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ITERATIVE ALGORITHMS AND FIXED POINT THEOREMS FOR SET-VALUED *G*-CONTRACTIONS IN GRAPHICAL CONVEX METRIC SPACES*

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Abstract In this article, we present a series of fixed point results of the Ishikawa iterative algorithm and the SP iterative algorithm in graphical convex metric spaces. First, we introduce the Ishikawa sequence and the SP sequence in the above space. Furthermore, we study the existence and uniqueness of fixed points for set-valued G-contractions in graphical convex metric spaces. Finally, by providing an example, we demonstrate the hypotheses of the existence theorem of fixed points for set-valued G-contractions in G-complete graphical convex metric spaces are sufficient but not necessary.

Keywords Ishikawa iterative scheme, SP iterative scheme, fixed point theorems, set-valued mappings, graphical convex metric spaces.

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1. Introduction

The fixed point theory has always been a crucial branch of functional analysis, which occupies a crucial position in the field of mathematics. Besides, it is also an important component of nonlinear functional analysis, which is closely related to many branches of modern mathematics. Especially, it plays an important role in establishing the existence and uniqueness of solutions to various equations. The fixed point theory can be applied to many fields, namely variational inequalities, initial and boundary value problems of differential equations, financial mathematics, biology, computer science, physics and other fields. For example, in 2022, Zoto *et al.* [46] certified the existence and uniqueness of solutions for a class of nonlinear integral equations. In 2023, Younis *et al.* [38] demonstrated the existence of a solution to a fourth-order two-point boundary value problem for elastic beam deformations by using the fixed point results studied. In 2023, Mani *et al.* [17] used fixed point theory to solve the integral equation and fractional differential equation. The research on it not only helps to solve the theoretical problems, but also helps

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to solve some practical application problems. In recent years, the fixed point theory has been widely developed [30]. Among them, Banach's fixed point theorem is of great significance in solving many nonlinear analysis problems and other mathematical fields [5–7, 18, 19, 27]. Since Banach's fixed point theorem was presented, fixed point problems have attracted the attention of many scholars at home and abroad. Later, they also proposed a series of generalized concepts of contractive mappings and fixed point theorems on this basis. In [21], Nadler extended Banach's fixed point theorem to the case of set-valued mappings, which led many researchers to study fixed point problems of set-valued mappings [23–25, 28, 39].

Mann iterative algorithm, Ishikawa iterative algorithm, and Halpern iterative algorithm are the basic iterative algorithms for solving fixed point problems of nonexpansive mappings. In recent years, a great deal of researchers constructed many different algorithms to approach fixed points of different types of nonlinear mappings, such as SP iteration [26], Normal-S iteration [31], Agarwal iteration [1] and so on. On the one hand, iterative algorithms can be chosen to approach the fixed points for nonexpansive mappings. On the other hand, iterative algorithms can also be used to solve the existence of solutions of some equations related to fixed point problems [32]. Recently, there have been some new developments in iterative algorithms. In [3], a new high-order and efficient iterative technique was constructed to solve a system of nonlinear equations. Garodia and Uddin [13] constructed a new iterative algorithm, and showed that the convergent rate of the new iterative algorithm is faster than many existing iterative algorithms by giving an example. Furthermore, they used the proposed algorithm to find a solution of a delay differential equation and prove that the sequence generated by the proposed algorithm converges to this solution. Yuanheng Wang *et al.* [37] proposed a new hybrid relaxed iterative algorithm to solve the fixed point problem and the split feasibility problem involving demicontractive mappings.

In 2008, Jachymski [16] introduced the concepts of graphical metric spaces, popularizing some important fixed point theorems. Later, some researchers in the study of fixed point theorems combined with graph theory [8,15,20,40–45]. In recent work, there have been some new developments in the combination of fixed point theory and graph theory. For example, Ahmad, Younis and Abdou [2] developed a new space—graphical bipolar b-metric space. Monica-felicia, Liliana and Gabriela [4] gave some existence and stability results for cyclic graphical contractions in complete metric spaces. Shukla, Dubey and Shukla [32] proposed the notions of graphical cone metric spaces on Banach algebra. In addition, they proved some fixed point results of a class of special contractive mappings which are defined on this kind of spaces.

In [22], the concept of the set-valued mapping was extended to graphical metric spaces. In 2013, a more general definition of the set-valued contractive mapping was given in the above mentioned space by Dinevari and Frigon [11].

A natural generalization of the Banach contractive mapping is the nonexpansive mapping. In 1970, the concepts of convex structures and convex metric spaces were proposed by Takahashi [36]. And he also gave the fixed point theorems of nonexpansive mappings in convex metric spaces. Besides, Goebel and Kirk [14] researched some iterative procedures of nonexpansive mappings in hyperbolic metric spaces in 1983. Nonexpansive iterations were proposed in hyperbolic metric spaces by Reich and Shafrir [29] in 1990. Actually, the Picard iterative algorithm has been widely used to study different kinds of fixed point theorems in graphical metric spaces. But because the graphical structure itself does not have a linear structure, the graphical metric spaces are more complex than the general metric spaces. So other iterative algorithms are difficult to be directly generalized to this space.

Based on the above related research, in this article, we present a series of fixed point results of the Ishikawa iterative algorithm and the SP iterative algorithm in graphical convex metric spaces. And the structure of the article is as shown below: Section 1 mainly introduces the history and research status of fixed point theory and iterative algorithms. And set-valued contractions and graphical convex metric spaces are introduced step by step. Section 2 introduces some elementary notations, concepts and results. Section 3 proposes the Ishikawa iterative algorithm. Furthermore, some results of fixed points theorems of the Ishikawa iterative algorithm for set-valued G-contractions are given in G-complete graphical convex metric spaces. And by providing an example, we demonstrate the hypotheses of the existence theorem of fixed points for set-valued G-contractions in the above space are sufficient but not necessary. Section 4 proposes the SP iterative algorithm. Likewise, we also give the fixed point results of the SP iterative algorithm.

2. Preliminaries

First of all, we enunciate some elementary notations, concepts and basic results which are helpful for this article.

Let the set of positive integers be represented by $\mathbb{Z}^+$. And in the following study, we presume that the graph $G = (\Omega(G), \Xi(G))$ does not have parallel edges. Among them, $\Omega(G)$ represents a set containing all vertices and $\Xi(G)$ represents a binary relation on $\Omega(G)$, where the elements in $\Xi(G)$ are said to be edges. We can say G is a directed graph when every edge of it has a direction. On the contrary, an undirected graph is every edge of G has no direction.

By reversing the direction of edges of a graph G, we can obtain the inverse of a directed graph G, which is denoted by G^{-1}. Therefore, we have

$$\Xi(G^{-1}) = \{(f, h) \in M \times M : (h, f) \in \Xi(G)\}.$$

We let a directed graph with symmetrical edges be denoted by $\widehat{G}$. And it is defined as follows:

$$\Xi(\widehat{G}) = \Xi(G) \cup \Xi(G^{-1}),$$

so we can see that $\widehat{G}$ is symmetrical. If all loops are contained in $\Xi(G)$, for every $f \in \Omega(G)$, there is $(f, f) \in \Xi(G)$, then the directed graph G is called reflexive. Furthermore, if the following condition is satisfied,

$$(f, h) \in \Xi(G), (h, s) \in \Xi(G) \Longrightarrow (f, s) \in \Xi(G),$$

for all $f, h, s \in \Xi(G)$, then G is said to be transitive.

Moreover, if the each edge of G is allocated by the distance between its edges, then G can be viewed as a graph with weights assigned to it. And in this paper, we presume the directed graph G is symmetric, reflexive and transitive.

Definition 2.1. [42] Let $m, n \in \Omega(G)$. A path (or directed path) of length $h \in \mathbb{Z}^+$ between m and n in G is defined as a sequence $\{f_k\}_{k=0}^{h}$ of vertices with $m = f_0$, $n = f_h$ and $(f_{k-1}, f_k) \in \Xi(G)$ for $k = 1, 2, ..., h$.

In [42], they also defined
$[m]_G^h = \{n \in \Omega(G)\colon$ there exists a path directing from m to n having length $h\}$.

Definition 2.2. [42] There is a relation R on $\Omega(G)$ satisfing $(mRn)_G$ if there is a path directing from m to n in G and $\zeta \in (mRn)_G$ if ζ is contained in $(mRn)_G$. For all $i \in \mathbb{Z}^+$, if $\{f_i\}$ satisfies $(f_i R f_{i+1})_G$, then the sequence $\{f_i\} \in \Omega(G)$ is called G-termwise connected $(G - TWC)$.

Definition 2.3. [33] Let $d\colon \Omega(G) \times \Omega(G) \longrightarrow [0, \infty)$ be a mapping and G be a graph, if
(i) $d(m,n) = 0 \iff m = n$ for all m, $n \in \Omega(G)$,
(ii) $d(m,n) = d(n,m)$ for all m, $n \in \Omega(G)$,
(iii) for $(mRn)_G$, $\zeta \in (mRn)_G$, we have $d(m,n) \leq d(m,\zeta) + d(\zeta,n)$, where m, n, $\zeta \in \Omega(G)$.
Then we can say the space (G,d) is a graphical metric space.

Definition 2.4. [42] In a graphical metric space (G,d), a sequence $\{f_i\}$ is called:
(i) a convergent sequence $\iff$ there is $a \in G$ making $\lim_{i\longrightarrow\infty} d(f_i, a) = 0$ hold,
(ii) a Cauchy sequence $\iff$ $\lim_{i,j\longrightarrow\infty} d(f_i, f_j) = 0$. Namely, for any $\epsilon > 0$, there is $i_0 \in \mathbb{Z}^+$ making $d(f_i, f_j) < \epsilon$ hold for all j, $i > i_0$.

Definition 2.5. [42] If every $G - TWC$ Cauchy sequence converges in G, then we can say the (G,d) is G-complete.

Definition 2.6. [34] Choose a graphical metric space (G,d). Besides, we also select two sets D, $E \subset \Omega(G)$. Then by:
(i) If D and E contain an edge, then $(D,E) \subset \Xi(G)$ for some $u \in D$ and $v \in E$,
(ii) If D and E contain a path, then there exists a path between some $u \in D$ and $v \in E$.

Moreover, we mean DRE by the relation R if and only if there exists a path between two sets D and E. In addition, if the relation R on $\Omega(G)$ satisfies the following:

$$DRE, ERF \Longrightarrow DRF,$$

then R is said to be transitive.

Definition 2.7. [35] Let Ψ be a set of all nonempty closed sets on a sphere V. For any $X, Y \in \Psi$, let

$$H(X,Y) = inf\{\nu; X \subset Y_\nu, Y \subset X_\nu\}.$$

Then $H(.,.)$ defines a distance called Hausdorff distance. And we say (Ψ, H) is a Hausdorff metric space.

Definition 2.8. [9] Let (M,d) be a metric space. Then we can say $\Gamma\colon M \to 2^M \setminus \{\emptyset\}$ is a set-valued contractive mapping when there is $\kappa \in (0,1)$ such that

$$H(\Gamma(f), \Gamma(h)) \leq \kappa d(f,h), f, h \in M,$$

where $H(X,Y)$ represents the Hausdorff distance between two elements X and Y.

Definition 2.9. [22] Define a set-valued mapping Γ on (G,d), where (G,d) is a graphical metric space. If
(i) there is $\kappa \in (0,1)$ making $H(\Gamma(f), \Gamma(h)) \leq \kappa d(f,h)$ hold for all $(f,h) \in \Xi(G)$,

(ii) there is $\kappa \in (0,1)$ making $d(m,n) \leq \kappa d(f,h)$ hold for each $(f,h) \in \Xi(G)$, $m \in \Gamma(f)$ and $n \in \Gamma(h)$, one has $(m,n) \in \Xi(G)$.

Then we can say Γ is a set-valued contraction in (G,d).

Definition 2.10. [11] Define a set-valued mapping Γ on (G,d), where (G,d) is a graphical metric space. Then the mapping Γ is said to be a G-contraction if there is $\kappa \in (0,1)$ such that for all $(f,h) \in \Xi(G)$ and $m \in \Gamma(f)$, there is $n \in \Gamma(h)$ such that

$$(m,n) \in \Xi(G) \quad \text{and} \quad d(m,n) \leq \kappa d(f,h). \tag{2.1}$$

Remark 2.1. [8] From the above definition, we can acquire that

$$H(\Gamma(f),\Gamma(h)) \leq \kappa d(f,h)$$

holds for all $(f,h) \in \Xi(G)$.

Definition 2.11. [36] Let (M,d) be a metric space and $U = [0,1]$. A mapping $W\colon M \times M \times U \to M$ is called the convex structure on M if for each $\tau \in M$ and $(f,h;\phi) \in M \times M \times U$,

$$d(\tau, W(f,h;\phi)) \leq \phi d(\tau,f) + (1-\phi)d(\tau,h).$$

Then we can say (M,d,W) is a convex metric space.

Definition 2.12. [42] If for any $G-TWC$ sequence $\{x_m\}$ which converges to some $a \in \Omega(G)$, and there is $m_0 \in \mathbb{Z}^+$ such that $(x_m,a) \in \Xi(G)$ for any $m \geq m_0$, then we can say the property $(\mathbb{P})$ holds on (G,d).

Definition 2.13. [8] Let (G,d) be a graphical metric space and $U = [0,1]$. If a mapping $W\colon \Omega(G) \times \Omega(G) \times U \to \Omega(G)$ satisfies

$$d(\tau, W(f,h;\sigma)) \leq (1-\sigma)d(\tau,f) + \sigma d(\tau,h), \tag{2.2}$$

for all f, h, $\tau \in \Omega(G)$ and $\sigma \in (0,1)$, then we can say (G,d,W) is a graphical convex metric space. And in the following discussion, we will use $GCMS$ to represent this space.

Meanwhile, a set is defined as follows:

$$L(\Omega(G)) = \{\ell \subseteq \Omega(G) : \ell \text{ is a closed subset of } \Omega(G)\}.$$

Definition 2.14. [8] If for any $(f,s) \in \Xi(G)$ and $h = W(f,s;\sigma)$, we have $(f,h) \in \Xi(G)$ and $(h,s) \in \Xi(G)$, then we can say the property $(\mathbb{Q})$ holds on (G,d,W).

Then, by introducing the concepts of Γ-Ishikawa sequence and Γ-SP sequence, the Ishikawa iterative algorithm and the SP iterative algorithm of set-valued mappings are extended to a graphical metric space.

3. Fixed point theorems of Γ-Ishikawa sequences

First of all, in [8], an example is given to prove that the property $(\mathbb{P})$ and the property $(\mathbb{Q})$ are both satisfied in a G-complete $GCMS$.

Next, the fixed point theorems of Γ-Ishikawa sequences will be given in the above mentioned space.

Definition 3.1. Suppose $\Gamma\colon \Omega(G) \to L(\Omega(G))$ is a set-valued mapping on a $GCMS$. Presume $f_0 \in \Omega(G)$ is the initial value. Then $\{f_n\}$ is said to be a Γ-Ishikawa sequence if it satisfies

$$\begin{cases} h_n = W(f_n, s_n; e_n), \\ f_{n+1} = W(f_n, s_n^{'}; \rho_n), \end{cases} \tag{3.1}$$

where $s_n \in \Gamma f_n$, $s_n^{'} \in \Gamma h_n$, and ρ_n, $e_n \in (0,1)$.

Theorem 3.1. *Let $\Gamma\colon \Omega(G) \to L(\Omega(G))$ be a G-contraction mapping on G-complete $GCMS$ satisfying properties $(\mathbb{P})$ and $(\mathbb{Q})$. Suppose that $\{\rho_n\}$ and $\{e_n\}$ satisfy $0 < 1-(1-\kappa)(\rho_n + \kappa\rho_n e_n) + 2\kappa e_n < 1-\theta$ where $\theta \in (0,1)$, $\{\rho_n\}$ and $\{e_n\}$ are monotonous. If*

$$E_\Gamma = \{f \in \Omega(G)\colon \text{ there is } h \in \Gamma f \text{ such that } (f,h) \in \Xi(G)\}$$

is nonempty, then the mapping Γ has a fixed point in G.

Proof. There is $s_0 \in \Gamma f_0$ making $(f_0, s_0) \in \Xi(G)$ hold for any $f_0 \in E_\Gamma$. Let $h_0 = W(f_0, s_0; e_0)$, according to the property $(\mathbb{Q})$, we have $(f_0, h_0) \in \Xi(G)$ and $(h_0, s_0) \in \Xi(G)$. From Definition 2.13, we can obtain that

$$d(h_0, s_0) = d(W(f_0, s_0; e_0), s_0) \le (1-e_0)d(f_0, s_0).$$

Since Γ is a G-contraction and $(f_0, h_0) \in \Xi(G)$, for $s_0 \in \Gamma f_0$, there is $s_0^{'} \in \Gamma h_0$ such that

$$(s_0, s_0^{'}) \in \Xi(G)$$

and

$$d(s_0, s_0^{'}) \le \kappa d(f_0, h_0).$$

And by the transitivity of G, we can also acquire $(h_0, s_0^{'}) \in \Xi(G)$ and $(f_0, s_0^{'}) \in \Xi(G)$.

Let $f_1 = W(f_0, s_0^{'}; \rho_0)$, by using the property $(\mathbb{Q})$, we have $(f_0, f_1) \in \Xi(G)$ and $(f_1, s_0^{'}) \in \Xi(G)$. Thanks to Definition 2.13, we can infer that

$$d(f_0, f_1) = d(f_0, W(f_0, s_0^{'}; \rho_0)) \le \rho_0 d(f_0, s_0^{'}),$$

and

$$d(f_1, s_0^{'}) = d(W(f_0, s_0^{'}; \rho_0), s_0^{'}) \le (1-\rho_0)d(f_0, s_0^{'}).$$

Since $(f_0, f_1) \in \Xi(G)$ and $(f_0, h_0) \in \Xi(G)$, we can obtain $(h_0, f_1) \in \Xi(G)$. And since Γ is a G-contraction and $(h_0, f_1) \in \Xi(G)$, for $s_0^{'} \in \Gamma h_0$, there is $s_1 \in \Gamma f_1$ such that

$$(s_0^{'}, s_1) \in \Xi(G)$$

and

$$d(s_0^{'}, s_1) \le \kappa d(h_0, f_1).$$

By using the transitivity of G, we claim $(s_1, f_1) \in \Xi(G)$ and $(s_0, s_1) \in \Xi(G)$. And by induction, we can acquire sequences $\{f_n\}$, $\{h_n\}$, $\{s_n\}$ and $\{s_n^{'}\}$, where $h_n = W(f_n, s_n; e_n)$, $f_{n+1} = W(f_n, s_n^{'}; \rho_n)$, $s_n \in \Gamma f_n$ and $s_n^{'} \in \Gamma h_n$. We still get that $(f_n, s_n) \in \Xi(G)$ and $(f_n, s_n^{'}) \in \Xi(G)$. From the property $(\mathbb{Q})$, we can see that $(f_n, h_n) \in \Xi(G)$, $(h_n, s_n) \in \Xi(G)$ and $(f_n, f_{n+1}) \in \Xi(G)$, $(f_{n+1}, s_n^{'}) \in \Xi(G)$.

Thanks to Definition 2.13, it is not hard to see

$$\begin{aligned}
&d(f_n, h_n) = d(f_n, W(f_n, s_n; e_n)) \leq e_n d(f_n, s_n),\\
&d(h_n, s_n) = d(W(f_n, s_n; e_n), s_n) \leq (1 - e_n) d(f_n, s_n),\\
&d(f_n, f_{n+1}) = d(f_n, W(f_n, s_n^{'}; \rho_n)) \leq \rho_n d(f_n, s_n^{'}),\\
&d(f_{n+1}, s_n^{'}) = d(W(f_n, s_n^{'}; \rho_n), s_n^{'}) \leq (1 - \rho_n) d(f_n, s_n^{'}),
\end{aligned}$$

and

$$\begin{aligned}
&(s_n, s_n^{'}) \in \Xi(G), d(s_n, s_n^{'}) \leq \kappa d(f_n, h_n),\\
&(s_n^{'}, s_{n+1}) \in \Xi(G), d(s_n^{'}, s_{n+1}) \leq \kappa d(h_n, f_{n+1}).
\end{aligned}$$

Moreover, we also notice that $\{f_n\}$ is $G - TWC$. Subsequently, we proclaim $\{d(f_n, s_n)\}$ is decreasing. Actually, we can acquire

$$\begin{aligned}
d(f_{n+1}, s_{n+1}) &\leq d(f_{n+1}, s_n^{'}) + d(s_n^{'}, s_{n+1})\\
&= d(W(f_n, s_n^{'}; \rho_n), s_n^{'}) + d(s_n^{'}, s_{n+1})\\
&\leq (1 - \rho_n) d(f_n, s_n^{'}) + \kappa d(h_n, f_{n+1})\\
&\leq (1 - \rho_n) d(f_n, s_n) + (1 - \rho_n) d(s_n, s_n^{'}) + \kappa d(h_n, f_{n+1})\\
&\leq (1 - \rho_n) d(f_n, s_n) + \kappa (1 - \rho_n) d(f_n, h_n) + \kappa d(h_n, f_{n+1}),
\end{aligned}$$

and

$$\begin{aligned}
d(h_n, f_{n+1}) &= d(W(f_n, s_n; e_n), W(f_n, s_n^{'}; \rho_n))\\
&\leq d(W(f_n, s_n; e_n), f_n) + d(f_n, W(f_n, s_n^{'}; \rho_n))\\
&\leq e_n d(f_n, s_n) + \rho_n d(f_n, s_n^{'})\\
&\leq (e_n + \rho_n) d(f_n, s_n) + \rho_n d(s_n, s_n^{'})\\
&\leq (e_n + \rho_n) d(f_n, s_n) + \kappa \rho_n d(f_n, h_n)\\
&= (e_n + \rho_n) d(f_n, s_n) + \kappa \rho_n d(f_n, W(f_n, s_n; e_n))\\
&\leq (\rho_n + e_n + \kappa \rho_n e_n) d(f_n, s_n).
\end{aligned}$$

It follows

$$\begin{aligned}
d(f_{n+1}, s_{n+1}) &\leq (1 - \rho_n) d(f_n, s_n) + \kappa e_n (1 - \rho_n) d(f_n, s_n)\\
&\quad + \kappa (\rho_n + e_n + \kappa \rho_n e_n) d(f_n, s_n)\\
&= (1 - \rho_n + \kappa e_n - \kappa \rho_n e_n + \kappa \rho_n + \kappa e_n + \kappa^2 \rho_n e_n) d(f_n, s_n)\\
&= [1 - (1 - \kappa)(\rho_n + \kappa \rho_n e_n) + 2\kappa e_n] d(f_n, s_n).
\end{aligned}$$

Since $0 < 1-(1-\kappa)(\rho_n+\kappa\rho_n e_n)+2\kappa e_n < 1-\theta$ where $\theta \in (0, 1)$, which indicates $\{d(f_n, s_n)\}$ is decreasing.

Let $t_n = 1 - (1 - \kappa)(\rho_n + \kappa \rho_n e_n) + 2\kappa e_n$, so we have

$$t_n \in (0, 1) \text{ and } d(f_{n+1}, s_{n+1}) \leq t_n d(f_n, s_n).$$

And we also find

$$d(f_n, f_{n+1}) = d(f_n, W(f_n, s_n^{'}; \rho_n))$$

$$\begin{aligned}
&\leq \rho_n d(f_n, s_n^{'}) \\
&\leq \rho_n d(f_n, s_n) + \rho_n d(s_n, s_n^{'}) \\
&\leq \rho_n d(f_n, s_n) + \kappa\rho_n d(f_n, h_n) \\
&= \rho_n d(f_n, s_n) + \kappa\rho_n d(f_n, W(f_n, s_n; e_n)) \\
&\leq \rho_n d(f_n, s_n) + \kappa\rho_n e_n d(f_n, s_n) \\
&= (\rho_n + \kappa\rho_n e_n) d(f_n, s_n).
\end{aligned}$$

Let $\rho_n + \kappa\rho_n e_n = \gamma_n$. Furthermore, for any $q \in \mathbb{Z}^+$, we can infer

$$\begin{aligned}
d(f_n, f_{n+q}) &\leq d(f_n, f_{n+1}) + d(f_{n+1}, f_{n+2}) + \cdots + d(f_{n+q-1}, f_{n+q}) \\
&\leq \gamma_n d(f_n, s_n) + \gamma_{n+1} d(f_{n+1}, s_{n+1}) + \cdots + \gamma_{n+q-1} d(f_{n+q-1}, s_{n+q-1}) \\
&\leq (\gamma_n \prod_{i=0}^{n-1} t_i + \gamma_{n+1} \prod_{i=0}^{n} t_i + \cdots + \gamma_{n+q-1} \prod_{i=0}^{n+q-2} t_i) d(f_0, s_0).
\end{aligned}$$

Let $D_{n+j} = \gamma_{n+j} \prod_{i=0}^{n+j-1} t_i$, $j = 0, 1, 2, \ldots, q-1$. Then we obtain

$$d(f_n, f_{n+q}) \leq (D_n + D_{n+1} + \cdots + D_{n+q-1}) d(f_0, s_0).$$

Since $0 < 1 - (1-\kappa)(\rho_n + \kappa\rho_n e_n) + 2\kappa e_n < 1 - \theta$ where $\theta \in (0,1)$, $\{\rho_n\}$ and $\{e_n\}$ are monotonous, we can get that

$$\begin{aligned}
\lim_{j\to\infty} sup \frac{D_{n+j+1}}{D_{n+j}} &= \lim_{j\to\infty} sup \frac{\gamma_{n+j+1} \prod_{i=0}^{n+j} t_i}{\gamma_{n+j} \prod_{i=0}^{n+j-1} t_i} \\
&= \lim_{j\to\infty} sup \frac{\gamma_{n+j+1} t_{n+j}}{\gamma_{n+j}} \\
&= \lim_{j\to\infty} sup \frac{\rho_{n+j+1} + \kappa\rho_{n+j+1} e_{n+j+1}}{\rho_{n+j} + \kappa\rho_{n+j} e_{n+i}} \\
&\quad \times [1 - (1-\kappa)(\rho_{n+j} + \kappa\rho_{n+j} e_{n+j}) + 2\kappa e_{n+j}] \\
&< 1.
\end{aligned}$$

According to the virtue of D'Alembert's test, we deduce $\sum_{j=0}^{\infty} D_j$ is convergent. Thus, we can draw a conclusion $\lim_{n\to\infty} d(f_n, f_{n+q}) = 0$ which indicates that $\{f_n\}$ is a Cauchy sequence. Since G is G-complete, we can find a $q \in \Omega(G)$ that makes $\lim_{n\to\infty} d(f_n, q) = 0$ hold. According to the property $(\mathbb{P})$, for large enough n, we can acquire $(f_n, q) \in \Xi(G)$, thus there is $q_n \in \Gamma q$ such that

$$d(f_n, q_n) \leq \kappa d(f_n, q),$$

which implies $d(f_n, q_n) \to 0$ as $n \to \infty$. Let $n \to +\infty$, then

$$d(q_n, q) \leq d(q_n, f_n) + d(f_n, q) \to 0,$$

which indicates $q \in \Gamma q$ since Γq is closed. □

Next, we will give an example to prove that it is sufficient but not necessary for the assumptions of the above theorem.

Example 3.1. Consider $M = [0,1]$, $X = \{\frac{1}{3^n} : n \in \mathbb{Z}^+ \cup \{0\}\}$, $Y = \{\frac{1}{3^{2n+1}} : n \in \mathbb{Z}^+ \cup \{0\}\}$. For any $f, h \in M$, we define

$$d(f,h) = \begin{cases} |f-h|, & f \neq h, \\ 0, & f = h. \end{cases}$$

Next, we give further consideration to G with $\Omega(G) = M$ and

$$\Xi(G) = A \cup B \cup C,$$

where

$$\begin{aligned} &A = \{(f,h) \in M \times M \colon f, h \in X \text{ or } f, h \in M \setminus X\}, \\ &B = \{(f,h) \in M \times M \colon f \in X \setminus Y \text{ and } h \in M \setminus X, \text{ or } h \in X \setminus Y \text{ and } f \in M \setminus X\}, \\ &C = \{(f,h) \in M \times M \colon f \in M \setminus X \text{ and } h \in Y, \\ &\quad \text{or } f \in Y \text{ and } h \in M \setminus X, \text{ then } \tfrac{3}{2}f \leq h \text{ or } h \leq \tfrac{2}{3}f\}. \end{aligned}$$

For any $\rho \in (0,1)$ and $f, h \in M$, we define $W(f,h;\rho) = (1-\rho)f + \rho h$, so we can see (G,d,W) is a $GCMS$. Subsequently, it will be demonstrated (G,d,W) does not have the property $(\mathbb{P})$ and the property $(\mathbb{Q})$.

Factually, we choose irrational number sequences $\{f_n\}$ and $\{\mu_n\}$ for any $n \in \mathbb{Z}^+$ in $\Omega(G)$, then we can obtain $(f_n, \mu_n) \in A$, that is $(f_n, \mu_n) \in \Xi(G)$, so the sequences $\{f_n\}$ and $\{\mu_n\}$ are $G-TWC$. Furthermore, for some $n_0 \in \mathbb{Z}^+$, let $f = \frac{1}{3^{2n_0+1}}$, and we choose the irrational number sequence $\{f_n\}$ in $\Omega(G)$ with $f_n > f$ for every $n \in \mathbb{Z}^+$ which converges to f. Therefore, we can acquire $(f_n, f) \notin A$ and $(f_n, f) \notin B$. For any n, since $f_n > f$, we have $f_n > \frac{2}{3}f$. And for large enough n, we have $f_n < \frac{3}{2}f$. Consequently, we can obtain $(f_n, f) \notin C$. Hence, from the above analysis, we can get that $(f_n, f) \notin \Xi(G)$, that is, (G,d,W) does not have the property $(\mathbb{P})$. Moreover, we take $f = \frac{1}{3^{2n+1}}$, $h = 0$. And we choose ρ making $W(f,h;\rho) = (1-\rho)f > \frac{2}{3}f$ and $W(f,h;\rho) \in [0,1] \setminus X$. Then we can acquire $(f, W(f,h;\rho)) \notin \Xi(G)$. This is equivalent to saying that (G,d,W) does not have the property $(\mathbb{Q})$.

Furthermore, we let Γ be a set-valued mapping which is defined as follows:

$$\Gamma f = \begin{cases} \{\frac{1}{3^{2n+3}}, \frac{1}{3^{2n+5}}\}, & f = \frac{1}{3^{2n+1}} \in Y, \\ \{0\}, & f = \frac{1}{3^{2n}} \in X \setminus Y, \\ \{0\}, & f \in M \setminus X. \end{cases}$$

Now we say Γ is a G-contraction with $\kappa = \frac{1}{3}$, namely, for all $(f,h) \in \Xi(G)$ and $a \in \Gamma f$, there is $b \in \Gamma h$ making $(a,b) \in \Xi(G)$ and $d(a,b) \leq \kappa d(f,h)$ hold. In the following, we will give the consideration to several cases:

Case 1. Choose f, $h \in X$ and $f = \frac{1}{3^{2n+1}}$, $h = \frac{1}{3^{2m+1}}$, without loss of generality, presuming $m > n$, we can obtain

$$\Gamma f = \{\frac{1}{3^{2n+3}}, \frac{1}{3^{2n+5}}\} \text{ and } \Gamma h = \{\frac{1}{3^{2m+3}}, \frac{1}{3^{2m+5}}\}.$$

We take $b = \frac{1}{3^{2m+5}}$, then

$$d(\frac{1}{3^{2n+3}}, \frac{1}{3^{2m+5}}) = \frac{1}{3^2} d(\frac{1}{3^{2n+1}}, \frac{1}{3^{2m+3}}) < \kappa d(\frac{1}{3^{2n+1}}, \frac{1}{3^{2m+1}}) = \kappa d(f, h),$$

and

$$d(\frac{1}{3^{2n+5}}, \frac{1}{3^{2m+5}}) = \frac{1}{3^4} d(\frac{1}{3^{2n+1}}, \frac{1}{3^{2m+1}}) < \kappa d(f, h).$$

Case 2. Choose $f, h \in X$ and $f = \frac{1}{3^{2n}}$, $h = \frac{1}{3^{2m}}$, then we get

$$\Gamma f = \{0\}, \Gamma h = \{0\}$$

and

$$0 = d(0, 0) \leq \kappa d(f, h).$$

Case 3. Choose $f, h \in X$ and $f = \frac{1}{3^{2n+1}}$, $h = \frac{1}{3^{2m}}$, then we acquire

$$\Gamma f = \{\frac{1}{3^{2n+3}}, \frac{1}{3^{2n+5}}\}, \Gamma h = \{0\},$$

and

$$d(\frac{1}{3^{2n+5}}, 0) = \frac{1}{3^2} d(\frac{1}{3^{2n+3}}, 0) < d(\frac{1}{3^{2n+3}}, 0) = \frac{1}{3^2} f.$$

If $m \leq n$, then we obtain

$$\kappa d(f, h) = \kappa d(\frac{1}{3^{2n+1}}, \frac{1}{3^{2m}}) = \kappa \frac{1}{3^{2m}} (1 - \frac{1}{3^{2(n-m)+1}}) \geq \kappa \frac{1}{3^{2n+1}} (1 - \frac{1}{3}) = \frac{2}{3^2} f > \frac{1}{3^2} f.$$

If $m > n$, then we obtain

$$\begin{aligned} \kappa d(f, h) &= \kappa d(\frac{1}{3^{2n+1}}, \frac{1}{3^{2m}}) \\ &= \kappa \frac{1}{3^{2n+1}} (1 - \frac{1}{3^{2(m-n)-1}}) \\ &\geq \kappa \frac{1}{3^{2n+1}} (1 - \frac{1}{3}) \\ &= \frac{2}{3^2} f \\ &> \frac{1}{3^2} f. \end{aligned}$$

Therefore,

$$d(\frac{1}{3^{2n+5}}, 0) < d(\frac{1}{3^{2n+3}}, 0) < \kappa d(f, h).$$

Case 4. Choose $f, h \in X$ and $f = \frac{1}{3^{2n}}$, $h = \frac{1}{3^{2m+1}}$, this case is similar to Case 3.

Case 5. Choose $f, h \in M \setminus X$, then we deduce

$$\Gamma f = \{0\}, \Gamma h = \{0\}$$

and

$$0 = d(0,0) \leq \kappa d(f,h).$$

Case 6. Choose $f \in X$, $h \in M \setminus X$ and $(f,h) \in \Xi(G)$.

If $f = \frac{1}{3^{2n}}$, then we acquire $\Gamma f = \{0\}$, $\Gamma h = \{0\}$ and $0 = d(0,0) \leq \kappa d(f,h)$.

If $f = \frac{1}{3^{2n+1}}$, then we acquire

$$\Gamma f = \{\frac{1}{3^{2n+3}}, \frac{1}{3^{2n+5}}\}, \Gamma h = \{0\},$$

and

$$d(\frac{1}{3^{2n+5}}, 0) < d(\frac{1}{3^{2n+3}}, 0) = \frac{1}{3^2} f.$$

Since $(f,h) \in \Xi(G)$, $f \in Y$ and $h \in M \setminus X$, so we have $\frac{3}{2} f \leq h$ or $h \leq \frac{2}{3} f$.

When $\frac{3}{2} f \leq h$, we get that

$$\kappa d(f,h) = \frac{1}{3}|f-h| = \frac{1}{3}(h-f) \geq \frac{1}{3} \times \frac{1}{2} f > \frac{1}{3^2} f.$$

When $h \leq \frac{2}{3} f$, we get that

$$\kappa d(f,h) = \frac{1}{3}|f-h| = \frac{1}{3}(f-h) \geq \frac{1}{3} \times \frac{1}{3} f = \frac{1}{3^2} f.$$

Thus

$$d(\frac{1}{3^{2n+5}}, 0) < d(\frac{1}{3^{2n+3}}, 0) \leq \kappa d(f,h).$$

Case 7. Choose $f \in M \setminus X$, $h \in X$, this case is similar to Case 6.

Hence, Γ is a G-contraction with $\kappa = \frac{1}{3}$. And there is no doubt that we have $0 \in \Gamma 0$, which indicates 0 is a fixed point of Γ.

Remark 3.1. In the proof procedure of Theorem 3.1, we can also gain

$$\lim_{n\to\infty} d(f_n, s_n) = 0 \text{ and } \lim_{n\to\infty} d(f_n, h_n) = 0.$$

Proof. Thanks to the definitions of $\{f_n\}$ and $\{h_n\}$, we can get

$$\begin{aligned} d(f_n, h_n) &= d(f_n, W(f_n, s_n^{'}; \rho_n)) \leq \rho_n d(f_n, s_n^{'}) \\ &\leq \rho_n [d(f_n, s_n) + d(s_n, s_n^{'})] \\ &\leq \rho_n d(f_n, s_n) + \kappa \rho_n d(f_n, h_n). \end{aligned}$$

Since $\kappa, \rho_n \in (0,1)$, we can acquire

$$d(f_n, h_n) \leq \frac{\rho_n}{1 - \kappa \rho_n} d(f_n, s_n).$$

Thus, from the above analysis, we only require to demenstrate $\lim_{n\to\infty} d(f_n, s_n) = 0$.

From the proof of Theorem 3.1, it can be found that

$$d(f_n, s_n) \leq \prod_{i=0}^{n-1} t_i d(f_0, s_0),$$

which indicates $\lim_{n\to\infty} d(f_n, s_n) = 0$ since $t_i \in (0,1)$. Furthermore, we can get $\lim_{n\to\infty} d(f_n, h_n) = 0$. □

Theorem 3.2. Presume all assumptions of Theorem 3.1 hold, and set

$$\begin{cases} h_n = W(f_n, s_n; e_n), \\ f_{n+1} = W(f_n, s_n^{'}; \rho_n), \end{cases}$$

where $s_n \in \Gamma f_n$, $s_n^{'} \in \Gamma h_n$, and ρ_n, $e_n \in (0,1)$, and

$$\begin{cases} \chi_n = W(\mu_n, g_n; \tau_n), \\ \mu_{n+1} = W(\mu_n, g_n^{'}; \psi_n), \end{cases}$$

where $g_n \in \Gamma\mu_n$, $g_n^{'} \in \Gamma\chi_n$, and τ_n, $\psi_n \in (0,1)$. In addition, $\{f_n\}$ and $\{\mu_n\}$ are generated from the above iterative process where $\{f_n\}$ converges to f and $\{\mu_n\}$ converges to μ, the sequence $\{\psi_n\}$ satisfies $\lim_{n\to\infty} \psi_n = \psi \neq 0$. Then $f = \mu$ provided that $(f_n, \mu_n) \in \Xi(G)$ for large enough $n \in \mathbb{Z}^+$.

Proof. According to Theorem 3.1, it follows f and μ are fixed points of Γ. Since Γ is a G-contraction, $(f_n, \mu_n) \in \Xi(G)$ and $(f_n, h_n) \in \Xi(G)$, for all $s_n \in \Gamma f_n$, there are $g_n \in \Gamma\mu_n$ and $s_n^{'} \in \Gamma h_n$ such that

$$(s_n, g_n) \in \Xi(G), d(s_n, g_n) \leq \kappa d(f_n, \mu_n),$$

and

$$(s_n, s_n^{'}) \in \Xi(G), d(s_n, s_n^{'}) \leq \kappa d(f_n, h_n).$$

From Remark 3.1, we deduce that $\lim_{n\to\infty} d(f_n, h_n) = 0$, $\lim_{n\to\infty} d(\mu_n, \chi_n) = 0$ and $\lim_{n\to\infty} d(f_n, s_n) = 0$. Combining the conditions $\lim_{n\to\infty} d(f_n, f) = 0$ and $\lim_{n\to\infty} d(\mu_n, \mu) = 0$, we can get $\lim_{n\to\infty} d(h_n, f) = 0$ and $\lim_{n\to\infty} d(\chi_n, \mu) = 0$.

By using the property $(\mathbb{P})$, we can acquire that $(f_n, f) \in \Xi(G)$, $(\mu_n, \mu) \in \Xi(G)$, $(h_n, f) \in \Xi(G)$ and $(\chi_n, \mu) \in \Xi(G)$ for large enough n.

From Theorem 3.1, it can be concluded that $(f_n, f_{n+1}) \in \Xi(G)$ and $(f_n, s_n^{'}) \in \Xi(G)$. Since $(f_n, f_{n+1}) \in \Xi(G)$, $(f_n, h_n) \in \Xi(G)$, we can obtain $(f_{n+1}, h_n) \in \Xi(G)$. Similarly, we also have $(\mu_{n+1}, \chi_n) \in \Xi(G)$. Combining with $(f_{n+1}, \mu_{n+1}) \in \Xi(G)$, we can get $(f_{n+1}, \chi_n) \in \Xi(G)$. And we also draw a conclusion that $(h_n, \chi_n) \in \Xi(G)$ due to the transitivity of G.

Because Γ is a G-contraction and $(h_n, \chi_n) \in \Xi(G)$, thus for any $s_n^{'} \in \Gamma h_n$, there exists $g_n^{'} \in \Gamma\chi_n$ such that

$$(s_n^{'}, g_n^{'}) \in \Xi(G) \text{ and } d(s_n^{'}, g_n^{'}) \leq \kappa d(h_n, \chi_n).$$

Since $(f_n, s_n^{'}) \in \Xi(G)$, $(s_n^{'}, g_n^{'}) \in \Xi(G)$, $(f_n, \mu_n) \in \Xi(G)$, according to the transitivity, we can acquire that $(f_n, g_n^{'}) \in \Xi(G)$ and $(s_n^{'}, \mu_n) \in \Xi(G)$.

Notice that

$$d(f,\mu) \le d(f,f_{n+1}) + d(f_{n+1},\mu_{n+1}) + d(\mu_{n+1},\mu), \tag{3.2}$$

and

$$\begin{aligned}
d(f_{n+1},\mu_{n+1}) &= d(W(f_n,s_n^{'};\rho_n), W(\mu_n,g_n^{'};\psi_n)) \\
&\le (1-\rho_n)(1-\psi_n)d(f_n,\mu_n) + (1-\rho_n)\psi_n d(f_n,g_n^{'}) \\
&\quad + \rho_n(1-\psi_n)d(s_n^{'},\mu_n) + \rho_n\psi_n d(s_n^{'},g_n^{'}) \\
&\le (1-\rho_n)(1-\psi_n)d(f_n,\mu_n) + (1-\rho_n)\psi_n[d(f_n,s_n^{'}) + d(s_n^{'},g_n^{'})] \\
&\quad + \rho_n(1-\psi_n)[d(s_n^{'},f_n) + d(f_n,\mu_n)] + \rho_n\psi_n d(s_n^{'},g_n^{'}) \\
&= (1-\psi_n)d(f_n,\mu_n) + [\rho_n+\psi_n-2\rho_n\psi_n]d(f_n,s_n^{'}) + \psi_n d(s_n^{'},g_n^{'}) \\
&\le (1-\psi_n)[d(f_n,f) + d(f,\mu) + d(\mu,\mu_n)] \\
&\quad + [\rho_n+\psi_n-2\rho_n\psi_n][d(f_n,s_n) + d(s_n,s_n^{'})] \\
&\quad + \kappa\psi_n[d(h_n,f) + d(f,\mu) + d(\mu,\chi_n)] \\
&< (1-\psi_n)[d(f_n,f) + d(\mu,\mu_n)] + 2[d(f_n,s_n) + d(s_n,s_n^{'})] \\
&\quad + \kappa\psi_n[d(h_n,f) + d(\mu,\chi_n)] + (1+\kappa\psi_n-\psi_n)d(f,\mu) \\
&< d(f_n,f) + d(\mu,\mu_n) + 2d(f_n,s_n) + 2d(s_n,s_n^{'}) + \kappa[d(h_n,f) \\
&\quad + d(\mu,\chi_n)] + (1+\kappa\psi_n-\psi_n)d(f,\mu).
\end{aligned} \tag{3.3}$$

Combining with (3.2) and (3.3), we can obtain

$$\begin{aligned}
(1-\kappa)\psi_n d(f,\mu) \le{}& d(f,f_{n+1}) + d(f_n,f) + d(\mu,\mu_n) + 2d(f_n,s_n) \\
&+ 2d(s_n,s_n^{'}) + \kappa[d(h_n,f) + d(\mu,\chi_n)] + d(\mu_{n+1},\mu).
\end{aligned}$$

Letting $n \to \infty$, we have $(1-\kappa)\psi d(f,\mu) \le 0$. Since $\kappa \in (0,1)$, $\lim_{n\to\infty}\psi_n = \psi \neq 0$, we can acquire $d(f,\mu) = 0$, that is $f = \mu$. □

4. Fixed point theorems of Γ-SP sequences

Next, on a G-complete $GCMS$, the fixed point results related to Γ-SP sequences will be presented.

Definition 4.1. Suppose $\Gamma\colon \Omega(G) \to L(\Omega(G))$ is a set-valued mapping on a $GCMS$. Presume $f_0 \in \Omega(G)$ is the initial value. Then $\{f_n\}$ is said to be a Γ-SP sequence if it satisfies

$$\begin{cases} s_n = W(f_n,\mu_n;c_n), \\ h_n = W(s_n,v_n;e_n), \\ f_{n+1} = W(h_n,\varphi_n;\rho_n), \end{cases} \tag{4.1}$$

where $\mu_n \in \Gamma f_n$, $v_n \in \Gamma s_n$, $\varphi_n \in \Gamma h_n$, and ρ_n, e_n, $c_n \in (0,1)$.

Theorem 4.1. Let $\Gamma\colon \Omega(G) \to L(\Omega(G))$ be a G-contraction mapping on G-complete $GCMS$ satisfying properties $(\mathbb{P})$ and $(\mathbb{Q})$. Suppose that $\{\rho_n\}$, $\{e_n\}$ and $\{c_n\}$ satisfy $\{\rho_n\}$, $\{e_n\}$ and $\{c_n\} \subset (0,1)$, $\{\rho_n\}$, $\{e_n\}$ and $\{c_n\}$ are monotonous. If the set

$$E_\Gamma = \{f \in \Omega(G)\colon \text{ there is } h \in \Gamma f \text{ such that } (f,h) \in \Xi(G)\}$$

is nonempty, then the mapping Γ has a fixed point in G.

Proof. There is $\mu_0 \in \Gamma f_0$ making $(f_0,\mu_0) \in \Xi(G)$ hold for any $f_0 \in E_\Gamma$. Let $s_0 = W(f_0,\mu_0;c_0)$, according to the property $(\mathbb{Q})$, we have $(f_0,s_0) \in \Xi(G)$ and $(s_0,\mu_0) \in \Xi(G)$. From Definition 2.13, we can obtain that

$$d(f_0,s_0) = d(f_0,W(f_0,\mu_0;c_0)) \le c_0 d(f_0,\mu_0),$$

and

$$d(s_0,\mu_0) = d(W(f_0,\mu_0;c_0),\mu_0) \le (1-c_0)d(f_0,\mu_0).$$

Since Γ is a G-contraction and $(f_0,s_0) \in \Xi(G)$, for $\mu_0 \in \Gamma f_0$, there is $v_0 \in \Gamma s_0$ such that

$$(\mu_0,v_0) \in \Xi(G) \text{ and } d(\mu_0,v_0) \le \kappa d(f_0,s_0).$$

Moreover, by the transitivity of G, we can also acquire $(f_0,\mu_0) \in \Xi(G)$, $(s_0,v_0) \in \Xi(G)$ and $(f_0,v_0) \in \Xi(G)$.

Let $h_0 = W(s_0,v_0;e_0)$, by using the property $(\mathbb{Q})$, we have $(s_0,h_0) \in \Xi(G)$ and $(h_0,v_0) \in \Xi(G)$. From Definition 2.13, we deduce that

$$d(s_0,h_0) = d(s_0,W(s_0,v_0;e_0)) \le e_0 d(s_0,v_0),$$

and

$$d(h_0,v_0) = d(W(s_0,v_0;e_0),v_0) \le (1-e_0)d(s_0,v_0).$$

Since $(s_0,\mu_0) \in \Xi(G)$ and $(s_0,h_0) \in \Xi(G)$, we can obtain $(h_0,\mu_0) \in \Xi(G)$. Since Γ is a G-contraction and $(s_0,h_0) \in \Xi(G)$, for $v_0 \in \Gamma s_0$, there is $\varphi_0 \in \Gamma h_0$ such that

$$(v_0,\varphi_0) \in \Xi(G) \text{ and } d(v_0,\varphi_0) \le \kappa d(s_0,h_0).$$

By using the transitivity of G, we also claim $(s_0,v_0) \in \Xi(G)$, $(h_0,\varphi_0) \in \Xi(G)$, $(s_0,\varphi_0) \in \Xi(G)$, $(f_0,\varphi_0) \in \Xi(G)$, $(\mu_0,\varphi_0) \in \Xi(G)$ and $(f_0,h_0) \in \Xi(G)$.

Let $f_1 = W(h_0,\varphi_0;\rho_0)$, by using the property $(\mathbb{Q})$, we have $(h_0,f_1) \in \Xi(G)$ and $(f_1,\varphi_0) \in \Xi(G)$. From Definition 2.13, we deduce that

$$d(h_0,f_1) = d(h_0,W(h_0,\varphi_0;\rho_0)) \le \rho_0 d(h_0,\varphi_0),$$

and

$$d(f_1,\varphi_0) = d(W(h_0,\varphi_0;\rho_0),\varphi_0) \le (1-\rho_0)d(h_0,\varphi_0).$$

Since $(s_0,h_0) \in \Xi(G)$ and $(h_0,f_1) \in \Xi(G)$, we can acquire that $(s_0,f_1) \in \Xi(G)$. Similarly, we can also get $(h_0,\varphi_0) \in \Xi(G)$, $(f_1,v_0) \in \Xi(G)$ and $(f_1,\mu_0) \in \Xi(G)$.

Since Γ is a G-contraction and $(h_0,f_1) \in \Xi(G)$ and $(s_0,f_1) \in \Xi(G)$, for $\varphi_0 \in \Gamma h_0$, there is $\mu_1 \in \Gamma f_1$ such that

$$(\mu_1,\varphi_0) \in \Xi(G) \text{ and } d(\mu_1,\varphi_0) \le \kappa d(f_1,h_0),$$

and for $v_0 \in \Gamma s_0$, there is $\mu_1 \in \Gamma f_1$ such that

$$(\mu_1,v_0) \in \Xi(G) \text{ and } d(\mu_1,v_0) \le \kappa d(f_1,s_0).$$

By the transitivity of G, we claim $(\mu_1, f_1) \in \Xi(G)$, $(\mu_1, \mu_0) \in \Xi(G)$ and $(\mu_1, h_0) \in \Xi(G)$. And by induction, we can acquire sequences $\{f_n\}$, $\{h_n\}$, $\{s_n\}$, $\{\mu_n\}$, $\{v_n\}$ and $\{\varphi_n\}$, where $s_n = W(f_n, \mu_n; c_n)$, $h_n = W(s_n, v_n; e_n)$, $f_{n+1} = W(h_n, \varphi_n; \rho_n)$, $\mu_n \in \Gamma f_n$ and $v_n \in \Gamma s_n$ and $\varphi_n \in \Gamma h_n$. We still get that $(f_n, \mu_n) \in \Xi(G)$, $(s_n, v_n) \in \Xi(G)$ and $(h_n, \varphi_n) \in \Xi(G)$. From the property $(\mathbb{Q})$, it follows $(f_n, s_n) \in \Xi(G)$, $(s_n, \mu_n) \in \Xi(G)$, $(s_n, h_n) \in \Xi(G)$, $(h_n, v_n) \in \Xi(G)$, $(h_n, f_{n+1}) \in \Xi(G)$, and $(f_{n+1}, \varphi_n) \in \Xi(G)$.

Thanks to Definition 2.13, it is not hard to see

$$\begin{aligned}
&d(f_n, s_n) = d(f_n, W(f_n, \mu_n; c_n)) \le c_n d(f_n, \mu_n),\\
&d(s_n, \mu_n) = d(W(f_n, \mu_n; c_n), \mu_n) \le (1 - c_n) d(f_n, \mu_n),\\
&d(s_n, h_n) = d(s_n, W(s_n, v_n; e_n)) \le e_n d(s_n, v_n),\\
&d(h_n, v_n) = d(W(s_n, v_n; e_n), v_n) \le (1 - e_n) d(s_n, v_n),\\
&d(h_n, f_{n+1}) = d(W(h_n, \varphi_n; \rho_n), h_n) \le \rho_n d(h_n, \varphi_n),\\
&d(f_{n+1}, \varphi_n) = d(W(h_n, \varphi_n; \rho_n), \varphi_n) \le (1 - \rho_n) d(h_n, \varphi_n),
\end{aligned}$$

and

$$\begin{aligned}
&(\mu_n, v_n) \in \Xi(G) \text{and} d(\mu_n, v_n) \le \kappa d(f_n, s_n),\\
&(v_n, \varphi_n) \in \Xi(G) \text{and} d(v_n, \varphi_n) \le \kappa d(h_n, s_n),\\
&(\mu_{n+1}, \varphi_n) \in \Xi(G) \text{and} d(\mu_{n+1}, \varphi_n) \le \kappa d(f_{n+1}, h_n),\\
&(\mu_{n+1}, v_n) \in \Xi(G) \text{and} d(\mu_{n+1}, v_n) \le \kappa d(f_{n+1}, s_n).
\end{aligned}$$

Moreover, we also notice that $\{f_n\}$ is $G - TWC$. Subsequently, we proclaim $\{d(f_n, \mu_n)\}$ is decreasing. Actually, we can acquire

$$\begin{aligned}
d(f_{n+1}, \mu_{n+1}) &\le d(f_{n+1}, \varphi_n) + d(\varphi_n, \mu_{n+1})\\
&= d(W(h_n, \varphi_n; \rho_n), \varphi_n) + d(\varphi_n, \mu_{n+1})\\
&\le (1 - \rho_n) d(h_n, \varphi_n) + \kappa d(f_{n+1}, h_n)\\
&\le (1 - \rho_n) d(h_n, \varphi_n) + \kappa \rho_n d(h_n, \varphi_n)\\
&= [1 + \kappa \rho_n - \rho_n] d(h_n, \varphi_n),
\end{aligned}$$

and

$$\begin{aligned}
d(h_n, \varphi_n) &\le d(h_n, v_n) + d(v_n, \varphi_n)\\
&= d(W(s_n, v_n; e_n), v_n) + d(v_n, \varphi_n)\\
&\le (1 - e_n) d(s_n, v_n) + \kappa d(h_n, s_n)\\
&\le (1 - e_n) d(s_n, v_n) + \kappa e_n d(s_n, v_n)\\
&= (1 + \kappa e_n - e_n) d(s_n, v_n)\\
&\le (1 + \kappa e_n - e_n)[d(s_n, \mu_n) + d(\mu_n, v_n)]\\
&\le (1 + \kappa e_n - e_n)(1 - c_n) d(f_n, \mu_n) + \kappa(1 + \kappa e_n - e_n) d(f_n, s_n)\\
&\le (1 + \kappa e_n - e_n)(1 - c_n) d(f_n, \mu_n) + \kappa c_n (1 + \kappa e_n - e_n) d(f_n, \mu_n)\\
&= (1 + \kappa e_n - e_n)(1 + \kappa c_n - c_n) d(f_n, \mu_n).
\end{aligned}$$

It follows

$$d(f_{n+1}, \mu_{n+1}) \le (1 + \kappa \rho_n - \rho_n) d(h_n, \varphi_n)$$

$$\begin{aligned}&\leq (1+\kappa\rho_n-\rho_n)(1+\kappa e_n-e_n)(1+\kappa c_n-c_n)d(f_n,\mu_n)\\&\leq d(f_n,\mu_n),\end{aligned}$$

which indicates the sequence $\{d(f_n,\mu_n)\}$ is decreasing.

Let $t_n=(1+\kappa\rho_n-\rho_n)(1+\kappa e_n-e_n)(1+\kappa c_n-c_n)$, so we have

$$t_n\in(0,1) \text{ and } d(f_{n+1},\mu_{n+1})\leq t_n d(f_n,\mu_n).$$

And we also find

$$\begin{aligned}d(f_n,f_{n+1})&=d(f_n,W(h_n,\varphi_n;\rho_n))\\&\leq(1-\rho_n)d(f_n,h_n)+\rho_n d(f_n,\varphi_n)\\&=(1-\rho_n)d(f_n,W(s_n,v_n;e_n))+\rho_n d(f_n,\varphi_n)\\&\leq(1-\rho_n)(1-e_n)d(f_n,s_n)+(1-\rho_n)e_n d(f_n,v_n)+\rho_n d(f_n,\varphi_n)\\&\leq(1-\rho_n)(1-e_n)c_n d(f_n,\mu_n)+(1-\rho_n)e_n[d(f_n,\mu_n)\\&\quad+d(\mu_n,v_n)]+\rho_n[d(f_n,\mu_n)+d(\mu_n,v_n)+d(v_n,\varphi_n)]\\&\leq(1-\rho_n)(1-e_n)c_n d(f_n,\mu_n)+(1-\rho_n)e_n d(f_n,\mu_n)\\&\quad+\kappa(1-\rho_n)e_n c_n d(f_n,\mu_n)+\rho_n d(f_n,\mu_n)\\&\quad+\kappa\rho_n c_n d(f_n,\mu_n)+\kappa\rho_n e_n d(s_n,v_n)\\&\leq(1-\rho_n)(1-e_n)c_n d(f_n,\mu_n)+(1-\rho_n)e_n d(f_n,\mu_n)\\&\quad+\kappa(1-\rho_n)e_n c_n d(f_n,\mu_n)+\rho_n d(f_n,\mu_n)+\kappa\rho_n c_n d(f_n,\mu_n)\\&\quad+\kappa\rho_n e_n[d(s_n,\mu_n)+d(\mu_n,v_n)]\\&\leq(1-\rho_n)(1-e_n)c_n d(f_n,\mu_n)+(1-\rho_n)e_n d(f_n,\mu_n)\\&\quad+\kappa(1-\rho_n)e_n c_n d(f_n,\mu_n)+\rho_n d(f_n,\mu_n)+\kappa\rho_n c_n d(f_n,\mu_n)\\&\quad+\kappa\rho_n e_n(1-c_n)d(f_n,\mu_n)+\kappa^2\rho_n e_n c_n d(f_n,\mu_n)\\&=[\rho_n+e_n+c_n+\kappa\rho_n e_n+\kappa\rho_n c_n+\kappa e_n c_n+\rho_n e_n c_n\\&\quad-\rho_n e_n-\rho_n c_n-e_n c_n-2\kappa\rho_n e_n c_n]d(f_n,\mu_n).\end{aligned}$$

Let $\rho_n+e_n+c_n+\kappa\rho_n e_n+\kappa\rho_n c_n+\kappa e_n c_n+\rho_n e_n c_n-\rho_n e_n-\rho_n c_n-e_n c_n-2\kappa\rho_n e_n c_n=\gamma_n$. Furthermore, for any $q\in\mathbb{Z}^+$, we can infer

$$\begin{aligned}d(f_n,f_{n+q})&\leq d(f_n,f_{n+1})+d(f_{n+1},f_{n+2})+\cdots+d(f_{n+q-1},f_{n+q})\\&\leq\gamma_n d(f_n,\mu_n)+\gamma_{n+1}d(f_{n+1},\mu_{n+1})+\cdots+\gamma_{n+q-1}d(f_{n+q-1},\mu_{n+q-1})\\&\leq(\gamma_n\prod_{i=0}^{n-1}t_i+\gamma_{n+1}\prod_{i=0}^{n}t_i+\cdots+\gamma_{n+q-1}\prod_{i=0}^{n+q-2}t_i)d(f_0,\mu_0).\end{aligned}$$

Let $D_{n+j}=\gamma_{n+j}\prod\limits_{i=0}^{n+j-1}t_i$, $j=0,1,2,\ldots,q-1$. Then we obtain

$$d(f_n,f_{n+q})\leq(D_n+D_{n+1}+\cdots+D_{n+q-1})d(f_0,\mu_0).$$

Since $\{\rho_n\}$, $\{e_n\}$ and $\{c_n\}$ are monotonous, we can get that $\{\gamma_n\}$ is also monotonous. Furthermore, we can acquire that

$$\lim_{j\to\infty}sup\frac{D_{n+j+1}}{D_{n+j}}=\lim_{j\to\infty}sup\frac{\gamma_{n+j+1}\prod\limits_{i=0}^{n+j}t_i}{\gamma_{n+j}\prod\limits_{i=0}^{n+j-1}t_i}$$

$$\begin{aligned}&= \lim_{j\to\infty} sup \frac{\gamma_{n+j+1} t_{n+j}}{\gamma_{n+j}}\\&= \lim_{j\to\infty} sup \frac{\gamma_{n+j+1}}{\gamma_{n+j}}[(1+\kappa\rho_{n+j}-\rho_{n+j})\\&\quad\times(1+\kappa e_{n+j}-e_{n+j})(1+\kappa c_{n+j}-c_{n+j})]\\&< 1.\end{aligned}$$

According to the virtue of D'Alembert's test, we deduce $\sum_{j=0}^{\infty} D_j$ is convergent. Thus, we can draw a conclusion $\lim_{n\to\infty} d(f_n, f_{n+q}) = 0$ which indicates that $\{f_n\}$ is a Cauchy sequence. Since G is G-complete, we can find a $q \in \Omega(G)$ that makes $\lim_{n\to\infty} d(f_n, q) = 0$ hold. According to the property $(\mathbb{P})$, for large enough n, we can acquire $(f_n, q) \in \Xi(G)$, thus there is $q_n \in \Gamma p$ such that

$$d(f_n, q_n) \le \kappa d(f_n, q),$$

which implies $d(f_n, q_n) \to 0$ as $n \to \infty$. Let $n \to +\infty$, then

$$d(q_n, q) \le d(q_n, f_n) + d(f_n, q) \to 0,$$

which indicates $q \in \Gamma q$ since Γq is closed. □

Remark 4.1. From the proof process of Theorem 4.1, we can also gain

$$\lim_{n\to\infty} d(f_n, \mu_n) = 0,\ \lim_{n\to\infty} d(f_n, h_n) = 0,$$

and

$$\lim_{n\to\infty} d(f_n, s_n) = 0,\ \lim_{n\to\infty} d(h_n, s_n) = 0.$$

Proof. Thanks to the definitions of $\{f_n\}$ and $\{h_n\}$, we can get

$$\begin{aligned}d(f_n, h_n) &= d(f_n, W(s_n, v_n; e_n))\\&\le (1-e_n)d(f_n, s_n) + e_n d(f_n, v_n)\\&\le (1-e_n)c_n d(f_n, \mu_n) + e_n[d(f_n, \mu_n) + d(\mu_n, v_n)]\\&\le (1-e_n)c_n d(f_n, \mu_n) + e_n d(f_n, \mu_n) + \kappa e_n d(f_n, s_n)\\&\le (1-e_n)c_n d(f_n, \mu_n) + e_n d(f_n, \mu_n) + \kappa e_n c_n d(f_n, \mu_n)\\&= [e_n + c_n + \kappa e_n c_n - e_n c_n] d(f_n, \mu_n).\end{aligned}$$

From the proof of Theorem 4.1, it can be found that

$$d(f_n, \mu_n) \le \prod_{i=0}^{n-1} t_i d(f_0, \mu_0),$$

which indicates $\lim_{n\to\infty} d(f_n, s_n) = 0$ since $t_i \in (0, 1)$. Furthermore, we can acquire that $\lim_{n\to\infty} d(f_n, h_n) = 0$.

From the definitions of $\{f_n\}$ and $\{s_n\}$, it follows

$$\begin{aligned}d(f_n, s_n) &= d(f_n, W(f_n, \mu_n; c_n))\\&\le c_n d(f_n, \mu_n),\end{aligned}$$

so we can obtain $\lim_{n\to\infty} d(f_n, s_n) = 0$.

From the definitions of $\{h_n\}$ and $\{s_n\}$, we have

$$\begin{aligned}
d(h_n, s_n) &= d(W(s_n, v_n; e_n), W(f_n, \mu_n; c_n)) \\
&\le d(W(s_n, v_n; e_n), f_n) + d(f_n, W(f_n, \mu_n; c_n)) \\
&\le (1 - e_n)d(f_n, s_n) + e_n d(f_n, v_n) + c_n d(f_n, \mu_n) \\
&\le (1 - e_n)c_n d(f_n, \mu_n) + e_n[d(f_n, s_n) \\
&\quad + d(s_n, \mu_n) + d(\mu_n, v_n)] + c_n d(f_n, \mu_n) \\
&\le (1 - e_n)c_n d(f_n, \mu_n) + e_n c_n d(f_n, \mu_n) + e_n(1 - c_n)d(f_n, \mu_n) \\
&\quad + \kappa c_n d(f_n, \mu_n) + c_n d(f_n, \mu_n) \\
&= [e_n + 2c_n + \kappa c_n - e_n c_n]d(f_n, \mu_n),
\end{aligned}$$

then we can acquire $\lim_{n\to\infty} d(h_n, s_n) = 0$. □

Theorem 4.2. Presume all assumptions of Theorem 4.1 hold, and set

$$\begin{cases} s_n = W(f_n, \mu_n; c_n), \\ h_n = W(s_n, v_n; e_n), \\ f_{n+1} = W(h_n, \varphi_n; \rho_n), \end{cases}$$

where $\mu_n \in \Gamma f_n$, $v_n \in \Gamma s_n$, $\varphi_n \in \Gamma h_n$, ρ_n, e_n, $c_n \in (0, 1)$, and

$$\begin{cases} d_n = W(a_n, \tau_n; \delta_n), \\ b_n = W(d_n, \xi_n; \omega_n), \\ a_{n+1} = W(b_n, g_n; \lambda_n), \end{cases}$$

where $\tau_n \in \Gamma a_n$, $\xi_n \in \Gamma d_n$, $g_n \in \Gamma b_n$, and λ_n, ω_n, $\delta_n \in (0, 1)$. In addition, $\{f_n\}$ and $\{a_n\}$ are generated from the above iterative process where $\{f_n\}$ converges to f and $\{a_n\}$ converges to a, the sequence $\{\lambda_n\}$ satisfies $\lim_{n\to\infty} \lambda_n = \lambda \ne 0$. Then $f = a$ provided that $(f_n, a_n) \in \Xi(G)$ for large enough $n \in \mathbb{Z}^+$.

Proof. According to Theorem 4.1, it follows f and a are fixed points of Γ. Since Γ is a G-contraction, $(f_n, a_n) \in \Xi(G)$ and $(f_n, h_n) \in \Xi(G)$, for all $\mu_n \in \Gamma f_n$, there are $\tau_n \in \Gamma a_n$, $\varphi_n \in \Gamma h_n$ such that

$$(\mu_n, \tau_n) \in \Xi(G),\ d(\mu_n, \tau_n) \le \kappa d(f_n, a_n),$$

and

$$(\mu_n, \varphi_n) \in \Xi(G),\ d(\mu_n, \varphi_n) \le \kappa d(f_n, h_n).$$

Similarly, for $(a_n, b_n) \in \Xi(G)$, we have that for all $\tau_n \in \Gamma a_n$, there is $g_n \in \Gamma b_n$ such that

$$(\tau_n, g_n) \in \Xi(G) \text{ and } d(\tau_n, g_n) \le \kappa d(a_n, b_n).$$

From Remark 4.1, we deduce $\lim_{n\to\infty} d(f_n, h_n) = 0$, $\lim_{n\to\infty} d(a_n, b_n) = 0$ and $\lim_{n\to\infty} d(f_n, \mu_n) = 0$. Combining the conditions $\lim_{n\to\infty} d(f_n, f) = 0$ and $\lim_{n\to\infty} d(a_n, a) = 0$, we can obtain $\lim_{n\to\infty} d(h_n, f) = 0$ and $\lim_{n\to\infty} d(b_n, a) = 0$. By using the property $(\mathbb{P})$, we can acquire that $(f_n, f) \in \Xi(G)$, $(a_n, a) \in \Xi(G)$, $(h_n, f) \in \Xi(G)$, $(b_n, a) \in \Xi(G)$ and $(f_n, \mu_n) \in \Xi(G)$ for large enough n. By the transitivity of the graph G, we also draw a conclusion that $(h_n, b_n) \in \Xi(G)$, $(h_n, g_n) \in \Xi(G)$, $(\varphi_n, b_n) \in \Xi(G)$ and $(\varphi_n, g_n) \in \Xi(G)$.

Notice that

$$d(f, a) \leq d(f, f_{n+1}) + d(f_{n+1}, a_{n+1}) + d(a_{n+1}, a), \tag{4.2}$$

and

$$\begin{aligned}
&d(f_{n+1}, a_{n+1}) \\
=& d(W(h_n, \varphi_n; \rho_n), W(b_n, g_n; \lambda_n)) \\
\leq& (1-\rho_n)(1-\lambda_n)d(h_n, b_n) + (1-\rho_n)\lambda_n d(h_n, g_n) \\
&+ \rho_n(1-\lambda_n)d(\varphi_n, b_n) + \rho_n\lambda_n d(\varphi_n, g_n) \\
\leq& (1-\rho_n)(1-\lambda_n)[d(h_n, f_n) + d(f_n, a_n) + d(a_n, a) + d(a, b_n)] \\
&+ (1-\rho_n)\lambda_n[d(h_n, f_n) + d(f_n, \mu_n) + d(\mu_n, \tau_n) + d(\tau_n, g_n)] \\
&+ \rho_n(1-\lambda_n)[d(\varphi_n, \mu_n) + d(\mu_n, f_n) + d(f_n, a_n) + d(a_n, a) + d(a, b_n)] \\
&+ \rho_n\lambda_n[d(\varphi_n, \mu_n) + d(\mu_n, \tau_n) + d(\tau_n, g_n)] \\
=& (1-\rho_n)d(h_n, f_n) + (1-\lambda_n)[d(f_n, a_n) + d(a_n, a) + d(a, b_n)] \\
&+ [(1-\rho_n)\lambda_n + \rho_n(1-\lambda_n)]d(f_n, \mu_n) + \lambda_n[d(\mu_n, \tau_n) + d(\tau_n, \varphi_n)] + \rho_n d(\varphi_n, \mu_n) \\
\leq& (1-\rho_n)d(h_n, f_n) + (1-\lambda_n)d(f_n, a_n) + (1-\lambda_n)[d(a_n, a) + d(a, b_n)] \\
&+ [(1-\rho_n)\lambda_n + \rho_n(1-\lambda_n)]d(f_n, \mu_n) + \kappa\lambda_n[d(f_n, a_n) + d(a_n, b_n)] \\
&+ \kappa\rho_n d(f_n, h_n) \\
\leq& (1-\rho_n)d(h_n, f_n) + (1-\lambda_n)[d(f_n, f) + d(f, a) + d(a, a_n)] \\
&+ (1-\lambda_n)[d(a_n, a) + d(a, b_n)] + [(1-\rho_n)\lambda_n + \rho_n(1-\lambda_n)]d(f_n, \mu_n) \\
&+ \kappa\lambda_n[d(f_n, f) + d(f, a) + d(a, a_n)] + \kappa\lambda_n[d(a_n, a) + d(a, b_n)] + \kappa\rho_n d(f_n, h_n) \\
=& (1+\kappa\rho_n-\rho_n)d(f_n, h_n) + (1+\kappa\lambda_n-\lambda_n)[d(f_n, f) + d(f, a) + d(a, a_n)] \\
&+ (1+\kappa\lambda_n-\lambda_n)[d(a_n, a) + d(a, b_n)] + (\rho_n+\lambda_n-2\rho_n\lambda_n)d(f_n, \mu_n) \\
<& (1+\kappa\rho_n-\rho_n)d(f_n, h_n) + (1+\kappa\lambda_n-\lambda_n)[d(f_n, f) + d(a, a_n)] \\
&+ (1+\kappa\lambda_n-\lambda_n)[d(a_n, a) + d(a, b_n)] + 2d(f_n, \mu_n) + (1+\kappa\lambda_n-\lambda_n)d(f, a).
\end{aligned} \tag{4.3}$$

Combining with (4.2) and (4.3), we can obtain

$$\begin{aligned}
(1-\kappa)\lambda_n d(f, a) \leq& d(f, f_{n+1}) + (1+\kappa\rho_n-\rho_n)d(f_n, h_n) \\
&+ (1+\kappa\lambda_n-\lambda_n)[d(f_n, f) + d(a, a_n)] \\
&+ (1+\kappa\lambda_n-\lambda_n)[d(a_n, a) + d(a, b_n)] + 2d(f_n, \mu_n) + d(a_{n+1}, a).
\end{aligned}$$

Letting $n \to \infty$, we have $(1-\kappa)\lambda d(f, a) \leq 0$. Since $\kappa \in (0, 1)$, $\lim_{n\to\infty} \lambda_n = \lambda \neq 0$, we can acquire $d(f, a) = 0$, that is $f = a$. $\square$

5. Conclusion

In this paper, by using the convex structure, we extended the Ishikawa iterative algorithm and the SP iterative algorithm to grapgical metric spaces. We obtained the existence and uniqueness of fixed points for set-valued G-contractions in the above space. And an example was explored to demonstrate the hypotheses of the existence theorem of fixed points for set-valued G-contractions are sufficient but not necessary.

Open problems

• Can the condition that GCMS in Theorem 3.1 and Theorem 4.1 satisfies properties ($\mathbb{P}$) and ($\mathbb{Q}$) be weakened? If this condition is weakened or even removed, can the corresponding conclusions still be reached?
• In the paper, the example is given without the conditions of Theorem 3.1, that is, GCMS satisfies properties ($\mathbb{P}$) and ($\mathbb{Q}$), and the theorem can still be established. Then, can we find an example that satisfies the conditions of Theorem 3.1?

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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