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RELATIONAL GERAGHTY CONTRACTIONS WITH AN APPLICATION TO A SINGULAR FRACTIONAL BOUNDARY VALUE PROBLEM

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Abstract In this article, we present some results on fixed points employing relational Geraghty contractions in the setting of metric space endued with a class of transitive binary relations. Our results complement, sharpen and improve several fixed point results of literature. By means of our findings, we discuss an existence and uniqueness theorem regarding the positive solutions of certain boundary value problems associated with a singular fractional differential equations.

Keywords Fixed points, binary relations, fractional differential equations, positive solution.

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1. Introduction

Banach Contraction principle (abbreviated as: BCP) is one of the fundamental and powerful results of metric fixed point theory. Within the foregoing centaury, this key result has been generalized by various authors employing different approaches. In 1973, Geraghty [10] obtained a natural extension of BCP by introducing a new family of test functions. Following Geraghty [10], $\mathfrak{S}$ will indicate the collection of functions $\alpha : [0, \infty) \to [0, 1)$ verifying

$$\alpha(t_n) \to 1 \Rightarrow t_n \to 0.$$

Typical examples of such functions are $\alpha(t) = e^t$ and $\alpha(t) = 1/(1+t)$. Using above family, Geraghty [10] proved the following variant of BCP.

Theorem 1.1. [10] *If* $(\mathbf{Z}, \sigma)$ *remains a complete metric space,* $\mathcal{T} : \mathbf{Z} \to \mathbf{Z}$ *is a map and* $\exists\ \alpha \in \mathfrak{S}$ *verifying*

$$\sigma(\mathcal{T}z, \mathcal{T}u) \leq \alpha(\sigma(z, u))\sigma(z, u), \quad \forall\ z, u \in \mathbf{Z},$$

then $\mathcal{T}$ *possesses a unique fixed point.*

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Alam and Imdad [3] presented a new variant of BCP in the framework of relational metric space. Afterwards, lots of fixed point results are obtained by many researchers (cf. [1, 2, 4, 5, 7, 8, 13]). In such results, the underling contractions are desired to satisfy for only comparative pair under the given binary relation. Because of the limiting nature, such results can be utilized to determine the unique solutions of special types of matrix equations, integral equations and ordinary differential equations, whereas classical fixed point theorems cannot be utilized. Several authors also studied for finding the unique solutions of certain fractional differential equations (abbreviated as: FDE) employing the certain fixed point theorems, e.g., [6, 11, 14, 20, 21].

The goal of the present paper is to prove the fixed point results for relational Geraghty contractions and to utilize these results in solving a boundary value problem (abbreviated as: BVP) associated with a FDE satisfying certain additional conditions. Intending to explain our finding, we incorporate an example.

2. Preliminaries

Throughout the paper, the set of: real numbers and natural numbers will be represented by $\mathbb{R}$ and $\mathbb{N}$, respectively. Moreover, we'll write $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Given a set $\mathbf{Z}$, any subset of $\mathbf{Z}^2$ is named as a binary relation (or, a relation) on $\mathbf{Z}$. Suppose that $\mathbf{Z}$ remains a set, $\mathcal{T} : \mathbf{Z} \to \mathbf{Z}$ is a map, $\mathcal{S}$ remains a relation on $\mathbf{Z}$ and σ is a metric on $\mathbf{Z}$. We call that:

Definition 2.1. [3] The elements $z, u \in \mathbf{Z}$ are $\mathcal{S}$-comparative if $(z, u) \in \mathcal{S}$ or $(u, z) \in \mathcal{S}$. Such a pair is often denoted by $[z, u] \in \mathcal{S}$.

Definition 2.2. [18] The relation $\mathcal{S}^{-1} := \{(z, u) \in \mathbf{Z}^2 : (u, z) \in \mathcal{S}\}$ is inverse of $\mathcal{S}$.

Definition 2.3. [18] The symmetric relation described by $\mathcal{S}^s := \mathcal{S} \cup \mathcal{S}^{-1}$ is symmetric closure of $\mathcal{S}$.

Definition 2.4. [5] $\mathcal{S}$ is locally $\mathcal{T}$-transitive if for any $\mathcal{S}$-preserving sequence $\{z_n\} \subset \mathcal{T}(\mathbf{Z})$ (having the range $\mathbf{Y} = \{z_n : n \in \mathbb{N}\}$), $\mathcal{S}|_{\mathbf{Y}}$ remains transitive.

Definition 2.5. [3] A sequence $\{z_n\} \subset \mathbf{Z}$ satisfying $(z_n, z_{n+1}) \in \mathcal{S}$, $\forall\ n \in \mathbb{N}$, is $\mathcal{S}$-preserving.

Definition 2.6. [3] $\mathcal{S}$ is $\mathcal{T}$-closed if for all $z, u \in \mathbf{Z}$ verifying $(z, u) \in \mathcal{S}$, we have

$$(\mathcal{T}z, \mathcal{T}u) \in \mathcal{S}.$$

Definition 2.7. [16] Given $z, u \in \mathbf{Z}$, a subset $\{\omega_0, \omega_1, \ldots, \omega_\ell\} \subset \mathbf{Z}$ is a path with length ℓ in $\mathcal{S}$ between z to u if $\omega_0 = z$, $\omega_\ell = u$ and $(\omega_i, \omega_{i+1}) \in \mathcal{S}$, $\quad 0 \leq i \leq \ell - 1$.

Definition 2.8. [5] The set $\mathbf{Y} \subseteq \mathbf{Z}$ is $\mathcal{S}$-connected if any two elements of $\mathbf{Z}$ joins a path in $\mathcal{S}$.

Definition 2.9. [16] Given $\mathbf{Y} \subseteq \mathbf{Z}$, the relation $\mathcal{S}|_{\mathbf{Y}}$ on $\mathbf{Y}$ is the restriction of $\mathcal{S}$ on $\mathbf{Y}$, whereas

$$\mathcal{S}|_{\mathbf{Y}} := \mathcal{S} \cap \mathbf{Y}^2.$$

For every fixed $z_0 \in \mathbf{Z}$, the set $\mathcal{O}_{\mathcal{T}}(z_0) := \{\mathcal{T}^n z_0 : n \in \mathbb{N}\}$ is termed as the orbit of z_0. If $\mathcal{T}$ is known, then we write $\mathcal{O}(z_0)$ instead of $\mathcal{O}_{\mathcal{T}}(z_0)$. A sequence is referred as a $\mathcal{T}$-orbital sequence if its range is $\mathcal{O}(z)$ for some $z \in \mathbf{Z}$ (c.f. [13]). In what follows, $\mathbf{Z}(\mathcal{O},\mathcal{S})$ denotes the set of $\mathcal{T}$-orbital $\mathcal{S}$-preserving sequences in $\mathbf{Z}$. We call that:

Definition 2.10. [13] $(\mathbf{Z},\sigma)$ is $(\mathcal{O},\mathcal{S})$-complete metric space if any Cauchy sequence in $\mathbf{Z}(\mathcal{O},\mathcal{S})$ converges.

Definition 2.11. [13] $\mathcal{T}$ is $(\mathcal{O},\mathcal{S})$-continuous map at $z \in \mathbf{Z}$ if for each sequence $\{z_n\} \subset \mathbf{Z}(\mathcal{O},\mathcal{S})$ verifying $z_n \xrightarrow{\sigma} z$, one has $\mathcal{T}(z_n) \xrightarrow{\sigma} \mathcal{T}(z)$.

Definition 2.12. [13] $\mathcal{S}$ is $(\mathcal{O},\sigma)$-self-closed if for each sequence $\{z_n\} \subset \mathbf{Z}(\mathcal{O},\mathcal{S})$ verifying $z_n \xrightarrow{\sigma} z$, $\exists$ a subsequence $\{z_{n_k}\}$ satisfying $[z_{n_k}, z] \in \mathcal{S}$, $\forall\ k \in \mathbb{N}$.

In Definitions 2.10,2.11,2.12, if the orbital concepts are ignored then we get the Definitions of '$\mathcal{S}$-complete metric space', '$\mathcal{S}$-continuous map' and 'σ-self-closed relation' respectively (cf. [4]).

Remark 2.1. [3] $(z,u) \in \mathcal{S}^s \iff [z,u] \in \mathcal{S}$.

Proposition 2.1. [5] *When $\mathcal{S}$ is $\mathcal{T}$-closed, $\mathcal{S}$ is also $\mathcal{T}^n$-closed, $\forall\ n \in \mathbb{N}_0$.*

Proposition 2.2. *For every $\alpha \in \mathfrak{S}$, the following are equivalent:*

(I) $\sigma(\mathcal{T}z,\mathcal{T}u) \le \alpha(\sigma(z,u))\sigma(z,u),\ \forall\ z,u \in \mathbf{Z}$ *with* $(z,u) \in \mathcal{S}$.

(II) $\sigma(\mathcal{T}z,\mathcal{T}u) \le \alpha(\sigma(z,u))\sigma(z,u),\ \forall\ z,u \in \mathbf{Z}$ *with* $[z,u] \in \mathcal{S}$.

Proof. The result (I) $\Longrightarrow$ (II) is straightforward. On the other hand, the conclusion (II) $\Longrightarrow$ (I) is immediate owing to symmetric property of metric σ. □

3. Main results

We shall prove the results on the existence and uniqueness of fixed point under relational Geraghty contraction.

Theorem 3.1. *Let $(\mathbf{Z},\sigma)$ be a metric space endued with a relation $\mathcal{S}$ while $\mathcal{T} : \mathbf{Z} \to \mathbf{Z}$ a map. Moreover, suppose that*

(a) $(\mathbf{Z},\sigma)$ is $(\mathcal{O},\mathcal{S})$-complete,

(b) $\mathcal{S}$ is locally $\mathcal{T}$-transitive and $\mathcal{T}$-closed,

(c) $\exists\ \zeta_0 \in \mathbf{Z}$ such that $(\zeta_0, \mathcal{T}\zeta_0) \in \mathcal{S}$,

(d) $\mathcal{T}$ remains $(\mathcal{O},\mathcal{S})$-continuous, or $\mathcal{S}$ is $(\mathcal{O},\sigma)$-self-closed,

(e) $\exists\ \alpha \in \mathfrak{S}$ verifying

$$\sigma(\mathcal{T}z,\mathcal{T}u) \le \alpha(\sigma(z,u))\sigma(z,u), \quad \forall\ z,u \in \mathbf{Z} \text{ with } (z,u) \in \mathcal{S}.$$

Then, $\mathcal{T}$ admits a fixed point.

Proof. In lieu of (c), we have $\zeta_0 \in \mathbf{Z}$. Define the following sequence $\{\zeta_n\} \subset \mathbf{Z}$:

$$\zeta_n := \mathcal{T}^n(\zeta_0) = \mathcal{T}(\zeta_{n-1}), \quad \forall\ n \in \mathbb{N}. \tag{3.1}$$

Owing to assumption (c), $\mathcal{T}$-closedness of $\mathcal{S}$ and Proposition 2.1, we obtain

$$(\mathcal{T}^n\zeta_0, \mathcal{T}^{n+1}\zeta_0) \in \mathcal{S},$$

which using (3.1) becomes

$$(\zeta_n, \zeta_{n+1}) \in \mathcal{S}, \quad \forall\, n \in \mathbb{N}_0. \tag{3.2}$$

Therefore, $\{\zeta_n\}$ is a $\mathcal{S}$-preserving sequence.

Write $\sigma_n := \sigma(\zeta_n, \zeta_{n+1})$. Using (3.1), (3.2) and condition (e), we obtain

$$\begin{aligned}\sigma(\zeta_{n+1}, \zeta_{n+2}) &= \sigma(\mathcal{T}\zeta_n, \mathcal{T}\zeta_{n+1})\\ &\leq \alpha(\sigma(\zeta_n, \zeta_{n+1}))\sigma(\zeta_n, \zeta_{n+1})\\ &\leq \sigma(\zeta_n, \zeta_{n+1})\end{aligned} \tag{3.3}$$

so that

$$\sigma_{n+1} \leq \sigma_n.$$

Thus, $\{\sigma_n\}$ is a monotonic decreasing sequence. As it is also bounded below, we have $\lim\limits_{n\to\infty} \sigma_n = r \geq 0$. We assert that $r = 0$. For if $r > 0$, then by (3.3), we have

$$\frac{\sigma_{n+1}}{\sigma_n} \leq \alpha(\sigma_n), \quad n = 1, 2, \cdots$$

implying thereby $\lim\limits_{n\to\infty} \alpha(\sigma_n) = 1$, which contradicts to $\sigma_n \to r \neq 0$. Therefore, we have

$$\lim_{n\to\infty} \sigma_n = 0. \tag{3.4}$$

Now, we claim that $\{\zeta_n\}$ is Cauchy. On the contrary, suppose that

$$\lim_{m,n\to\infty} \sup \sigma(\zeta_n, \zeta_m) > 0. \tag{3.5}$$

Using triangle inequality, we get

$$\sigma(\zeta_n, \zeta_m) \leq \sigma(\zeta_n, \zeta_{n+1}) + \sigma(\zeta_{n+1}, \zeta_{m+1}) + \sigma(\zeta_{m+1}, \zeta_m). \tag{3.6}$$

By (3.1), (3.2) and locally $\mathcal{T}$-transitivity of $\mathcal{S}$, we obtain $(\zeta_n, \zeta_m) \in \mathcal{S}$. On applying the contractivity condition (e), we get

$$\sigma(\zeta_{n+1}, \zeta_{m+1}) = \sigma(\mathcal{T}\zeta_n, \mathcal{T}\zeta_m) \leq \alpha(\sigma(\zeta_n, \zeta_m))\sigma(\zeta_n, \zeta_m). \tag{3.7}$$

From (3.6) and (3.7), we have

$$\sigma(\zeta_n, \zeta_m) \leq [1 - \alpha(\sigma(\zeta_n, \zeta_m))]^{-1}[\sigma(\zeta_n, \zeta_{n+1}) + \sigma(\zeta_{m+1}, \zeta_m)].$$

Making use of (3.4) and (3.5), we find

$$\limsup_{m,n\to\infty} [1 - \alpha(\sigma(\zeta_n, \zeta_m))]^{-1} = \infty,$$

implying thereby $\limsup\limits_{m,n\to\infty} \alpha(\sigma(\zeta_n, \zeta_m) = 1$. Further as $\alpha \in \mathfrak{S}$, we have $\limsup\limits_{m,n\to\infty} \sigma(\zeta_n, \zeta_m) = 0$, which is a contradiction. Therefore, $\{\zeta_n\}$ is Cauchy. Since $\{\zeta_n\}$ is also $\mathcal{T}$-orbital and $\mathcal{S}$-preserving, therefore using $(\mathcal{O}, \mathcal{S})$-completeness of $\mathbf{Z}$, we can find $\bar{z} \in \mathbf{Z}$ such that $\zeta_n \xrightarrow{\sigma} \bar{z}$.

Finally, we conclude the proof utilizing the assumption (d). Assume the map $\mathcal{T}$ is $(\mathcal{O},\mathcal{S})$-continuous. As $\{\zeta_n\}$ remains $\mathcal{T}$-orbital and $\mathcal{S}$-preserving satisfying $\zeta_n \xrightarrow{\sigma} \bar{z}$, by $(\mathcal{O},\mathcal{S})$-continuity of $\mathcal{T}$, we get $\zeta_{n+1} = \mathcal{T}(\zeta_n) \xrightarrow{\sigma} \mathcal{T}(\bar{z})$. By uniqueness of convergence limit, we obtain $\mathcal{T}(\bar{z}) = \bar{z}$.

Otherwise, let $\mathcal{S}$ be $(\mathcal{O},\sigma)$-self closed. As $\{\zeta_n\}$ remains $\mathcal{T}$-orbital and $\mathcal{S}$-preserving satisfying $\zeta_n \xrightarrow{\sigma} \bar{z}$, $\exists$ a subsequence $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$ such that $[\zeta_{n_k}, \bar{z}] \in \mathcal{S}$, $\forall\ k \in \mathbb{N}$. By condition (e), Proposition 2.2 and $[\zeta_{n_k}, \bar{z}] \in \mathcal{S}$, we get

$$\begin{aligned}\sigma(\zeta_{n_k+1}, \mathcal{T}\bar{z}) &= \sigma(\mathcal{T}\zeta_{n_k}, \mathcal{T}\bar{z})\\ &\le \alpha(\sigma(\zeta_{n_k}, \bar{z}))\sigma(\zeta_{n_k}, \bar{z})\\ &\le \sigma(\zeta_{n_k}, \bar{z}).\end{aligned}$$

Making use of limit in above inequality and due to $\zeta_{n_k} \xrightarrow{\sigma} \bar{z}$, we obtain $\zeta_{n_k+1} \xrightarrow{\sigma} \mathcal{T}(\bar{z})$. By uniqueness of convergence limit, we conclude that $\mathcal{T}(\bar{z}) = \bar{z}$. Thus, $\bar{z}$ remains a fixed point of $\mathcal{T}$. □

Theorem 3.2. *Under the assumptions of Theorem 3.1, if $\mathcal{T}(\mathbf{Z})$ is $\mathcal{S}^s$-connected, then $\mathcal{T}$ admits a unique fixed point.*

Proof. Following Theorem 3.1, if $\exists\ \bar{z}, \bar{u} \in \mathbf{Z}$ verifying

$$\mathcal{T}(\bar{z}) = \bar{z} \text{ and } \mathcal{T}(\bar{u}) = \bar{u}. \tag{3.8}$$

Since $\bar{z}, \bar{u} \in \mathcal{T}(\mathbf{Z})$, therefore by $\mathcal{S}^s$-connectedness of $\mathcal{T}(\mathbf{Z})$, $\exists$ a path $\{\omega_0, \omega_1, \omega_2, \ldots, \omega_\ell\}$ with length ℓ in $\mathcal{S}^s$ between $\bar{z}$ to $\bar{u}$ so that

$$\omega_0 = \bar{z}, \omega_\ell = \bar{u} \text{ and } [\omega_i, \omega_{i+1}] \in \mathcal{S}, \text{ for each } 0 \le i \le \ell - 1. \tag{3.9}$$

As $\mathcal{S}$ is $\mathcal{T}$-closed, we find

$$[\mathcal{T}^n\omega_i, \mathcal{T}^n\omega_{i+1}] \in \mathcal{S}, \quad \forall\ n \in \mathbb{N}_0 \text{ and for each } 0 \le i \le \ell - 1. \tag{3.10}$$

Let us denote

$$\tau_n^i = \sigma(\mathcal{T}^n\omega_i, \mathcal{T}^n\omega_{i+1}).$$

We shall show that

$$\lim_{n\to\infty} \tau_n^i = 0. \tag{3.11}$$

If for some $n_0 \in \mathbb{N}_0$, $\tau_{n_0}^i = \sigma(\mathcal{T}^{n_0}\omega_i, \mathcal{T}^{n_0}\omega_{i+1}) = 0$, then we have $\mathcal{T}^{n_0}(\omega_i) = \mathcal{T}^{n_0}(\omega_{i+1})$, which yields that $\mathcal{T}^{n_0+1}(\omega_i) = \mathcal{T}^{n_0+1}(\omega_{i+1})$ and hence we obtain $\tau_{n_0+1}^i = \sigma(\mathcal{T}^{n_0+1}\omega_i, \mathcal{T}^{n_0+1}\omega_{i+1}) = 0$. Using induction, we obtain $\tau_n^i = 0\ \forall\ n \ge n_0$, implying thereby $\lim\limits_{n\to\infty} \tau_n^i = 0$. Otherwise, we have $\tau_n^i > 0$, $\forall\ n \in \mathbb{N}_0$. In this case, by (3.10) condition (e) and Proposition 2.2, we find

$$\begin{aligned}\tau_{n+1}^i &= \sigma(\mathcal{T}^{n+1}\omega_i, \mathcal{T}^{n+1}\omega_{i+1})\\ &\le \alpha(\sigma(\mathcal{T}^n\omega_i, \mathcal{T}^n\omega_{i+1}))\sigma(\mathcal{T}^n\omega_i, \mathcal{T}^n\omega_{i+1})\\ &= \alpha(\tau_n^i)\tau_n^i\\ &\le \tau_n^i.\end{aligned} \tag{3.12}$$

Thus for each $0 \leq i \leq \ell - 1$, $\{\tau_n^i\}$ is a monotonic decreasing sequence. As it is also bounded below, we have $\lim\limits_{n\to\infty} \tau_n^i = \epsilon_i \geq 0$. Suppose that $\epsilon_i > 0$. From (3.12) we have

$$\frac{\tau_{n+1}^i}{\tau_n^i} \leq \alpha(\tau_n^i), \quad n = 1, 2, \cdots .$$

The above inequality implies that $\lim\limits_{n\to\infty} \alpha(\tau_n^i) = 1$. Further as $\alpha \in \mathfrak{S}$, we obtain $\epsilon_i = 0$. Thus in all, (3.11) is verified for each $0 \leq i \leq \ell - 1$. By triangle inequality and (3.11), we conclude

$$\begin{aligned}\sigma(\bar{z}, \bar{u}) &= \sigma(\mathcal{T}^n\omega_0, \mathcal{T}^n\omega_\ell)\\ &\leq \tau_n^0 + \tau_n^1 + \cdots + \tau_n^{\ell-1}\\ &\to 0 \text{ as } n \to \infty\end{aligned}$$

implying thereby $\bar{z} = \bar{u}$. Therefore, $\mathcal{S}$ possesses a unique fixed point. □

4. Applications to fractional differential equations

Let us consider the following singular fractional three-point BVP:

$$\begin{cases}\mathfrak{D}_{0+}^{\kappa} v(s) + h(s, v(s)) = 0, & \forall\ s \in (0, 1),\\ v(0) = v'(0) = v''(0) = 0, & v''(1) = \lambda v''(\gamma),\end{cases} \tag{4.1}$$

where $3 < \kappa \leq 4$, $0 < \gamma < 1$, $\lambda\gamma^{\kappa-3} < 1$ and $h : [0, 1] \times [0, \infty) \to [0, \infty)$ remains continuous.

As usual, the classical gamma function and the classical beta function will be represented by $\Gamma(\cdot)$ and $\beta(\cdot, \cdot)$ respectively. Motivated by [17] and [9], we shall compute the unique positive solution of (4.1) under the assumption that h is singular at $s = 0$ (i.e., $\lim\limits_{s\to 0^+} h(s, \cdot) = \infty$).

Definition 4.1. [19] Given a real valued function ϕ defined in $(0, \infty)$, the Riemann-Liouville fractional integral of ϕ of order $\kappa > 0$ is

$$\mathfrak{I}_{0+}^{\kappa} \phi(s) = \frac{1}{\Gamma(\kappa)} \int_0^s (s - \xi)^{\kappa-1} \phi(\xi) d\xi,$$

for which R.H.S. is pointwise defined on $(0, \infty)$.

Definition 4.2. [15] Given a real valued function ϕ defined in $(0, \infty)$, the Riemann-Liouville fraction derivative of ϕ of order $\kappa > 0$ is

$$\mathfrak{D}_{0+}^{\kappa} \phi(s) = \frac{1}{\Gamma(n - \kappa)} \left(\frac{d}{ds}\right)^n \int_0^s \frac{\phi(\xi)}{(s - \xi)^{\kappa-n+1}} d\xi.$$

Here $n = [\kappa] + 1$ and $[\kappa]$ represents the integer part of κ.

Lemma 4.1. [15] *If $v \in L^1(0, 1) \cap \mathcal{C}(0, 1)$ and $\kappa > 0$ then the FDE*

$$\mathfrak{D}_{0+}^{\kappa} v(s) = 0$$

admits a unique solution of the form:

$$v(s) = a_1 s^{\kappa-1} + a_2 s^{\kappa-2} + \cdots + a_n s^{\kappa-n},$$

where $a_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$ and $n = [\kappa] + 1$.

Lemma 4.2. [15] *If $v \in L^1(0,1) \cap \mathcal{C}(0,1)$ has a κ^{th} order fractional derivative (where $\kappa > 0$) which also belongs to $L^1(0,1) \cap \mathcal{C}(0,1)$, then $\exists\, a_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$ verifying*

$$\mathfrak{I}^{\kappa}_{0+}\mathfrak{D}^{\kappa}_{0+}v(s) = v(s) + a_1 s^{\kappa-1} + a_2 s^{\kappa-2} + \cdots + a_n s^{\kappa-n},$$

where $n = [\kappa] + 1$.

Employing the Lemma 4.2, Liang and Zhang [17] proved the following result.

Lemma 4.3. [17] *The BVP*

$$\begin{aligned} &\mathfrak{D}^{\kappa}_{0+}v(s) + f(s) = 0, \;\; 0 < s < 1, \\ &v(0) = v'(0) = v''(0), \;\; v''(1) = \lambda v''(\gamma), \end{aligned} \tag{4.2}$$

where $3 < \kappa \leq 4$, $0 < \gamma < 1$, $\lambda\gamma^{\kappa-3} < 1$ and $f : [0,1] \to [0,\infty)$ remains continuous, admits a unique solution of the form:

$$v(s) = \int_0^1 \mathcal{G}(s,\xi) f(\xi) d\xi + \frac{\lambda s^{\kappa-1}}{(\kappa-1)(\kappa-2)(1-\lambda\gamma^{\kappa-3})} \int_0^1 \mathcal{H}(\gamma,\xi) f(\xi) d\xi,$$

where

$$\mathcal{G}(s,\xi) = \begin{cases} \dfrac{s^{\kappa-1}(1-\xi)^{\kappa-3} - (s-\xi)^{\kappa-1}}{\Gamma(\kappa)}, & 0 \leq \xi \leq s \leq 1, \\ \dfrac{s^{\kappa-1}(1-\xi)^{\kappa-3}}{\Gamma(\kappa)}, & 0 \leq s \leq \xi \leq 1 \end{cases}$$

and

$$\begin{aligned} \mathcal{H}(s,\xi) &= \frac{\partial^2 \mathcal{G}(s,\xi)}{\partial s^2} \\ &= \begin{cases} \dfrac{(\kappa-1)(\kappa-2)}{\Gamma(\kappa)} \left[s^{\kappa-3}(1-\xi)^{\kappa-3} - (s-\xi)^{\kappa-3} \right], & 0 \leq \xi \leq s \leq 1, \\ \dfrac{(\kappa-1)(\kappa-2)}{\Gamma(\kappa)} s^{\kappa-3}(1-\xi)^{\kappa-3}, & 0 \leq s \leq \xi \leq 1. \end{cases} \end{aligned}$$

Remark 4.1. Under the hypotheses of Lemma 4.3, $\mathcal{G}$ and $\mathcal{H}$ admits the following properties:

- $\mathcal{G}$ remains continuous on $[0,1] \times [0,1]$,
- $\mathcal{G}(s,\xi) \geq 0$,
- $\mathcal{G}(s,1) = 0$,
- $\sup\limits_{0 \leq s \leq 1} \int_0^1 \mathcal{G}(s,\xi) d\xi = \frac{2}{(\kappa-2)\Gamma(\kappa+1)}$,
- $\int_0^1 \mathcal{H}(\gamma,\xi) d\xi = \frac{\gamma^{\kappa-3}(\kappa-1)(1-\gamma)}{\Gamma(\kappa)}$.

Lemma 4.4. [9] *Assume $0 < \rho < 1$, $3 < \kappa \leq 4$ and $F : (0,1] \to \mathbb{R}$ remains continuous such that $\lim\limits_{s \to 0^+} F(s) = \infty$. If the function $s^{\rho} F(s)$ is continuous on $[0,1]$, then*

$$\mathcal{L}(s) = \int_0^1 \mathcal{G}(s,\xi) F(\xi) d\xi \tag{4.3}$$

is also continuous on $[0,1]$, whereas $\mathcal{G}(s,\xi)$ remains the Green's function given in Lemma 4.3.

Lemma 4.5. [9] *Assume* $0 < \rho < 1,\ 3 < \kappa \le 4,\ 0 < \beta\gamma^{\kappa-3} < 1$, *and* F *remains a continuous real function defined on* $(0, 1]$ *for which* $\lim\limits_{s\to 0^+} F(s) = \infty$. *If* $s^\rho F(s)$ *remains a continuous function on* $[0, 1]$, *then*

$$\mathcal{N}(s) = \frac{\beta s^{\kappa-1}}{(\kappa-1)(\kappa-2)(1-\beta\gamma^{\kappa-3})} \int_0^1 \mathcal{H}(\gamma, \xi) F(\xi) d\xi \tag{4.4}$$

is also a continuous function on $[0, 1]$, *whereas* $\mathcal{H}(s, \xi)$ *is the function given in Lemma 4.3.*

Remark 4.2. [9] The function $\mathcal{H}(s, \xi)$ (given in Lemma 4.3) defined by

$$\mathcal{H}(s,\xi) = \begin{cases} \dfrac{(\kappa-1)(\kappa-2)}{\Gamma(\kappa)} \left[s^{\kappa-3}(1-\xi)^{\kappa-3} - (s-\xi)^{\kappa-3}\right], & 0 \le \xi \le s \le 1, \\ \dfrac{(\kappa-1)(\kappa-2)}{\Gamma(\kappa)} s^{\kappa-3}(1-\xi)^{\kappa-3}, & 0 \le s \le \xi \le 1, \end{cases} \tag{4.5}$$

is continuous on $[0, 1]^2$. Also, we have $\mathcal{H}(s, \xi) \ge 0$.

For $0 \le s \le \xi \le 1$, obviously one has $\mathcal{H}(s, \xi) \ge 0$. Also in case $0 \le \xi \le s \le 1$, we have

$$\begin{aligned} \mathcal{H}(s,\xi) &= \frac{(\kappa-1)\kappa-2}{\Gamma(\kappa)} \left[s^{\kappa-3}(1-\xi)^{\kappa-3} - (s-\xi)^{\kappa-3}\right] \\ &= \frac{(\kappa-1)(\kappa-2)}{\Gamma(\kappa)} \left[(s-ts)^{\kappa-3} - (s-\xi)^{\kappa-3}\right] \\ &\ge 0. \end{aligned}$$

This shows the nonnegative character of $\mathcal{H}$ on $[0, 1]^2$.

Lemma 4.6. [9] *If* $0 < \rho < 1$, *then*

$$\sup_{0\le s\le 1} \int_0^1 \mathcal{G}(s,\xi)\xi^{-\rho} d\xi = \frac{1}{\Gamma(\kappa)} (\beta(1-\rho, \kappa-2) - \beta(1-\rho, \kappa)), \tag{4.6}$$

whereas $\mathcal{G}(s, \xi)$ *remains the Green's function given in Lemma 4.3*

Lemma 4.7. [9] *If* $0 < \rho < 1$, *then*

$$\int_0^1 \mathcal{H}(\gamma,\xi)\xi^{-\rho} d\xi = \frac{(\kappa-1(\kappa-2)}{\Gamma(\kappa)} \big(\gamma^{\kappa-3} - \gamma^{\kappa-\rho-2}\beta(1-\rho,\kappa-2)\big), \tag{4.7}$$

whereas $\mathcal{H}(s, \xi)$ *remains the function given in Lemma 4.3.*

Remark 4.3. Denote

$$K := \frac{1}{\Gamma(\kappa)} \left[\left(1 + \frac{\beta(\gamma^{\kappa-3} - \gamma^{\kappa-\rho-2})}{1-\beta\gamma^{\kappa-3}} \right) \beta(1-\rho,\kappa-2) - \beta(1-\rho,\kappa) \right].$$

In what follows, by Ψ, we denote the family of functions $\psi : [0, \infty) \to [0, \infty)$ verifying that:

(i) ψ is monotone increasing,

(ii) $\psi(t) < t, \quad \forall\, t > 0$,

(iii) $\forall\ t \in (0,\infty)$, $\alpha(t) := \psi(t)/t$, where $\alpha \in \mathfrak{S}$.

Now, we'll show the main result of this section.

Theorem 4.1. *Let* $0 < \rho < 1$, $3 < \kappa \leq 4$, $0 < \gamma < 1$, $0 < \lambda\gamma^{\kappa-3} < 1$. *Also, suppose that* $h : (0,1] \times [0,\infty) \to [0,\infty)$ *remains a continuous function verifying* $\lim\limits_{s\to 0^+} h(s,\cdot) = \infty$ *and that* $s^\rho h(s,y)$ *is continuous on* $[0,1]\times[0,\infty)$. *If* $\exists\ 0 < \mu \leq 1/K$ *and* $\psi \in \Psi$ *such that* $\forall\ x,y \in [0,\infty)$ *verifying* $x \geq y$ *and* $\forall\ s \in [0,1]$, *one has*

$$0 \leq s^\rho[h(s,x) - h(s,y)] \leq \mu\psi(x-y), \tag{4.8}$$

then BVP (4.1) *has a unique positive solution.*

Proof. Consider the following metric on Banach space $\mathcal{C}[0,1]$

$$\sigma(z,u) = \sup_{0\leq s\leq 1} |z(s) - u(s)|.$$

Define the cone

$$\mathbf{Z} = \{z \in C[0,1] : z(s) \geq 0\}.$$

On $\mathbf{Z}$, consider the relation

$$\mathcal{S} = \{(z,u) \in \mathbf{Z}^2 : z(s) \leq u(s),\ \text{for each } s \in [0,1]\}.$$

Now, define the map $\mathcal{T} : \mathbf{Z} \to \mathbf{Z}$ by

$$\begin{aligned}(\mathcal{T}z)(s) = &\int_0^1 \mathcal{G}(s,\xi)h(\xi,z(\xi))d\xi \\ &+ \frac{\lambda s^{\kappa-1}}{(\kappa-1)(\kappa-2)(1-\lambda\gamma^{\kappa-3})}\int_0^1 \mathcal{H}(\gamma,\xi)h(\xi,z(\xi))d\xi.\end{aligned}$$

We'll now verify all the hypotheses of Theorems 3.1 and 3.2.

(a) Clearly, $\mathbf{Z}$ being a closed set of $\mathcal{C}[0,1]$ forms a complete metric space under the metric σ and hence an $(\mathcal{O},\mathcal{S})$-complete metric space.

(b) Take $(z,u) \in \mathcal{S}$ implying thereby $z(s) \leq u(s)$, for each $s \in [0,1]$. Consequently, we have

$$\begin{aligned}(\mathcal{T}z)(s) &= \int_0^1 \mathcal{G}(s,\xi)h(\xi,z(\xi))d\xi \\ &\quad + \frac{\lambda s^{\kappa-1}}{(\kappa-1)(\kappa-2)(1-\lambda\gamma^{\kappa-3})}\int_0^1 \mathcal{H}(\gamma,\xi)h(\xi,z(\xi))d\xi. \\ &= \int_0^1 \mathcal{G}(s,\xi)\xi^{-\rho}\xi^\rho h(x,z(\xi))d\xi \\ &\quad + \frac{\lambda s^{\kappa-1}}{(\kappa-1)(\kappa-2)(1-\lambda\gamma^{\kappa-3})}\int_0^1 \mathcal{H}(\gamma,\xi)\xi^{-\rho}\xi^\rho h(\xi,z(\xi))d\xi \\ &\leq \int_0^1 \mathcal{G}(s,\xi)\xi^{-\rho}\xi^\rho h(\xi,u(\xi))d\xi \\ &\quad + \frac{\lambda s^{\kappa-1}}{(\kappa-1)(\kappa-2)(1-\lambda\gamma^{\kappa-3})}\int_0^1 \mathcal{H}(\gamma,\xi)\xi^{-\rho}\xi^\rho h(\xi,u(\xi))d\xi\end{aligned}$$

$$
\begin{aligned}
=&\int_0^1 \mathcal{G}(s,\xi)h(\xi,u(\xi))d\xi \\
&+\frac{\lambda^{\kappa-1}}{(\kappa-1)(\kappa-2)(1-\lambda\gamma^{\kappa-3})}\int_0^1 \mathcal{H}(\gamma,\xi)h(\xi,u(\xi))d\xi \\
=&(\mathcal{T}u)(s)
\end{aligned}
$$

yielding $(\mathcal{T}z,\mathcal{T}u)\in\mathcal{S}$. Therefore, $\mathcal{S}$ is $\mathcal{T}$-closed.

(c) Let $\mathbf{0}\in\mathbf{Z}$ be zero function. Then for each $s\in[0,1]$, we have $\mathbf{0}(s)\le(\mathcal{T}\mathbf{0})(s)$ yielding thereby $(\mathbf{0},\mathcal{T}\mathbf{0})\in\mathcal{S}$.

(d) Let $\{z_n\}\subset\mathbf{Z}$ be an $\mathcal{T}$-orbital $\mathcal{S}$-preserving sequence converging to $z\in\mathbf{Z}$. Then $\forall\ s\in[0,1]$, $\{z_n(s)\}$ is an increasing sequence of reals converging to $z(s)$. This implies that $\forall\ n\in\mathbb{N}$ and $\forall\ s\in[0,1]$, $z_n(s)\le z(s)$ so that $(z_n,z)\in\mathcal{S},\ \forall\ n\in\mathbb{N}$. Consequently, $\mathcal{S}$ remains $(\mathcal{O},\sigma)$-self-closed.

(e) Take $(z,u)\in\mathcal{S}$ implying thereby $z(s)\le u(s)$, for each $s\in[0,1]$. Thus

$$
\begin{aligned}
\sigma(\mathcal{T}z,\mathcal{T}u)&=\sup_{0\le s\le 1}|(\mathcal{T}z)(s)-(\mathcal{T}u)(s)| \\
&=\sup_{0\le s\le 1}[(\mathcal{T}u)(s)-(\mathcal{T}z)(s)] \\
&=\sup_{0\le s\le 1}\left[\int_0^1 \mathcal{G}(s,\xi)(h(\xi,u(\xi))-h(\xi,z(\xi)))\,d\xi\right. \\
&\quad\left.+\frac{\lambda s^{\kappa-1}}{(\kappa-1)(\kappa-2)(1-\lambda\gamma^{\kappa-3})}\int_0^1 \mathcal{H}(\gamma,\xi)(h(\xi,u(\xi))-h(\xi,z(\xi))d\xi\right] \\
&\le\sup_{0\le s\le 1}\int_0^1 \mathcal{G}(s,\xi)\xi^{-\rho}\xi^{\rho}[h(\xi,u(\xi))-h(\xi,z(\xi))]d\xi \\
&\quad+\frac{\lambda}{(\kappa-1)(\kappa-2)(1-\lambda\gamma^{\kappa-3})}\int_0^1 \mathcal{H}(\gamma,\xi)\xi^{-\rho}\xi^{\rho}[h(\xi,u(\xi))-h(\xi,z(\xi))]d\xi \\
&\le\sup\int_0^1 \mathcal{G}(s,\xi)\xi^{-\rho}\mu\psi(u(\xi)-z(\xi))d\xi \\
&\quad+\frac{\lambda}{(\kappa-1)(\kappa-2)(1-\lambda\gamma^{\kappa-3})}\int_0^1 \mathcal{H}(\gamma,\xi)\xi^{-\rho}\mu\psi(u(\xi)-z(\xi))d\xi.
\end{aligned}
$$

Using increasing property of ψ, above relation becomes

$$
\begin{aligned}
\sigma(\mathcal{T}z,\mathcal{T}u)\le{}&\mu\psi(\sigma(z,u))\sup_{0\le s\le 0}\int_0^1 \mathcal{G}(s,\xi)\xi^{-\rho}d\xi \\
&+\frac{\lambda}{(\kappa-1)(\kappa-2)(1-\lambda\gamma\kappa-3)}\mu\psi(\sigma(u,v))\int_0^1 \mathcal{H}(\gamma,\xi)\xi^{\rho}d\xi \\
={}&\mu\psi(\sigma(z,u))\left[\sup_{0\le s\le 0}\int_0^1 \mathcal{G}(s,\xi)\xi^{-\rho}d\xi\right. \\
&\left.+\frac{\lambda}{(\kappa-1)(\kappa-2)(1-\lambda\gamma^{\kappa-3})}\int_0^1 \mathcal{H}(\gamma,\xi)\xi^{-\rho}d\xi\right]. \qquad (4.9)
\end{aligned}
$$

Using Lemmas 4.6 and 4.7, (4.9) reduces to

$$
\begin{aligned}
&\sigma(\mathcal{T}z,\mathcal{T}u)\\
\leq&\mu\psi(\sigma(z,u))\left[\frac{1}{\Gamma(\kappa)}(\beta(1-\rho,\kappa-2)-\beta(1-\rho\kappa))\right.\\
&\left.+\frac{\lambda}{(\kappa-1)(\kappa-2)(1-\lambda\gamma^{\kappa-3})}\times\frac{(\kappa-1)(\kappa-2)}{\Gamma(\kappa)}\left(\gamma^{\kappa-3}-\gamma^{\kappa-\rho-2}\right)\right]\\
=&\mu\psi(\sigma(z,u))\left[\frac{1}{\Gamma(\kappa)}(\beta(1-\rho,\kappa-2)-\beta(1-\rho,\kappa))\right.\\
&\left.+\frac{\lambda(\gamma^{\kappa-3}-\gamma^{\kappa-\rho-2})}{(1-\lambda\gamma^{\kappa-3})\Gamma(\kappa)}\beta(1-\rho,\kappa-2)\right]\\
=&\mu\psi(\sigma(z,u))\left[\frac{1}{\Gamma(\kappa)}\left[\left(1+\frac{\lambda(\gamma^{\kappa-3}-\gamma^{\kappa-\rho-2})}{1-\lambda-\lambda\gamma^{\kappa-3}}\right)\beta(1-\rho,\kappa-2)-\beta(1-\rho,\kappa)\right]\right]\\
=&\mu\psi(\sigma(z,u))K.
\end{aligned}
$$

As $0<\mu\leq 1/K$, the last inequality becomes

$$\sigma(\mathcal{T}z,\mathcal{T}u)\leq\mu\psi(\sigma(z,u))K\leq\psi(\sigma(z,u)). \tag{4.10}$$

If $z\neq u$, then (4.10) can be expressed as

$$\sigma(\mathcal{T}z,\mathcal{T}u)\leq\frac{\psi(\sigma(z,u))}{\sigma(z,u)}\sigma(z,u)$$

so that

$$\sigma(\mathcal{T}z,\mathcal{T}u)\leq\alpha(\sigma(z,u))\sigma(z,u),\quad\forall\ z,u\in\mathbf{Z}\text{ with }(z,u)\in\mathcal{S}.$$

In case $z=u$, above inequality is obviously satisfied.

All assumptions of Theorem 3.1 thus hold; subsequently $\mathcal{T}$ possesses a fixed point. Choose $z,u\in\mathcal{T}(\mathbf{Z})$. Define $\omega:=\max\{z,u\}\in\mathbf{Z}$. Then $\{z,\omega,u\}$ forms a path in $\mathcal{S}^s$ from z to u. It follows that $\mathcal{T}(\mathbf{Z})$ is $\mathcal{S}^s$-connected. Consequently in view of Theorem 3.2, the fixed point of $\mathcal{T}$, say $\bar{v}\in(\mathbf{Z}$, remains unique, which indeed forms a unique nonnegative solution of BVP (4.1).

Finally, we shall have to prove that $\bar{v}$ is a positive solution. On contrary, suppose that $\exists\ 0<s^*<1$ verifying $\bar{v}(s^*)=0$. As, the nonnegative solution $\bar{v}$ of (4.1) remains fixed point of $\mathcal{T}$, we have

$$\bar{v}(s)=\int_0^1\mathcal{G}(s,\xi)h(\xi,x(\xi))d\xi+\frac{\lambda s^{\kappa-1}}{(\kappa-1)(\kappa-2)(1-\lambda\gamma^{\kappa-3})}\int_0^1\mathcal{H}(\gamma,\xi)h(\xi,\bar{v}(\xi))d\xi.$$

In particular, we have

$$
\begin{aligned}
\bar{v}(s^*)=&\int_0^1\mathcal{G}(s^*,\xi)h(\xi,\bar{v}(\xi))d\xi\\
&+\frac{\lambda s^{*\kappa-1}}{(\kappa-1)(\kappa-2)(1-\lambda\gamma^{\kappa-3})}\int_0^1\mathcal{H}(\gamma,\xi)h(\xi,x(\xi))d\xi\\
=&\,0.
\end{aligned}
$$

As both summands in RHS are nonnegative, by Remarks 4.1 and 4.2, we get

$$\int_0^1\mathcal{G}(s^*,\xi)\cdot h(\xi,\bar{v}(\xi))d\xi=0,$$

$$\int_0^1 \mathcal{H}(\gamma,\xi)\cdot h(\xi,\xi(\xi))d\xi = 0.$$

Due to nonnegative character of $\mathcal{G}(s,\xi)$, $\mathcal{H}(\gamma,\xi)$ and $h(\xi,\bar{v})$, we get

$$\begin{cases} \mathcal{G}(s^*,\xi)\cdot h(\xi,\bar{v}(\xi)) = 0, & a.e. \;\; (\xi), \\ \mathcal{H}(\gamma,\xi)\cdot h(\xi,\bar{v}(\xi)) = 0, & a.e \;\; (\xi). \end{cases} \tag{4.11}$$

Using the fact $\lim_{s\to 0^+} h(s,0) = \infty$, it follows that for given $M > 0$, $\exists\ \delta > 0$ such that $h(\xi,0) > M$, for every $\xi \in J := [0,1]\cap(0,\delta)$. Note that

$$J \subset \{\xi \in [0,1] : h(\xi,x(\xi)) > M\}$$

and

$$m(J) > 0,$$

where m remains the Lebesque measure. Therefore, (4.11) gives rise to

$$\begin{cases} \mathcal{G}(s^*,\xi) = 0, & a.e. \;\; (\xi), \\ \mathcal{H}(\gamma,\xi) = 0, & a.e. \;\; (\xi) \end{cases}$$

which arises a contradiction as $\mathcal{G}(s^*,\xi)$ and $\mathcal{H}(\gamma,\xi)$ are rational functions of ξ. Consequently, we have $\bar{v}(s) > 0$ for all $s \in (0,1)$. This completes the proof. □

5. An illustrative example

Now, we present an example intending to illustrate Theorem 4.1. Before consider this, we are required to prove some results on hyperbolic tangent function. Let us recall some definitions.

Definition 5.1. A function $\zeta : [0,\infty)$ is called subadditive if

$$\zeta(t+t') \le \zeta(t) + \zeta(t'), \quad \forall\ t,t' \in [0,\infty), \tag{5.1}$$

e.g., the square root function $\zeta(t) = \sqrt{t}$ is a subadditive function.

Remark 5.1. If $\zeta : [0,\infty) \to [0,\infty)$ is subadditive and $t \ge t'$, then

$$\zeta(t) - \zeta(t') \le \zeta(t-t').$$

Obviously, we have

$$\zeta(t) = \zeta(t-t'+t') \le \zeta(t-t') + \zeta(t')$$

so that

$$\zeta(t) - \zeta(t') \le t - t'.$$

Definition 5.2. We call that a function $\zeta : [0,\infty) \to [0,\infty)$ is concave if for each $\lambda \in [0,1]$.

$$\zeta(\lambda t + (1-\lambda)t') \ge \lambda\zeta(t) + (1-\lambda)\zeta(t'), \quad \forall\ t,t' \in [0,\infty).$$

Proposition 5.1. *Assume that $\zeta : [0,\infty) \to [0,\infty)$ is a concave function such that $\zeta(0) = 0$. Then ζ remains subadditive.*

Proof. Take $t, t' \in [0,\infty)$. As ζ is concave and $\zeta(0) = 0$, we have

$$\zeta(t) = \zeta\left(\frac{t'}{t+t'}0 + \frac{t}{t+t'}(t+t')\right) \geq \frac{t'}{t+t'}\zeta(0) + \frac{t}{t+t'}\zeta(t+t') = \frac{t}{t+t'}\zeta(t+t'),$$
$$\zeta(t') = \zeta\left(\frac{t}{t+t'}0 + \frac{t'}{t+t'}(t+t')\right) \geq \frac{t}{t+t'}\zeta(0) + \frac{t'}{t+t'}\zeta(t+t') = \frac{t'}{t+t'}\zeta(t+t').$$

Adding above, we get

$$\zeta(t) + \zeta(t') \geq \frac{t}{t+t'}\zeta(t+t') + \frac{t'}{t+t'}\zeta(t+t') = \zeta(t+t').$$

□

Lemma 5.1. *The hyperbolic tangent function $\zeta(t) = \tanh t$ verifies the following properties:*

(i) $\zeta \in \Psi$.

(ii) ζ *is subadditive.*

Proof. (i) By definition, we have

$$\tanh t = \frac{\sinh t}{\cosh t} = \frac{(e^t - e^{-t})/2}{(e^t + e^{-t})/2} = \frac{e^{2t} - 1}{e^{2t} + 1}$$

so that $\zeta(t) = \frac{e^{2t}-1}{e^{2t}+1}$. Now, as $\zeta'(t) = 4e^{2t}/(e^{2t}+1)^2 > 0, \quad \forall\, t > 0$, ζ is monotonic increasing. Also, the function $\theta(t) = t - \tan ht = t - \frac{e^{2t}-1}{e^{2t}+1}$ has as derivative $\theta'(t) = \frac{(e^{2t}-1)^2}{(e^{2t}+1)^2} > 0, \quad \forall\, t > 0$. It follows that θ is strictly increasing on $(0,\infty)$. As $\theta(0) = 0$, we have $0 = \theta(0) < \zeta(t), \quad \forall\, t > 0$. Consequently, one has $\zeta(t) = \tanh t < t, \quad \forall\, t > 0$.

Next, we verify that $\alpha(t) = \tanh t/t \in \mathfrak{S}$. If $\alpha(s_n) \to 1$, then we shall have to verify that the sequence $\{s_n\}$ must be bounded. Let on contrary $s_n \to \infty$. Then we have

$$\alpha(s_n) = \frac{\tanh s_n}{s_n} \to 0$$

which contradicts the fact that $\alpha(s_n) \to 1$.

Now, we assume that $\alpha(s_n) \to 1$ and $s_n \to 0$. Then, $\exists\, \varepsilon > 0$ such that for every $n \in \mathbb{N}$, we can find $\tau_n \geq n$ verifying $\tau_n \geq \varepsilon$. As the sequence (s_n) is bounded (due to $\lambda(s_n) \to 1$), $\exists$ a subsequence of s_{τ_n} (denoting in same way), verifying $s_{\tau_n} \to l$. Since $\alpha(s_n) \to 1$, therefore we get

$$\alpha(s_{\varrho_n}) = \frac{\tanh s_{\varrho_n}}{s_{\varrho_n}} \to \frac{\tanh l}{l} = 1.$$

As $t_0 = 0$ remains the (unique) solution of $\tanh t = t$ on $[0,\infty)$, we have $l = 0$. Hence, $s_{\varrho_n} \to 0$, which yields that $\exists\, n_0 \in \mathbb{N}$ verifying $s_{\varrho_n} < \varepsilon$ for $n \geq n_0$. This contradicts $s_{\varrho_n} \geq \varepsilon$, $\forall\, n \in \mathbb{N}$. Thus, $s_n \to 0$. This concludes that $\alpha(t) = \tanh t/t \in \mathfrak{S}$ and hence $\zeta \in \Phi$.

(ii) As $\tanh 0 = 0$ and $(\tanh t)'' = \frac{8e^{2t}(1-e^{2t})}{(e^{2t}+1)^3} < 0, \quad \forall\, t > 0$. This implies that $\zeta(t) = \tanh t$ is concave $\tanh 0 = 0$. Consequently by Proposition 5.1, $\zeta(t) = \tanh t$ is subadditive. □

Remark 5.2. Owing to Remark 5.1 and item (ii) of Lemma 5.1, $\forall\ t,t' \in [0,\infty)$ with $t \geq t'$, one has

$$\tanh t - \tanh t' \leq \tanh(t-t'). \tag{5.2}$$

Example 5.1. Let us consider the following singular fractional BVP:

$$\begin{aligned} &\mathfrak{D}_{0^+}^{7/2} v(s) + \frac{\mu(s^2+1)\tan hv(s)}{s^1/2} = 0, 0 < s < 1, \\ &v(0) = v'(0) = v''(0) = 0, v''(1) = v''\left(\frac{1}{4}\right). \end{aligned} \tag{5.3}$$

Here, $\rho = 1/2,\ \gamma = 1/4,\ \lambda = 1$ and $\kappa = 7/2$. Also, we have

$$h(s,x) = \mu(s^2+1)\tanh x/s^2, \quad \forall\ (s,x) \in (0,1] \times [0,\infty).$$

Clearly, h is continuous in $(0,1] \times [0,\infty)$ and $\lim\limits_{s\to 0^+} h(s,\cdot) = \infty$.

Now, we shall verify that h satisfies assumptions mentioned in Theorem 4.1. Clearly, the function $s^{1/2}h(s,x) = \mu(s^2+1)\tanh x$ remains continuous on $[0,1] \times [0,\infty)$. Also, by Lemma 5.1 and Remark 5.2, for all $x \geq y$ and $s \in [0,1]$, we have

$$\begin{aligned} 0 &\leq s^{1/2}\big[h(s,x) - h(s,y)\big] \\ &= \mu\left(s^2+1\right)(\tanh x - \tanh y) \\ &\leq \mu\big(s^2+1\big)\tanh(x-y) \\ &\leq 2\mu\tanh(x-y), \end{aligned}$$

where the function $\zeta(t) = \tanh t$ belongs to $\mathfrak{S}$ (due to Lemma 5.1). Now,

$$\begin{aligned} 2\mu &\leq \frac{1}{K} \\ &= \frac{\Gamma(7/2)}{\big[1 + \big((1/4)^{1/2} - 1/4\big) / \big(1 - (1/4)^{1/2}\big)\big] \cdot \beta(1/2,3/2) - \beta(1/2,7/2)} \\ &\Rightarrow \mu \leq 15\sqrt{\pi}/7. \end{aligned}$$

Thus, in view of Theorem 4.1, BVP (5.3) enjoys a unique positive solution for $\mu \leq 15\sqrt{\pi}/7$.

6. Discussions

This article concludes the results on fixed points employing relational Geraghty contractions in the setting of metric space indued with a locally $\mathcal{T}$-transitive relation. Under full relation $\mathcal{S} = \mathbf{Z}^2$, Theorem 3.2 deduces Theorem 1.1. In case $\mathcal{S} :=\preceq$ (a partial ordering), Theorems 3.1 and 3.2 deduce the fixed point theorems of Harandi and Emami [12]. For $\alpha(t) := \frac{1}{t}\phi(t)$ for $t > 0$, where ϕ is comparison function, our results reduce to corresponding results of Arif et al. [8]. Furthermore, our results sharpen the recent results of Almarri et al. [7] in the following respects:

- '$\mathcal{S}$-completeness' of metric space is replaced by '$(\mathcal{O},\mathcal{S})$-completeness';
- '$\mathcal{S}$-continuity' requirement on map is replaced by '$(\mathcal{O},\mathcal{S})$-continuity';
- 'σ-self-closed relation' is replaced by '$(\mathcal{O},\sigma)$-self-closed relation'.

7. Conclusions

In the hypotheses of our results, the orbital as well as relational analogues of frequently used metrical concepts are adopted. In practice, we determined a unique positive solution for certain singular nonlinear FDE prescribed by three-point boundary conditions, which shown the efficiency of our findings. Using our results, we deduce several known fixed point theorems, which substantiates the utility of our results over certain existent results in the literature. Inspired by the novelty of the relation metric space, in future, readers can prove the same results for different types of contractions or in the frameworks of generalized metrical structures (e.g., dislocated space, symmetric space etc).

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