic engineering, nanotechnology, and many nonlinear disciplines based on the study of fractional time and its results related to materials. In many cases, the primary the problem may be treated as it is presented. Therefore, we will transform the problem from fractional differential-integral equations to an equivalent integral equation, and we will then shed light on how to solve this integral equation.

Integrating (1.1) and using (1.2), we have

$$[\frac{\Phi(x,t)-f(x,t)}{\delta(G(t),\Phi(x,t))}]_0^t$$

$$=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}F(\tau)\int_{-1}^1 k(x,y)\gamma(\Phi(y,\tau))dyd\tau,$$

$$[\frac{\Phi(x,t)-f(x,t)}{\delta(G(t),\Phi(x,t))}]-[\frac{\Phi(x,0)-f(x,0)}{\delta(G(0),\Phi(x,0))}]$$

$$=\frac{1}{\Gamma(\alpha)}\int_0^t\int_{-1}^1(t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi(y,\tau))dyd\tau,$$

$$[\frac{\Phi(x,t)-f(x,t)}{\delta(G(t),\Phi(x,t))}]$$

$$=[\frac{\Phi(x,0)-f(x,0)}{\delta(G(0),\Phi(x,0))}]+\frac{1}{\Gamma(\alpha)}\int_0^t\int_{-1}^1(t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi(y,\tau))dyd\tau.$$

Then

$$\Phi(x,t)=f(x,t)+\delta(G(t),\Phi(x,t))[\frac{u(x)-f(x,0)}{\delta(G(0),\Phi(x,0))}]$$

$$+\frac{\delta(G(t),\Phi(x,t))}{\Gamma(\alpha)}\int_0^t\int_{-1}^1(t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi(y,\tau))dyd\tau,$$

$$\Phi(x,t)=f(x,t)+D(x)\delta(G(t),\Phi(x,t))$$

$$+\frac{\delta(G(t),\Phi(x,t))}{\Gamma(\alpha)}\int_0^t\int_{-1}^1(t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi(y,\tau))dyd\tau, \tag{1.3}$$

where $D(x)=[\frac{u(x)-f(x,0)}{\delta(G(0),\Phi(x,0))}]$.

The formula (1.3) represents a nonlinear mixed integral equation.

Special Situations

(I) In the case where $u(x)=f(x,0)$, the quadratic nonlinear IE is as described

below:

$$\Phi(x,t) = f(x,t)$$
$$+ \frac{\delta(G(t), \Phi(x,t))}{\Gamma(\alpha)} \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau) k(x,y) \gamma(\Phi(y,\tau)) dy d\tau. \quad (1.4)$$

(II) Assuming that $f(x,t) = 0, \Phi(x,0) = u(x)$, we obtain

$$\Phi(x,t) = D^*(x)\delta(G(t), \Phi(x,t))$$
$$+ \frac{\delta(G(t), \Phi(x,t))}{\Gamma(\alpha)} \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau) k(x,y) \gamma(\Phi(y,\tau)) dy d\tau, \quad (1.5)$$

where $D^*(x) = [\frac{u(x)}{\delta(G(0), \Phi(x,0))}]$.
(III) Assuming that $f(x,t) = 0, \Phi(x,0) = 0$, we obtain

$$\Phi(x,t) = \frac{\delta(G(t), \Phi(x,t))}{\Gamma(\alpha)} \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau) k(x,y) \gamma(\Phi(y,\tau)) dy d\tau. \quad (1.6)$$

The rest of this study's parts are organized as follows. In section 2, we will discuss several characteristics that apply to fractional derivatives. In section 3, we will discuss the existence of the nonlinear fractional MIE as well as the unique solution to it. The proposed solution's convergence will be studied in section 4, which is dedicated to doing so. Furthermore taken into account is the stability of the generated error in section 5. The nonlinear fractional MIE is converted into the NFIE in section 6, which can be completed by the use of the algorithm of separation of variables. The NFIE was solved using CP6K in section 7. An illustrative example is presented in section 8. Finally, the characterization of fractional time is displayed in section 9.

2. Definitions of fractional calculus in basic terms

This section provides an overview of fundamental concepts and characteristics dealing with the theory of fractional calculus.

Definition 2.1. [21, 26] Fractional integral of function based on the Riemann-Liouville formula $f : (0, \infty) \longrightarrow R$, for f$\alpha \in R^+$ consists of the definition that

$$I^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t \dfrac{f(\tau)}{(t-\tau)^{1-\alpha}} \, d\tau, & \alpha > 0, \\ f(t), & \alpha = 0. \end{cases} \quad (2.1)$$

Definition 2.2. [21, 26] The fractional derivative of the Caputo function $f : (0, \infty) \longrightarrow R$, consists of a description that

$$D^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t f^{(m)}(\tau)(t-\tau)^{m-\alpha-1} \, d\tau, & \alpha \neq m \in R - M, \\ f^{(m)}(t), & \alpha = m \in M. \end{cases} \quad (2.2)$$

Therefore, we have the following characteristics:

1. $I^\alpha \ I^\nu g = I^{\alpha+\nu} g$, $\qquad\qquad\qquad\qquad\qquad \alpha, \nu > 0, \ \mu > 0,$ $\qquad$ (2.3)

2. $I^\alpha \ x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} \ x^{\alpha+\gamma},$ $\qquad\qquad \alpha > 0, \ \gamma > -1, \ x > 0,$

$$(2.4)$$

3. $I^\alpha \ D^\alpha g(x) = g(x) - \displaystyle\sum_{k=0}^{m-1} g^{(k)}(0^+) \dfrac{x^k}{k!},$ $\qquad x > 0, \ \ m-1 < \alpha \le m,$

$$(2.5)$$

4. $D^\alpha \ I^\alpha \ g(x) = g(x)$, $\qquad\qquad\qquad\qquad x > 0, \ \ m-1 < \alpha \le m.$

$$(2.6)$$

We must state Banach's fixed point theorem.

3. Equation (1.3) existence and unique solution

To discuss whether or not Equation (1.3) has a single, unique solution in light using the theorem of Banach fixed points, according to the integral operator form, we shall express (1.3) as follows:

$$\Phi(x,t) = \overline{W}\Phi(x,t) = f(x,t) + D(x)\delta(G(t),\Phi(x,t)) + \delta(G(t),\Phi(x,t))W\Phi(x,t),$$

$$(3.1)$$

$$W\Phi(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau)k(x,y)\gamma(\Phi(y,\tau))dy d\tau. \qquad (3.2)$$

Thereafter, let us suppose the conditions that are listed below:
$(i) \ \forall C > C_1, C_2, C_3, (\int_a^b \int_a^b k^2(x,y)dxdy)^{1/2} \le C_1, \ \ | F(t) | \le C_2, \ \ | G(t) | \le C_3.$
$(ii) \ f(x,t) \in L_2[-1,1] \times C[-1,1]$ and it's norm is

$$\| f(x,t) \| = \max_t \int_0^t \{\int_a^b f^2(x,\tau)dx\}^{\frac{1}{2}} d\tau = E.$$

$(iii) \ \forall A > A_1, A_2$, we have
$(iii - a) \ \ | \gamma(\Phi(x,t)) | < A_1 | \Phi(x,t) |,$
$(iii - b) \ | \gamma(\Phi_1(x,y)) - \gamma(\Phi_2(x,y)) | < A_2 | \Phi_1(x,t) - \Phi_2(x,t) | .$
$(iv) \ \forall B > B_1, B_2$, we have
$(iv - a) \ \ | \delta(G(t),\Phi(x,t)) | < B_1 | \Phi(x,t) |,$
$(iv - b) \ \ | \delta(G(t),\Phi_1(x,t)) - \delta(G(t),\Phi_2(x,t)) | < B_2 | \Phi_1(x,t) - \Phi_1(x,t) | .$
(v) If the researchers follow the way of integration with respect to time, they must consider that: the kernel $(t-s)^{\alpha-1} \forall t, \tau \in [0,T], 0 \le \tau \le T < 1$, is fulfilling for each continuous function $F(t), \forall 0 \le \tau_1 \le \tau_2 \le t$ and each type of constant $M > 0$, the integrals $\int_{\tau_1}^{\tau_2}(t-\tau)^{\alpha-1}F(\tau)d\tau, \ \ \int_0^t(t-\tau)^{\alpha-1}F(\tau)d\tau, \ \ \max_{0 \le t \le T} \int_0^t(t-\tau)^{\alpha-1}d\tau,$
are continuous functions of t. In general, we let $\| \int_0^t(t-\tau)^{\alpha-1}F(\tau)d\tau \| \le H$.
The integral operator's normality and continuity are both discussed.

Lemma 3.1. *We are going to demonstrate that the integral the operator is normal. The operator, which translates the space $L_2[-1,1] \times C[0,T]$ into itself must be implemented to take into account the constraints (i)–(iii-a) $\overline{W}\Phi(x,t)$ described using (3.1).*

Proof. Since

$$\overline{W}\Phi(x,t) = f(x,t) + D(x)\delta(G(t),\Phi(x,t)) + \delta(G(t),\Phi(x,t))W\Phi(x,t).$$

Then

$$\| \overline{W}\Phi(x,t) \|$$
$$\leq \| f(x,t) \| + \| D(x)\delta(G(t),\Phi(x,t)) \| + \| \delta(G(t),\Phi(x,t)) \| \| W\Phi(x,t) \|,$$

$$\| f(x,t) \| = \max_{0 \leq t \leq T} \int_0^t \{\int_{-1}^1 f^2(x,\tau)dx\}^{\frac{1}{2}} d\tau = E,$$

$$\| D(x)\delta(G(t),\Phi(x,t)) \| \leq DB \| \Phi(x,t)) \|,$$
$$\| \delta(G(t),\Phi(x,t)) \| \leq B \| \Phi(x,t)) \|,$$

$$\| W\Phi(x,t) \| = \frac{1}{\Gamma(\alpha)} \| \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi(y,\tau))dyd\tau \|$$

$$\leq \frac{A}{\Gamma(\alpha)}\{ \max_{0 \leq t \leq T} \int_0^t (t-\tau)^{\alpha-1}d\tau \mid F(\tau) \mid\}$$

$$\times \{(\int_{-1}^1 \int_{-1}^1 k^2(x,y)dxdy)^{1/2}\}(\max_{0 \leq t \leq T} \int_0^t [\int_{-1}^1 \Phi^2(x,\tau)dx]^{\frac{1}{2}} d\tau).$$

We have, by applying the conditions (i) through (iii),

$$\| W\Phi(x,t) \| \leq \frac{ACHB}{\Gamma(\alpha)} \| \Phi(x,t)) \|,$$

$$\| \overline{W}\Phi(x,t) \| \leq E + DB \| \Phi(x,t)) \| + \frac{ACHB}{\Gamma(\alpha)} \| \Phi(x,t)) \| . \qquad (3.3)$$

In the final inequality (3.3), it is demonstrated that the operator $\overline{W}$ is responsible for mapping the ball S_r into itself, where

$$r = \frac{E\Gamma(\alpha)}{\Gamma(\alpha)(1-DB) - ACHB}.$$

Since $r > 0$, therefore we have $\sigma < 1, \sigma = (DB + \frac{ACHB}{\Gamma(\alpha)})$. $\qquad \square$

Lemma 3.2. *We shall establish the integral operator's continuity. The operator in the space $L_2[-1,1] \times C[0,T]$ is continuous and must take into account the requirements (i)-(iii-a) $\overline{W}\Phi(x,t)$ given by (1.3).*

Proof. For the continuity, we consider

$$\overline{W}(\Phi_1(x,t) - \Phi_2(x,t))$$
$$= D(x)\{\delta(G(t),\Phi_1(x,t)) - \delta(G(t),\Phi_2(x,t))\}$$
$$+ \frac{(\delta(G(t),\Phi_1(x,t))}{\Gamma(\alpha)} \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi_1(y,\tau))dyd\tau$$
$$- \frac{\delta(G(t),\Phi_2(x,t))}{\Gamma(\alpha)} \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1}F(\tau)k(x,y)\gamma(\Phi_2(y,\tau))dyd\tau\}.$$

Then, we follow

$$\mid \overline{W}(\Phi_1(x,t) - \Phi_2(x,t)) \mid$$

$$\leq \mid D(x) \mid \times \mid \{\delta(G(t), \Phi_1(x,t)) - \delta(G(t), \Phi_2(x,t))\} \mid$$

$$+ \frac{1}{\Gamma(\alpha)} \mid (\delta(G(t), \Phi_1(x,t)) \mid$$

$$\times \mid \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau) k(x,y) \{\gamma(\Phi_1(y,\tau)) - \gamma(\Phi_2(y,\tau))\} dy d\tau \mid$$

$$+ \frac{1}{\Gamma(\alpha)} \mid (\delta(G(t), \Phi_1(x,t)) - (\delta(G(t), \Phi_2(x,t)) \mid$$

$$\times \mid \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau) k(x,y) \gamma(\Phi_2(y,\tau)) dy d\tau \mid .$$

Finally, we have

$$\| \overline{W}(\Phi_1(x,t) - \Phi_2(x,t)) \|$$
$$\leq DB \| \Phi_1(x,t) - \Phi_2(x,t) \|$$
$$+ \frac{ACHB}{\Gamma(\alpha)} \| \Phi_1(x,t) - \Phi_2(x,t) \| \| \Phi_1(x,t) \|$$
$$+ \frac{ACHB}{\Gamma(\alpha)} \| \Phi_1(x,t) - \Phi_2(x,t) \| \| \Phi_2(x,t) \| .$$

Consider $\| \Phi_1(x,t) \| = Q_1, \| \Phi_2(x,t) \| = Q_2$, hence, we have

$$\| \overline{W}(\Phi_1(x,t) - \Phi_2(x,t)) \| \leq (DB + \frac{ACHBQ_1 + ACHBQ_2}{\Gamma(\alpha)}) \| \Phi_1(x,t) - \Phi_2(x,t) \| .$$

Hence the operator $\overline{W}$ is continuous and under the condition $\sigma = (DB + \frac{ACHB}{\Gamma(\alpha)})$, we have a contraction mapping. $\qquad \square$

4. Convergence of solutions

The form of the family of solutions should be supposed

$$\Phi(x,t) = \{\Phi_0(x,t), \Phi_1(x,t), \ldots, \Phi_{n-1}(x,t), \Phi_n(x,t), \ldots\} = \{\Phi_i(x,t)\}_{i=0}^\infty,$$
$$\Phi_0(x,t) = f(x,t) + D(x) \times \delta(G(t), \Phi_0(x,t)),$$
$$\mid \Phi_0(x,t) \mid \leq \mid f(x,t) \mid + B \mid D(x) \mid \| \Phi_0(x,t) \|$$
$$\leq E + BD \| \Phi_0(x,t) \|$$
$$\leq \frac{E}{1-BD}, \ (1-BD \neq 0), \ \mid D(x) \mid = D.$$

Pick up two functions $\{\Phi_n(x,t), \Phi_{n-1}(x,t)\} \in \{\Phi_i(x,t)\}_{i=0}^\infty$

$$\Phi_n(x,t)$$
$$= f(x,t) + D(x) \times \delta(G(t), \Phi_n(x,t)) \tag{4.1}$$
$$+ \frac{\delta(G(t), \Phi_n(x,t))}{\Gamma(\alpha)} \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau) k(x,y) \gamma(\Phi_{n-1}(y,\tau)) dy d\tau,$$
$$\Phi_{n-1}(x,t)$$
$$= f(x,t) + D(x) \times \delta(G(t), \Phi_{n-1}(x,t)) \tag{4.2}$$

$$+ \frac{\delta(G(t), \Phi_n(x,t))}{\Gamma(\alpha)} \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau) k(x,y) \gamma(\Phi_{n-2}(y,\tau)) dy d\tau.$$

For this, we write

$$\mid \Phi_n(x,t) - \Phi_{n-1}(x,t) \mid$$
$$\leq BD \mid \Phi_n(x,t) - \Phi_{n-1}(x,t) \mid + \frac{AB}{\Gamma(\alpha)} \mid \Phi_n(x,t) \mid \quad (4.3)$$
$$\times \mid \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau) k(x,y) \mid \Phi_{n-1}(y,\tau) - \Phi_{n-2}(y,\tau) \mid dy d\tau \mid$$
$$+ \frac{AB}{\Gamma(\alpha)} \mid \Phi_n(x,t) - \Phi_{n-1}(x,t) \mid$$
$$\times \mid \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau) k(x,y) \mid \Phi_{n-2}(y,\tau) \mid dy d\tau \mid .$$

Hence,

$$\mid \Psi_n(x,t) \mid$$
$$\leq BD \mid \Psi_n(x,t) \mid \quad (4.4)$$
$$+ \frac{AB}{\Gamma(\alpha)} \mid \Phi_n(x,t) \mid \times \mid \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau) k(x,y) \mid \Psi_{n-1}(y,\tau) \mid dy d\tau \mid$$
$$+ \frac{AB}{\Gamma(\alpha)} \mid \Psi_n(x,t) \mid \times \mid \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau) k(x,y) \mid \Phi_{n-2}(y,\tau) \mid dy d\tau \mid .$$

Let

$$\mid L \mid = \mid \int_0^t \int_{-1}^1 F(\tau) k(x,y) \mid \Phi_{n-2}(y,\tau) \mid dy d\tau \mid .$$

Since

$$\mid \Phi_{n-2}(y,\tau) \mid \leq \mid \Psi_{n-2}(y,\tau) \mid + \mid \Psi_{n-3}(y,\tau) \mid + \ldots + \mid \Psi_1(y,\tau) \mid + \mid \Psi_0(y,\tau) \mid,$$

i.e

$$\mid \Phi_{n-2}(y,\tau) \mid \leq \sum_{i=0}^{n-2} \mid \Psi_i(y,\tau) \mid . \quad (4.5)$$

Then

$$\mid L \mid \leq \mid \int_0^t \int_{-1}^1 F(\tau) k(x,y) \sum_{i=0}^{n-2} \mid \Psi_i(y,\tau) \mid dy d\tau \mid .$$

Hence, we have

$$\parallel L \parallel \leq \frac{C_1 C_2 T^\alpha}{\Gamma(1+\alpha)} \sum_{i=0}^{n-2} \parallel \Psi_i(y,\tau) \parallel . \quad (4.6)$$

Using (4.5) and (4.6) in (4.4), we follow

$$(1-BD) \parallel \Psi_n(x,t) \parallel \leq \frac{C_1 C_2 ABT^\alpha}{\Gamma(1+\alpha)} \sum_{i=0}^{n} \parallel \Psi_i(x,t) \parallel \parallel \Psi_{n-1}(x,t) \parallel$$

$$+ \frac{C_1 C_2 A B T^\alpha}{\Gamma(1+\alpha)} \parallel \Psi_n(x,t) \parallel [\sum_{i=0}^{n-2} \parallel \Psi_i(x,t) \parallel].$$

This equation can be modified to have the following format:

$$(1 - BD) \parallel \Psi_n(x,t) \parallel \leq \frac{C_1 C_2 A B T^\alpha}{\Gamma(1+\alpha)} \parallel \Psi_n(x,t) \parallel [\sum_{i=0}^{n-1} \parallel \Psi_i(x,t) \parallel]$$

$$+ \frac{C_1 C_2 A B T^\alpha}{\Gamma(1+\alpha)} \sum_{i=0}^{n-1} \parallel \Psi_i(x,t) \parallel \parallel \Psi_{n-1}(x,t) \parallel,$$

$$(1 - BD) \parallel \Psi_n(x,t) \parallel \leq \frac{C_1 C_2 A B T^\alpha}{\Gamma(1+\alpha)} [\parallel \Psi_n(x,t) \parallel$$

$$+ \parallel \Psi_{n-1}(x,t) \parallel] \times [\sum_{i=0}^{n-1} \parallel \Psi_i(x,t) \parallel].$$

To obtain the format, rewrite the equation used previously.

$$(1 - BD - \frac{C_1 C_2 A B T^\alpha}{\Gamma(1+\alpha)} [\sum_{i=0}^{n-1} \parallel \Psi_i(x,t) \parallel]) \parallel \Psi_n(x,t) \parallel$$

$$\leq \frac{C_1 C_2 A B T^\alpha}{\Gamma(1+\alpha)} \parallel \Psi_{n-1}(x,t) \parallel] \times [\sum_{i=0}^{n-1} \parallel \Psi_i(x,t) \parallel].$$

Lastly, we have

$$\parallel \Psi_n(x,t) \parallel \leq \frac{C_1 C_2 A B T^\alpha \parallel \Psi_{n-1}(x,t) \parallel] \times [\sum_{i=0}^{n-1} \parallel \Psi_i(x,t) \parallel]}{(1 - BD)\Gamma(1+\alpha) - C_1 C_2 A B T^\alpha [\sum_{i=0}^{n-1} \parallel \Psi_i(x,t) \parallel]},$$

$$L \geq [\sum_{i=0}^{n-1} \parallel \Psi_i(x,t) \parallel].$$

Let $n = 0$,
$$\parallel \Psi_0(x,t) \parallel \leq \frac{E}{1 - BD}, (1 - BD \neq 0).$$

Let $n = 1$,

$$\parallel \Psi_1(x,t) \parallel \leq \frac{C^2 T^\alpha B A \{\parallel \Psi_0(x,t) \parallel\} \parallel \Psi_0(x,t) \parallel}{\{(1 - BD)\Gamma(1+\alpha) - C^2 T^\alpha B A \{\parallel \Psi_0(x,t) \parallel\}\}},$$

$$\parallel \Psi_0(x,t) \parallel \leq \frac{E}{1 - BD},$$

$$\parallel \Psi_1(x,t) \parallel \leq \frac{(C^2 T^\alpha B A E^2)}{(1 - BD)\Delta_1}, \Delta_1 = \{(1 - BD)^2 \Gamma(1+\alpha) - C^2 T^\alpha B A E\}.$$

Let $n = 2$,

$$\parallel \Psi_2(x,t) \parallel \leq \frac{C^2 T^\alpha B A \{\sum_{i=0}^{1} \parallel \Psi_i(x,t) \parallel\} \parallel \Psi_1(x,t) \parallel}{\{(1 - BD)\Gamma(1+\alpha) - C^2 B A T^\alpha [\sum_{i=0}^{1} \parallel \Psi_i(x,t) \parallel]\}},$$

$$\parallel \Psi_2(x,t) \parallel \leq \frac{E^2 (C^2 T^\alpha B A)^2 K_2}{[(1 - BD)\Delta_1]\Delta_2}, \text{ where } K_2 = \{E\Delta_1 + C^2 T^\alpha B A E^2\},$$

$$\Delta_1 = \{(1 - BD)^2\Gamma(1 + \alpha) - C^2T^\alpha BAE\},$$
$$\Delta_2 = \{(1 - BD)^2\Gamma(1 + \alpha)\Delta_1 - C^2T^\alpha BAK_2\}.$$

Let $n = 3$,

$$\parallel \Psi_3(x,t) \parallel$$
$$\leq \frac{C^2T^\alpha BA \parallel \Psi_2(x,t) \parallel \{\parallel \Psi_0(x,t) \parallel + \parallel \Psi_1(x,t) \parallel + \parallel \Psi_2(x,t) \parallel\}}{\{(1 - BD)\Gamma(1 + \alpha) - C^2BAT^\alpha[\parallel \Psi_0(x,t) \parallel + \parallel \Psi_1(x,t) \parallel + \parallel \Psi_2(x,t) \parallel\}}.$$

Since

$$\{\parallel \Psi_0(x,t) \parallel + \parallel \Psi_1(x,t) \parallel + \parallel \Psi_2(x,t) \parallel\}$$
$$= \frac{E}{1 - BD} + \frac{(C^2T^\alpha BAE^2)}{(1 - BD)\Delta_1} + \frac{E^2(C^2T^\alpha BA)^2 K_2}{(1 - BD)\Delta_1\Delta_2},$$
$$C^2T^\alpha BA\{\parallel \Psi_0(x,t) \parallel + \parallel \Psi_1(x,t) \parallel + \parallel \Psi_2(x,t) \parallel\}$$
$$= \frac{(C^2T^\alpha BA)[E\Delta_1\Delta_2 + (C^2T^\alpha BAE^2)\Delta_2 + E^2(C^2T^\alpha BA)^2 K_2]}{(1 - BD)\Delta_1\Delta_2},$$
$$\{(1 - BD)\Gamma(1 + \alpha) - C^2BAT^\alpha[\parallel \Psi_0(x,t) \parallel + \parallel \Psi_1(x,t) \parallel + \parallel \Psi_2(x,t) \parallel\}$$
$$= (1 - BD)\Gamma(1 + \alpha)$$
$$\quad - \frac{(C^2T^\alpha BA)[E\Delta_1\Delta_2 + (C^2T^\alpha BAE^2)\Delta_2 + E^2(C^2T^\alpha BA)^2 K_2]}{(1 - BD)\Delta_1\Delta_2}$$
$$= \frac{(1 - BD)^2\Gamma(1 + \alpha)\Delta_1\Delta_2 - (C^2T^\alpha BA)[E\Delta_1\Delta_2 + Z_1]}{(1 - BD)\Delta_1\Delta_2},$$

where

$$Z_1 = (C^2T^\alpha BAE^2)\Delta_2 + E^2(C^2T^\alpha BA)^2 K_2.$$

Then

$$\parallel \Psi_3(x,t) \parallel$$
$$\leq \frac{C^2T^\alpha BA \parallel \Psi_2(x,t) \parallel \{\parallel \Psi_0(x,t) \parallel + \parallel \Psi_1(x,t) \parallel + \parallel \Psi_2(x,t) \parallel\}}{\{(1 - BD)\Gamma(1 + \alpha) - C^2BAT^\alpha[\parallel \Psi_0(x,t) \parallel + \parallel \Psi_1(x,t) \parallel + \parallel \Psi_2(x,t) \parallel\}}$$
$$= \frac{\frac{(C^2T^\alpha BA)[E\Delta_1\Delta_2 + (C^2T^\alpha BAE^2)\Delta_2 + E^2(C^2T^\alpha BA)^2 K_2]}{(1 - BD)\Delta_1\Delta_2}}{\frac{(1 - BD)^2\Gamma(1 + \alpha)\Delta_1\Delta_2 - (C^2T^\alpha BA)[E\Delta_1\Delta_2 + (C^2T^\alpha BAE^2)\Delta_2 + E^2(C^2T^\alpha BA)^2 K_2]}{(1 - BD)\Delta_1\Delta_2}}.$$

Hence,

$$\parallel \Psi_3(x,t) \parallel$$
$$\leq \frac{E^2(C^2T^\alpha BA)^3 K_2[E\Delta_1\Delta_2 + (C^2T^\alpha BAE^2)\Delta_2 + E^2(C^2T^\alpha BA)^2 K_2]}{(1 - BD)\Delta_1\Delta_2\{(1 - BD)^2\Gamma(1 + \alpha)\Delta_1\Delta_2 - (C^2T^\alpha BA)Z_2\}},$$

where

$$Z_2 = [E\Delta_1\Delta_2 + (C^2T^\alpha BAE^2)\Delta_2 + E^2(C^2T^\alpha BA)^2 K_2],$$
$$\parallel \Psi_3(x,t) \parallel$$
$$\leq \frac{E^2(C^2T^\alpha BA)^3 K_2 K_3}{(1 - BD)\Gamma(1 + \alpha)[(1 - BD)\Delta_1\Delta_2]^2 - (C^2T^\alpha BA)(1 - BD)\Delta_1\Delta_2 K_3}.$$

Finally, we have

$$\| \Psi_3(x,t) \| \leq \frac{E^2(C^2T^\alpha BA)^3 K_2 K_3}{(1-BD)\Gamma(1+\alpha)\Delta_1\Delta_2[(1-BD)^2\Delta_1\Delta_2 - (C^2T^\alpha BA)K_3]},$$

$$\| \Psi_3(x,t) \| \leq \frac{E^2(C^2T^\alpha BA)^3 K_2 K_3}{(1-BD)\Gamma(1+\alpha)\Delta_1\Delta_2\Delta_3},$$

where

$$K_3 = [E\Delta_1\Delta_2 + (C^2T^\alpha BAE^2)\Delta_2 + E^2(C^2T^\alpha BA)^2 K_2],$$

$$\Delta_3 = [(1-BD)^2\Delta_1\Delta_2 - (C^2T^\alpha BA)K_3].$$

By induction, we have

$$\| \Psi_n(x,t) \| \leq \frac{E^2(C^2T^\alpha BA)^3 K_2 K_3 \ldots K_n}{(1-BD)\Gamma(1+\alpha)\Delta_1\Delta_2\Delta_3 \ldots \Delta_n},$$

$$K_n = [E\Delta_1\Delta_2\Delta_3 \ldots \Delta_{n-1} + E^2(C^2T^\alpha BA)\Delta_2 \ldots \Delta_{n-1}$$
$$+ E^2(C^2T^\alpha BA)^2 K_2\Delta_3 \ldots \Delta_{n-1} + \ldots + E^2(C^2T^\alpha BA)^{n-1} K_{n-1}],$$

$$\Delta_{n-1} = [(1-BD)^2]\Delta_1\Delta_2\Delta_3 \ldots \Delta_{n-1} - (\alpha\beta TBA)K_{n-1}].$$

Then, under the condition $\frac{C^2T^\alpha BA}{\Gamma(1+\alpha)} < (1-BD)$, the function $\Psi_n(x,t), \forall n$ convergence

$$C^2T^\alpha BA + \Gamma(1+\alpha)(1-DB) < \Gamma(1+\alpha).$$

5. The stability of error

The numerical solution $\Phi_n(x,t)$, of Equation (1.3) becomes

$$\Phi_n(x,t) = f_n(x,t) + D(x)\delta(G(t), \Phi_n(x,t))$$
$$+ \frac{\delta(G(t), \Phi_n(x,t))}{\Gamma(\alpha)} \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau)k(x,y)\gamma(\Phi_n(y,\tau))dyd\tau. \tag{5.1}$$

Therefore, we have the error $R_n(x,t)$ after subtracting (1.3) and (5.1) to have

$$R_n(x,t) = h_n(x,t) + D(x)\{\delta(G(t), \Phi(x,t)) - \delta(G(t), \Phi_n(x,t))\}$$
$$+ \{\frac{\delta(G(t), \Phi(x,t))}{\Gamma(\alpha)} \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau)k(x,y)\gamma(\Phi(y,\tau))dyd\tau$$
$$- \frac{\delta(G(t), \Phi_n(x,t))}{\Gamma(\alpha)} \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau)k(x,y)\gamma(\Phi_n(y,\tau))dyd\tau\},$$

$$(R_n(x,t) = \Phi(x,t) - \Phi_n(x,t), \quad h_n(x,t) = f(x,t) - f_n(x,t)).$$

The equation above can be modified to take the form

$$R_n(x,t)$$
$$= h_n(x,t) + D(x)\{\delta(G(t), \Phi(x,t)) - \delta(G(t), \Phi_n(x,t))\}$$

$$+ \{\frac{\delta(G(t), \Phi(x,t))}{\Gamma(\alpha)} \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau)k(x,y)[\gamma(\Phi(y,\tau)) - \gamma(\Phi_n(y,\tau)]dyd\tau\}$$

$$+ \{\frac{\delta(G(t), \Phi(x,t)) - \delta(G(t), \Phi_n(x,t))}{\Gamma(\alpha)}$$

$$\times \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau)k(x,y)\gamma(\Phi_n(y,\tau)dyd\tau\},$$

$$\mid R_n(x,t) \mid$$

$$\leq \mid h_n(x,t) \mid + \mid D(x) \parallel \delta(G(t), \Phi(x,t)) - \delta(G(t), \Phi_n(x,t)) \mid$$

$$+ \frac{\mid \delta(G(t), \Phi(x,t)) \mid}{\Gamma(\alpha)}$$

$$\times \mid \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau)k(x,y)[\gamma(\Phi(y,\tau)) - \gamma(\Phi_n(y,\tau)]dyd\tau \mid$$

$$+ \frac{\mid \delta(G(t), \Phi(x,t)) - \delta(G(t), \Phi_n(x,t)) \mid}{\Gamma(\alpha)}$$

$$\times \mid \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau)k(x,y)\gamma(\Phi_n(y,\tau)dyd\tau \mid.$$

Using the conditions (i)-(v), we follow

$$\mid R_n(x,t) \mid$$

$$\leq \mid h_n(x,t) \mid + DB_2 \mid R_n(x,t) \mid$$

$$+ \frac{B_1}{\Gamma(\alpha)} \mid \Phi(x,t) \parallel \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau)k(x,y)[\gamma(\Phi(y,\tau)) - \gamma(\Phi_n(y,\tau)]dy \mid$$

$$+ \frac{B_1}{\Gamma(\alpha)} \mid R_n(x,t) \parallel \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau)k(x,y)\gamma(\Phi_n(y,\tau)dyd\tau \mid.$$

Hence, we have

$$\parallel R_n(x,t) \parallel$$

$$\leq \parallel h_n(x,t) \parallel + DB \parallel R_n(x,t) \parallel$$

$$+ \frac{BCHA}{\Gamma(\alpha)} \parallel \Phi(x,t) \parallel \parallel R_n(x,t) \parallel \frac{BCHA}{\Gamma(\alpha)} \parallel \Phi_n(x,t) \parallel \parallel R_n(x,t) \parallel.$$

Assuming that $\parallel \Phi(x,t) \parallel = Q, \parallel \Phi_n(x,t) \parallel = Q_n$, hence we have

$$\parallel R_n(x,t) \parallel \leq \parallel h_n(x,t) \parallel + \frac{BD\Gamma(\alpha) + BCHAQ + BCHAQ_n}{\Gamma(\alpha)} \parallel R_n(x,t) \parallel.$$

Hence, the error is stable under the condition

$$\Gamma(\alpha) > \frac{BCHA(Q + Q_n)}{1 - BD}, (BD < 1).$$

6. The separation of the variables

It is in this section that the approach of separation of variables is applied the fractional MIE (1.3) needs to be transformed into the FIE in position. Suppose

that the functions in Equation (1.3) that are unknown and known are, in turn, the following:

$$\Phi(x,t) = \Phi(x)\Psi(t), \quad f(x,t) = f(x)\Psi(t), \quad \Psi(0) \neq 0, \quad (6.1)$$

where $\Psi(t), f(x)$ are known functions and $\Phi(x)$ has unknown function.

We get the following result by substituting (6.1) with (1.3):

$$\Phi(x)\Psi(t) = f(x)\Psi(t) + D(x)\delta(G(t), \Phi(x)\Psi(t)) \quad (6.2)$$
$$+ \frac{\delta(G(t), \Phi(x)\Psi(t))}{\Gamma(\alpha)} \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau) k(x,y) \gamma(\Phi(y)\Psi(\tau)) dy d\tau.$$

By dividing both sides of Equation (6.2) via $\Psi(t)$, we are able to generate the following result:

$$\Phi(x) = f(x) + D(x)\frac{\delta(G(t), \Phi(x)\Psi(t))}{\Psi(t)} \quad (6.3)$$
$$+ \frac{\delta(G(t), \Phi(x)\Psi(t))}{\Gamma(\alpha)\Psi(t)} \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1} F(\tau) k(x,y) \gamma(\Phi(y)\Psi(\tau)) dy d\tau.$$

The expression for formula (6.3) is

$$\Phi(x) = f(x) + D(x)\delta(\Phi(x))\delta_1(t) + \delta(\Phi(x))\delta_1(t)\lambda(t) \int_{-1}^1 k(x,y)\gamma(\Phi(y)) dy, \quad (6.4)$$

where

$$\delta_1(t) = \frac{\delta(G(t), \Phi(x)}{\Psi(t))},$$

$$\lambda(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F(\tau)\gamma(\Psi(\tau)) d\tau.$$

Equation (6.4) represents NFIE of the second type.

7. Chebyshev polynomials of the sixth type

This section presents an application of CP6K for the purpose of solving NFIE (6.4).

There are orthogonal functions that are Chebyshev polynomials of the sixth type, denoted by $Y_j(\xi)$ on the interval $[-1,1]$ described using the next recurrence relation:

$$Y_j(\xi) = \xi Y_{j-1} - \frac{j(j+1) + (-1)^j(2j+1) + 1}{4j(j+1)} Y_{j-2}(\xi), \quad Y_0(\xi) = 1, \quad Y_1(\xi) = \xi, \quad j \geq 2. \quad (7.1)$$

On the interval $[-1, 1]$, the following information polynomials are orthogonal to the weight function according to the weight function $\omega_2(\xi) = \xi^2\sqrt{1-\xi^2}$ and satisfies:

$$\int_{-1}^1 \xi^2\sqrt{1-\xi^2}Y_i(\xi)Y_j(\xi)d\xi = \begin{cases} \Theta_i, & i = j, \\ \\ 0, & i \neq j, \end{cases}$$

where

$$
\Theta_i = \frac{\pi}{2^{2i+3}} \begin{cases} 1, & i \ \text{even}, \\[2ex] \dfrac{i+3}{i+1}, & i \ \text{odd}. \end{cases}
$$

The trigonometric representation of CP6K was demonstrated by Abd-Elhameed and Youssri in [1] as follows:

$$
Y_j(\cos\theta) = \begin{cases} \dfrac{\sin((j+2)\theta)}{2^j \sin(2\theta)}, & j \ \text{even}, \\[3ex] \dfrac{\sin((j+1)\theta) + (j+1)\cos(\theta)\sin((j+2)\theta)}{2^{j+1}(j+1)\cos^2(\theta)\sin(\theta)}, & j \ \text{odd}. \end{cases}
\tag{7.2}
$$

Lemma 7.1. [2] *Let m be any non-negative integer. Therefore, the power form that can be used to describe CP6K is as explained below:*

$$
Y_{2m}(\xi) = \frac{\Gamma(m+\frac{3}{2})}{(2m+1)!} \sum_{i=0}^{m} \frac{(-1)^i \binom{m}{i}(2m-i+1)!}{\Gamma(m-i+\frac{3}{2})} \xi^{2m-2i},
\tag{7.3}
$$

and

$$
Y_{2m+1}(\xi) = \frac{\Gamma(m+\frac{5}{2})}{(2m+2)!} \sum_{i=0}^{m} \frac{(-1)^i \binom{m}{i}(2m-i+2)!}{\Gamma(m-i+\frac{5}{2})} \xi^{2m-2i+1}.
\tag{7.4}
$$

Lemma 7.2. [2] *Let p be any non-negative integer. Following that, the formulas for the inversion of CP6K are as follows:*

$$
\xi^{2p} = (2p+1)! \sum_{i=0}^{p} \frac{2^{1-2i}(p-i+1)!}{i!(2p-i+2)!} Y^{2p-2i}(\xi),
\tag{7.5}
$$

and

$$
\xi^{2p+1} = (2p+1)!(2p+3) \sum_{i=0}^{p} \frac{2^{1-2i}(p-i+1)!}{i!(2p-i+3)!} Y^{2p-2i+1}(\xi).
\tag{7.6}
$$

Let $\Phi(x) \in L_2[-1,1]$, hence the following $Y_n(x)$ approximation for $\Phi(x)$ can be made:

$$
\Phi(x) \approx \Phi_N(x) = \sum_{n=0}^{N} a_n \, Y_n(x),
\tag{7.7}
$$

where the constants a_n are $n = 0, 1, \ldots, N$.

Substituting from (7.7) into (6.4) we get

$$
\sum_{n=0}^{N} a_n \, Y_n(x) - D(x)\delta\left(\sum_{n=0}^{N} a_n \, Y_n(x)\right)\delta_1(t)
$$

$$
+ \delta\left(\sum_{n=0}^{N} a_n \, Y_n(x)\right)\delta_1(t)\lambda(t) \int_{-1}^{1} k(x,y)\gamma\left(\sum_{n=0}^{N} a_n \, Y_n(y)\right)dy = f(x).
\tag{7.8}
$$

Table 1. The error of absolute value of Example 8.1, $T = 0.001$

x	Error, $\alpha = 0.0004$,	Error, $\alpha = \frac{1}{5}$	Error, $\alpha = \frac{1}{3}$	Error, $\alpha = \frac{1}{2}$	Error, $\alpha = 0.8$	Error, $\alpha = 0.9$
-1	2.7×10^{-13}	1.9×10^{-12}	4.61×10^{-26}	1.9×10^{-12}	1.01×10^{-29}	2.2×10^{-13}
-0.8	9.7×10^{-14}	1.2×10^{-12}	3.6×10^{-26}	1.2×10^{-12}	2.9×10^{-30}	7.3×10^{-14}
-0.6	5.8×10^{-14}	7.1×10^{-13}	2.6×10^{-26}	7.1×10^{-13}	6.1×10^{-32}	1.3×10^{-14}
-0.4	5.2×10^{-14}	3.2×10^{-13}	1.7×10^{-26}	3.2×10^{-13}	4.2×10^{-30}	6.2×10^{-15}
-0.2	3.4×10^{-14}	8.6×10^{-14}	8.6×10^{-27}	8.6×10^{-14}	4.01×10^{-30}	6.9×10^{-15}
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.2	3.4×10^{-14}	7.3×10^{-14}	8.6×10^{-27}	7.3×10^{-14}	4.01×10^{-30}	6.9×10^{-15}
0.4	5.2×10^{-14}	3.1×10^{-13}	1.7×10^{-26}	3.1×10^{-13}	4.2×10^{-30}	7.10×10^{-15}
0.6	5.8×10^{-14}	7.2×10^{-13}	2.6×10^{-26}	7.2×10^{-13}	6.1×10^{-32}	3.1×10^{-15}
0.8	9.7×10^{-14}	1.2×10^{-12}	3.6×10^{-26}	1.2×10^{-12}	2.9×10^{-30}	1.6×10^{-14}
1	2.7×10^{-13}	2.01×10^{-12}	4.6×10^{-26}	2.01×10^{-12}	1.01×10^{-29}	1.6×10^{-15}

Now, by employing the collection of points

$$x_r = a + \frac{(b-a)r}{N}, \quad r = 0, 1, 2, \ldots, N. \tag{7.9}$$

For $N + 1$ unknowns, we get the following $N + 1$ SNAE:

$$\sum_{n=0}^{N} a_n \, Y_n(x_r) - D(x)\delta(\sum_{n=0}^{N} a_n \, Y_n(x_r))\delta_1(t)$$

$$+ \delta(\sum_{n=0}^{N} a_n \, Y_n(x_r))\delta_1(t)\lambda(t) \int_{-1}^{1} k(x_r, y)\gamma(\sum_{n=0}^{N} a_n \, Y_n(y))dy = f(x_r). \tag{7.10}$$

As a result of solving the the system described above, we present the approximate solution and employing the use of (6.1) through numerical analysis, we are able to get the solution to equation (1.3), which corresponds to the NfrPIo-DE equation (1.1).

8. Illustrative example

This section will illustrate the outcomes presented above by the following example given several kinds of values of time and various values of α.

Example 8.1. Suppose that the NfrPIo-DE:

$$\Phi(x,t) = f(x,y) + (e^{-t}\Phi(x,t))[\frac{(u(x) - f(x,0))}{\delta(G(0), u(x))}]$$

$$+ \frac{1}{\Gamma(\alpha)}(e^{-t}\Phi(x,t)) \int_0^t \int_{-1}^1 (t-\tau)^{\alpha-1}(0.1 + \tau)x^2y^2\Phi^2(y,\tau)dyd\tau, \tag{8.1}$$

with exact solution $\Phi(x,t) = x^4(0.01 + t^2), G(t) = e^{-t}, \delta(G(t), \Phi(x,t)) = e^{-t}\Phi(x,t)$.

At $N = 5$ and different values of T and α are used with separation of variables and CP6K for Equation (8.1), the numbers in Tables 1-5 and Figures 1-5 show the results.

When $T = 0.001, T = 0.2, \alpha = 0.0004, \frac{1}{5}, \frac{1}{3}$, we obtain the estimated solution as the accurate solution and the error results are demonstrated in Table 1 and Table 2.

Table 2. At $T = 0.2$, clarification of the absolute error for Example 8.1

x	Error, $\alpha = 0.0004$,	Error, $\alpha = \frac{1}{5}$	Error, $\alpha = \frac{1}{3}$	Error, $\alpha = \frac{1}{2}$	Error, $\alpha = 0.8$	Error, $\alpha = 0.9$
-1	1.7×10^{-13}	5.3×10^{-14}	4.6×10^{-14}	5.09×10^{-12}	2.1×10^{-13}	5.03×10^{-12}
-0.8	4.9×10^{-14}	1.5×10^{-14}	3.3×10^{-14}	3.2×10^{-12}	8.1×10^{-14}	3.2×10^{-12}
-0.6	2.4×10^{-14}	5.8×10^{-14}	1.3×10^{-14}	1.7×10^{-12}	2.6×10^{-14}	1.8×10^{-12}
-0.4	2.4×10^{-14}	6.6×10^{-14}	1.52×10^{-14}	7.87×10^{-13}	6.9×10^{-15}	8.1×10^{-13}
-0.2	1.7×10^{-14}	4.2×10^{-14}	1.8×10^{-14}	1.9×10^{-13}	1.6×10^{-15}	2.09×10^{-13}
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.2	1.7×10^{-14}	4.2×10^{-14}	1.8×10^{-14}	2.08×10^{-13}	1.6×10^{-15}	1.9×10^{-13}
0.4	2.4×10^{-14}	6.6×10^{-14}	1.5×10^{-14}	8.1×10^{-13}	6.9×10^{-15}	7.8×10^{-13}
0.6	2.4×10^{-14}	5.8×10^{-14}	1.3×10^{-14}	1.8×10^{-12}	2.6×10^{-14}	1.7×10^{-12}
0.8	4.9×10^{-14}	1.5×10^{-14}	3.3×10^{-14}	3.1×10^{-12}	8.1×10^{-14}	3.1×10^{-12}
1	1.7×10^{-13}	5.3×10^{-14}	4.6×10^{-14}	4.9×10^{-12}	2.1×10^{-13}	4.9×10^{-12}

Table 3. The error of absolute value for Example 8.1, $T = 0.4$

x	Error, $\alpha = \frac{1}{2}$	Error, $\alpha = 0.8$	Error, $\alpha = 0.9$
-1	5.09×10^{-25}	5.005×10^{-13}	6.5×10^{-13}
-0.8	6.3×10^{-25}	1.4×10^{-15}	1.4×10^{-14}
-0.6	6.04×10^{-25}	7.3×10^{-15}	4.2×10^{-15}
-0.4	4.6×10^{-25}	9.3×10^{-14}	1.03×10^{-13}
-0.2	2.5×10^{-25}	9.01×10^{-14}	1.03×10^{-13}
0	0.000000	0.000000	0.000000
0.2	2.5×10^{-25}	9.01×10^{-14}	1.03×10^{-13}
0.4	4.6×10^{-25}	9.3×10^{-14}	1.03×10^{-13}
0.6	6.04×10^{-25}	7.3×10^{-15}	4.24×10^{-15}
0.8	4.24×10^{-25}	1.4×10^{-15}	1.4×10^{-14}
1	5.09×10^{-25}	5.005×10^{-13}	6.5×10^{-12}

When $T = 0.4$, we obtain the error of absolute value at $\alpha = \frac{1}{2}, 0.8, 0.9$ the numerical results are demonstrated in Table 3.

When $T = 0.6$, we obtain the error of absolute value at $\alpha = 0.0004, \frac{1}{5}, \frac{1}{2}, 0.8$ the numerical results are displayed in Table 4.

When $T = 0.8$, we obtain the error of absolute value at $\alpha = 0.0004, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 0.8, 0.9$ the numerical results are displayed in Table 5.

9. Conclusion and characterization of fractional time

In the present study, CP6K is used to solve NfrPIo-DE with a continuous kernel with the assistance of variable separation. We illustrate the theoretical results with numerical examples for different values of time and α. The results were analyzed as follows:

The fractional tense is related to the kernel of Apple on the form $F(t, \tau) = \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$.

This nucleus has only one anomalous point on the path of integration when

Table 4. The error of absolute value for Example 8.1, $T = 0.6$

x	Error, $\alpha = 0.0004$	Error, $\alpha = \frac{1}{5}$	Error, $\alpha = \frac{1}{2}$	Error, $\alpha = 0.8$
-1	3.7×10^{-11}	3.4×10^{-11}	5.5×10^{-12}	1.04×10^{-12}
-0.8	2.3×10^{-11}	2.3×10^{-11}	1.4×10^{-12}	1.2×10^{-12}
-0.6	1.3×10^{-11}	1.3×10^{-11}	4.6×10^{-13}	4.6×10^{-13}
-0.4	5.9×10^{-14}	6.1×10^{-12}	8.4×10^{-14}	1.4×10^{-13}
-0.2	1.4×10^{-12}	1.5×10^{-12}	2.9×10^{-14}	2.5×10^{-13}
0	0.000000	0.000000	0.000000	0.000000
0.2	1.4×10^{-12}	1.3×10^{-12}	2.9×10^{-14}	2.5×10^{-13}
0.4	5.9×10^{-12}	5.7×10^{-12}	8.4×10^{-14}	1.4×10^{-13}
0.6	1.3×10^{-11}	1.3×10^{-11}	1.08×10^{-13}	4.6×10^{-13}
0.8	2.3×10^{-11}	2.4×10^{-11}	1.4×10^{-12}	1.2×10^{-12}
1	3.7×10^{-11}	3.9×10^{-11}	5.5×10^{-12}	1.04×10^{-12}

Table 5. Absolute Error of Example 8.1, $T = 0.8$

x	Error, $\alpha = 0.0004$,	Error, $\alpha = \frac{1}{4}$	Error, $\alpha = \frac{1}{3}$	Error, $\alpha = \frac{1}{2}$	Error, $\alpha = 0.8$	Error, $\alpha = 0.9$
-1	2.4×10^{-10}	1.7×10^{-10}	1.7×10^{-10}	1.7×10^{-10}	1.05×10^{-10}	1.7×10^{-10}
-0.8	3.8×10^{-11}	3.9×10^{-11}	3.9×10^{-11}	3.9×10^{-11}	2.5×10^{-11}	3.2×10^{-12}
-0.6	2.8×10^{-11}	7.9×10^{-12}	7.9×10^{-12}	7.9×10^{-12}	7.2×10^{-11}	5.2×10^{-11}
-0.4	2.8×10^{-14}	1.3×10^{-11}	1.3×10^{-11}	1.3×10^{-11}	7.8×10^{-11}	3.9×10^{-11}
-0.2	9.6×10^{-12}	4.7×10^{-12}	4.7×10^{-12}	4.7×10^{-14}	6.9×10^{-11}	1.2×10^{-11}
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.2	9.6×10^{-12}	4.7×10^{-12}	4.7×10^{-12}	4.7×10^{-12}	6.9×10^{-11}	1.2×10^{-11}
0.4	2.8×10^{-11}	1.3×10^{-11}	1.3×10^{-11}	1.3×10^{-11}	7.8×10^{-11}	3.9×10^{-11}
0.6	2.8×10^{-11}	7.9×10^{-12}	7.9×10^{-12}	7.9×10^{-12}	7.2×10^{-11}	5.2×10^{-11}
0.8	3.8×10^{-11}	3.9×10^{-11}	3.9×10^{-11}	3.9×10^{-11}	2.5×10^{-11}	3.20×10^{-12}
1	2.4×10^{-10}	1.7×10^{-10}	1.7×10^{-10}	1.7×10^{-10}	1.05×10^{-10}	1.7×10^{-10}
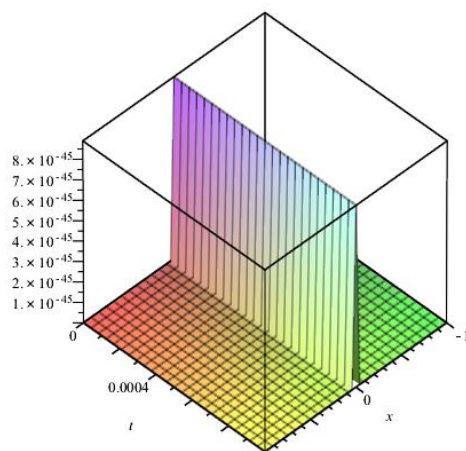
Figure 1. Clarification of the error of absolute value of Example 8.1, $T = 0.001, \alpha = 0.8$.
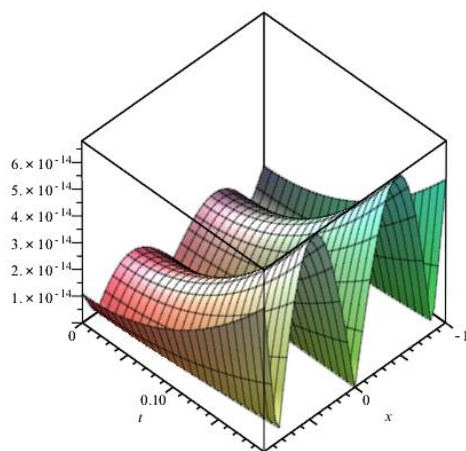
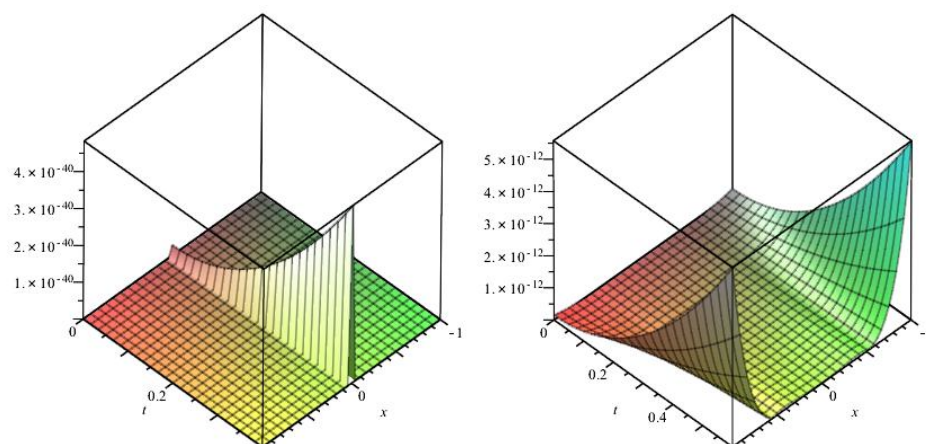**Figure 2.** Clarification of the error of absolute value Example 8.1 at, $T = 0.2, \alpha = \frac{1}{5}$.

Figure 3. At $T = 0.4, \alpha = \frac{1}{2}$, clarification of the absolute error of Example 8.1.

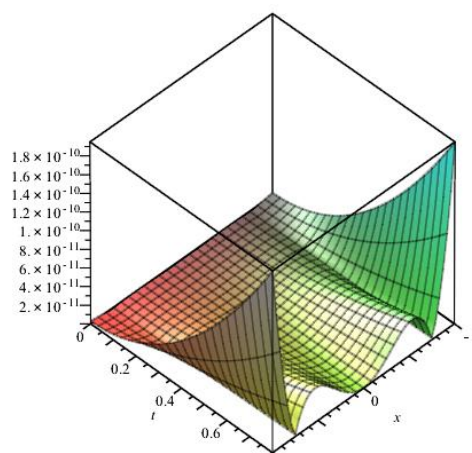**Figure 4.** At $T = 0.6, \alpha = \frac{1}{2}$, clarification of the error of absolute value of Example 8.1.

Figure 5. At $T = 0.8, \alpha = 0.0004$, clarification of the error of absolute value of Example 8.1.

$t = \tau$. Knowing the form of the time function, researchers can convert this abnormal integral into a form of the beta function.

Therefore, researchers prefer to convert FDEs into fractional integral equations, and in the end, the situation will be, knowing the time function in one of the forms of the beta function. Also, the results depend on our first choice of the form of the time function. In economics in general, it means the time required to reach the best product. The authors indicated that the best general form of the time function is as follows $G(t) = e^{-t}$. This choice is often used by researchers in the science of thermoelasticity or the science of atmospheric differential equations emerging in the science of mathematical physics, in general, and is known as the normal mode method. The second aspect of solving the problems of fractional equations in general, is represented in our choice of the numerical method. It has become clear through the applications and the choice of methods that the traditional methods lead to the accumulation of error and it increases with the increase in the time of

Table 6

T	α	Less value for error
0.001	0.8	6.1×10^{-32}
0.2	0.8	1.6×10^{-15}
0.4	$\frac{1}{2}$	2.8×10^{-25}
0.6	$\frac{1}{2}$	2.9×10^{-14}
0.8	0.0004	2.8×10^{-14}

Table 7

T	α	The largest error value
0.001	$\frac{1}{2}$	1.2×10^{-12}
0.2	0.9	1.8×10^{-12}
0.4	0.9	6.5×10^{-12}
0.6	$\frac{1}{5}$	1.3×10^{-11}
0.8	0.0004	2.4×10^{-10}

the experiment. However, it turns out that the best methods to use are the methods derived from the polynomials.

It is noted that the value of α plays a critical role in the description and flow of the line of integrations, and the following tables 6 and 7 show the relationship of α, time, and absolute error.

It is known that the authors, during their discussion of the research, dealt with the time function as follows:

If $G(t) = e^{-t}, t \in [0, T], T < 1$. This choice was based on the the fact that the time function should be decreasing and not increasing. And so we find on for example

$$G(t) = e^t \implies G(0.5) = e^{0.5} = 1.652271164185831,$$
$$G(0.25) = e^{0.25} = 1.285407003320672.$$

This means that the positive exponential function when the fractional time is greater than zero is the time function Often greater than the correct one, which makes the error increase in cumulative proportions.

As for when the function $G(t) = e^{-t}, t \in [0, T], T < 1$ the highest value at zero is the correct one and at the fractional time we find that

$$G(0.5) = e^{-0.5} = 0.6052275326688024,$$
$$G(0) = 1, \quad G(0.24) = 0.7779637090949696.$$

These results show that the most appropriate choice of time is the negative exponential function or choosing a part of it.

Future work. The technique described above can be expanded in further work to solve NfrPIo-DE with a discontinuous kernel.

Acknowledgement

References

[1] W. M. Abd-Elhameed and Y. H. Youssri, *Sixth-kind Chebyshev spectral approach for solving fractional differential equations*, International Journal of Nonlinear Sciences and Numerical Simulation, 2019, 20(2), 191–203.

[2] W. M. Abd-Elhameed and Y. H. Youssri, *Neoteric formulas of the monic orthogonal Chebyshev polynomials of the sixth kind involving moments and linearization formulas*, Advances in Difference Equations, 2021, 2021, 1–19.

[3] M. A. Abdou and M. Basseem, *Numerical treatments for solving nonlinear mixed integral equation*, Alexandria Engineering Journal, 2016, 55, 3247-3251.

[4] M. A. Abdou and M. I. Youssef, *A new model for solving three mixed integral equations with continuous and discontinuous kernels*, Asian Research Journal of Mathematics, 2021, 1–13.

[5] A. M. Al-Bugami, M. A. Abdou and A. M. S. Mahdy, *Sixth kind Chebyshev and Bernoulli polynomial numerical methods for solving nonlinear mixed partial integro-differential equations with continuous kernels*, Journal of Function Spaces, 2023, 2023, 1–14.

[6] F. M. Alharbi, A. M. Zidan, M. Naeem, R. Shah and K. Nonlaopon, *Numerical investigation of fractional-order differential equations via φ Haar-Wavelet method*, Journal of Function Spaces, 2021, 2021, 1–14.

[7] Sh. E. Alhazmi and M. A. Abdou, *A physical phenomenon for the fractional nonlinear mixed integro-differential equation using a general discontinuous kernel*, Fractal and Fractional, 2023, 7(173), 1–19.

[8] Sh. E. Alhazmi, A. M. S. Mahdy, M. A. Abdou and D. Sh. Mohamed, *Computational techniques for solving mixed (1+1) dimensional integral equations with strongly symmetric singular kernel*, Symmetry, 2023, 15(6), 1284, 1–27.

[9] R. E. Alsulaiman, M. A. Abdou, E. M. Youssef and M. Taha, *Solvability of a nonlinear integro-differential equation with fractional order using the Bernoulli matrix approach*, AIMS Mathematics, 2023, 8(3), 7515–7534.

[10] A. G. Atta, E. W. M. Abd-Elhameed, E G. M. Moatimid and E Y. H. Youssri, *Advanced shifted sixth kind Chebyshev tau approach for solving linear one-dimensional hyperbolic telegraph type problem*, Mathematical Sciences, 2022, 1–15.

[11] M. Basseem and A. Alalyani, *On the solution of quadratic nonlinear integral equation with different singular kernels*, Mathematical Problems in Engineering, 2020, 2020, 1–7.

[12] M. M. Bekkouche, I. Mansouri and A. A. Azeb Ahmed, *Numerical solution of fractional boundary value problem with Caputo-Fabrizio and its fractional integral*, Journal of Applied Mathematics and Computing, 2022, 68, 4305–4316.

[13] D. Guo, V. Lakshmikantham and X. Liu, *Nonlinear Integral Equations in Abstract Spaces*, Springer Science and Business Media, Springer New York, NY, 2013, 373.

[14] R. M. Hafez and Y. H. Youssri, *Spectral Legendre-Chebyshev treatment of 2D linear and nonlinear mixed Volterra-Fredholm integral equation*, Mathematical Sciences Letters, 2020, 9(2), 37–47.

[15] S. Hamdan, N. Qatanani and A. Daraghmeh, *Numerical techniques for solving linear Volterra fractional integral equation*, Journal of Applied Mathematics, 2019, 2019, 1–9.

[16] A. R. Jan, *An asymptotic model for solving mixed integral equation in position and time*, Journal of Mathematics, 2022, 2022, 1–11.

[17] A. Khalouta, *A new general integral transform for solving Caputo fractional-order differential equations*, International Journal of Nonlinear Analysis and Applications, 2023, 14(1), 67–78.

[18] A. M. S. Mahdy, M. A. Abdou and D. Sh. Mohamed, *Computational methods for solving higher-order (1+1) dimensional mixed-difference integro-differential equations with variable coefficients*, Mathematics, 2023, 11(9), 1–25.

[19] A. M. S. Mahdy, M. A. Abdou and D. Sh. Mohamed, *A computational technique for computing second-type mixed integral equations with singular kernels*, Journal of Mathematics and Computer Sciences, 2024, 32(2), 137–151.

[20] A. M. S. Mahdy, A. S. Nagdy, K. M. Hashem and D. S. Mohamed, *A computational technique for solving three-dimensional mixed Volterra-Fredholm integral equations*, Fractal and Fractional, 2023, 7(2), 1–14.

[21] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons. New York. 1993.

[22] M. Mohammad, A. Trounev and M. Alshbool, *A novel numerical method for solving fractional diffusion-wave and nonlinear Fredholm and Volterra integral equations with zero absolute error*, Axioms, 2021, 10(3), 1–16.

[23] A. Panda and J. Mohapatra, *Analysis of some semi-analytical methods for the solutions of a class of time-fractional partial integro-differential equations*, International Journal of Applied and Computational Mathematics, 2024, 10(54), 1–20.

[24] P. Polyanin and A. V. Manzhirov, *Handbook of Integral Equations*, Chapman and Hall/CRC, New York, 2008.

[25] Kh. Sadri and H. Aminikhah, *Chebyshev polynomials of sixth kind for solving nonlinear fractional PDEs with proportional delay and its convergence analysis*, Journal of Function Spaces, 2022, 1–20.

[26] S. G. Samko, A. A. Kilbas and O. I. Marchev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Switzerland, Reading, 1993.

[27] S. Yaghoubi, H. Aminikhah and K. Sadri, *A new efficient method for solving system of weakly singular fractional integro-differential equations by shifted sixth kind Chebyshev polynomials*, Journal of Mathematics, 2022, 2022, 1–18.

[28] Y. H. Youssri and A. G. Atta, *Fejér-quadrature collocation algorithm for solving fractional integro-differential equations via Fibonacci polynomials*, Contemporary Mathematics, 2024, 5(1), 296–308.

[29] Y. H. Youssri and R. M. Hafez, *Chebyshev collocation treatment of Volterra-Fredholm integral equation with error analysis*, Arabian Journal of Mathematics, 2020, 9, 471–480.