Journal of Applied Analysis and Computation
Volume 15, Number 1, February 2025, 1–8
Website:http://www.jaac-online.com
DOI:10.11948/20240319

ON SYMMETRY PROPERTY OF CENTER MANIFOLDS OF DIFFERENTIAL SYSTEMS*

Maoan Han[1,†] and Ai Ke[1]

Abstract We present a simple and new proof on some symmetry properties of center manifolds of differential systems under certain symmetry conditions in a given differential system. These properties are fundamental to study local behavior of orbits, including stability of singular points, bifurcation of periodic solutions and homoclinic orbits of the reduced equations.

Keywords Center manifold, symmetrical property, invariance.

MSC(2010) 34C45, 37D10.

1. Introduction on center manifolds

As we know, the theory of center manifolds plays an important role in the study of differential systems. It is a valid tool to reduce the dimension of the phase space. See [1–9] for general theory of center manifolds and applications of the theory. For a given system, using its center manifold we can obtain a reduced system under certain conditions. If we know more information about the center manifold we can find more properties of the reduced system and therefore make a more precise analysis on local behavior of orbits. There have been many studies on this aspect, see [3,6] and references therein. In this paper we provide a simple and new proof on some symmetry properties of center manifolds of differential systems under certain symmetry conditions in a given differential system in a similar manner to the proof of local center manifold theorem given in [1]. In this section we list two well-known theorems on the existence and uniqueness of center manifolds and a result obtained in [6].

Consider a differential system of the form

$$\dot{x} = Ax + f(x),\ x \in \mathbb{R}^n, \tag{1.1}$$

where $n \geq 1$, A is an $n \times n$ matrix, and $f \in C^k(\mathbb{R}^n)$ for some $k \geq 1$ with

$$f(0) = 0, \quad Df(0) = 0. \tag{1.2}$$

Before stating center manifold theorems, we first introduce some notations. Let

$$x = (x_1, x_2, \cdots, x_n),\ \|x\| = \Big(\sum_{i=1}^{n} x_i^2\Big)^{\frac{1}{2}},\ f(x) = (f_1(x), f_2(x), \cdots, f_n(x)).$$

†The corresponding author.
[1]School of Mathematical Sciences, Zhejiang Normal University, Jinhua 321004, China
*This work is supported by the National Key R&D Program of China (No. 2022YFA1005900) and the National Natural Science Foundation of China (No. 12301214).
Email: mahan@zjnu.edu.cn(M. Han), aike_math@zjnu.edu.cn(A. Ke)

Define

$$\|f\|_k = \max_{\substack{0 \le j \le k \\ 1 \le i,\, l \le n}} \sup_{x \in \mathbb{R}^n} \left| \frac{\partial^j f_i(x)}{\partial^j x_l} \right|$$

and

$$C_b^k(\mathbb{R}^n) = \{ f \in C^k(\mathbb{R}^n) \mid \|f\|_k < \infty \}.$$

Also, let

$$\|Df\|_0 = \max_{1 \le i,\, l \le n} \sup_{x \in \mathbb{R}^n} \left| \frac{\partial f_i(x)}{\partial x_l} \right|.$$

To state center manifold theorems clearly, we let further

$$Ax = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad u \in \mathbb{R}^{n_1}, \quad v \in \mathbb{R}^{n_2}, \tag{1.3}$$

where $n = n_1 + n_2$, A_1 is an $n_1 \times n_1$ matrix with eigenvalues having zero real part and A_2 an $n_2 \times n_2$ matrix with each eigenvalue having nonzero real part. Then by Theorem 1.1 and Theorem 2.1 in Chapter one of [1], we have the following theorem which is called the global center manifold theorem.

Theorem 1.1. *Suppose that* (1.2) *and* (1.3) *hold. Let* $f \in C_b^k(\mathbb{R}^n)$, $k \ge 1$. *Then there is a number* $\delta_k > 0$ *such that if* $\|Df\|_0 < \delta_k$ *then* (1.1) *has a unique global center manifold* W^c *of class* C^k *which is invariant and has the form*

$$W^c = \{ x = (u, v) \in \mathbb{R}^n \mid v = \psi(u),\ u \in \mathbb{R}^{n_1} \}, \tag{1.4}$$

where

$$\psi \in C^k,\ \ \psi(0) = 0,\ \ D\psi(0) = 0,\ \ Lip(\psi) < 1 \text{ and } \sup_{u \in \mathbb{R}^{n_1}} \|\psi(u)\| < \infty. \tag{1.5}$$

We remark that the uniqueness of W^c of the form (1.4) means that the function ψ satisfying (1.5) is unique.

By Theorem 3.2 of Chapter one in [1], we have the following local manifold theorem.

Theorem 1.2. *Let* (1.2) *and* (1.3) *hold. Suppose* $f \in C^k(\mathbb{R}^n)$ *with* $k \ge 1$. *Then* (1.1) *has a local* C^k *center manifold* W_l^c *of the form*

$$W_l^c = \{ x = (u, v) \mid v = \psi(u),\ u \in V \}, \tag{1.6}$$

where V *is an open neighborhood of the origin in* $\mathbb{R}^{n_1}$, ψ *is a* C^k *function on* V, *and*

$$\psi(0) = 0, \quad D\psi(0) = 0. \tag{1.7}$$

As we know (see [1]), Theorem 1.2 can be obtained by applying Theorem 1.1 to a system of the form

$$\dot{x} = Ax + f(x)\varphi\Big(\frac{x}{\rho}\Big), \quad x \in \mathbb{R}^n,$$

where $\rho > 0$ is a small constant, φ is a C^∞ cut-off function from $\mathbb{R}^n$ to $[0, 1]$ satisfying

$$\varphi(x) = 1\,(= 0,\ \text{resp.}) \text{ for } \|x\| \le 1\,(\|x\| \ge 2,\ \text{resp.}).$$

The author [6] gave a theorem without proof which concerns bifurcations of equivariant vector field on a Banach space supposing that the vector field is invariant under a group which acts linearly and isometrically on the space. In particular, the following theorem implied from [6] (also see Theorem 7.6 in [3]).

Theorem 1.3. *Consider* (1.1) *with* (1.2) *and* (1.3) *satisfied. Let* $\Lambda = \{T_g | g \in G\}$ *be a linear representation of a group* G *by isometries of* $\mathbb{R}^n$ *such that for all* $g \in G$

$$T_g(Ax + f(x)) = AT_g x + f(T_g x).$$

Then any center manifold W^c *of the equilibrium of* (1.1) *at the origin is locally* G*-invariant. Moreover, there are local coordinates* $\xi \in \mathbb{R}^{n_1}$ *on* W^c *in which the restriction of* (1.1) *to* W^c

$$\dot{\xi} = f^c(\xi), \ \xi \in \mathbb{R}^{n_1}$$

is invarian with respect to the restriction Λ^c *of* Λ *to* W^c*, i.e., for all* $T_g^c \in \Lambda^c$

$$T_g^c f^c(\xi) = f^c(T_g^c \xi).$$

In section 2 we present some results on symmetry properties of center manifolds and the reduced systems under certain symmetry conditions supposed for a given system of the form (1.1) and provide a simple and new proof of them.

2. Main results and proof

Consider system (1.1). Let (1.2) and (1.3) hold. Then (1.1) can be rewritten as

$$\begin{aligned} \dot{u} &= A_1 u + \tilde{f}(u, v), \\ \dot{v} &= A_2 v + \tilde{g}(u, v), \end{aligned} \quad (u, v) \in \mathbb{R}^n.$$

First, we have

Theorem 2.1. *Consider system* (1.1)*, satisfying* (1.2) *and* (1.3)*. Then we have*

(a) If

$$\begin{aligned} \tilde{f}(-u, v) &= -\tilde{f}(u, v), \\ \tilde{g}(-u, v) &= \tilde{g}(u, v) \end{aligned} \tag{2.1}$$

for $(u, v) \in \mathbb{R}^n$*, then* (1.1) *has a local* C^k *center manifold* W_l^c *of the form* (1.6)*, where* ψ *is a* C^k *function on* V *satisfying* (1.7) *and*

$$\psi(-u) = \psi(u) \text{ for } \pm u \in V. \tag{2.2}$$

(b) If $f(-x) = -f(x)$ *or equivalently*

$$\begin{aligned} \tilde{f}(-u, -v) &= -\tilde{f}(u, v), \\ \tilde{g}(-u, -v) &= -\tilde{g}(u, v), \end{aligned} \tag{2.3}$$

then (1.1) *has a local* C^k *center manifold* W_l^c *of the form* (1.6)*, where* ψ *is a* C^k *function on* V *satisfying* (1.7) *and*

$$\psi(-u) = -\psi(u) \text{ for } \pm u \in V. \tag{2.4}$$

Proof. Let $\varphi_0 : (-\infty, +\infty) \to [0,1]$ be a C^∞ function satisfying $\varphi_0(x) = 1$ (= 0, resp.) for $|x| \leq 1$ ($|x| \geq 2$, resp.). Based on φ_0 we introduce a C^∞ function φ on $\mathbb{R}^n$ as follows

$$\varphi(x) = \varphi_0(\|x\|), \quad x \in \mathbb{R}^n, \tag{2.5}$$

where $\|x\|$ is the Euclidean norm of x.

Consider a system of the form

$$\dot{x} = Ax + f_\rho(x), \quad x \in \mathbb{R}^n, \tag{2.6}$$

where

$$f_\rho(x) = f(x)\varphi\Big(\frac{x}{\rho}\Big), \quad x \in \mathbb{R}^n, \tag{2.7}$$

and $\rho > 0$ is a constant.

By Lemma 3.1 in [1] and (1.2) we have $f_\rho \in C_b^k(\mathbb{R}^n)$ and $\|Df_\rho\|_0 < \delta_k$ as $\rho > 0$ is sufficiently small, where δ_k is the constant in Theorem 1.1. Then by Theorem 1.1, (2.6) has a unique global center manifold W^c of the form (1.4), where ψ satisfies (1.5).

Let

$$f_\rho(x) = (\tilde{f}_\rho(u,v), \tilde{g}_\rho(u,v)), \quad (u,v) \in \mathbb{R}^n,$$

where

$$\tilde{f}_\rho(u,v) = \tilde{f}(u,v)\varphi\Big(\frac{x}{\rho}\Big), \ \tilde{g}_\rho(u,v) = \tilde{g}(u,v)\varphi\Big(\frac{x}{\rho}\Big).$$

Note that (2.6) can be rewritten as

$$\begin{aligned} \dot{u} &= A_1 u + \tilde{f}_\rho(u,v), \\ \dot{v} &= A_2 v + \tilde{g}_\rho(u,v). \end{aligned} \tag{2.8}$$

If (2.1) is satisfied, then by (2.5) and (2.7)

$$\tilde{f}_\rho(-u,v) = -\tilde{f}_\rho(u,v), \ \tilde{g}_\rho(-u,v) = \tilde{g}(u,v)$$

for all $(u,v) \in \mathbb{R}^n$, which implies that (2.8) is invariant under the change $(u,v) \to (-u,v)$.

Note that the manifold W^c becomes

$$\tilde{W}^c = \{x = (u,v) \in \mathbb{R}^n \,|\, v = \psi(-u), \, u \in \mathbb{R}^{n_1}\}$$

under the change $(u,v) \to (-u,v)$. On the other hand, $\tilde{W}^c$ is also a global center manifold of (2.8). Thus, the uniqueness of global center manifold implies that

$$\psi(u) = \psi(-u) \text{ for } u \in \mathbb{R}^{n_1}.$$

Obviously, there exists $\varepsilon_0 = \varepsilon_0(\rho) > 0$ such that $\|(u, \psi(u))\| < \rho$ for $|u| < \varepsilon_0$. Then we can take

$$W_l^c = \{(u,v) \,|\, v = \psi(u), \, |u| < \varepsilon_0\},$$

and the conclusion (a) follows. The conclusion (b) can be obtained in the same way since under (2.3) the system (2.8) is invariant under the change $(u,v) \to (-u,-v)$. □

As we know, the flow of (1.1) on the manifold W_l^c is determined by the following reduced system

$$\dot{u} = A_1 u + \tilde{f}(u, \psi(u)) \equiv f_l(u), \ |u| < \varepsilon_0. \tag{2.9}$$

If (2.1) or (2.3) holds, then by (2.2) and (2.4) we have $f_l(-u) = -f_l(u)$ for $|u|$ small. Thus, (2.9) is centrally symmetric with respect to the origin.

We remark that the two conclusions in Theorem 2.1 can be obtained from Theorems 1.2 and 1.3. Here, we present a clear and precise statement on symmetry properties of center manifolds and provide a simple and new proof for the conclusions in two concrete cases which are very typical.

By the invariance of W_l^c we have for $|u|$ small

$$A_2\psi(u) + \tilde{g}(u, \psi(u)) = \psi'(u)[A_1u + \tilde{f}(u, \psi(u))],$$

which can be used to compute expansions of $\psi(u)$ at $u = 0$.

For example, by Theorem 2.1, the system

$$\dot{x} = x^3 + 2xy, \quad \dot{y} = y + x^2$$

has a local center manifold of the form

$$y = -x^2 + 2x^4 + O(x^6) = \psi(x).$$

The corresponding reduced system is

$$\dot{x} = x^3 + 2x\psi(x) = -x^3 + O(x^5).$$

Similarly, the system

$$\begin{aligned} \dot{x} &= y + xz, \\ \dot{y} &= -x + yz, \\ \dot{z} &= -z + xy + x^2 \end{aligned}$$

has a local center manifold of the form

$$z = \frac{1}{5}(4x^2 - xy + y^2) + O(|x, y|^4),$$

which is even in (x, y).

Now we generalize Theorem 2.1 to a more general case by a similar proof.

Let further

$$A_1u = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_j \in \mathbb{R}^{m_j}, \quad j = 1, 2, \tag{2.10}$$

where $m_1 + m_2 = n_1$, $u = (u_1, u_2)^T$. Then we can rewrite (1.1) as

$$\begin{aligned} \dot{u_1} &= B_1u_1 + \tilde{f}_1(u_1, u_2, v), \\ \dot{u_2} &= B_2u_2 + \tilde{f}_2(u_1, u_2, v), \\ \dot{v} &= A_2v + \tilde{g}(u_1, u_2, v), \end{aligned} \tag{2.11}$$

where $(u_1, u_2, v) \in G \subset \mathbb{R}^n$ with G the domain of (1.1). By a very similar way to the proof of Theorem 2.1 we can prove the following theorem.

Theorem 2.2. *Consider system* (1.1), *where the function* f *is defined on an open set* G *of the form*

$$G = \{x \in \mathbb{R}^n \mid \|x\| < \varepsilon_0\},\ \ \varepsilon_0 > 0.$$

Let (1.2), (1.3) *and* (2.10) *hold.*
(a) If (2.11) *satisfies that*

$$\begin{aligned}&\tilde{f}_1(-u_1, u_2, v) = -\tilde{f}_1(u_1, u_2, v),\\&\tilde{f}_2(-u_1, u_2, v) = \tilde{f}_2(u_1, u_2, v),\\&\tilde{g}(-u_1, u_2, v) = \tilde{g}(u_1, u_2, v)\end{aligned}$$

for $(u_1, u_2, v) \in G$, *then* (1.1) *has a local* C^k *center manifold* W_l^c *of the form* (1.6), *where* ψ *is a* C^k *function on* V *satisfying* (1.7) *and*

$$\psi(-u_1, u_2) = \psi(u_1, u_2) \text{ for } (\pm u_1, u_2) \in V.$$

(b) If (2.11) *satisfies that*

$$\begin{aligned}&\tilde{f}_1(-u_1, u_2, -v) = -\tilde{f}_1(u_1, u_2, v),\\&\tilde{f}_2(-u_1, u_2, -v) = \tilde{f}_2(u_1, u_2, v),\\&\tilde{g}(-u_1, u_2, -v) = -\tilde{g}(u_1, u_2, v)\end{aligned}$$

for $(u_1, u_2, v) \in G$, *then* (1.1) *has a local* C^k *center manifold* W_l^c *of the form* (1.6), *where* ψ *is a* C^k *function on* V *satisfying* (1.7) *and*

$$\psi(-u_1, u_2) = -\psi(u_1, u_2) \text{ for } (\pm u_1, u_2) \in V.$$

Under the conditions of the above theorem, instead of (2.7) we define the function f_ρ in (2.6) as

$$f_\rho(x) = \begin{cases} f(x)\varphi(\frac{x}{\rho}), & \|x\| < \varepsilon_0,\\ 0, & \|x\| \geq \varepsilon_0.\end{cases}$$

Then as before, when $\rho \in (0, \varepsilon_0/2)$ is sufficiently small, we have

$$f_\rho \in C_b^k(\mathbb{R}^n),\ \ \|Df_\rho\|_0 < \delta_k.$$

Hence, Theorem 2.2 can be obtained by applying Theorem 1.1 in the same way as Theorem 2.1.

The above theorem can be used to study center manifolds of differential systems with parameters. For simplicity, consider a three dimensional system of the form

$$\begin{aligned}&\dot{x} = y + f_1(x, y, z, \varepsilon),\\&\dot{y} = -a_n x^{2n+1} + f_2(x, y, z, \varepsilon),\\&\dot{z} = \lambda z + f_3(x, y, z, \varepsilon),\end{aligned} \tag{2.12}$$

where $\lambda \neq 0$ is a constant, $a_n = 1\,(\neq 0)$ for $n = 0\,(n \geq 1)$, f_1, f_2 and f_3 are C^k functions with $k \geq 2$ for $(x, y, z, \varepsilon) \in G \times U$ with $G \subset \mathbb{R}^3$ containing the origin in $\mathbb{R}^3$ and $U \subset \mathbb{R}^m$ a neighborhood of $\varepsilon = 0$ in $\mathbb{R}^m$ for some $m \geq 1$. Further, suppose

$$f_j(x, y, z, 0) = O(|x, y, z|^2), \quad j = 1, 2, 3. \tag{2.13}$$

By adding the equation $\dot{\varepsilon} = 0$ to (2.12), taking $u_1 = (x, y)$, $u_2 = \varepsilon$, $v = z$ and applying Theorem 2.2 to the resulting system, we obtain immediately

Theorem 2.3. *Let* (2.12) *satisfy* (2.13).
(a) If

$$\begin{aligned} f_1(-x,-y,z,\varepsilon) &= -f_1(x,y,z,\varepsilon), \\ f_2(-x,-y,z,\varepsilon) &= -f_2(x,y,z,\varepsilon), \\ f_3(-x,-y,z,\varepsilon) &= f_3(x,y,z,\varepsilon) \end{aligned} \tag{2.14}$$

for $|x|+|y|+|z|+|\varepsilon|$ *small, then* (2.12) *has a local* C^k *center manifold* W_ε^c *of the form*

$$W_\varepsilon^c = \{(x,y,z) \mid z = \psi(x,y,\varepsilon),\ x^2+y^2<\delta\} \tag{2.15}$$

for $|\varepsilon|<\delta$ *with* $\delta>0$ *a small constant, where* $\psi \in C^k$ *and satisfies*

$$\psi(x,y,\varepsilon) = O(|\varepsilon|+|x,y|^2), \quad \psi(-x,-y,\varepsilon) = \psi(x,y,\varepsilon).$$

(b) If

$$f_j(-x,-y,-z,\varepsilon) = -f_j(x,y,z,\varepsilon), \quad j=1,2,3 \tag{2.16}$$

for $|x|+|y|+|z|+|\varepsilon|$ *small, then* (2.12) *has a local* C^k *center manifold* W_ε^c *of the form* (2.15) *for* $|\varepsilon|<\delta$ *with* $\delta>0$ *a small constant, where* $\psi\in C^k$ *and satisfies*

$$\psi(x,y,\varepsilon) = O(|\varepsilon||x,y|+|x,y|^3), \quad \psi(-x,-y,\varepsilon) = -\psi(x,y,\varepsilon).$$

(c) Under (2.14) *or* (2.16) *the reduced system of* (2.12) *is of the form*

$$\begin{aligned} \dot{x} &= y + f_1(x,y,\psi(x,y,\varepsilon),\varepsilon), \\ \dot{y} &= -a_n x^{2n+1} + f_2(x,y,\psi(x,y,\varepsilon),\varepsilon) \end{aligned}$$

and is centrally symmetric with respect to the origin.

In fact, when (2.16) holds, one has

$$f_j(0,0,0,\varepsilon) = 0, \quad j=1,2,3,$$

which ensure $\psi(0,0,\varepsilon)=0$ since the singular point at the origin must lie on the local center manifold W_ε^c. When (2.14) holds, system (2.12) has a singular point $(0,0,z_0(\varepsilon))$ near the origin. Then one can make a change of variables $(x,y,z)\to(x,y,z-z_0(\varepsilon))$ to move the singular point to the origin before applying Theorem 1.1.

The third conclusion of Theorem 2.3 provides an important property of the reduced system of (2.12) which is really useful in the study of limit cycle bifurcation of Bogdanov-Takens type near the origin for (2.12).

Finally we remark that the function ψ in (2.15) satisfies the equation

$$\lambda\psi + f_3(x,y,\psi,\varepsilon) = \psi_x(y+f_1(x,y,\psi,\varepsilon)) + \psi_y(-a_n x^{2n+1} + f_2(x,y,\psi,\varepsilon)),$$

which can be used to compute expansions of ψ in (x,y) near $(x,y)=(0,0)$ for $|\varepsilon|$ small. In particular, it follows $\psi(0,0,\varepsilon)=0$ if $f_j(0,0,0,\varepsilon)=0$ for $j=1,2,3$.

References

[1] S.-N. Chow, C. Li and D. Wang, *Normal Forms and Bifurcation of Planar Vector Fields*, Cambridge University Press, New York, 1994.

[2] E. Fontich and A. Vieiro, *Dynamics near the invariant manifolds after a Hamiltonian-Hopf bifurcation*, Communications in Nonlinear Science and Numerical Simulation, 2023, 117, 106971.

[3] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory*, Applied Mathematical Sciences, Springer Verlag, 2004.

[4] J. Li, K. Lu and P. W. Bates, *Invariant foliations for random dynamical systems*, Discrete and Continuous Dynamical Systems, 2014, 34(9), 3639–3666.

[5] X. Liu and M. Han, *Poincaré bifurcation of a three-dimensional system*, Chaos, Solitons and Fractals, 2005, 23, 1385–1398.

[6] D. Ruelle, *Bifurcation in the presence of a symmetry group*, Archive for Rational Mechanics and Analysis, 1973, 51, 136–152.

[7] Y. Tian and P. Yu, *An explicit recursive formula for computing the normal form and center manifold of general n-dimensional differential systems associated with Hopf bifurcation*, International Journal of Bifurcation and Chaos, 2013, 23(6), 1350104.

[8] Y. Tian and P. Yu, *Seven limit cycles around a focus point in a simple three-dimensional quadratic vector field*, International Journal of Bifurcation and Chaos, 2014, 24(6), 1450083.

[9] W. Zhang and W. Zhang, *On invariant manifolds and invariant foliations without a spectral gap*, Advances in Mathematics, 2016, 303, 549–610.