Journal of Applied Analysis and Computation
Volume 14, Number 6, December 2024, 3496–3519
Website:http://www.jaac-online.com
DOI:10.11948/20240068

A BOUNDARY VALUE PROBLEM WITH IMPULSIVE EFFECTS AND RIEMANN-LIOUVILLE TEMPERED FRACTIONAL DERIVATIVES

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Abstract In this paper, we study a fractional impulsive differential equation with mixed tempered fractional derivatives. We justify some fundamental properties in the variational structure to fractional impulsive differential equations with the tempered fractional derivative operator. Finally, we study the existence of weak solutions with critical point theory and variational methods for the proposed problem. To prove the effectiveness of our main result, we investigate an interesting example.

Keywords Riemann-Liouville and Caputo tempered fractional derivatives, impulsive effects, tempered fractional space of Sobolev type, variational methods.

MSC(2010) 26A33, 34A08, 34B15, 35J20.

1. Introduction

In applied science such as biology, physics, control theory, economics and mechanics among other areas processes are frequently simulated using fractional differential equations, for details, see [12, 22, 24, 29, 30, 47] and the references therein. The theory of fractional differential equations has consequently attracted a lot of attention in recent years. For example, existence and stability are studied in [1, 5], and several resolution strategies are in [33, 36, 40]. In [13] the authors studied the nonlinear time-fractional gas dynamics equation. On the other hand, in [28], it is proposed an impulsive nonlinear differential equation with fractional derivative with interesting applications to pest management and, besides, some contributions to the study of option price governed by a Black-Scholes equation with a time-fractional derivative can be found in [8]. Some recent important applications of fractional differential models are those about time-fractional Schrödinger equation [16], Schrödinger equa-

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tion with fractional Laplacian [18], fractional wave equations [11], fractional damped dynamical systems [4] and fractional Euler-Lagrange equation modeling a fractional oscillator [6]. Some other recent application, these such equations are important and are considered as a novel subject in the theory of fractional differential equations. Using analytical methods such as fixed point theory, there have many results dealing with the existence of solutions to nonlinear fractional differential equations in this subject. For instance, we name here critical point theory and variational methods [3,20,21,46,48,49,51], topological degree theory [19], Leray-Schauder nonlinear alternative [53], and so on.

The tempered fractional calculus is the generalized version of fractional calculus. The tempered fractional derivatives and integrals are obtained when the fractional derivatives and integrals are multiplied by an exponential factor [27,43]. Recently, it has been observed that the use of tempered fractional derivatives, which leads to the so-called truncated Lévy flight, exhibits some important advantages compared to the usual fractional derivatives, especially with regard to the spatial moments [15, 23, 42, 54]. In this context, it should be noted that solutions obtained for the tempered fractional derivative contain those for the untempered one as a special case. Therefore, the truncated Lévy process can be seen as a generalization of the conventional untempered one. Tempered fractional differential equations have been applied in different fields of physics such as in geophysics, statistical physics, plasma physics or in the context of astrophysics [10, 32, 45, 54]. Apart from the physics field, the tempered fractional derivatives have also been applied in finance for modeling price fluctuations with semi-heavy tails [42]. Recently, Almeida and Morgado [2] studied variational problems where the cost functional involves the tempered Caputo fractional derivative.

Differential equations with impulsive effects arise from many phenomena in the real world and describe the dynamics of processes in which sudden, discontinuous jumps occur. We refer to [9, 17, 25, 26, 44, 52] for some monographs and papers including relevant information about this topic. In [7, 34, 35, 41], it is proved the existence and multiplicity of weak solutions for a class of Dirichlet's boundary value problems for fractional differential equations with impulses by using a critical point theory. More precisely, in [50], the authors studied the existence of weak solutions by using new linking theorem due to Schechter included in [38].

The used method in this paper is standard, but its configuration and relations of variational methods in the present paper is new. The obtained results in tempered fractional derivatives are new and contribute to this new research topic concerning the study of positive boundary value problems.

Motivated by these previous works, we would like to study variational structure for the tempered fractional derivative operator and we have justified some fundamental properties in the variational structure. Also, we deal with the following tempered fractional boundary value problem

$$\begin{cases} \mathbb{D}_{b^-}^{\alpha,\sigma}({}^C\mathbb{D}_{a^+}^{\alpha,\sigma}u(x)) = f(x,u), \quad x \neq x_j \ \text{ a.e. } x \in (0,T), \\ u(0) = u(T) = 0, \\ \Delta\left(\mathbb{I}_{T^-}^{1-\alpha,\sigma}{}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u\right)(x_j) = I_j(u(x_j)), \quad j = 1,2,\cdots,n, \end{cases} \tag{1.1}$$

where $\alpha \in (\frac{1}{2}, 1)$ and $\sigma > 0$,

$$0 = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = T,$$

$$\Delta(\mathbb{I}_{T^-}^{1-\alpha,\sigma C}\mathbb{D}_{0^+}^{\alpha,\sigma}u)(x_j) = \lim_{x\to x_j^+} \mathbb{I}_{T^-}^{1-\alpha,\sigma C}\mathbb{D}_{0^+}^{\alpha,\sigma}u(x) - \lim_{x\to x_j^-} \mathbb{I}_{T^-}^{1-\alpha,\sigma C}\mathbb{D}_{0^+}^{\alpha,\sigma}u(x),$$

$f : (0,T)\times\mathbb{R}\to\mathbb{R}$ and $I_j : \mathbb{R}\to\mathbb{R}$, $j = 1,2,\cdots,n$ are continuous functions satisfying some suitable conditions. More precisely, we assume that $I_j(s)$ and f satisfy the following hypotheses:

(F_1) There exists a constant $\eta > 2$ such that $f(x,u) = o(|u|^\eta)$ as $|u|\to\infty$ and $f(x,u) = o(|u|)$ as $|u|\to 0$ uniformly for $x\in[0,T]$.

(F_2) There exist constant $\gamma > 2$ and $\vartheta_0 > 0$ such that

$$0 < \gamma F(x,u) \leq uf(x,u), \quad \text{for every } \ (x,u)\in[0,T]\times\mathbb{R}, \ \ |\tau|\geq\vartheta_0.$$

(I_1) There exists a constant $1 < \varpi < \gamma - 1$ such that $I_j(u) = o(|u|^\varpi)$ as $|u|\to\infty$ and $I_j(u) = o(|u|)$ as $|u|\to 0$.

(I_2) There exist constant $0 < \gamma_j < 2$ and $\vartheta > 0$ for any $j = 1,\ldots,n$, such that

$$0 < \gamma_j\int_0^\tau I_j(s)ds \leq I_j(\tau)\tau, \quad \text{for every } \ \tau\in\mathbb{R}, \ \ |\tau|\geq\vartheta.$$

Our main results read as follows:

Theorem 1.1. *Assume that the conditions (F_1), (F_2), (I_1) and (I_2) hold. Moreover, $I_j(t)$ and $F(x,t)$ about t are evens. Then problem (1.1) has infinitely many weak solutions.*

If we choose $\sigma = 0$, problem (1.1) reduce to the following boundary value problem

$$\begin{cases} D_{b^-}^\alpha({}^C D_{a^+}^\alpha u(x)) = f(x,u), \ \ x\neq x_j \ \text{ a.e. } x\in(0,T), \\ u(0) = u(T) = 0, \\ \Delta\left(I_{T^-}^{1-\alpha C}D_{0^+}^\alpha u\right)(x_j) = I_j(u(x_j)), \ \ j = 1,2,\cdots,n, \end{cases} \tag{1.2}$$

where $D_{b^-}^\alpha u$ and ${}^C D_{a^+}^\alpha u$ are the right Riemann-Liouville fractional derivative and the left Caputo fractional derivative respectively. As a consequence of Theorem 4.1 we have the following result.

Theorem 1.2. *Assume that the conditions (F_1), (F_2), (I_1) and (I_2) hold. Moreover, $I_j(t)$ and $F(x,t)$ about t are evens. Then problem (1.2) has infinitely many weak solutions.*

Remark 1.1. We recall that in Theorem 1.1 and Theorem 1.2 we just consider the case $\alpha\in(\frac{1}{2},1)$, because, in this case we can consider the classical Dirichlet boundary conditions and we have the following characterization of our fractional space

$$\mathbb{H}_0^{\alpha,\sigma}(a,b) = \{u\in L^2(a,b) : \ {}^C\mathbb{D}_{a^+}^{\alpha,\sigma}u\in L^2(a,b) \ \text{ and } \ u(a) = u(b) = 0\}.$$

The case $\alpha\in(0,\frac{1}{2})$ is an open problem yet.

2. Some previous results

Let (a,b) be a bounded interval. For $\alpha \in (0,1)$, $\sigma > 0$ and a suitable function u, the left and right Riemann-Liouville tempered fractional derivatives of order α are defined as

$$\mathbb{D}^{\alpha,\sigma}_{a^+}u(x) = \left(\frac{d}{dx} + \sigma\right)\mathbb{I}^{1-\alpha,\sigma}_{a^+}u(x), \quad x > a, \tag{2.1}$$

and

$$\mathbb{D}^{\alpha;\sigma}_{b^-}u(x) = -\left(\frac{d}{dx} - \sigma\right)\mathbb{I}^{1-\alpha,\sigma}_{b^-}u(x), \quad x < b, \tag{2.2}$$

respectively, where $\mathbb{I}^{\alpha,\sigma}_{a^+}u, \mathbb{I}^{\alpha,\sigma}_{b^-}u$ are the left and right Riemann-Liouville tempered fractional integrals of order α defined as

$$\mathbb{I}^{\alpha,\sigma}_{a^+}u(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}e^{-\sigma(x-s)}u(s)ds, \quad x > a, \tag{2.3}$$

and

$$\mathbb{I}^{\alpha,\sigma}_{b^-}u(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (s-x)^{\alpha-1}e^{-\sigma(s-x)}u(s)ds, \quad x < b, \tag{2.4}$$

respectively. An alternative approach in defining the tempered fractional derivatives is based on the left-sided and right-sided tempered Caputo fractional derivatives of order α, defined, respectively, as

$${}^C\mathbb{D}^{\alpha,\sigma}_{a^+}u(x) = \mathbb{I}^{1-\alpha,\sigma}_{a^+}\left(\frac{d}{dx} + \sigma\right)u(x), \quad x > a, \tag{2.5}$$

and

$${}^C\mathbb{D}^{\alpha,\sigma}_{b^-}u(x) = -\mathbb{I}^{1-\alpha,\sigma}_{b^-}\left(\frac{d}{dx} - \sigma\right)u(x), \quad x < b, \tag{2.6}$$

respectively. Note that, if $u \in AC[a,b]$, then the following identities holds for Riemann-Liouville and Caputo tempered fractional derivatives

$$\mathbb{D}^{\alpha,\sigma}_{a^+}u(x) = e^{-\sigma x}{}_aD^\alpha_x e^{\sigma x}u(x) \quad \text{and} \quad \mathbb{D}^{\alpha,\sigma}_{b^-}u(x) = e^{\sigma x}{}_xD^\alpha_b e^{-\sigma x}u(x), \tag{2.7}$$

and

$${}^C\mathbb{D}^{\alpha,\sigma}_{a^+}u(x) = e^{-\sigma x}\cdot {}^C_aD^\alpha_x e^{\sigma x}u(x) \quad \text{and} \quad \mathbb{D}^{\alpha,\sigma}_{b^-}u(x) = e^{\sigma x}\cdot {}^C_xD^\alpha_b e^{-\sigma x}u(x). \tag{2.8}$$

In what follows we consider some properties of tempered fractional operators which are know in the literature.

Lemma 2.1. [48] *Let $\alpha > 0$, $\sigma > 0$ and $u \in AC[a,b]$. Then $\mathbb{I}^{\alpha,\sigma}_{a^+}u, \mathbb{I}^{\alpha,\sigma}_{b^-}u$ are well defined. Moreover*

$$\mathbb{I}^{\alpha,\sigma}_{a^+}u(x) = \frac{u(a)}{\sigma^\alpha\Gamma(\alpha)}\gamma(\alpha,\sigma(x-a)) + \frac{1}{\sigma^\alpha\Gamma(\alpha)}\int_a^x \gamma(\alpha,\sigma(x-t))u'(t)dt, \tag{2.9}$$

and

$$\mathbb{I}^{\alpha,\sigma}_{b^-}u(x) = \frac{u(b)}{\sigma^\alpha\Gamma(\alpha)}\gamma(\alpha,\sigma(b-x)) - \frac{1}{\sigma^\alpha\Gamma(\alpha)}\int_x^b \gamma(\alpha,\sigma(t-x))u'(t)dt, \tag{2.10}$$

where

$$\gamma(\alpha,x) = \int_0^x t^{\alpha-1}e^{-t}dt$$

is the lower incomplete Gamma function.

Remark 2.1. The incomplete Gamma function has the following bounds

$$e^{-x}\frac{x^\alpha}{\alpha} \leq \gamma(\alpha, x) \leq \frac{x^\alpha}{\alpha}. \tag{2.11}$$

Moreover, for each $n \in \mathbb{N}$ and $-n < \alpha < -n+1$, using integration by parts and induction, we obtain

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1}\left(e^{-t} - \sum_{k=0}^{n-1}\frac{(-t)^k}{k!}\right)dt + \sum_{k=0}^{n-1}\frac{(-1)^k}{(\alpha+k)k!}x^{\alpha+k}. \tag{2.12}$$

This equality can be used to extend the definition of $\gamma(\alpha, x)$ to negative, non integer values of α. For example, if $\alpha \in (-1, 0)$ and $x > 0$, then

$$\gamma(\alpha, x) = \frac{1}{\alpha}\gamma(\alpha+1, x) + \frac{1}{\alpha}x^\alpha e^{-x}. \tag{2.13}$$

For more details the reader's can see [14].

Theorem 2.1. [48] *Let $\alpha \in (0,1)$, $\sigma > 0$, $p \in [1, \infty]$. Then, the tempered fractional integrals of Riemann-Liouville $\mathbb{I}^{\alpha,\sigma}_{a^+}, \mathbb{I}^{\alpha,\sigma}_{b^-} : L^p(a,b) \to L^p(a,b)$ are bounded. Moreover*

$$\|\mathbb{I}^{\alpha,\sigma}_{a^+}u\|_{L^p(a,b)} \leq \frac{\gamma(\alpha, \sigma(b-a))}{\sigma^\alpha\Gamma(\alpha)}\|u\|_{L^p(a,b)}, \tag{2.14}$$

and

$$\|\mathbb{I}^{\alpha,\sigma}_{b^-}u\|_{L^p(a,b)} \leq \frac{\gamma(\alpha, \sigma(b-a))}{\sigma^\alpha\Gamma(\alpha)}\|u\|_{L^p(a,b)}. \tag{2.15}$$

Lemma 2.2. [37] *For $\alpha_1, \alpha_2 > 0$, $\sigma \geq 0$ and for all $u \in L^p(a,b)$ with $p \in [1, \infty]$ we have*

$$\mathbb{I}^{\alpha_1,\sigma}_{a^+}\cdot\mathbb{I}^{\alpha_2,\sigma}_{a^+}u(x) = \mathbb{I}^{\alpha_1+\alpha_2,\sigma}_{a^+}u(x) \quad \text{and} \quad \mathbb{I}^{\alpha_1,\sigma}_{b^-}\cdot\mathbb{I}^{\alpha_2,\sigma}_{b^-}u(x) = \mathbb{I}^{\alpha_1+\alpha_2,\sigma}_{b^-}u(x).$$

Theorem 2.2. *Let $\alpha \in (0,1)$, $\sigma > 0$, $p \in (1,\infty)$, $q \in (1, \infty)$ and*

$$\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha.$$

If $u \in L^p(a,b)$ and $v \in L^q(a,b)$, then

$$\int_a^b \mathbb{I}^{\alpha,\sigma}_{a^+}u(x)v(x)dx = \int_a^b u(x)\mathbb{I}^{\alpha,\sigma}_{b^-}v(x)dx. \tag{2.16}$$

Now we consider some smoothness properties of the Riemann-Liouville tempered fractional integrals.

Theorem 2.3. [48] *Let $\alpha \in (0,1)$, $\sigma > 0$ and $u \in C[a,b]$. Then the tempered fractional integrals of Riemann-Liouville $\mathbb{I}^{\alpha,\sigma}_{a^+}u, \mathbb{I}^{\alpha,\sigma}_{b^-}$ are continuos on $[a,b]$ and*

$$\lim_{x\to a^+}\mathbb{I}^{\alpha,\sigma}_{a^+}u(x) = 0 \quad \text{and} \quad \lim_{x\to b^-}\mathbb{I}^{\alpha,\sigma}_{b^-}u(x) = 0. \tag{2.17}$$

Moreover

$$\|\mathbb{I}^{\alpha,\sigma}_{a^+}u\|_\infty \leq \frac{1}{\sigma^\alpha\Gamma(\alpha)}\gamma(\alpha, \sigma(b-a))\|u\|_\infty,$$

and

$$\|\mathbb{I}^{\alpha,\sigma}_{b^-}u\|_\infty \leq \frac{1}{\sigma^\alpha\Gamma(\alpha)}\gamma(\alpha, \sigma(b-a))\|u\|_\infty.$$

The following result were considered by Torres et al. [48, Theorem 3.9]. More precisely, assuming that $\alpha \in (\frac{1}{2}, 1)$, $\sigma > 0$ and $u \in L^2(a,b)$, then Torres et al. proved that $\mathbb{I}^{\alpha,\sigma}_{a^+}u, \mathbb{I}^{\alpha,\sigma}_{b^-}u \in C[a,b]$. We note that under a carefully analysis we are able to prove that the Riemann-Liouville fractional tempered integrals $\mathbb{I}^{\alpha,\sigma}_{a^+}u, \mathbb{I}^{\alpha,\sigma}_{b^-}u$ are Hölder continuous with order $\alpha - \frac{1}{2}$. To state our result we need the following inequality: For any $x_1 \geq x_2 \geq 0$ and $q \geq 1$

$$(x_1 - x_2)^q \leq x_1^q - x_2^q. \tag{2.18}$$

Theorem 2.4. *Let $\alpha \in (\frac{1}{2}, 1)$ and $\sigma > 0$. Then, for each $u \in L^2(a,b)$, $\mathbb{I}^{\alpha,\sigma}_{a^+}u \in \mathbb{H}_0^{\alpha-\frac{1}{2},\sigma}(a,b)$ Defined in Section 3) and*

$$\lim_{x\to a^+} \mathbb{I}^{\alpha,\sigma}_{a^+}u(x) = 0,$$

where $\mathbb{H}_0^{\alpha-\frac{1}{2},\sigma}(a,b)$ denotes the Hölder space of order $\alpha - \frac{1}{2} > 0$.

Proof. Let $a < x_1 < x_1 \leq b$ and $u \in L^p(a,b)$, then by Hölder inequality

$$\begin{aligned}
&|\mathbb{I}^{\alpha,\sigma}_{a^+}u(x_1) - \mathbb{I}^{\alpha,\sigma}_{a^+}u(x_2)|\\
\leq&\frac{1}{\Gamma(\alpha)}\left[\int_a^{x_1}\left|(x_1-s)^{\alpha-1}e^{-\sigma(x_1-s)} - (x_2-s)^{\alpha-1}e^{-\sigma(x_2-s)}\right||u(s)|ds\right.\\
&\left.+\int_{x_1}^{x_2}(x_2-s)^{\alpha-1}e^{-\sigma(x_2-s)}|u(s)|ds\right]\\
\leq&\frac{1}{\Gamma(\alpha)}\left(\int_a^{x_1}\left|(x_1-s)^{\alpha-1}e^{-\sigma(x-s)} - (x_2-s)^{\alpha-1}e^{-\sigma(x_2-s)}\right|^2 ds\right)^{1/2}\|u\|_{L^2(a,x_1)}\\
&+\frac{1}{\Gamma(\alpha)}\left(\int_{x_1}^{x_2}(x_2-s)^{2\alpha-2}e^{-2\sigma(x_2-s)}ds\right)^{1/2}\left(\int_{x_1}^{x_2}|u(s)|^2ds\right)^{1/2}.
\end{aligned} \tag{2.19}$$

Doing the change of variable $t = 2\sigma(x_2 - s)$ and using (2.11) we get

$$\int_{x_1}^{x_2}(x_2-s)^{2\alpha-2}e^{-2\sigma(x_2-s)}ds = \frac{1}{(2\sigma)^{2\alpha-1}}\gamma(2\alpha-1, 2\sigma(x_2-x_1)) \leq \frac{(x_2-x_1)^{2\alpha-1}}{2\alpha-1}.$$

Hence

$$\left(\int_{x_1}^{x_2}(x_2-s)^{2\alpha-2}e^{-2\sigma(x_2-s)}ds\right)^{1/2} \leq \frac{1}{(2\alpha-1)^{1/2}}(x_2-x_1)^{\alpha-\frac{1}{2}}. \tag{2.20}$$

On the other hand, the change of variable $t = \frac{x_1-s}{x_2-x_1}$ yields that

$$\begin{aligned}
&\int_a^{x_1}\left|(x_1-s)^{\alpha-1}e^{-\sigma(x_1-s)} - (x_2-s)^{\alpha-1}e^{-\sigma(x_2-s)}\right|^2 ds\\
=&(x_2-x_1)^{2\alpha-1}\int_0^{\frac{x_1-a}{x_2-x_1}}\left|t^{\alpha-1}e^{-\sigma t(x_2-x_1)} - (1+t)^{\alpha-1}e^{-\sigma(1+t)(x_2-x_1)}\right|^2 dt.
\end{aligned} \tag{2.21}$$

So, if $\frac{x_1-a}{x_2-x_1} \leq 1$, by (2.18) we derive

$$\begin{aligned}
&\int_0^{\frac{x_1-a}{x_2-x_1}} \left| t^{\alpha-1} e^{-\sigma t(x_2-x_1)} - (1+t)^{\alpha-1} e^{-\sigma(1+t)(x_2-x_1)} \right|^2 dt \\
\leq &\int_0^1 \left| t^{\alpha-1} e^{-\sigma t(x_2-x_1)} - (1+t)^{\alpha-1} e^{-\sigma(1+t)(x_2-x_1)} \right|^2 dt \\
\leq &\int_0^1 \left(t^{2\alpha-2} e^{-2\sigma t(x_2-x_1)} - (1+t)^{2\alpha-2} e^{-2\sigma(1+t)(x_2-x_1)} \right) dt \\
= &\frac{1}{[2\sigma(x_2-x_1)]^{2\alpha-1}} \Big(2\gamma(2\alpha-1, 2\sigma(x_2-x_1)) - \gamma(2\alpha-1, 4\sigma(x_2-x_1)) \Big).
\end{aligned}$$

Now, note that by (2.11) we obtain

$$\gamma(2\alpha-1, 2\sigma(x_2-x_1)) \leq \frac{[2\sigma(x_2-x_1)]^{2\alpha-1}}{2\alpha-1}$$

and

$$e^{-4\sigma(x_2-x_1)} \frac{[4\sigma(x_2-x_1)]^{2\alpha-1}}{2\alpha-1} \leq \gamma(2\alpha-1, 4\sigma(x_2-x_1)),$$

consequently

$$\begin{aligned}
&2\gamma(2\alpha-1, 2\sigma(x_2-x_1)) - \gamma(2\alpha-1, 4\sigma(x_2-x_1)) \\
\leq &\frac{[2\sigma(x_2-x_1)]^{2\alpha-1}}{2\alpha-1} \left(2 - 2^{2\alpha-1} e^{-4\sigma(x_2-x_1)} \right).
\end{aligned}$$

Therefore, replacing in (2.19) we obtain

$$\begin{aligned}
&\int_a^{x_1} \left| (x_1-s)^{\alpha-1} e^{-\sigma(x_1-s)} - (x_2-s)^{\alpha-1} e^{-\sigma(x_2-s)} \right|^2 ds \\
\leq &\frac{(x_2-x_1)^{2\alpha-1}}{2\alpha-1} \left(2 - 2^{2\alpha-1} e^{-4\sigma(x_2-x_1)} \right) \\
\leq &\frac{2}{2\alpha-1} (x_2-x_1)^{2\alpha-1}.
\end{aligned} \tag{2.22}$$

On the other hand, if $\frac{x_1-a}{x_2-x_1} > 1$, then

$$\begin{aligned}
&\int_0^{\frac{x_1-a}{x_2-x_1}} \left| t^{\alpha-1} e^{-\sigma t(x_2-x_1)} - (1+t)^{\alpha-1} e^{-\sigma(1+t)(x_2-x_1)} \right|^2 dt \\
= &\int_0^1 \left| t^{\alpha-1} e^{-\sigma t(x_2-x_1)} - (1+t)^{\alpha-1} e^{-\sigma(1+t)(x_2-x_1)} \right|^2 dt \\
&+ \int_1^{\frac{x_1-a}{x_2-x_1}} \left| t^{\alpha-1} e^{-\sigma t(x_2-x_1)} - (1+t)^{\alpha-1} e^{-\sigma(1+t)(x_2-x_1)} \right|^2 dt \\
\leq &\frac{2}{2\alpha-1} + \int_1^{\frac{x_1-a}{x_2-x_1}} \left| t^{\alpha-1} e^{-\sigma t(x_2-x_1)} - (1+t)^{\alpha-1} e^{-\sigma(1+t)(x_2-x_1)} \right|^2 dt.
\end{aligned} \tag{2.23}$$

The mean value theorem and the change of variable $\lambda = 2\sigma t(x_2 - x_1)$ yield that

$$
\begin{aligned}
&\int_1^{\frac{x_1-a}{x_2-x_1}} \left| t^{\alpha-1}e^{-\sigma t(x_2-x_1)} - (1+t)^{\alpha-1}e^{-\sigma(1+t)(x_2-x_1)} \right|^2 dt \\
=&\int_1^{\frac{x_1-a}{x_2-x_1}} \Big((1-\alpha)r^{\alpha-2}e^{-\sigma t(x_2-x_1)} + \sigma(x_2-x_1)t^{\alpha-1}e^{-\sigma t(x_2-x_1)} \Big)^2 dt \\
\leq& 2(1-\alpha)^2 \int_1^{\frac{x_1-a}{x_2-x_1}} t^{2\alpha-4}e^{-2\sigma t(x_2-x_1)}dt \\
&+ 2\sigma^2(x_2-x_1)^2 \int_1^{\frac{x_1-a}{x_2-x_1}} t^{2\alpha-2}e^{-2\sigma t(x_2-x_1)}dt \\
=&\frac{2(1-\alpha)^2}{(2\sigma)^{2\alpha-3}}(x_2-x_1)^{3-2\alpha} \int_{2\sigma(x_2-x_1)}^{2\sigma(x_1-a)} \lambda^{2\alpha-4}e^{-\lambda}d\lambda \\
&+ \frac{2\sigma^2}{(2\sigma)^{2\alpha-1}}(x_2-x_3)^{3-2\alpha} \int_{2\sigma(x_2-x_1)}^{2\sigma(x_1-a)} \lambda^{2\alpha-2}e^{-\lambda}d\lambda.
\end{aligned}
$$

Note that, integrating by parts the first integral of the last expression we get

$$
\begin{aligned}
&\int_{2\sigma(x_2-x_1)}^{2\sigma(x_1-a)} \lambda^{2\alpha-4}e^{-\lambda}d\lambda \\
=&\frac{(2\sigma)^{2\alpha-3}}{2\alpha-3} \Big((x_1-a)^{2\alpha-3}e^{-2\sigma(x_1-a)} - (x_2-x_1)^{2\alpha-3}e^{-2\sigma(x_2-x_1)} \Big) \\
&+ \frac{(2\sigma)^{2\alpha-2}}{(2\alpha-3)(2\alpha-2)} \Big((x_1-a)^{2\alpha-2}e^{-2\sigma(x_1-a)} - (x_2-x_1)^{2\alpha-2}e^{-2\sigma(x_2-x_1)} \Big) \\
&+ \frac{1}{(2\alpha-2)(2\alpha-3)} \int_{2\sigma(x_2-x_1)}^{2\sigma(x_1-a)} \lambda^{2\alpha-2}e^{-\lambda}d\lambda.
\end{aligned}
$$

Consequently, replacing in the last inequality we derive

$$
\begin{aligned}
&\int_1^{\frac{x_1-a}{x_2-x_1}} \left| t^{\alpha-1}e^{-\sigma t(x_2-x_1)} - (1+t)^{\alpha-1}e^{-\sigma(1+t)(x_2-x_1)} \right|^2 dt \\
\leq&\frac{2(1-\alpha)^2}{(2\sigma)^{2\alpha-3}}(x_2-x_1)^{3-2\alpha} \int_{2\sigma(x_2-x_1)}^{2\sigma(x_1-a)} \lambda^{2\alpha-4}e^{-\lambda}d\lambda \\
&+ \frac{2\sigma^2}{(2\sigma)^{2\alpha-1}}(x_2-x_3)^{3-2\alpha} \int_{2\sigma(x_2-x_1)}^{2\sigma(x_1-a)} \lambda^{2\alpha-2}e^{-\lambda}d\lambda \\
=&\frac{2(1-\alpha)^2}{(2\sigma)^{2\alpha-3}}(x_2-x_1)^{3-2\alpha} \\
&\times \left[\frac{(2\sigma)^{2\alpha-3}}{2\alpha-3} \Big((x_1-a)^{2\alpha-3}e^{-2\sigma(x_1-a)} - (x_2-x_1)^{2\alpha-3}e^{-2\sigma(x_2-x_1)} \Big) \right. \\
&\left. + \frac{(2\sigma)^{2\alpha-2}}{(2\alpha-3)(2\alpha-2)} \Big((x_1-a)^{2\alpha-2}e^{-2\sigma(x_1-a)} - (x_2-x_1)^{2\alpha-2}e^{-2\sigma(x_2-x_1)} \Big) \right] \\
&+ \left(\frac{2\sigma^2}{(2\sigma)^{2\alpha-1}} + \frac{2(1-\alpha)^2}{(2\sigma)^{2\alpha-3}(2\alpha-2)(2\alpha-3)} \right)(x_2-x_3)^{3-2\alpha} \\
&\times \int_{2\sigma(x_2-x_1)}^{2\sigma(x_1-a)} \lambda^{2\alpha-2}e^{-\lambda}d\lambda
\end{aligned}
$$

$$
\begin{aligned}
\leq & \frac{2(1-\alpha)^2}{2\alpha-3}(x_2-x_1)^{3-2\alpha}\left((x_1-a)^{2\alpha-3}e^{-2\sigma(x_1-a)}-(x_2-x_1)^{2\alpha-3}e^{-2\sigma(x_2-x_1)}\right)\\
& +\left(\frac{2\sigma^2}{(2\sigma)^{2\alpha-1}}+\frac{2(1-\alpha)^2}{(2\sigma)^{2\alpha-3}(2\alpha-2)(2\alpha-3)}\right)(x_2-x_3)^{3-2\alpha}\\
& \times\int_{2\sigma(x_2-x_1)}^{2\sigma(x_1-a)}\lambda^{2\alpha-2}e^{-\lambda}d\lambda.
\end{aligned}
$$

By other side, (2.11) yields that

$$
\begin{aligned}
& \int_{2\sigma(x_2-x_1)}^{2\sigma(x_1-a)}\lambda^{2\alpha-2}e^{-\lambda}d\lambda\\
= & \gamma(2\alpha-1,2\sigma(x_1-a))-\gamma(2\alpha-1,2\sigma(x_2-x_1))\\
\leq & \frac{(2\sigma)^{2\alpha-1}}{2\alpha-1}(x_2-x_1)^{2\alpha-1}\left(\left(\frac{x_1-a}{x_2-x_1}\right)^{2\alpha-1}-e^{-2\sigma(x_2-x_1)}\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
& \int_1^{\frac{x_1-a}{x_2-x_1}}\left|t^{\alpha-1}e^{-\sigma t(x_2-x_1)}-(1+t)^{\alpha-1}e^{-\sigma(1+t)(x_2-x_1)}\right|^2dt\\
\leq & \frac{2(1-\alpha)^2}{2\alpha-3}\left(\left(\frac{x_1-a}{x_2-x_1}\right)^{2\alpha-3}e^{-2\sigma(x_1-a)}-e^{-2\sigma(x_2-x_1)}\right)\\
& +\left(\frac{2\sigma^2}{2\alpha-1}+\frac{8(1-\alpha)^2\sigma^2}{(2\alpha-1)(2\alpha-2)(2\alpha-3)}\right)(x_2-x_1)^2\\
& \times\left(\left(\frac{x_1-a}{x_2-x_1}\right)^{2\alpha-1}-e^{-2\sigma(x_2-x_1)}\right).
\end{aligned}
\tag{2.24}
$$

Finally, combining (2.21) with (2.23) and (2.24) we derive

$$
\int_a^{x_1}\left|(x_1-s)^{\alpha-1}e^{-\sigma(x_1-s)}-(x_2-s)^{\alpha-1}e^{-\sigma(x_2-s)}\right|^2ds\leq\mathfrak{M}(x_2-x_1)^{2\alpha-1}, \tag{2.25}
$$

where

$$
\begin{aligned}
\mathfrak{M}= & \frac{2}{2\alpha-1}+\frac{2(1-\alpha)^2}{2\alpha-3}\left(\left(\frac{x_1-a}{x_2-x_1}\right)^{2\alpha-3}e^{-2\sigma(x_1-a)}-e^{-2\sigma(x_2-x_1)}\right)\\
& +\left(\frac{2\sigma^2}{2\alpha-1}+\frac{8(1-\alpha)^2\sigma^2}{(2\alpha-1)(2\alpha-2)(2\alpha-3)}\right)(x_2-x_1)^2\\
& \times\left(\left(\frac{x_1-a}{x_2-x_1}\right)^{2\alpha-1}-e^{-2\sigma(x_2-x_1)}\right).
\end{aligned}
$$

Therefore, by (2.19), (2.20), (2.22) and (2.25) we get

$$
\begin{aligned}
& |\mathbb{I}_{a^+}^{\alpha,\sigma}u(x_1)-\mathbb{I}_{a^+}^{\alpha,\sigma}u(x_2)|\\
\leq & \frac{1}{\Gamma(\alpha)}\left(\int_a^{x_1}\left|(x_1-s)^{\alpha-1}e^{-\sigma(x-s)}-(x_2-s)^{\alpha-1}e^{-\sigma(x_2-s)}\right|^2ds\right)^{1/2}\\
& \times\left(\int_a^{x_1}|u(s)|^2ds\right)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\left(\int_{x_1}^{x_2}(x_2-s)^{2\alpha-2}e^{-2\sigma(x_2-s)}ds\right)^{1/2}\left(\int_{x_1}^{x_2}|u(s)|^2ds\right)^{1/2}\\
\leq&\frac{\mathfrak{M}^{1/2}}{\Gamma(\alpha)}(x_2-x_1)^{\alpha-\frac{1}{2}}\|u\|_{L^2(a,b)}+\frac{1}{(2\alpha-1)^{1/2}\Gamma(\alpha)}(x_2-x_1)^{\alpha-\frac{1}{2}}\|u\|_{L^2(a,b)}\\
=&\frac{1}{\Gamma(\alpha)}\left(\mathfrak{M}^{1/2}+\frac{1}{(2\alpha-1)^{1/2}}\right)\|u\|_{L^2(a,b)}(x_2-x_1)^{\alpha-\frac{1}{2}},
\end{aligned}
$$

which implies that $\mathbb{I}_{x^+}^{\alpha,\sigma}u\in\mathbb{H}_0^{\alpha-\frac{1}{2},\sigma}(a,b)$.

To finish with the proof, note that for any $u\in L^2(a,b)$ and Hölder inequality we get

$$
\begin{aligned}
|\mathbb{I}_{a^+}^{\alpha,\sigma}u(x)|&\leq\frac{1}{\Gamma(\alpha)}\int_a^x(x-s)^{\alpha-1}e^{-\sigma(x-s)}|u(s)|ds\\
&\leq\frac{1}{\Gamma(\alpha)}\left(\int_a^x(x-s)^{2(\alpha-1)}e^{-\sigma2(x-s)}ds\right)^{1/2}\left(\int_a^x|u(s)|^2ds\right)^{1/2}\\
&\leq\frac{1}{2^{\alpha-\frac{1}{2}}\Gamma(\alpha)}\frac{[\gamma(2\alpha-1,2\sigma(x-a))]^{1/2}}{\sigma^{\alpha-\frac{1}{2}}}\|u\|_{L^2(a,b)}.
\end{aligned}
$$

Furthermore, (2.11) yields that

$$
\begin{aligned}
e^{-\sigma(x-a)}\frac{2^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{1/2}}(x-a)^{\alpha-\frac{1}{2}}&\leq\frac{(\gamma(2\alpha-1,2\sigma(x-a)))^{1/2}}{\sigma^{\alpha-\frac{1}{2}}}\\
&\leq\frac{2^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{1/2}}(x-a)^{\alpha-\frac{1}{2}},
\end{aligned}
$$

which implies

$$\lim_{x\to a^+}\frac{(\gamma(2\alpha-1,2\sigma(x-a)))^{1/2}}{\sigma^{\alpha-\frac{1}{2}}}=0.$$

So, combining this limit with the last inequality we get

$$\lim_{x\to a^+}\mathbb{I}_{a^+}^{\alpha,\sigma}u(x)=0.$$

□

Considering the Riemann-Liouville and Caputo tempered fractional derivative we have the following result:

Theorem 2.5. *Let $\alpha\in(0,1)$, $\sigma>0$ and $u\in AC[a,b]$. Then*

$$\mathbb{D}_{a^+}^{\alpha,\sigma}u(x)=\frac{u(a)}{\Gamma(1-\alpha)}(x-a)^{-\alpha}e^{-\sigma(x-a)}+{}^C\mathbb{D}_{a^+}^{\alpha,\sigma}u(x),\tag{2.26}$$

and

$$\mathbb{D}_{b^-}^{\alpha,\sigma}u(x)=\frac{u(b)}{\Gamma(1-\alpha)}(b-x)^{-\alpha}e^{-\sigma(b-x)}+{}^C\mathbb{D}_{b^-}^{\alpha,\sigma}u(x).\tag{2.27}$$

The following result are the fundamental theorem of calculus for Caputo tempered fractional derivative.

Theorem 2.6. *For $\alpha\in(0,1)$, $\sigma>0$ and $u\in AC[a,b]$, we have*

1.
$$
{}^{C}\mathbb{D}^{\alpha,\sigma}_{a^+}\cdot\mathbb{I}^{\alpha,\sigma}_{a^+}u(x)=u(x),
$$
$$
{}^{C}\mathbb{D}^{\alpha,\sigma}_{b^-}\cdot\mathbb{I}^{\alpha,\sigma}_{b^-}u(x)=u(x).
$$

2.
$$
\mathbb{I}^{\alpha,\sigma}_{a^+}\cdot{}^{C}\mathbb{D}^{\alpha,\sigma}_{a^+}u(x)=u(x)-e^{-\sigma(x-a)}u(a),
$$
$$
\mathbb{I}^{\alpha,\sigma}_{b^-}\cdot{}^{C}\mathbb{D}^{\alpha,\sigma}_{b^-}u(x)=u(x)-e^{-\sigma(x-b)}u(b).
$$

In our next result we are dealing with the integration by parts theorem for Riemann-Liouville tempered fractional derivative.

Theorem 2.7. *Let $\alpha\in(0,1)$, $\sigma>0$ and $u,v\in AC[a,b]$, then*

$$
\int_a^b u(x)\mathbb{D}^{\alpha,\sigma}_{b^-}v(x)dx
$$
$$
=\lim_{x\to a^+}u(x)\mathbb{I}^{1-\alpha,\sigma}_{b^-}v(x)-\lim_{x\to b^-}u(x)\mathbb{I}^{1-\alpha,\sigma}_{b^-}v(x)+\int_a^b {}^{C}\mathbb{D}^{\alpha,\sigma}_{a^+}u(x)v(x)dx. \tag{2.28}
$$

Proof. Note that, as in Lemma 2.1 we can show that, if $\varphi\in AC[a,b]$, then ${}_aI^{\alpha}_x\varphi, {}_xI^{\alpha}_b\varphi\in AC[a,b]$. Hence, ${}_xI^{1-\alpha}_b e^{-\sigma\cdot}v\in AC[a,b]$ and then

$$
{}_xD^{\alpha}_b e^{-\sigma x}v(x)=-\frac{d}{dx}{}_xI^{1-\alpha}_b e^{-\sigma x}v(x)\in L^1[a,b].
$$

Consequently

$$
\int_a^b|\mathbb{D}^{\alpha,\sigma}_{b^-}v(x)|dx=\int_a^b|e^{\sigma x}{}_xD^{\alpha}_b e^{-\sigma x}v(x)|dx\le e^{\sigma b}\int_a^b|{}_xD^{\alpha}_b e^{-\sigma x}v(x)|dx<\infty.
$$

By other side, as $u\in AC[a,b]$, then $u\in C[a,b]$. Therefore

$$
\int_a^b u(x)\mathbb{D}^{\alpha,\sigma}_{b^-}v(x)dx\le\left|\int_a^b u(x)\mathbb{D}^{\alpha,\sigma}_{b^-}v(x)dx\right|\le\|u\|_\infty\int_a^b|\mathbb{D}^{\alpha,\sigma}_{b^-}v(x)|dx<\infty.
$$

Now we are going to show (2.28). In fact, by using integration by parts and Theorem 2.2 we get

$$
\begin{aligned}
\int_a^b u(x)\mathbb{D}^{\alpha,\sigma}_{b^-}v(x)dx&=\int_a^b u(x)e^{\sigma x}{}_xD^{\alpha}_b e^{-\sigma x}v(x)dx\\
&=-\int_a^b u(x)e^{\sigma x}\frac{d}{dx}{}_xI^{1-\alpha}_b e^{-\sigma x}v(x)dx\\
&=-\left[u(x)\mathbb{I}^{1-\alpha,\sigma}_{b^-}v(x)\Big|_a^b-\int_a^b[e^{\sigma x}u(x)]'{}_xI^{1-\alpha}_b e^{-\sigma x}v(x)dx\right]\\
&=\lim_{x\to a^+}u(x)\mathbb{I}^{1-\alpha,\sigma}_{b^-}v(x)-\lim_{x\to b^-}u(x)\mathbb{I}^{1-\alpha,\sigma}_{b^-}v(x)\\
&\quad+\int_a^b {}^{C}\mathbb{D}^{\alpha,\sigma}_{a^+}u(x)v(x)dx.
\end{aligned}
$$

□

3. Tempered fractional space of Sobolev type

In this section we introduce the tempered fractional space of Sobolev type $\mathbb{H}_0^{\alpha,\sigma}(a,b)$ defined as

$$\mathbb{H}_0^{\alpha,\sigma}(a,b)=\overline{C_0^\infty(a,b)}^{\|\cdot\|_{\alpha,\sigma}},$$

where

$$\|u\|_{\alpha,\sigma}=\left(\int_a^b|u(x)|^2dx+\int_a^b|{}^C\mathbb{D}_{a^+}^{\alpha,\sigma}u(x)|^2dx\right)^{1/2},\tag{3.1}$$

and endowed with the inner product

$$\langle u,v\rangle_{\alpha,\sigma}=\int_a^b u\ vdx+\int_a^b{}^C\mathbb{D}_{a^+}^{\alpha,\sigma}u\ {}^C\mathbb{D}_{a^+}^{\alpha,\sigma}vdx.\tag{3.2}$$

$\mathbb{H}_0^{\alpha,\sigma}(a,b)$ is a Hilbert space. In fact. Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{H}_0^{\alpha,\sigma}(a,b)$. Then $(u_n)_{n\in\mathbb{N}}$ and $({}^C\mathbb{D}_{a^+}^{\alpha,\sigma}u_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $L^2(a,b)$ and there are $u,v\in L^2(a,b)$ such that

$$u_n\to u\quad\text{and}\quad {}^C\mathbb{D}_{a^+}^{\alpha,\sigma}u_n\to v\ \text{ in } L^2(a,b)\text{ as } n\to\infty.$$

Let $\varphi\in C_0^\infty(a,b)$, then by Theorem 2.2 and definition of Caputo tempered fractional derivative we have

$$\begin{aligned}\int_a^b{}^C\mathbb{D}_{a^+}^{\alpha,\sigma}u_n(x)\varphi(x)dx&=\int_a^b\left(e^{-\sigma x}\cdot{}_a^CD_x^\alpha e^{\sigma x}u_n(x)\right)\varphi(x)dx\\&=\int_a^b{}_aI_x^{1-\alpha}(e^{\sigma x}u_n(x))'e^{-\sigma x}\varphi(x)dx\\&=\int_a^b(e^{\sigma x}u_n(x))'{}_xI_b^{1-\alpha}e^{-\sigma x}\varphi(x)dx\\&=e^{\sigma x}u_n(x){}_xI_b^{1-\alpha}e^{-\sigma x}\varphi(x)\Big|_a^b\\&\quad-\int_a^b e^{\sigma x}u_n(x)\frac{d}{dx}{}_xI_b^{1-\alpha}e^{-\sigma x}\varphi(x)dx\\&=\int_a^b u_n(x)\mathbb{D}_{b^-}^{\alpha,\sigma}\varphi(x)dx\\&\to\int_a^b u(x)\mathbb{D}_{b^-}^{\alpha,\sigma}\varphi(x)dx\\&=\int_a^b{}^C\mathbb{D}_{a^+}^{\alpha,\sigma}u(x)\varphi(x)dx,\end{aligned}$$

as $n\to\infty$. Then $u\in\mathbb{H}_0^{\alpha,\sigma}(a,b)$, ${}^C\mathbb{D}_{a^+}^{\alpha,\sigma}u=v$ and

$$\|u_n-u\|_{\alpha,\sigma}\to 0\ \text{ as }\ n\to\infty.$$

Considering this function space, we have the following properties.

Lemma 3.1. *For any $u\in\mathbb{H}_0^{\alpha,\sigma}(a,b)$, we have*

$$\mathbb{I}_{a^+}^{\alpha,\sigma}\cdot{}^C\mathbb{D}_{a^+}^{\alpha,\sigma}u(x)=u(x),\quad a.e.\ in\ (a,b).$$

Proof. By definition, there exists $(\varphi_n)_{n\in\mathbb{N}} \subset C_0^\infty(a,b)$ such that

$$\lim_{n\to\infty} \|u-\varphi_n\|_{\alpha,\sigma} = 0.$$

Hence

$$\lim_{n\to\infty} \|u-\varphi_n\|_{L^2(a,b)} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|{}^C\mathbb{D}^{\alpha,\sigma}_{a^+}(u-\varphi_n)\|_{L^2(a,b)} = 0. \tag{3.3}$$

Fatou's Lemma yields that

$$\begin{aligned} &\int_a^b |u(x)|^2dx \le \liminf_{n\to\infty} \int_a^b |\varphi_n(x)|^2 dx < \infty \quad \text{and} \\ &\int_a^b |{}^C\mathbb{D}^{\alpha,\sigma}_{a^+}u(x)|^2dx \le \liminf_{n\to\infty} \int_a^b |{}^C\mathbb{D}^{\alpha,\sigma}_{a^+}\varphi_n(x)|^2dx < +\infty. \end{aligned} \tag{3.4}$$

By other side

$$\begin{aligned} &\|\mathbb{I}^{\alpha,\sigma}_{a^+}\cdot {}^C\mathbb{D}^{\alpha,\sigma}_{a^+}u-u\|_{L^2(a,b)} \\ \le &\|\mathbb{I}^{\alpha,\sigma}_{a^+}\cdot {}^C\mathbb{D}^{\alpha,\sigma}_{a^+}(u-\varphi_n)\|_{L^2(a,b)} + \|\mathbb{I}^{\alpha,\sigma}_{a^+}\cdot {}^C\mathbb{D}^{\alpha,\sigma}_{a^+}\varphi_n-\varphi_n\|_{L^2(a,b)} + \|\varphi_n-u\|_{L^2(a,b)}. \end{aligned} \tag{3.5}$$

Since $\varphi_n(a)=0$, Theorem 2.6 implies that

$$\mathbb{I}^{\alpha,\sigma}_{a^+}\cdot {}^C\mathbb{D}^{\alpha,\sigma}_{a^+}\varphi_n(x) = \varphi_n(x),$$

next

$$\|\mathbb{I}^{\alpha,\sigma}_{a^+}\cdot {}^C\mathbb{D}^{\alpha,\sigma}_{a^+}\varphi_n-\varphi_n\|_{L^2(a,b)} = 0 \ \ \forall n\in\mathbb{N}. \tag{3.6}$$

By other side, Theorem 2.1 yields that

$$\|\mathbb{I}^{\alpha,\sigma}_{a^+}\cdot {}^C\mathbb{D}^{\alpha,\sigma}_{a^+}(u-\varphi_n)\|_{L^2(a,b)} \le \frac{\gamma(\alpha,\sigma(b-a))}{\sigma^\alpha\Gamma(\alpha)}\|{}^C\mathbb{D}^{\alpha,\sigma}_{a^+}(u-\varphi_n)\|_{L^2(a,b)}. \tag{3.7}$$

Therefore, by (3.3), (3.6), (3.7) and (3.5) we obtain that

$$\|\mathbb{I}^{\alpha,\sigma}_{a^+}\cdot {}^C\mathbb{D}^{\alpha,\sigma}_{a^+}u-u\|_{L^2(a,b)} = 0,$$

which implies the desired result. □

As an immediate consequence of this result we have the following version of Poincaré inequality:

Corollary 3.1. *Let $\alpha\in(0,1)$, $\sigma>0$. Then,*

$$\|u\|_{L^2(a,b)} \le \frac{\gamma(\alpha,\sigma(b-a))}{\sigma^\alpha\Gamma(\alpha)}\|{}^C\mathbb{D}^{\alpha,\sigma}_{a^+}u\|_{L^2(a,b)}, \tag{3.8}$$

for any $u\in\mathbb{H}^{\alpha,\sigma}_0(a,b)$.

Remark 3.1. By Corollary 3.1 we can endowed $\mathbb{H}^{\alpha,\sigma}_0(a,b)$ with the norm

$$\|u\| = \left(\int_a^b |{}^C\mathbb{D}^{\alpha,\sigma}_{a^+}u(x)|^2dx\right)^{1/2},$$

which is equivalent with $\|\cdot\|_{\alpha,\sigma}$. In fact, note that

$$\|u\| \leq \|u\|_{\alpha,\sigma}.$$

By other side, Corollary 3.1 yields that

$$\begin{aligned}\|u\|_{\alpha,\sigma}^2 &= \int_a^b |u(x)|^2 dx + \int_a^b |{}^C\mathbb{D}_{a^+}^{\alpha,\sigma} u(x)|^2 dx \\ &\leq \left(\left[\frac{\gamma(\alpha, \sigma(b-a))}{\sigma^\alpha \Gamma(\alpha)} \right]^2 + 1 \right) \int_a^b |{}^C\mathbb{D}_{a^+}^{\alpha,\sigma} u(x)|^2 dx.\end{aligned}$$

Therefore, we get the desired result.

In the following result we are able to show that $\mathbb{H}_0^{\alpha,\sigma}(a,b)$ is continuously embedded into $C(a,b)$, more precisely we have:

Theorem 3.1. *Let $\alpha \in (\frac{1}{2}, 1)$ and $\sigma > 0$, then $\mathbb{H}_0^{\alpha,\sigma}(a,b)$ is continuously embedded into $C(a,b)$.*

Proof. Let $u \in \mathbb{H}_0^{\alpha,\sigma}(a,b)$, Lemma 3.1 yields that $u, {}^C\mathbb{D}_{a^+}^{\alpha,\sigma} u \in L^2(a,b)$ and

$$u(x) = \mathbb{I}_{a^+}^{\alpha,\sigma} \cdot {}^C\mathbb{D}_{a^+}^{\alpha,\sigma} u(x) \ \text{ a.e. } x \in (a,b).$$

Hence, by Theorem 2.4 we obtain

$$\begin{aligned}\|u\|_\infty &= \|\mathbb{I}_{a^+}^{\alpha,\sigma} \cdot {}^C\mathbb{D}_{a^+}^{\alpha,\sigma} u\|_\infty \\ &\leq \frac{1}{(2\sigma)^{\alpha-\frac{1}{2}}\Gamma(\alpha)} [\gamma(2\alpha-1, 2\sigma(b-a))]^{1/2} \|{}^C\mathbb{D}_{a^+}^{\alpha,\sigma} u\|_{L^2(a,b)} \\ &= \frac{1}{(2\sigma)^{\alpha-\frac{1}{2}}\Gamma(\alpha)} [\gamma(2\alpha-1, 2\sigma(b-a))]^{1/2} \|u\|,\end{aligned}$$

which implies the desired result. □

The following compactness result will be crucial for our purpose.

Theorem 3.2. *Let $\alpha \in (\frac{1}{2}, 1)$ and $\sigma > 0$. Then the embedding*

$$\mathbb{H}_0^{\alpha,\sigma}(a,b) \hookrightarrow C\overline{(a,b)}$$

is compact.

Proof. Let B be a bounded subset of $\mathbb{H}_0^{\alpha,\sigma}(a,b)$, then we need to show that B is relative compact in $C\overline{(a,b)}$. By virtue of the Arzelá-Ascoli theorem, the conclusion will be achieved by proving that B is equibounded and equicontinuous in $C\overline{(a,b)}$. In fact, by the previous theorem $\mathbb{H}_0^{\alpha,\sigma}(a,b)$ is continuously embedded in $C\overline{(a,b)}$, and

$$\|u\|_\infty \leq \frac{[\gamma(2\alpha-1, 2\sigma(b-a))]^{1/2}}{(2\sigma)^{\alpha-\frac{1}{2}}\Gamma(\alpha)} \|u\|, \ \text{ for every } u \in B. \tag{3.9}$$

Hence, the set B is equibounded in $C\overline{(a,b)}$. Moreover, by Theorem 2.4 and Lemma 3.1 there is a positive constant $\mathcal{K}$ such that

$$\begin{aligned}|u(x) - u(y)| &= |\mathbb{I}_{a^+}^{\alpha,\sigma} \cdot {}^C\mathbb{D}_{a^+}^{\alpha,\sigma} u(x) - \mathbb{I}_{a^+}^{\alpha,\sigma} \cdot {}^C\mathbb{D}_{a^+}^{\alpha,\sigma} u(y)| \\ &\leq \mathcal{K} \|{}^C\mathbb{D}_{a^+}^{\alpha,\sigma} u\|_{L^2(a,b)} |x-y|^{\alpha-\frac{1}{2}},\end{aligned}$$

which implies the equicontinuity of B. This completes the proof of Theorem 3.2. □

Remark 3.2. If $\alpha \in (\frac{1}{2}, 1)$ and $\sigma > 0$, then for every $u \in \mathbb{H}_0^{\alpha,\sigma}(a,b)$, there exists $(\phi_n)_{n\in\mathbb{N}} \subset C_0^\infty(a,b)$ such that

$$\lim_{n\to\infty} \|u - \phi_n\| = 0.$$

Combining this limit with Theorem 3.1 we arrive to

$$\begin{aligned}
0 &\leq |u(a)| \\
&= |u(a) - \phi_n(a)| \\
&\leq \frac{1}{(2\sigma)^{\alpha-\frac{1}{2}}\Gamma(\alpha)}[\gamma(2\alpha-1, 2\sigma(b-a))]^{1/2}\|u-\phi_n\| \\
&\to 0 \ \text{ as } \ n \to \infty.
\end{aligned}$$

Consequently $u(a) = 0$. Similarly we can obtain that $u(b) = 0$.

By other side, Lemma 3.1 yields that

$$^{C}\mathbb{D}_{a^+}^{\alpha,\sigma} u \in L^2(a,b).$$

Therefore, $\mathbb{H}_0^{\alpha,\sigma}(a,b)$ can be rewritten as

$$\mathbb{H}_0^{\alpha,\sigma}(a,b) = \{u \in L^2(a,b) : \ ^{C}\mathbb{D}_{a^+}^{\alpha,\sigma} u \in L^2(a,b) \ \text{ and } \ u(a) = u(b) = 0\}.$$

4. Proof of Theorem 1.1

In this section we are going to give the prove of Theorem 1.1. In this direction, in what follows we consider $a = 0$, $b = T$ and

$$0 = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = T.$$

Hence, for $u \in \mathbb{H}_0^{\alpha,\sigma}(0,T)$ next

$$\begin{aligned}
&\int_0^T {}^{C}\mathbb{D}_{0^+}^{\alpha,\sigma} u(x) {}^{C}\mathbb{D}_{0^+}^{\alpha,\sigma} \varphi(x) dx \\
&= \int_0^T {}^{C}\mathbb{D}_{0^+}^{\alpha,\sigma} u(x) \left(e^{-\sigma x} \cdot {}_0^C D_x^\alpha e^{\sigma x}\varphi(x)\right) dx \\
&= \int_0^T e^{-\sigma x} \cdot {}^{C}\mathbb{D}_{0^+}^{\alpha,\sigma} u(x) {}_0 I_x^{1-\alpha} \left(e^{\sigma x}\varphi(x)\right)' dx \\
&= \int_0^T {}_x I_T^{1-\alpha} \left(e^{-\sigma x} \cdot {}^{C}\mathbb{D}_{0^+}^{\alpha,\sigma} u(x)\right) \left(e^{\sigma x}\varphi(x)\right)' dx \\
&= \sum_{j=0}^n \int_{x_j}^{x_{j+1}} {}_x I_T^{1-\alpha} \left(e^{-\sigma x} \cdot {}^{C}\mathbb{D}_{0^+}^{\alpha,\sigma} u(x)\right) \left(e^{\sigma x}\varphi(x)\right)' dx \\
&= \sum_{j=0}^n \left(\lim_{x\to x_{j+1}^-} \mathbb{I}_{T^-}^{1-\alpha,\sigma} \cdot {}^{C}\mathbb{D}_{0^+}^{\alpha,\sigma} u(x)\varphi(x) - \lim_{x\to x_j^+} \mathbb{I}_{T^-}^{1-\alpha,\sigma} \cdot {}^{C}\mathbb{D}_{0^+}^{\alpha,\sigma} u(x)\varphi(x) \right.
\end{aligned}$$

$$+\int_{x_j}^{x_{j+1}} \mathbb{D}_{T^-}^{\alpha,\sigma}\left({}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u(x)\right)\varphi(x)dx\Bigg). \tag{4.1}$$

Note that

$$\begin{aligned}
&\sum_{j=0}^{n}\left(\lim_{x\to x_{j+1}^-}\mathbb{I}_{T^-}^{1-\alpha,\sigma}\cdot {}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u(x)\varphi(x)-\lim_{x\to x_j^+}\mathbb{I}_{T^-}^{1-\alpha,\sigma}\cdot {}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u(x)\varphi(x)\right)\\
=&\lim_{x\to x_{n+1}^-}\varphi(x)\mathbb{I}_{T^-}^{1-\alpha,\sigma}\cdot {}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u(x)-\lim_{x\to x_n^+}\varphi(x)\mathbb{I}_{T^-}^{1-\alpha,\sigma}\cdot {}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u(x)\\
&+\lim_{x\to x_1^-}\varphi(x)\mathbb{I}_{T^-}^{1-\alpha,\sigma}\cdot {}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u(x)-\lim_{x\to x_0^+}\varphi(x)\mathbb{I}_{T^-}^{1-\alpha,\sigma}\cdot {}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u(x)\\
&+\sum_{j=1}^{n-1}\left(\lim_{x\to x_{j+1}^-}\mathbb{I}_{T^-}^{1-\alpha,\sigma}\cdot {}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u(x)\varphi(x)-\lim_{x\to x_j^+}\mathbb{I}_{T^-}^{1-\alpha,\sigma}\cdot {}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u(x)\varphi(x)\right)\\
=&-\lim_{x\to x_n^+}\varphi(x)\mathbb{I}_{T^-}^{1-\alpha,\sigma}\cdot {}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u(x)+\lim_{x\to x_1^-}\varphi(x)\mathbb{I}_{T^-}^{1-\alpha,\sigma}\cdot {}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u(x)\\
&+\sum_{j=1}^{n-1}\left(\lim_{x\to x_{j+1}^-}\mathbb{I}_{T^-}^{1-\alpha,\sigma}\cdot {}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u(x)\varphi(x)-\lim_{x\to x_j^+}\mathbb{I}_{T^-}^{1-\alpha,\sigma}\cdot {}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u(x)\varphi(x)\right)\\
=&-\sum_{j=1}^{n}\left(\lim_{x\to x_j^+}\mathbb{I}_{T^-}^{1-\alpha,\sigma}\cdot {}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u(x)\varphi(x)-\lim_{x\to x_j^-}\mathbb{I}_{T^-}^{1-\alpha,\sigma}\cdot {}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u(x)\varphi(x)\right).
\end{aligned}$$

Combining this equality with (4.1) we derive

$$\begin{aligned}
&\int_0^T {}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u(x){}^C\mathbb{D}_{0^+}^{\alpha,\sigma}\varphi(x)dx\\
=&\int_0^T \mathbb{D}_{T^-}^{\alpha,\sigma}\left({}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u(x)\right)\varphi(x)dx-\sum_{j=1}^{n}\Delta\left(\mathbb{I}_{T^-}^{1-\alpha,\sigma}\cdot {}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u\right)(x_j)\varphi(x_j).
\end{aligned}$$

Now we introduce the notion of solutions that we consider in this paper.

Definition 4.1. A function

$$u\in\left\{u\in AC[0,T]:\int_{x_j}^{x_{j+1}}|{}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u(x)|^2dx<\infty,\ \ j=0,\cdots,n\right\}$$

is said to be a classical solution of problem (1.1), if u satisfies the equation a.e. on $(0,T)\setminus\{x_1,x_2,\cdots,x_n\}$, the limits

$$\lim_{x\to x_j^+}\mathbb{I}_{T^-}^{1-\alpha,\sigma}\cdot {}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u(x)\quad\text{and}\quad\lim_{x\to x_j^-}\mathbb{I}_{T^-}^{1-\alpha,\sigma}\cdot {}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u(x)$$

exist and satisfy the impulsive condition $\Delta\left(\mathbb{I}_{T^-}^{1-\alpha,\sigma}\cdot {}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u\right)(x_j)=I_j(u(x_j))$ and boundary condition $u(0)=u(T)=0$.

Definition 4.2. A function $u\in\mathbb{H}_0^{\alpha,\sigma}(0,T)$ is said to be a weak solution of problem (1.1), if for every $\varphi\in\mathbb{H}_0^{\alpha,\sigma}(0,T)$, the following identity holds

$$\int_0^T {}^C\mathbb{D}_{0^+}^{\alpha,\sigma}u(x){}^C\mathbb{D}_{0^+}^{\alpha,\sigma}\varphi(x)dx+\sum_{j=1}^{n}I_j(u(x_j))\varphi(x_j)=\int_0^T f(x,u(x))\varphi(x)dx.$$

As in [7, Lemma 2.1] we can show the following result.

Lemma 4.1. *The function $u \in \mathbb{H}_0^{\alpha,\sigma}(0,T)$ is a weak solution of (1.1), if and only if u is a classical solution of (1.1).*

Note that the problem (1.1) has a variational structure and its solution are critical points of a suitable functional I defined on the fractional space $\mathbb{H}_0^{\alpha,\sigma}(0,T)$ as follows

$$I(u) = \frac{1}{2}\|u\|^2 - \int_0^T F(x,u(x))dx + \sum_{j=1}^n \int_0^{u(x_j)} I_j(s)ds. \tag{4.2}$$

To prove our main result we need the following lemma:

Lemma 4.2. ([39, Theorem 9.12]). *Let X be an infinite dimensional real Banach space and $\Phi \in C^1(X,\mathbb{R})$ be even, satisfying the (PS)-condition and $\Phi(0) = 0$. If $X = X_1 \oplus X_2$ with $k := \dim X_2 < \infty$ and Φ satisfies the following conditions:*
(i) There exist constants $\rho, \sigma > 0$ such that $\Phi|_{\partial B_\rho \cap X_1} \geq \sigma$;
(ii) For each finite dimensional subspace $V \subset X$, there is an $R = R(V)$ such that $\Phi(u) \leq 0$ for every $u \in V$ with $\|u\| > R$.
Then Φ has an unbounded sequence of critical values.

Proof of Theorem 1.1. We shall apply Lemma 4.2 to I. We know that $\mathbb{H}_0^{\alpha,\sigma}(0,T)$ is a Banach space and $I \in C^1(\mathbb{H}_0^{\alpha,\sigma}(0,T),\mathbb{R})$. We can easily that, $I(0) = 0$ and I is even. Next, we prove that I satisfies the (PS)-condition. Assume that $\{u_n\}$ is a (PS)-sequence of I such that $\{I(u_n)\}$ is bounded and $I'(u_n) \longrightarrow 0$, as $n \longrightarrow \infty$.

By (I_2), we have

$$\frac{\gamma_j}{\tau} \leq \frac{I_j(\tau)}{\int_0^\tau I_j(s)ds}, \quad \forall\, \tau \geq \vartheta,$$
$$\frac{\gamma_j}{\tau} \geq \frac{I_j(\tau)}{\int_0^\tau I_j(s)ds}, \quad \forall\, \tau \leq -\vartheta.$$

By integrating the above relations respect to τ on $[\vartheta,\tau]$ and $[\tau,-\vartheta]$, respectively, one can get

$$\gamma_j \ln\frac{\tau}{\vartheta} \leq \ln\frac{\int_0^\tau I_j(s)ds}{\int_0^\vartheta I_j(s)ds}, \quad \forall\, \tau \geq \vartheta,$$
$$\gamma_j \ln\frac{\vartheta}{-\tau} \geq \ln\frac{\int_0^{-\vartheta} I_j(s)ds}{\int_0^\tau I_j(s)ds}, \quad \forall\, \tau \leq -\vartheta.$$

Consequently,

$$\int_0^\tau I_j(s)ds \geq \left(\frac{\tau}{\vartheta}\right)^{\gamma_j} \int_0^\vartheta I_j(s)ds, \quad \forall\, \tau \geq \vartheta,$$
$$\int_0^\tau I_j(s)ds \geq \left(\frac{-\tau}{\vartheta}\right)^{\gamma_j} \int_0^{-\vartheta} I_j(s)ds, \quad \forall\, \tau \leq -\vartheta.$$

So there exist constants $m_j = m_j(\gamma_j,\vartheta) > 0$ such that $\int_0^\tau I_j(s)ds \geq m_j|\tau|^{\gamma_j}$, for all $|\tau| \geq \vartheta$. By the continuity of $\int_0^\tau I_j(s)ds$, there exist positive constants K_j, such that

$$\int_0^\tau I_j(s)ds \geq -K_j \geq m_j|\tau|^{\gamma_j} - m_j\vartheta^{\gamma_j} - K_j, \quad \forall\, |\tau| \leq \vartheta.$$

Therefore, we get

$$\int_0^{\tau} I_j(s)ds \geq m_j|\tau|^{\gamma_j} - \widehat{m}_j, \ \ \forall\, \tau \in \mathbb{R}, \tag{4.3}$$

where $\widehat{m}_j = m_j\vartheta^{\gamma_j} + K_j$.

Suppose that $\{I(u_n)\}$ be a bounded sequence and $I'(u_n) \longrightarrow 0$ as $n \to \infty$. Hence

$$\begin{aligned} I(u_n) - \frac{1}{\gamma}\langle I'(u_n), u_n\rangle = & \left(\frac{1}{2} - \frac{1}{\gamma}\right)\|u_n\|^2 \\ & + \frac{1}{\gamma}\int_0^T f(x, u_n(x))u_n(x)dx - \int_0^T F(x, u_n(x))u_n(x)dx \\ & + \frac{1}{\gamma}\sum_{j=1}^{n} I_j(u_n(x_j))u_n(x_j) - \sum_{j=1}^{n}\int_0^{u_n(x_j)} I_j(s)ds. \end{aligned}$$

So, by straightforward calculation, for some positive constant $C_0 > 0$, we obtain

$$I(u_n) \geq \left(\frac{1}{2} - \frac{1}{\gamma}\right)\|u_n\|^2 + \frac{1}{\gamma}\|I'(u_n)\|_{(\mathbb{H}_0^{\alpha,\sigma}(0,T)^*)}\|u_n\| + C_0.$$

Since $\{I(u_n)\}$ is bounded then the sequence $\{u_n\} \subset \mathbb{H}_0^{\alpha,\sigma}(0,T)$ is bounded. Since $\mathbb{H}_0^{\alpha,\sigma}(0,T)$ is a reflexive Banach space and so by passing to a subsequence (for simplicity denoted again by $\{u_n\}$) if necessary, by Theorem 3.2, we may assume that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } \mathbb{H}_0^{\alpha,\sigma}(0,T), \\ u_n \longrightarrow u, & \text{strongly in } C\overline{(0,T)}. \end{cases} \tag{4.4}$$

So, we get

$$\begin{aligned} \|u_n - u\|^2 \leq & \langle I'(u_n) - I'(u), u_n - u\rangle \\ & - \int_0^T \Big(f(x, u_n(x)) - f(x, u(x)), u_n(x) - u(x)\Big)dx \\ & + \sum_{j=1}^{n}\Big[I_j(u_n(x_j)) - I_j(u(x_j))\Big](u_n(x_j) - u(x_j)). \end{aligned} \tag{4.5}$$

For any $j = 1, \ldots, n$, we have that $u_n(x_j) \longrightarrow u(x_j)$ as $n \longrightarrow \infty$. Thus it follows from the continuity of all I_i that

$$\sum_{j=1}^{n}\Big[I_j(u_n(x_j)) - I_j(u(x_j))\Big](u_n(x_j) - u(x_j)) \longrightarrow 0 \quad \text{as} \ \ n \longrightarrow \infty. \tag{4.6}$$

By (4.4), we have

$$\int_0^T (f(x, u_n(x)) - f(x, u(x)))(u_n(x) - u(x))dx \longrightarrow 0, \ \ \text{as} \ \ n \longrightarrow \infty. \tag{4.7}$$

Since $I'(u_n) \longrightarrow 0$, then by using (4.5), (4.6) and (4.7), we have that $\|u_n - u\| \longrightarrow 0$, which means that $I(u)$ satisfies the (PS)-condition.

On the other hand, by (F_1), for any $\varepsilon > 0$, there exists $C_0(\varepsilon)$ such that

$$F(x,u) \leq \varepsilon |u|^2 + C_0(\varepsilon)|u|^{\eta+1}, \ \ \forall\ (x,u) \in [0,T]\times\mathbb{R}. \tag{4.8}$$

In view of (3.9), (4.3) and (4.8), we get

$$\begin{aligned}
c &\geq I(u)\\
&= \frac{1}{2}\|u\|^2 + \sum_{j=1}^{n}\int_0^{u(x_j)} I_j(s)ds - \int_0^T F(x,u(x))dx\\
&\geq \frac{1}{2}\|u\|^2 + \sum_{j=1}^{n}(m_j|u(x_j)|^{\gamma_j} - \widehat{m}_j) - \int_0^T (\varepsilon|u|^2 + C_0(\varepsilon)|u|^{\eta+1})dx\\
&\geq \frac{1}{2}\|u\|^2 - \sum_{j=1}^{n}(m_j\|u\|_\infty^{\gamma_j} - \widehat{m}_j) - \varepsilon T\|u\|_\infty^2 - C_0(\varepsilon)T\|u\|_\infty^{\eta+1})dx\\
&\geq \left(\frac{1}{2} - \varepsilon T\left(\frac{[\gamma(2\alpha-1,2\sigma(b-a))]^{1/2}}{(2\sigma)^{\alpha-\frac{1}{2}}\Gamma(\alpha)}\right)^2\right)\|u\|^2\\
&\quad -\sum_{j=1}^{n} m_j\left(\frac{[\gamma(2\alpha-1,2\sigma(b-a))]^{1/2}}{(2\sigma)^{\alpha-\frac{1}{2}}\Gamma(\alpha)}\right)^{\gamma_j}\|u\|^{\gamma_j} + \sum_{j=1}^{n}\widehat{m}_j\\
&\quad -C_0(\varepsilon)T\left(\frac{[\gamma(2\alpha-1,2\sigma(b-a))]^{1/2}}{(2\sigma)^{\alpha-\frac{1}{2}}\Gamma(\alpha)}\right)^{\eta+1}\|u\|^{\eta+1}dx.
\end{aligned} \tag{4.9}$$

By (4.9) and (I_2), one can get

$$\begin{aligned}
I(u) &= \frac{1}{2}\|u\|^2 - \int_0^T F(x,u(x))dx + \sum_{j=1}^{n}\int_0^{u(x_j)} I_j(s)ds\\
&\geq \frac{1}{2}\|u\|^2 - \int_0^T F(x,u(x))dx\\
&\geq \left(\frac{1}{2} - \varepsilon T\left(\frac{[\gamma(2\alpha-1,2\sigma(b-a))]^{1/2}}{(2\sigma)^{\alpha-\frac{1}{2}}\Gamma(\alpha)}\right)^2\right)\|u\|^2\\
&\quad -C_0(\varepsilon)T\left(\frac{[\gamma(2\alpha-1,2\sigma(b-a))]^{1/2}}{(2\sigma)^{\alpha-\frac{1}{2}}\Gamma(\alpha)}\right)^{\eta+1}\|u\|^{\eta+1}.
\end{aligned} \tag{4.10}$$

Let

$$\begin{aligned}
\varepsilon &= \frac{1}{4T}\left(\frac{[\gamma(2\alpha-1,2\sigma(b-a))]^{1/2}}{(2\sigma)^{\alpha-\frac{1}{2}}\Gamma(\alpha)}\right)^{-2},\\
\rho &= \left(8C_0(\varepsilon)T\left(\frac{[\gamma(2\alpha-1,2\sigma(b-a))]^{1/2}}{(2\sigma)^{\alpha-\frac{1}{2}}\Gamma(\alpha)}\right)^{\eta+1}\right)^{\frac{1}{1-\eta}},
\end{aligned}$$

and $B_\rho = \{u \in E^\alpha : \|u\|_\alpha < \rho\}$. Therefore,

$$I|_{\partial B_\rho} \geq \frac{1}{8}\rho^2 := \sigma > 0. \tag{4.11}$$

$\mathbb{H}_0^{\alpha,\sigma}(0,T)$ has a countable orthogonal basis $\{e_i\}$. Set $Y_k = span\{e_1, e_2, \dots e_k\}$ and $Z_k = Y_k^{\perp}$. Then $\mathbb{H}_0^{\alpha,\sigma}(0,T) = Y_k \oplus Z_k$. Hence,

$$I|_{\partial B_\rho \cap Z_k} \geq \sigma > 0. \tag{4.12}$$

Furthermore, for any finite dimensional subspace $V \subset \mathbb{H}_0^{\alpha,\sigma}(0,T)$, there is a positive constant m such that $V \subset Y_m$. Since all norms in a finite dimensional space are equivalent, then there is a constant $\varrho > 0$ such that

$$\|u\|_\gamma \geq \varrho\|u\|, \quad \forall u \in Y_m. \tag{4.13}$$

By similar method in (4.3) and (F_2), we can get

$$F(x,u) \geq m|u|^\gamma - \widehat{m}, \ \ \forall\ (x,u) \in [0,T] \times \mathbb{R}. \tag{4.14}$$

From (I_1), for any $\varepsilon > 0$, there exists $C_1(\varepsilon)$ such that

$$\int_0^u I_j(s)ds \leq \varepsilon|u|^2 + C_1(\varepsilon)|u|^{\varpi+1}, \ \ \forall\ u \in \mathbb{R}. \tag{4.15}$$

In view of (3.9), (4.15) and (4.14) , we have

$$\begin{aligned}
I(u) &= \frac{1}{2}\|u\|^2 - \int_0^T F(x,u(x))dx + \sum_{j=1}^n \int_0^{u(x_j)} I_j(s)ds \\
&\leq \frac{1}{2}\|u\|^2 + \sum_{j=1}^n (\varepsilon|u(x_j)|^2 + C_1(\varepsilon)|u(x_j)|^{\varpi+1}) - m\int_0^T |u(x)|^\gamma dx + \widehat{m}T \\
&\leq \left(\frac{1}{2} + \varepsilon n\left(\frac{[\gamma(2\alpha-1, 2\sigma(b-a))]^{1/2}}{(2\sigma)^{\alpha-\frac{1}{2}}\Gamma(\alpha)}\right)^2\right)\|u\|^2 \\
&\quad + C_1(\varepsilon)n\left(\frac{[\gamma(2\alpha-1, 2\sigma(b-a))]^{1/2}}{(2\sigma)^{\alpha-\frac{1}{2}}\Gamma(\alpha)}\right)^{\varpi+1}\|u\|^{\varpi+1} \\
&\quad - m\varrho^\gamma\|u\|^\gamma + \widehat{m}T,
\end{aligned} \tag{4.16}$$

for all $u \in Y_m$. Since $2 < \varpi + 1 < \gamma$ then there is a large $r_1 > 0$ such that $I < 0$ on $V \backslash B_{r_1}$. Consequently, there is a point $e \in \mathbb{H}_0^{\alpha,\sigma}(0,T)$ with $\|e\|_\alpha > \rho$ such that $I(e) < 0$. By Lemma 4.2, I possesses infinitely many critical points, i.e. the problem (1.1) has infinitely many weak solutions. □

Example 4.1. Let $\alpha = \frac{3}{4}$, $\sigma = 1$ and $n = T = 1$. Consider the boundary value problem of the fractional differential equation with impulsive effects

$$\begin{cases}
\mathbb{D}_{b^-}^{\frac{3}{4},1}({}^C\mathbb{D}_{a^+}^{\frac{3}{4},1}u(x)) = f(x,u), \ \ x \neq x_j \ \text{ a.e. } x \in (0,1), \\
u(0) = u(T) = 0, \\
\Delta\left(\mathbb{I}_{1^-}^{1-\alpha,\sigma\,C}\mathbb{D}_{0^+}^{\alpha,\sigma}u\right)(x_1) = I_1(u(x_1)).
\end{cases} \tag{4.17}$$

Let

$$F(x,u) = \begin{cases} (x+2)|u|^8 & |u| > 1, \\ (x+3)(|u|^4 - |u|^6) & |u| \leq 1, \end{cases}$$

and $I_1(t) = t^4$. we have that (F_2) holds with $\gamma = 7$ and $\vartheta_0 = 1$. By choosing $\eta = 9$, (F_1) holds. Also (I_1) and (I_2) hold by choosing $\varpi = 5$ and $\vartheta = 1$. Hence, Theorem yields that the problem (4.17) has infinitely many weak solutions.

Acknowledgements

The author warmly thanks the anonymous referees for their useful and nice comments on the paper.

Contribution. All authors contributed equally to this work.

Conflict of interest. This work does not have any conflicts of interest.

Data supporting. The manuscript has no associated data.

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