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MULTIPLICITY AND CONCENTRATION OF SOLUTIONS TO A SINGULAR CHOQUARD EQUATION WITH CRITICAL SOBOLEV EXPONENT*

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Abstract In this paper, we consider a nonautonomous singular Choquard equation with critical exponent

$$\begin{cases} -\Delta u + V(x)u + \lambda(I_\alpha * |u|^p)|u|^{p-2}u = f(x)u^{-\gamma} + |u|^4 u, & x \in \mathbb{R}^3, \\ u > 0, & x \in \mathbb{R}^3, \end{cases}$$

where I_α is the Riesz potential of order $\alpha \in (0,3)$ and $1+\frac{\alpha}{3} \le p < 3$, $0 < \gamma < 1$. Under certain assumptions on V and f, we show the existence and multiplicity of positive solutions for $\lambda > 0$ by using variational method and Nehari type constraint. We also study concentration of solutions as $\lambda \to 0^+$.

Keywords Singular Choquard equation, variational method, concentration, critical Sobolev exponent.

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1. Introduction

In this paper, we are interested in the nonautonomous Choquard equation

$$\begin{cases} -\Delta u + V(x)u + \lambda(I_\alpha * |u|^p)|u|^{p-2}u = f(x)u^{-\gamma} + |u|^4 u, & x \in \mathbb{R}^3, \\ u > 0, & x \in \mathbb{R}^3, \end{cases} \tag{P_λ}$$

where $1 + \frac{\alpha}{3} \le p < 3$, $0 < \gamma < 1$, $\lambda > 0$ and I_α with $\alpha \in (0,3)$ is the Riesz potential defined by $I_\alpha = \frac{\Gamma(\frac{3-\alpha}{2})}{\Gamma(\frac{\alpha}{2})2^\alpha \pi^{3/2}|x|^{3-\alpha}}$, $x \in \mathbb{R}^3 \backslash \{0\}$. Here, Γ denotes the Gamma function. Throughout the paper, we suppose V and f satisfy:

(V_1) $V \in C(\mathbb{R}^3)$ satisfies $\inf_{x\in\mathbb{R}^3} V(x) > V_0 > 0$, where V_0 is a constant.

(V_2) $\operatorname{meas}\{x \in \mathbb{R}^3 : -\infty < V(x) \le \nu\} < +\infty$ for all $\nu \in \mathbb{R}$.

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(f_1) $f \in L^{\frac{6}{5+\gamma}}(\mathbb{R}^3)$ is a positive function.

(f_2) There are $\delta_1 > 0$, $\max\{\frac{3+\gamma}{2}, \frac{5+\gamma-2\alpha}{2}\} < \beta_1 < \frac{5+\gamma}{2}$ and $\rho_1 > 0$ such that $f(x) \geq \rho_1|x|^{-\beta_1}$ for $|x| < \delta_1$.

Recently, many scholars pay attentions to the following more general Choquard equation

$$-\Delta u + V(x)u + \lambda(I_\alpha * |u|^p)|u|^{p-2}u = h(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \in \mathbb{N}$ and $\alpha \in (0, N)$. Problem (1.1) with $N = 3$, $V(x) = 1$, $\lambda = -1$, $p = \alpha = 2$ and $h(x,u) = 0$ was proposed by Pekar [26] to describe the quantum theory of a polaron at rest and as an approximation to Hartree-Fock theory of one component plasma by Choquard (see [19]). Many papers considered problem (1.1) with $\lambda = -1$: when $V(x) = 1$, $2 \leq p \leq \frac{N+\alpha}{N-2}$ and $h(x,u) = 0$, Ruiz and Van Schaftingen [27] proved that least energy nodal solutions for problem (1.1) have an odd symmetry with respect to a hyperplane when $\alpha \to 0^+$ or $\alpha \to N^-$. Based on [27], Seok [29] further studied limit profiles of ground states as $\alpha \to 0^+$ or $\alpha \to N^-$. When $N \geq 3$,$V(x) = 1$ and $p > 1$, Seok [30] considered problem (1.1) with a critical local term and showed the existence of radially symmetric nontrivial solution and concentration results as $\alpha \to 0^+$. When $N \geq 3$ and $V(x) = 1 + \mu g(x)$ is a potential well, Lü [21] obtained the existence of ground state solutions and concentration results as $\mu \to +\infty$ for problem (1.1) with subcritical exponents and $h(x,u) = 0$. Li et al. [15] extended the results of Lü [21] to critical case and obtained the existence of ground state solutions and concentration results as $\alpha \to 0$. Ghimenti, Moroz and Van Schaftingen [6] got the existence of least action sign-changing radial solutions for problem (1.1) with $V(x) = 1$, $p = 2$ and $h(x,u) = 0$. The solution is constructed as the limit of least action sign-changing radial solutions when $p \searrow 2$. When $V(x) = 1$, Van Schaftingen and Xia [28], Ao [1], Li and Ma [17], Li and Tang [16], Seok [31], Su and Chen [32] further investigated the existence of solutions for problem (1.1) with lower and upper critical exponents. When $V(x) = 1 + \mu g(x)$ satisfying some conditions and $\mu < 0$, Zhong and Tang [42] investigated the existence of ground state sign-changing solutions for problem (1.1) with a critical pure power nonlinearity. As for $\lambda = 1$, Mercuri et al. [23] obtained the existence and regularity of ground state solutions and radial solutions for problem (1.1) with $V(x) = 0$, $p > 1$ and $h(x,u) = |u|^{q-2}u$, $q > 1$. When $N \geq 3$, $p \in \left[1 + \frac{\alpha}{N}, \frac{N}{N-2}\right)$, Wu [36] investigated the existence, multiplicity and asymptotic behavior of positive solution for problem (1.1) with $V(x)$ and $h(x,u)$ satisfying some suitable conditions. Lü [22] and Li et al. [14] discussed the existence and concentration of solutions for problem (1.1) with Kirchhoff term in $\mathbb{R}^3$. We [38] obtained the existence, uniqueness and asymptotical behavior of solutions to problem (1.1) with $N = 3$ and $h(x,u) = f(x)u^{-\gamma}$ i.e. a singular nonlinearity. Mukherjee and Sreenadh [25] investigated a nonlinear Choquard equation with upper critical exponent and singularity. For more related topics, we refer to the survey paper [24] and the references therein.

On the bounded domains $\Omega \subset \mathbb{R}^N$ with $N \geq 3$, problem (1.1) without convolution term i.e. $\lambda = 0$ is related to the following equation

$$\begin{cases} -\Delta u = \mu f(x)u^{-\gamma} + |u|^{2^*-2}u, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $2^* = \frac{2N}{N-2}$ is a critical Sobolev exponent. When $f(x) = 1$, Coclite and Palmieri [4] showed the existence of a solution of (1.2); Yang [37] improved the result of [4] and obtained multiplicity and asymptotic behavior of positive solutions for (1.2); Hirano et al. [8] further established the multiplicity and regularity of positive solutions for (1.2) with $\gamma > 0$; Giacomoni and Saoudi [7] proved a multiplicity result for a more general critical and singular problem, involving also a subcritical term and $0 < \gamma < 3$; Mukherjee and Sreenadh [25] investigated existence, multiplicity and regularity of positive solutions for a nonlinear singular Choquard equation with upper critical exponent. Consider (1.2) with parameter μ multiplying the critical term, Hirano et al. [9] studied multiplicity of positive solutions for the problem; Wang et al. [35], Sun and Wu [34] obtained existence and multiplicity of positive solutions and an exact estimate result for the problem. We [41] investigated the relation between the number of the maxima of the coefficient function of the critical term and the number of the positive solutions for elliptic equations with singularity in $\mathbb{R}^3$. Both Lei et al. [12] and Liu et al. [20] got two positive solutions for problem (1.2) with Kirchhoff term. When $N = 3$ and $f(x)$ satisfying some suitable conditions, Lei and Liao [11] obtained two positive solutions for problem (1.2) with Poisson term i.e. a singular Schrödinger-Poisson system. Lei, Suo and Chu [13] studied a Schrödinger-Newton system with singularity and critical growth terms in $\mathbb{R}^N$. We [40] obtained existence, uniqueness and asymptotic behaviour of positive solutions for fractional Schrödinger-Poisson system with singularity in $\mathbb{R}^3$.

To the best of our knowledge, many works which considered concentration of solutions for Choquard equations [6, 14, 15, 21, 27, 29, 30, 36, 38–40] mainly focus on convergence property of one solution such as one ground state positive or sign-changing (nodal) solution and so on, there are few papers investigated convergence property of multiple solutions. Moreover, comparing problem (P_λ) with the previous mentioned works, we need to overcome the lack of compactness as well as the non-differentiability of the functional of the problem and indirect availability of critical point theory due to the presence of singular term.

Define the function space $E = \{u \in L^2(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3), \|u\|_E < +\infty\}$, where $\|u\|_E = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)\mathrm{d}x\right)^{1/2}$ and $L^s(\mathbb{R}^3)$ is a Lebesgue space with the norm $\|u\|_s = (\int_{\mathbb{R}^3} |u|^s \mathrm{d}x)^{\frac{1}{s}}$. Then E is a Hilbert space with the inner product $\langle u, \psi\rangle_E = \int_{\mathbb{R}^3} (\nabla u \nabla \psi + V(x)u\psi)\mathrm{d}x$. Obviously, for $s \in [2, 6]$, the embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ is continuous. By [2], we can further get that under assumptions (V_1) and (V_2), the embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ is compact for any $s \in [2, 6)$.

The energy functional corresponding to problem (P_λ) given by

$$J_\lambda(u) = \frac{1}{2}\|u\|_E^2 + \frac{\lambda}{2p}\int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p \mathrm{d}x - \frac{1}{1-\gamma}\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 \mathrm{d}x, \tag{1.3}$$

and a function $u \in E$ is called a solution of problem (P_λ) if $u > 0$ in $\mathbb{R}^3$ and for every $\psi \in E$,

$$\langle u, \psi\rangle_E + \lambda \int_{\mathbb{R}^3} (I_\alpha * u^p)u^{p-2}u\psi \mathrm{d}x - \int_{\mathbb{R}^3} f(x)u^{-\gamma}\psi \mathrm{d}x - \int_{\mathbb{R}^3} u^5\psi \mathrm{d}x = 0. \tag{1.4}$$

By using variational method and Nehari type constraint, our main results on existence, multiplicity and concentration of solutions with respect to the parameter λ for problem (P_λ) can be stated as follows.

Theorem 1.1. *Suppose $\lambda > 0$, $0 < \gamma < 1$, $1 + \frac{\alpha}{3} \le p < 3$ and (V_1), (V_2), (f_1) hold, then there exists $T_0 > 0$ such that for all $0 < \|f\|_{\frac{6}{5+\gamma}} < T_0$, problem (P_λ) admits a positive ground state solution u_λ satisfying u_λ tends to u_0 in E as $\lambda \to 0^+$, where u_0 is a positive ground state solution of the limit problem*

$$\begin{cases} -\Delta u + V(x)u = f(x)u^{-\gamma} + |u|^4 u, & x \in \mathbb{R}^3, \\ u > 0, & x \in \mathbb{R}^3. \end{cases} \tag{P_0}$$

Theorem 1.2. *Suppose $\lambda > 0$, $0 < \gamma < 1$, $1 + \frac{\alpha}{3} \le p < 3$ and (V_1), (V_2), (f_1), (f_2) hold, then there exists $0 < T_{00} < T_0$ such that for all $0 < \|f\|_{\frac{6}{5+\gamma}} < T_{00}$, problem (P_λ) has at least two solutions: a positive ground state solution u_λ and a positive solution v_λ. Moreover, as $\lambda \to 0^+$, these solutions have the following convergence:*

(i) u_λ tends to u_0 in E, where u_0 is a positive ground state solution of problem (P_0);

(ii) v_λ tends to v_0 in E, where v_0 is a positive solution of problem (P_0) and $\|u_0\|_E^2 < \|v_0\|_E^2$.

Throughout the paper, we use the following notations.

• $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with the norm $\|\cdot\|^2 = \int_{\mathbb{R}^3} |\nabla \cdot|^2 \mathrm{d}x$.

• Denote $d_\alpha := \frac{\Gamma(\frac{3-\alpha}{2})}{2^\alpha \pi^{3/2} \Gamma(\frac{3+\alpha}{2})} \Big(\frac{\Gamma(\frac{3}{2})}{\Gamma(3)}\Big)^{\frac{\alpha}{3}}$ and $\mathbb{D}(u) := \int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p \mathrm{d}x$, then it holds

$$\langle \mathbb{D}'(u), \psi \rangle = 2p \int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^{p-2} u\psi \mathrm{d}x, \ \forall \psi \in E.$$

• $B_r(x)$ is a ball centered at x with radius r.

• C and C_i denotes various positive constants, which may vary from line to line.

• $\to$ (resp. $\rightharpoonup$) denotes the strong (resp. weak) convergence.

• $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$ for any function u.

• S is the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$, namely,

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x}{\big(\int_{\mathbb{R}^3} |u|^6 \mathrm{d}x\big)^{\frac{1}{3}}} > 0. \tag{1.5}$$

Hence, $\int_{\mathbb{R}^3} |u|^6 \mathrm{d}x \le S^{-3} \|u\|^6 \le S^{-3} \|u\|_E^6$.

2. Preliminary results

In this and next section, we always assume that all assumptions in Theorem 1.1 hold. It follows from Hardy-Littlewood-Sobolev inequality (see [24]) that

$$\mathbb{D}(u) = \int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^p \mathrm{d}x \le d_\alpha \Big(\int_{\mathbb{R}^3} |u|^{\frac{6p}{3+\alpha}} \mathrm{d}x \Big)^{\frac{3+\alpha}{3}}. \tag{2.1}$$

Moreover, since $0 < \gamma < 1$, by Hölder's inequality, (f_1) and (1.5), we have

$$\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma} \mathrm{d}x \le \|f\|_{\frac{6}{5+\gamma}} \Big[\int_{\mathbb{R}^3} |u|^6 \mathrm{d}x \Big]^{\frac{1-\gamma}{6}} \le \|f\|_{\frac{6}{5+\gamma}} S^{\frac{\gamma-1}{2}} \|u\|_E^{1-\gamma}, \tag{2.2}$$

and for any $u,\, v\in E$, it holds

$$\begin{aligned}\left|\int_{\mathbb{R}^3} f(x)\Big(|u|^{1-\gamma}-|v|^{1-\gamma}\Big)\mathrm{d}x\right| &\le \int_{\mathbb{R}^3} f(x)\,|u-v|^{1-\gamma}\,\mathrm{d}x\\ &\le \|f\|_{\frac{6}{5+\gamma}}\Big[\int_{\mathbb{R}^3}|u-v|^6\,\mathrm{d}x\Big]^{\frac{1-\gamma}{6}}. \end{aligned}\tag{2.3}$$

In order to prove our results, we first consider the following constrained set:

$$\mathcal{N}_\lambda=\left\{u\in E\setminus\{0\}\,:\,\|u\|_E^2+\lambda\mathbb{D}(u)-\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x-\int_{\mathbb{R}^3}|u|^6\mathrm{d}x=0\right\},$$

and split $\mathcal{N}_\lambda$ as follows

$$\mathcal{N}_\lambda^+=\left\{u\in\mathcal{N}_\lambda\,:\,2\|u\|_E^2+2p\lambda\mathbb{D}(u)-(1-\gamma)\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x>6\int_{\mathbb{R}^3}|u|^6\mathrm{d}x\right\},$$

$$\mathcal{N}_\lambda^-=\left\{u\in\mathcal{N}_\lambda\,:\,2\|u\|_E^2+2p\lambda\mathbb{D}(u)-(1-\gamma)\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x<6\int_{\mathbb{R}^3}|u|^6\mathrm{d}x\right\},$$

$$\mathcal{N}_\lambda^0=\left\{u\in\mathcal{N}_\lambda\,:\,2\|u\|_E^2+2p\lambda\mathbb{D}(u)-(1-\gamma)\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x=6\int_{\mathbb{R}^3}|u|^6\mathrm{d}x\right\},$$

for any $\lambda>0$. One can easily see that for $u\in\mathcal{N}_\lambda$,

$$\begin{aligned}&2\|u\|_E^2+2p\lambda\mathbb{D}(u)-(1-\gamma)\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x-6\int_{\mathbb{R}^3}|u|^6\mathrm{d}x\\ =&2\lambda(p-1)\mathbb{D}(u)+(1+\gamma)\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x-4\int_{\mathbb{R}^3}|u|^6\mathrm{d}x\\ =&(2-2p)\|u\|_E^2+(2p-1+\gamma)\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x-(6-2p)\int_{\mathbb{R}^3}|u|^6\mathrm{d}x\\ =&(1+\gamma)\|u\|_E^2+\lambda(2p-1+\gamma)\mathbb{D}(u)-(5+\gamma)\int_{\mathbb{R}^3}|u|^6\mathrm{d}x\\ =&-4\|u\|_E^2-(6-2p)\lambda\mathbb{D}(u)+(5+\gamma)\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x.\end{aligned}\tag{2.4}$$

We also recall the following lemma on the properties of $\mathbb{D}(u)$ from [14,18], etc.

Lemma 2.1. *For $0<\alpha<3$ and $1+\frac{\alpha}{3}\le p<3$, assume that $u_n\rightharpoonup u$ in E, then for any $\psi\in E$, we have $\lim\limits_{n\to\infty}\mathbb{D}(u_n)=\mathbb{D}(u)$ and $\lim\limits_{n\to\infty}\langle\mathbb{D}'(u_n),\psi\rangle=\langle\mathbb{D}'(u),\psi\rangle$.*

Set

$$T_1=\frac{4}{5+\gamma}S^{\frac{1-\gamma}{2}}\Big[\frac{(1+\gamma)S^3}{5+\gamma}\Big]^{\frac{1+\gamma}{4}}.\tag{2.5}$$

Lemma 2.2. *Suppose $0<\|f\|_{\frac{6}{5+\gamma}}<T_1$, where T_1 is defined in (2.5), then for any $u\in E\setminus\{0\}$, there exist unique $t_{max}=t_{max}(u)>0$, $t^+=t^+(u)>0$ and $t^-=t^-(u)>0$ with $t^+<t_{max}<t^-$, such that $t^+u\in\mathcal{N}_\lambda^+$, $t^-u\in\mathcal{N}_\lambda^-$, $J_\lambda(t^+u)=\inf\limits_{0<t\le t^-}J_\lambda(tu)$ and $J_\lambda(t^-u)=\sup\limits_{t\ge t_{max}}J_\lambda(tu)$. Furthermore, $\mathcal{N}_\lambda^0=\emptyset$ for $0<\|f\|_{\frac{6}{5+\gamma}}<T_1$.*

Proof. For any $u \in E \setminus \{0\}$ and $t > 0$, we have

$$\begin{aligned} t\frac{\mathrm{d}J_\lambda(tu)}{\mathrm{d}t} =& t^2\|u\|_E^2 + \lambda t^{2p}\mathbb{D}(u) - t^{1-\gamma}\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x - t^6\int_{\mathbb{R}^3}|u|^6\mathrm{d}x \\ =& t^{1-\gamma}\Big[t^{1+\gamma}\|u\|_E^2 - t^{5+\gamma}\int_{\mathbb{R}^3}|u|^6\mathrm{d}x + \lambda t^{2p-1+\gamma}\mathbb{D}(u) - \int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x\Big] \\ \equiv& t^{1-\gamma}\Big[g(t) - \int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x\Big], \end{aligned} \tag{2.6}$$

where $g(t)=t^{1+\gamma}\|u\|_E^2-t^{5+\gamma}\int_{\mathbb{R}^3}|u|^6\mathrm{d}x+\lambda t^{2p-1+\gamma}\mathbb{D}(u)$. Rewrite $g'(t)=t^{2p-2+\gamma}g_1(t)$ with

$$g_1(t) = (1+\gamma)t^{2-2p}\|u\|_E^2 - (5+\gamma)t^{6-2p}\int_{\mathbb{R}^3}|u|^6\mathrm{d}x + \lambda(2p-1+\gamma)\mathbb{D}(u). \tag{2.7}$$

Since $\alpha \in (0,3)$ and $1+\frac{\alpha}{3} \le p < 3$, we have $\lim\limits_{t\to 0^+} g_1(t) = +\infty$, $\lim\limits_{t\to+\infty} g_1(t) = -\infty$ and

$$g_1'(t) = (1+\gamma)(2-2p)t^{1-2p}\|u\|_E^2 - (5+\gamma)(6-2p)t^{5-2p}\int_{\mathbb{R}^3}|u|^6\mathrm{d}x < 0,$$

for all $t > 0$. Thus, $g(t)$ admits a global maximum point t_{max} which is the unique zero point of $g_1(t)$ and $g(t)$ is increasing on $(0, t_{max})$, decreasing on $(t_{max}, +\infty)$. Set $g_2(t) = t^{1+\gamma}\|u\|_E^2 - t^{5+\gamma}\int_{\mathbb{R}^3}|u|^6\mathrm{d}x$. Obviously, $g_2(0) = 0$, $\lim\limits_{t\to+\infty} g_2(t) = -\infty$ and $g_2(t)$ achieves its maximum at $t_{g_2} = \Big[\frac{(1+\gamma)\|u\|_E^2}{(5+\gamma)\int_{\mathbb{R}^3}|u|^6\mathrm{d}x}\Big]^{\frac{1}{4}}$ with

$$\max_{t\in[0,+\infty)} g_2(t) = g_2(t_{g_2}) = \frac{4}{5+\gamma}\|u\|_E^2\Big[\frac{(1+\gamma)\|u\|_E^2}{(5+\gamma)\int_{\mathbb{R}^3}|u|^6\mathrm{d}x}\Big]^{\frac{1+\gamma}{4}}.$$

It follows from (1.5) and (2.2) that

$$\begin{aligned} & g(t_{max}) - \int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x \\ \ge& \max_{t\in(0,+\infty)} g_2(t) - \int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x \\ \ge& \frac{4}{5+\gamma}\|u\|_E^2\Big[\frac{(1+\gamma)\|u\|_E^2}{(5+\gamma)\int_{\mathbb{R}^3}|u|^6\mathrm{d}x}\Big]^{\frac{1+\gamma}{4}} - \|f\|_{\frac{6}{5+\gamma}} S^{\frac{\gamma-1}{2}}\|u\|_E^{1-\gamma} \\ \ge& \Big[\frac{4}{5+\gamma}\Big(\frac{1+\gamma}{(5+\gamma)S^{-3}}\Big)^{\frac{1+\gamma}{4}} - \|f\|_{\frac{6}{5+\gamma}} S^{\frac{\gamma-1}{2}}\Big]\|u\|_E^{1-\gamma} \\ >& 0, \end{aligned} \tag{2.8}$$

since $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$. Consequently, there exist two points $0 < t^+ < t_{max} < t^-$ such that

$$g(t^+) = g(t^-) = \int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x \ \text{ and } \ g'(t^+) > 0 > g'(t^-).$$

That is $t^+u \in \mathcal{N}_\lambda^+$ and $t^-u \in \mathcal{N}_\lambda^-$. Hence, $\mathcal{N}_\lambda^\pm \neq \emptyset$ when $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$. We can further obtain from (2.6) that $\frac{\mathrm{d}J_\lambda(tu)}{\mathrm{d}t} > 0$ for all $t \in (t^+, t^-)$, $\frac{\mathrm{d}J_\lambda(tu)}{\mathrm{d}t} < 0$

for all $t \in (0, t^+)$ and $t \in (t^-, \infty)$. Thus, $J_\lambda(t^+u) = \inf_{0<t\leq t^-} J_\lambda(tu)$ and $J_\lambda(t^-u) = \sup_{t\geq t_{max}} J_\lambda(tu)$.

Now, we come to show that $\mathcal{N}_\lambda^0 = \emptyset$ for $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$. By contradiction, assume that there exists $u_0 \in \mathcal{N}_\lambda^0$ and $u_0 \neq 0$. Similarly to (2.8), we can obtain from (2.4) that

$$\begin{aligned}
0 <& \frac{4}{5+\gamma}\|u_0\|_E^2\Big[\frac{(1+\gamma)\|u_0\|_E^2}{(5+\gamma)\int_{\mathbb{R}^3}|u_0|^6 \mathrm{d}x}\Big]^{\frac{1+\gamma}{4}} - \int_{\mathbb{R}^3} f(x)|u_0|^{1-\gamma}dx \\
\leq& \frac{4}{5+\gamma}\|u_0\|_E^2 - \int_{\mathbb{R}^3} f(x)|u_0|^{1-\gamma}dx \\
\leq& 0,
\end{aligned}$$

which is a contradiction. Hence, $\mathcal{N}_\lambda^0 = \emptyset$ for $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$. $\square$

Lemma 2.3. *Suppose* $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$*, then there exists a gap structure in* $\mathcal{N}_\lambda$*:*

$$\|U\|_E > A^* > A_* > \|u\|_E,\ u \in \mathcal{N}_\lambda^+,\ U \in \mathcal{N}_\lambda^-,$$

where

$$A_* = \Big(\frac{5+\gamma}{4}\|f\|_{\frac{6}{5+\gamma}} S^{\frac{\gamma-1}{2}}\Big)^{\frac{1}{1+\gamma}},\ \ A^* = \Big[\frac{(1+\gamma)S^3}{5+\gamma}\Big]^{\frac{1}{4}}.$$

Proof. Since $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$, we have $\mathcal{N}_\lambda^\pm \neq \emptyset$ by Lemma 2.2. For any $u \in \mathcal{N}_\lambda^+$, it follows from (2.2) and (2.4) that

$$\|u\|_E^2 < \frac{5+\gamma}{4}\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x \leq \frac{5+\gamma}{4}\|f\|_{\frac{6}{5+\gamma}} S^{\frac{\gamma-1}{2}}\|u\|_E^{1-\gamma},$$

which yields $\|u\|_E < A_*$.

For any $U \in \mathcal{N}_\lambda^-$, it follows from (1.5) and (2.4) that

$$(1+\gamma)\|U\|_E^2 < (5+\gamma)\int_{\mathbb{R}^3}|U|^6\mathrm{d}x \leq (5+\gamma)S^{-3}\|U\|_E^6,$$

which yields $\|U\|_E > A^*$.

Using $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$ and the definition of T_1, one can further obtain $A_* < \Big(\frac{5+\gamma}{4}T_1 S^{\frac{\gamma-1}{2}}\Big)^{\frac{1}{1+\gamma}} = A^*$. So the proof is completed. $\square$

Lemma 2.4. *Suppose* $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$*, then* $\mathcal{N}_\lambda^-$ *is a closed set in* E*.*

Proof. Since $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$, by Lemma 2.2, one has $\mathcal{N}_\lambda^- \neq \emptyset$ and $\mathcal{N}_\lambda^0 = \emptyset$. Let $\{U_n\}$ be a sequence in $\mathcal{N}_\lambda^-$ with $U_n \to U_0$ in E, then $U_n \to U_0$ in $L^6(\mathbb{R}^3)$. Since $\mathcal{N}_\lambda^- \subset \mathcal{N}_\lambda$, one can obtain from Lemma 2.1, (2.3) and (2.4) that

$$\begin{aligned}
\|U_0\|_E^2 &= \lim_{n\to\infty}\|U_n\|_E^2 \\
&= \lim_{n\to\infty}\Big[\int_{\mathbb{R}^3} f(x)|U_n|^{1-\gamma}\mathrm{d}x + \int_{\mathbb{R}^3}|U_n|^6\mathrm{d}x - \lambda\mathbb{D}(U_n)\Big] \\
&= \int_{\mathbb{R}^3} f(x)|U_0|^{1-\gamma}\mathrm{d}x + \int_{\mathbb{R}^3}|U_0|^6\mathrm{d}x - \lambda\mathbb{D}(U_0)
\end{aligned}$$

and

$$
\begin{aligned}
&-4\|U_0\|_E^2-(6-2p)\lambda\mathbb{D}(U_0)+(5+\gamma)\int_{\mathbb{R}^3}f(x)|U_0|^{1-\gamma}\mathrm{d}x\\
=&\lim_{n\to\infty}\Big[-4\|U_n\|_E^2-(6-2p)\lambda\mathbb{D}(U_n)+(5+\gamma)\int_{\mathbb{R}^3}f(x)|U_n|^{1-\gamma}\mathrm{d}x\Big]\\
\le&0,
\end{aligned}
$$

so $U_0\in\mathcal{N}_\lambda^-\cup\{0\}$. It follows from $\{U_n\}\subset\mathcal{N}_\lambda^-$ and Lemma 2.3 that

$$\|U_0\|_E^2=\lim_{n\to\infty}\|U_n\|_E^2\ge A^*>0,$$

that is, $U_0\neq 0$. Hence, $U_0\in\mathcal{N}_\lambda^-$ and then $\mathcal{N}_\lambda^-$ is a closed set in E. □

Lemma 2.5. *Let $0<\|f\|_{\frac{6}{5+\gamma}}<T_1$, given $u\in\mathcal{N}_\lambda^\pm$, then there exist $\varepsilon>0$ and a continuous function $H(w)>0$, $w\in E$, $\|w\|_E<\varepsilon$ satisfying that*

$$H(0)=1,\ H(w)(u+w)\in\mathcal{N}_\lambda^\pm,\ \forall w\in E,\ \|w\|_E<\varepsilon.$$

Proof. We only prove the case $u\in\mathcal{N}_\lambda^+$. Define $F:E\times\mathbb{R}\to\mathbb{R}$ by

$$F(w,t)=t^2\|u+w\|_E^2+\lambda t^{2p}\mathbb{D}(u+w)-t^{1-\gamma}\int_{\mathbb{R}^3}f(x)|u+w|^{1-\gamma}\mathrm{d}x-t^6\int_{\mathbb{R}^3}|u+w|^6\mathrm{d}x.$$

In view of $u\in\mathcal{N}_\lambda^+\subset\mathcal{N}_\lambda$, we obtain $F(0,1)=0$ and

$$F_t(0,1)=2\|u\|_E^2+2p\lambda\mathbb{D}(u)-(1-\gamma)\int_{\mathbb{R}^3}f(x)|u|^{1-\gamma}\mathrm{d}x-6\int_\Omega|u|^6\mathrm{d}x>0.$$

By applying Implicit function Theorem for F at the point $(0,1)$, we get that there exists $\bar\varepsilon>0$ such that for $w\in E$, $\|w\|_E<\bar\varepsilon$, the equation $F(w,t)=0$ has a unique continuous solution $t=H(w)>0$ satisfying that $H(0)=1$ and $F(w,H(w))=0$ i.e. $H(w)(u+w)\in\mathcal{N}_\lambda$. Moreover, since $F_t(0,1)>0$ and

$$
\begin{aligned}
F_t(w,H(w))=&2H(w)\|u+w\|_E^2+2p\lambda H^{2p-1}(w)\mathbb{D}(u+w)\\
&-(1-\gamma)H^{-\gamma}(w)\int_{\mathbb{R}^3}f(x)|u+w|^{1-\gamma}\mathrm{d}x-6H^5(w)\int_{\mathbb{R}^3}|u+w|^6\mathrm{d}x\\
=&H^{-1}(w)\Big[2H^2(w)\|u+w\|_E^2+2p\lambda H^{2p}(w)\mathbb{D}(u+w)\\
&-(1-\gamma)H^{1-\gamma}(w)\int_{\mathbb{R}^3}f(x)|u+w|^{1-\gamma}\mathrm{d}x-6H^6(w)\int_{\mathbb{R}^3}|u+w|^6\mathrm{d}x\Big],
\end{aligned}
$$

we can choose $\varepsilon>0$ possibly small ($\varepsilon<\bar\varepsilon$) such that for $w\in E$ and $\|w\|_E<\varepsilon$,

$$
\begin{aligned}
&2H^2(w)\|u+w\|_E^2+2p\lambda H^{2p}(w)\mathbb{D}(u+w)-(1-\gamma)H^{1-\gamma}(w)\int_{\mathbb{R}^3}f(x)|u+w|^{1-\gamma}\mathrm{d}x\\
&-6H^6(w)\int_{\mathbb{R}^3}|u+w|^6\mathrm{d}x>0,
\end{aligned}
$$

that is

$$H(w)(u+w)\in\mathcal{N}_\lambda^+,\quad\text{for any } w\in E,\ \|w\|_E<\varepsilon.$$

This ends the proof of Lemma 2.5. □

Lemma 2.6. *J_λ is coercive and bounded below on $\mathcal{N}_\lambda$. Moreover,*

(i) if $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$, then $\inf_{\mathcal{N}_\lambda^+ \cup\{0\}} J_\lambda = \inf_{\mathcal{N}_\lambda^+} J_\lambda < 0$;

(ii) if $0 < \|f\|_{\frac{6}{5+\gamma}} < \dfrac{1-\gamma}{2}T_1$, then $\inf_{\mathcal{N}_\lambda^-} J_\lambda \geq \beta_0 > 0$ for some constant $\beta_0 = \beta_0(\gamma, S, \|f\|_{\frac{6}{5+\gamma}})$.

Proof. For any $u \in \mathcal{N}_\lambda$, we can obtain from (1.3), $\lambda > 0$, $0 < \gamma < 1$, $1 + \frac{\alpha}{3} \leq p < 3$ and (2.2) that

$$\begin{aligned} J_\lambda(u) =& \Big(\frac{1}{2} - \frac{1}{6}\Big)\|u\|_E^2 + \lambda\Big(\frac{1}{2p} - \frac{1}{6}\Big)\mathbb{D}(u) - \Big(\frac{1}{1-\gamma} - \frac{1}{6}\Big)\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x \\ \geq& \frac{1}{3}\|u\|_E^2 - \frac{5+\gamma}{6(1-\gamma)}\|f\|_{\frac{6}{5+\gamma}} S^{\frac{\gamma-1}{2}}\|u\|_E^{1-\gamma} \\ \geq& \frac{1+\gamma}{3(\gamma-1)}A_*^2, \end{aligned} \tag{2.9}$$

where A_* is defined in Lemma 2.3. Due to $0 < \gamma < 1$, J_λ is coercive and bounded from below on $\mathcal{N}_\lambda$.

(i) When $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$, $\mathcal{N}_\lambda^\pm \neq \emptyset$ from Lemma 2.2, also $\mathcal{N}_\lambda^-$ and $\mathcal{N}_\lambda^+ \cup \{0\}$ are two closed sets in E from Lemma 2.4. Hence, $\inf_{\mathcal{N}_\lambda^-} J_\lambda$ and $\inf_{\mathcal{N}_\lambda^+\cup\{0\}} J_\lambda$ are well defined. For any $u \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, we can get from $0 < \gamma < 1$, $1 + \frac{\alpha}{3} \leq p < 3$, (2.4) and (2.9) that

$$\begin{aligned} J_\lambda(u) =& \frac{1}{3}\|u\|_E^2 + \lambda\frac{3-p}{6p}\mathbb{D}(u) - \frac{5+\gamma}{6(1-\gamma)}\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x \\ <& \frac{1}{3}\|u\|_E^2 - \frac{2}{3(1-\gamma)}\|u\|_E^2 + \lambda(3-p)\Big(\frac{1}{6p} - \frac{1}{3(1-\gamma)}\Big)\mathbb{D}(u) \\ =& -\frac{1+\gamma}{3(1-\gamma)}\|u\|_E^2 + \lambda(3-p)\frac{1-\gamma-2p}{6p(1-\gamma)}\mathbb{D}(u) \\ <& 0, \end{aligned} \tag{2.10}$$

which yields $\inf_{\mathcal{N}_\lambda^+} J_\lambda < 0$. Since $J_\lambda(0) = 0$, we can further get $\inf_{\mathcal{N}_\lambda^+\cup\{0\}} J_\lambda = \inf_{\mathcal{N}_\lambda^+} J_\lambda < 0$.

(ii) Let $u \in \mathcal{N}_\lambda^-$, it follows from Lemma 2.3 that $\|u\|_E > A^*$. Using this and (2.9), $\|f\|_{\frac{6}{5+\gamma}} \in (0, \dfrac{1-\gamma}{2}T_1)$, we can obtain that

$$\begin{aligned} J_\lambda(u) \geq& \|u\|_E^{1-\gamma}\Big[\frac{1}{3}\|u\|_E^{1+\gamma} - \frac{5+\gamma}{6(1-\gamma)}\|f\|_{\frac{6}{5+\gamma}} S^{\frac{\gamma-1}{2}}\Big] \\ \geq& \Big[\frac{(1+\gamma)S^3}{5+\gamma}\Big]^{\frac{1-\gamma}{4}}\Big\{\frac{1}{3}\Big[\frac{(1+\gamma)S^3}{5+\gamma}\Big]^{\frac{1+\gamma}{4}} - \frac{5+\gamma}{6(1-\gamma)}\|f\|_{\frac{6}{5+\gamma}} S^{\frac{\gamma-1}{2}}\Big\} \\ >& 0, \end{aligned}$$

which implies that there exists a constant $\beta_0 = \beta_0(\gamma, S, \|f\|_{\frac{6}{5+\gamma}})$ such that $\inf_{\mathcal{N}_\lambda^-} J_\lambda \geq \beta_0 > 0$ for $\|f\|_{\frac{6}{5+\gamma}} \in (0, \dfrac{1-\gamma}{2}T_1)$. □

According to Lemma 2.2 and Lemma 2.4, for $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$, $\mathcal{V}_\lambda^- := \mathcal{N}_\lambda^-$ and $\mathcal{V}_\lambda^+ := \mathcal{N}_\lambda^+ \cup \{0\}$ are two closed sets in E, then we can apply Ekeland variational principle to find the minimums of functional J_λ on both $\mathcal{V}_\lambda^+$ and $\mathcal{V}_\lambda^-$. Let $\{u_n\} \subset \mathcal{V}_\lambda^\pm$ be a minimizing sequence for J_λ on $\mathcal{V}_\lambda^\pm$. That is, $\{u_n\} \subset \mathcal{V}_\lambda^\pm$ satisfy

$$\tau_\lambda^\pm < J_\lambda(u_n) < \tau_\lambda^\pm + \frac{1}{n} \tag{2.11}$$

and

$$J_\lambda(z) \geq J_\lambda(u_n) - \frac{1}{n}\|u_n - z\|_E, \forall z \in \mathcal{V}_\lambda^\pm, \tag{2.12}$$

where

$$\tau_\lambda^+ = \inf_{u\in\mathcal{V}_\lambda^+} J_\lambda(u) = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u),\ \tau_\lambda^- = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u) \text{ and } \tau_\lambda = \inf_{u\in\mathcal{N}_\lambda} J_\lambda(u).$$

From $J_\lambda(|u_n|) = J_\lambda(u_n)$, we could assume that $u_n \geq 0$. Moreover, Lemma 2.6 shows that $\|u_n\|_E \leq C_0$ for some suitable positive constant C_0, so there exists a nonnegative function $u_\lambda \in E$ such that

$$\begin{aligned} &u_n \rightharpoonup u_\lambda, \quad \text{in } E,\\ &u_n \to u_\lambda, \quad \text{in } L^s(\mathbb{R}^3),\ s \in [2,6),\\ &u_n \to u_\lambda, \quad \text{a.e. in } \mathbb{R}^3. \end{aligned} \tag{2.13}$$

By Vitali Convergence Theorem, similarly to the proof of [13, Lemma 2.7], we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} f(x)|u_n|^{1-\gamma}\mathrm{d}x = \int_{\mathbb{R}^3} f(x)|u_\lambda|^{1-\gamma}\mathrm{d}x, \tag{2.14}$$

when $\{u_n\}$ is bounded in E. In order to show that all convergence in (2.13) hold true on a strong sense, inspired by [5,34], we need following Lemmas.

Lemma 2.7. *Assume* $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$. *Suppose* $\{u_n\} \subset \mathcal{N}_\lambda^\pm$ *satisfy* (2.13) *with* $u_\lambda \not\equiv 0$, *then there exists a constant* $C_1 > 0$ *such that for* n *large enough, the following alternative holds true:*

(i) if $\{u_n\} \subset \mathcal{N}_\lambda^+$, *we have*

$$(1+\gamma)\|u_n\|_E^2 + \lambda(2p-1+\gamma)\mathbb{D}(u_n) - (5+\gamma)\int_{\mathbb{R}^3} |u_n|^6\mathrm{d}x \geq C_1;$$

(ii) if $\{u_n\} \subset \mathcal{N}_\lambda^-$, *we have*

$$(1+\gamma)\|u_n\|_E^2 + \lambda(2p-1+\gamma)\mathbb{D}(u_n) - (5+\gamma)\int_{\mathbb{R}^3} |u_n|^6\mathrm{d}x \leq -C_1.$$

Proof. We only prove (i), since (ii) follows similarly. Using $u_n \in \mathcal{N}_\lambda^+$, (2.4), Lemma 2.1, (2.14) and $u_\lambda \not\equiv 0$, it is enough to show that

$$(5+\gamma)\int_{\mathbb{R}^3} f(x)|u_\lambda|^{1-\gamma}\mathrm{d}x - (6-2p)\lambda\mathbb{D}(u_\lambda) > \liminf_{n\to\infty}\left[4\|u_n\|_E^2\right]. \tag{2.15}$$

Arguing by contradiction, assume that

$$(5+\gamma)\int_{\mathbb{R}^3} f(x)|u_\lambda|^{1-\gamma}\mathrm{d}x - (6-2p)\lambda\mathbb{D}(u_\lambda) = \liminf_{n\to\infty}\left[4\|u_n\|_E^2\right]. \tag{2.16}$$

Since $u_n \in \mathcal{N}_\lambda^+$, one has

$$(5+\gamma)\int_{\mathbb{R}^3} f(x)|u_n|^{1-\gamma}\mathrm{d}x - (6-2p)\lambda\mathbb{D}(u_n) > 4\|u_n\|_E^2.$$

According to (2.14) and Lemma 2.1, we can further obtain

$$(5+\gamma)\int_{\mathbb{R}^3} f(x)|u_\lambda|^{1-\gamma}\mathrm{d}x - (6-2p)\lambda\mathbb{D}(u_\lambda) \geq \limsup_{n\to\infty}\left[4\|u_n\|_E^2\right] \geq \liminf_{n\to\infty}\left[4\|u_n\|_E^2\right]. \tag{2.17}$$

It follows from (2.16) and (2.17) that

$$(5+\gamma)\int_{\mathbb{R}^3} f(x)|u_\lambda|^{1-\gamma}\mathrm{d}x - (6-2p)\lambda\mathbb{D}(u_\lambda) = \lim_{n\to\infty}\left[4\|u_n\|_E^2\right]. \tag{2.18}$$

Since $u_n \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, i.e. $\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x = \|u_n\|_E^2 + \lambda\mathbb{D}(u_n) - \int_{\mathbb{R}^3} f(x)|u_n|^{1-\gamma}\mathrm{d}x$, passing to the limit as $n\to\infty$ and using (2.14), (2.18) and Lemma 2.1 lead to

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x = \frac{1+\gamma}{4}\int_{\mathbb{R}^3} f(x)|u_\lambda|^{1-\gamma}\mathrm{d}x + \frac{p-1}{2}\lambda\mathbb{D}(u_\lambda). \tag{2.19}$$

Therefore, it follows from (2.18), (2.19), $\lambda>0$ and $u_\lambda \not\equiv 0$ that

$$\lim_{n\to\infty}\frac{(1+\gamma)\|u_n\|_E^2}{(5+\gamma)\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x} < 1. \tag{2.20}$$

Similarly to (2.8), for $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$, one can get from (2.2), (2.14), (2.18) and (2.20) that

$$\begin{aligned}
0 \leq & \left[\frac{4}{5+\gamma}\left(\frac{1+\gamma}{(5+\gamma)S^{-3}}\right)^{\frac{1+\gamma}{4}} - \|f\|_{\frac{6}{5+\gamma}} S^{\frac{\gamma-1}{2}}\right]\lim_{n\to\infty}\|u_n\|_E^{1-\gamma} \\
\leq & \lim_{n\to\infty}\frac{4}{5+\gamma}\|u_n\|_E^2\left(\frac{(1+\gamma)\|u_n\|_E^2}{(5+\gamma)\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x}\right)^{\frac{1+\gamma}{4}} - \lim_{n\to\infty}\int_{\mathbb{R}^3} f(x)u_n^{1-\gamma}dx \\
< & \frac{4}{5+\gamma}\lim_{n\to\infty}\|u_n\|_E^2 - \int_{\mathbb{R}^3} f(x)|u_\lambda|^{1-\gamma}\mathrm{d}x \\
= & -\frac{6-2p}{5+\gamma}\lambda\mathbb{D}(u_\lambda) \\
< & 0,
\end{aligned}$$

which is clearly impossible. So (2.15) holds and this ends the proof. □

For any $0 \leq \psi \in E$, we apply Lemma 2.5 with $u = u_n \in \mathcal{N}_\lambda^\pm$ (n large enough such that $\frac{(1-\gamma)C_0}{n} < C_1$) and $w = \eta\psi$, $\eta > 0$ small enough, we can find $h_{n,\psi}(\eta) = H(\eta\psi)$ such that $h_{n,\psi}(0) = 1$ and $h_{n,\psi}(\eta)(u_n + \eta\psi) \in \mathcal{N}_\lambda^\pm$. However, we have no idea whether or not $h_{n,\psi}(\eta)$ is differentiable. For the sake of proof, we set

$$h'_{n,\psi}(0) = \lim_{\eta\to 0^+}\frac{h_{n,\psi}(\eta)-1}{\eta} \in [-\infty,+\infty].$$

If the above limit does not exist, we choose $\eta_k \to 0$ (instead of $\eta\to 0$) with $\eta_k > 0$ such that $h'_{n,\psi}(0) = \lim_{k\to\infty}\frac{h_{n,\psi}(\eta_k)-1}{\eta_k} \in [-\infty,+\infty]$.

Lemma 2.8. *Assume* $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$. *Suppose* $\{u_n\} \subset \mathcal{N}_\lambda^{\pm}$ *satisfy* (2.12) *and* (2.13) *with* $u_\lambda \not\equiv 0$, *then* $h'_{n,\psi}(0)$ *is uniformly bounded for any* $0 \le \psi \in E$ *and* n *large enough.*

Proof. We only consider that u_n, $h_{n,\psi}(\eta)(u_n + \eta\psi) \in \mathcal{N}_\lambda^+$ since the situation on $\mathcal{N}_\lambda^-$ can be proved similarly. By u_n, $h_{n,\psi}(\eta)(u_n + \eta\psi) \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, we have

$$\begin{aligned}
&\|u_n\|_E^2 + \lambda\mathbb{D}(u_n) - \int_{\mathbb{R}^3} f(x)u_n^{1-\gamma}\mathrm{d}x = \int_{\mathbb{R}^3} u_n^6\mathrm{d}x,\\
&\quad h_{n,\psi}^2(\eta)\|u_n + \eta\psi\|_E^2 + \lambda h_{n,\psi}^{2p}\mathbb{D}(u_n + \eta\psi) - h_{n,\psi}^{1-\gamma}(\eta)\int_{\mathbb{R}^3} f(x)(u_n + \eta\psi)^{1-\gamma}\mathrm{d}x\\
&= h_{n,\psi}^6(\eta)\int_{\mathbb{R}^3}(u_n + \eta\psi)^6\mathrm{d}x.
\end{aligned}$$

Using $0 < \gamma < 1$ and $\lambda > 0$, the above two equalities yield

$$\begin{aligned}
0 =& \Big[h_{n,\psi}^2(\eta) - 1\Big]\|u_n + \eta\psi\|_E^2 + \lambda\Big[h_{n,\psi}^{2p}(\eta) - 1\Big]\mathbb{D}(u_n + \eta\psi)\\
&- \Big[h_{n,\psi}^{1-\gamma}(\eta) - 1\Big]\int_{\mathbb{R}^3} f(x)(u_n + \eta\psi)^{1-\gamma}\mathrm{d}x - \Big[h_{n,\psi}^6(\eta) - 1\Big]\int_{\mathbb{R}^3}(u_n + \eta\psi)^6\mathrm{d}x\\
&+ \Big[\|u_n + \eta\psi\|_E^2 - \|u_n\|_E^2\Big] + \lambda\Big[\mathbb{D}(u_n + \eta\psi) - \mathbb{D}(u_n)\Big]\\
&- \int_{\mathbb{R}^3} f(x)\Big[(u_n + \eta\psi)^{1-\gamma} - u_n^{1-\gamma}\Big]\mathrm{d}x - \int_{\mathbb{R}^3}\Big[(u_n + \eta\psi)^6 - u_n^6\Big]\mathrm{d}x\\
\le& \Big[h_{n,\psi}(\eta) - 1\Big]\Big\{\Big[h_{n,\psi}(\eta) + 1\Big]\|u_n + \eta\psi\|_E^2 + \lambda\frac{h_{n,\psi}^{2p}(\eta) - 1}{h_{n,\psi}(\eta) - 1}\mathbb{D}(u_n + \eta\psi)\\
&- \frac{h_{n,\psi}^{1-\gamma}(\eta) - 1}{h_{n,\psi}(\eta) - 1}\int_{\mathbb{R}^3} f(x)(u_n + \eta\psi)^{1-\gamma}\mathrm{d}x - \frac{h_{n,\psi}^6(\eta) - 1}{h_{n,\psi}(\eta) - 1}\int_{\mathbb{R}^3}(u_n + \eta\psi)^6\mathrm{d}x\Big\}\\
&+ \Big[\|u_n + \eta\psi\|_E^2 - \|u_n\|_E^2\Big] + \lambda\Big[\mathbb{D}(u_n + \eta\psi) - \mathbb{D}(u_n)\Big].
\end{aligned}$$

Dividing by $\eta > 0$ and passing to the limit as $\eta \to 0^+$, it follows from (2.4) and the continuity of $h_{n,\psi}(\eta)$ that

$$\begin{aligned}
0 \le& h'_{n,\psi}(0)\Big\{2\|u_n\|_E^2 + 2p\lambda\mathbb{D}(u_n) - (1-\gamma)\int_{\mathbb{R}^3} f(x)u_n^{1-\gamma}\mathrm{d}x - 6\int_{\mathbb{R}^3} u_n^6\mathrm{d}x\Big\}\\
&+ 2\langle u_n, \psi\rangle_E + \lambda\langle\mathbb{D}'(u_n), \psi\rangle\\
=& h'_{n,\psi}(0)\Big\{(1+\gamma)\|u_n\|_E^2 + \lambda(2p - 1 + \gamma)\mathbb{D}(u_n) - (5+\gamma)\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x\Big\}\\
&+ 2\langle u_n, \psi\rangle_E + \lambda\langle\mathbb{D}'(u_n), \psi\rangle,
\end{aligned} \tag{2.21}$$

which implies that $h'_{n,\psi}(0) \ne -\infty$ according to Lemma 2.7 and the boundedness of $\{u_n\}$. Now we show that $h'_{n,\psi}(0) \ne +\infty$. Arguing by contradiction, we assume that $h'_{n,\psi}(0) = +\infty$ and so $h_{n,\psi}(\eta) > 1$ for n sufficiently large and $\eta > 0$ small. Applying condition (2.12) with $z = h_{n,\psi}(\eta)(u_n + \eta\psi)$ leads to

$$\begin{aligned}
\frac{1}{n}[h_{n,\psi}(\eta) - 1]\|u_n\|_E + \frac{\eta}{n}h_{n,\psi}(\eta)\|\psi\|_E \ge& \frac{1}{n}\|u_n - h_{n,\psi}(\eta)(u_n + \eta\psi)\|_E\\
\ge& J_\lambda(u_n) - J_\lambda[h_{n,\psi}(\eta)(u_n + \eta\psi)].
\end{aligned} \tag{2.22}$$

Since $u_n \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, then one can get from (1.3) and (2.22) that

$$\begin{aligned}\frac{\|\psi\|_E}{n} h_{n,\psi}(\eta) \geq & \frac{h_{n,\psi}(\eta)-1}{\eta}\Big\{ -\frac{\|u_n\|_E}{n} - \Big(\frac{1}{2}-\frac{1}{1-\gamma}\Big)[h_{n,\psi}(\eta)+1]\|u_n+\eta\psi\|_E^2 \\ & - \lambda\Big(\frac{1}{2p}-\frac{1}{1-\gamma}\Big)\frac{h_{n,\psi}^{2p}(\eta)-1}{h_{n,\psi}(\eta)-1}\mathbb{D}(u_n+\eta\psi) \\ & + \Big(\frac{1}{6}-\frac{1}{1-\gamma}\Big)\frac{h_{n,\psi}^{6}(\eta)-1}{h_{n,\psi}(\eta)-1}\int_{\mathbb{R}^3}(u_n+\eta\psi)^6 \mathrm{d}x\Big\} \\ & - \Big(\frac{1}{2}-\frac{1}{1-\gamma}\Big)\frac{\|u_n+\eta\psi\|^2-\|u_n\|^2}{\eta} \\ & + \Big(\frac{1}{6}-\frac{1}{1-\gamma}\Big)\int_{\mathbb{R}^3}\frac{(u_n+\eta\psi)^6-u_n^6}{\eta}\mathrm{d}x \\ & - \lambda\Big(\frac{1}{2p}-\frac{1}{1-\gamma}\Big)\frac{\mathbb{D}(u_n+\eta\psi)-\mathbb{D}(u_n)}{\eta}.\end{aligned}$$

Letting $\eta \to 0^+$, using the continuity of $h_{n,\psi}(\eta)$, Lemma 2.7 and $\|u_n\|_E \leq C_0$, we obtain

$$\begin{aligned}\frac{\|\psi\|_E}{n} \geq & h'_{n,\psi}(0)\Big\{ -\frac{\|u_n\|_E}{n} - \Big(1-\frac{2}{1-\gamma}\Big)\|u_n\|_E^2 - \lambda\Big(1-\frac{2p}{1-\gamma}\Big)\mathbb{D}(u_n) \\ & + \Big(1-\frac{6}{1-\gamma}\Big)\int_{\mathbb{R}^3}u_n^6\mathrm{d}x\Big\} - \Big(1-\frac{2}{1-\gamma}\Big)\langle u_n,\psi\rangle_E \\ & + \Big(1-\frac{6}{1-\gamma}\Big)\int_{\mathbb{R}^3}u_n^5\psi\mathrm{d}x - \lambda\Big(\frac{1}{2p}-\frac{1}{1-\gamma}\Big)\langle\mathbb{D}'(u_n),\psi\rangle \\ = & h'_{n,\psi}(0)\Big\{ -\frac{\|u_n\|_E}{n} + \frac{1}{1-\gamma}\Big[(\gamma+1)\|u_n\|_E^2 + \lambda(2p-1+\gamma)\mathbb{D}(u_n) \\ & - (5+\gamma)\int_{\mathbb{R}^3}u_n^6\mathrm{d}x\Big]\Big\} - \Big(1-\frac{2}{1-\gamma}\Big)\langle u_n,\psi\rangle_E \\ & + \Big(1-\frac{6}{1-\gamma}\Big)\int_{\mathbb{R}^3}u_n^5\psi\mathrm{d}x - \lambda\Big(\frac{1}{2p}-\frac{1}{1-\gamma}\Big)\langle\mathbb{D}'(u_n),\psi\rangle \\ \geq & h'_{n,\psi}(0)\Big(-\frac{C_0}{n}+\frac{C_1}{1-\gamma}\Big) - \Big(1-\frac{2}{1-\gamma}\Big)\langle u_n,\psi\rangle_E \\ & + \Big(1-\frac{6}{1-\gamma}\Big)\int_{\mathbb{R}^3}u_n^5\psi\mathrm{d}x - \lambda\Big(\frac{1}{2p}-\frac{1}{1-\gamma}\Big)\langle\mathbb{D}'(u_n),\psi\rangle\end{aligned} \tag{2.23}$$

which is impossible because $h'_{n,\psi}(0) = +\infty$ and $-\frac{C_0}{n}+\frac{C_1}{1-\gamma} > 0$ for n large enough. Hence, $h'_{n,\psi}(0) \neq +\infty$. To sum up, $|h'_{n,\psi}(0)| < +\infty$. Moreover, Lemma 2.7, (2.21) and (2.23) with $\|u_n\| \leq C_0$ also imply that

$$|h'_{n,\psi}(0)| \leq C_2, \tag{2.24}$$

for n sufficiently large and a suitable positive constant C_2. □

Lemma 2.9. *Assume* $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$. *Suppose* $\{u_n\} \subset \mathcal{N}_\lambda^\pm$ *satisfy* (2.12) *and* (2.13) *with* $u_\lambda \not\equiv 0$, *then for any* $\psi \in E$, *we have as* $n \to \infty$,

$$\langle u_n,\psi\rangle_E + \frac{\lambda}{2p}\langle\mathbb{D}'(u_n),\psi\rangle - \int_{\mathbb{R}^3} f(x)u_n^{-\gamma}\psi\mathrm{d}x - \int_{\mathbb{R}^3}u_n^5\psi\mathrm{d}x = o(1). \tag{2.25}$$

Proof. For any $0 \le \psi \in E$, applying condition (2.12) with $z = h_{n,\psi}(\eta)(u_n + \eta\psi)$ leads to

$$\begin{aligned}
&\frac{|1-h_{n,\psi}(\eta)|}{\eta}\frac{\|u_n\|_E}{n} + \frac{\|\psi\|_E}{n}h_{n,\psi}(\eta)\\
\ge&\frac{1}{n\eta}\|u_n - h_{n,\psi}(\eta)(u_n+\eta\psi)\|_E\\
\ge&\frac{1}{\eta}\big\{J_\lambda(u_n) - J_\lambda[h_{n,\psi}(\eta)(u_n+\eta\psi)]\big\}\\
=&\frac{h_{n,\psi}(\eta)-1}{\eta}\Big\{-\frac{h_{n,\psi}(\eta)+1}{2}\|u_n+\eta\psi\|_E^2 - \frac{\lambda[h_{n,\psi}^{2p}(\eta)-1]}{2p[h_{n,\psi}(\eta)-1]}\mathbb{D}(u_n+\eta\psi)\\
&+\frac{h_{n,\psi}^{1-\gamma}(\eta)-1}{(1-\gamma)[h_{n,\psi}(\eta)-1]}\int_{\mathbb{R}^3} f(x)(u_n+\eta\psi)^{1-\gamma}\mathrm{d}x\\
&+\frac{h_{n,\psi}^{6}(\eta)-1}{6[h_{n,\psi}(\eta)-1]}\int_{\mathbb{R}^3}(u_n+\eta\psi)^6\mathrm{d}x\Big\}\\
&-\frac{1}{2}\frac{\|u_n+\eta\psi\|_E^2-\|u_n\|_E^2}{\eta} - \frac{\lambda[\mathbb{D}(u_n+\eta\psi)-\mathbb{D}(u_n)]}{2p\eta}\\
&+\frac{1}{6}\int_{\mathbb{R}^3}\frac{(u_n+\eta\psi)^6-u_n^6}{\eta}\mathrm{d}x + \frac{1}{1-\gamma}\int_{\mathbb{R}^3} f(x)\frac{(u_n+\eta\psi)^{1-\gamma}-u_n^{1-\gamma}}{\eta}\mathrm{d}x.
\end{aligned}$$

Passing to the liminf as $\eta \to 0^+$ and using the continuity of $h_{n,\psi}(\eta)$, Fatou's Lemma, $0<\gamma<1$ lead to

$$\begin{aligned}
&\frac{|h'_{n,\psi}(0)|\cdot\|u_n\|_E}{n} + \frac{\|\psi\|_E}{n}\\
\ge&h'_{n,\psi}(0)\Big\{-\|u_n\|_E^2 + \int_{\mathbb{R}^3} f(x)u_n^{1-\gamma}\mathrm{d}x - \lambda\mathbb{D}(u_n) + \int_{\mathbb{R}^3}u_n^6\mathrm{d}x\Big\} - \langle u_n,\psi\rangle_E\\
&-\frac{\lambda}{2p}\langle\mathbb{D}'(u_n),\psi\rangle + \int_{\mathbb{R}^3}u_n^5\psi\mathrm{d}x + \liminf_{\eta\to0^+}\frac{1}{1-\gamma}\int_{\mathbb{R}^3} f(x)\frac{(u_n+\eta\psi)^{1-\gamma}-u_n^{1-\gamma}}{\eta}\mathrm{d}x\\
\ge&-\langle u_n,\psi\rangle_E - \frac{\lambda}{2p}\langle\mathbb{D}'(u_n),\psi\rangle + \int_{\mathbb{R}^3}u_n^5\psi\mathrm{d}x\\
&+\int_{\mathbb{R}^3}\frac{f(x)}{1-\gamma}\liminf_{\eta\to0^+}\frac{(u_n+\eta\psi)^{1-\gamma}-u_n^{1-\gamma}}{\eta}\mathrm{d}x\\
=&-\langle u_n,\psi\rangle_E - \frac{\lambda}{2p}\langle\mathbb{D}'(u_n),\psi\rangle + \int_{\mathbb{R}^3}u_n^5\psi\mathrm{d}x + \int_{\mathbb{R}^3}f(x)u_n^{-\gamma}\psi\mathrm{d}x,
\end{aligned}$$

since $u_n \in \mathcal{N}_\lambda^\pm \subset \mathcal{N}_\lambda$. Hence, for n large, we have

$$\begin{aligned}
\int_{\mathbb{R}^3}f(x)u_n^{-\gamma}\psi\mathrm{d}x \le&\frac{|h'_{n,\psi}(0)|\cdot\|u_n\|_E}{n} + \frac{\|\psi\|_E}{n}\\
&+\langle u_n,\psi\rangle_E + \frac{\lambda}{2p}\langle\mathbb{D}'(u_n),\psi\rangle - \int_{\mathbb{R}^3}u_n^5\psi\mathrm{d}x\\
\le&\frac{C_0\cdot C_2+\|\psi\|_E}{n} + \langle u_n,\psi\rangle_E + \frac{\lambda}{2p}\langle\mathbb{D}'(u_n),\psi\rangle - \int_{\mathbb{R}^3}u_n^5\psi\mathrm{d}x,
\end{aligned}$$

thanks to $\|u_n\|_E \le C_0$ and $|h'_{n,\varphi}(0)| \le C_2$ by (2.24). Thus, for any $0 \le \psi \in E$, we can get as $n \to \infty$,

$$\langle u_n, \psi\rangle_E + \frac{\lambda}{2p}\langle \mathbb{D}'(u_n), \psi\rangle - \int_{\mathbb{R}^3} f(x)u_n^{-\gamma}\psi \mathrm{d}x - \int_{\mathbb{R}^3} u_n^5\psi \mathrm{d}x \geq o(1). \tag{2.26}$$

Now, we come to show that (2.26) holds for every $\psi \in E$. For any $\psi \in E$ and $\varepsilon > 0$, set $\psi_\varepsilon = u_n + \varepsilon\psi$ and $\Omega_\varepsilon = \{x \in \mathbb{R}^3 : \psi_\varepsilon \leq 0\}$. Since $u_n \in \mathcal{N}_\lambda$, by applying inequality (2.26) with $\psi = \psi_\varepsilon^+$, we have

$$\begin{aligned}
o(1) \leq& \frac{1}{\varepsilon}\Big\{\langle u_n, \psi_\varepsilon^+\rangle_E + \frac{\lambda}{2p}\langle \mathbb{D}'(u_n), \psi_\varepsilon^+\rangle - \int_{\mathbb{R}^3} f(x)u_n^{-\gamma}\psi_\varepsilon^+ \mathrm{d}x - \int_{\mathbb{R}^3} u_n^5\psi_\varepsilon^+ \mathrm{d}x\Big\}\\
=& \frac{1}{\varepsilon}\int_{\mathbb{R}^3\setminus\Omega_\varepsilon} \Big\{\nabla u_n \nabla\psi_\varepsilon + V(x)u_n\psi_\varepsilon\\
&+ \lambda(I_\alpha * u_n^p)u_n^{p-2}u_n\psi_\varepsilon - f(x)u_n^{-\gamma}\psi_\varepsilon - u_n^5\psi_\varepsilon\Big\}\mathrm{d}x\\
=& \frac{1}{\varepsilon}\Big\{\|u_n\|_E^2 + \lambda\mathbb{D}(u_n) - \int_{\mathbb{R}^3} f(x)u_n^{1-\gamma}\mathrm{d}x - \int_{\mathbb{R}^3} u_n^6 \mathrm{d}x\Big\}\\
&+ \Big\{\langle u_n, \psi\rangle_E + \frac{\lambda}{2p}\langle \mathbb{D}'(u_n), \psi\rangle - \int_{\mathbb{R}^3} f(x)u_n^{-\gamma}\psi \mathrm{d}x - \int_{\mathbb{R}^3} u_n^5\psi \mathrm{d}x\Big\}\\
&- \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} \Big\{\nabla u_n\nabla\psi_\varepsilon + V(x)u_n\psi_\varepsilon + \lambda(I_\alpha * u_n^p)u_n^{p-2}u_n\psi_\varepsilon\\
&- f(x)u_n^{-\gamma}\psi_\varepsilon - u_n^5\psi_\varepsilon\Big\}\mathrm{d}x\\
=& \Big\{\langle u_n, \psi\rangle_E + \frac{\lambda}{2p}\langle \mathbb{D}'(u_n), \psi\rangle - \int_{\mathbb{R}^3} f(x)u_n^{-\gamma}\psi \mathrm{d}x - \int_{\mathbb{R}^3} u_n^5\psi \mathrm{d}x\Big\}\\
&- \frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\Big[|\nabla u_n|^2 + V(x)u_n^2 + \lambda(I_\alpha * u_n^p)u_n^p\Big]\mathrm{d}x\\
&- \int_{\Omega_\varepsilon}\Big[\nabla u_n\nabla\psi + V(x)u_n\psi + \lambda(I_\alpha * u_n^p)u_n^{p-2}u_n\psi\Big]\mathrm{d}x\\
&+ \frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\Big[f(x)u_n^{-\gamma}\psi_\varepsilon + u_n^5\psi_\varepsilon\Big]\mathrm{d}x\\
\leq& \Big\{\langle u_n, \psi\rangle_E + \frac{\lambda}{2p}\langle \mathbb{D}'(u_n), \psi\rangle - \int_{\mathbb{R}^3} f(x)u_n^{-\gamma}\psi \mathrm{d}x - \int_{\mathbb{R}^3} u_n^5\psi \mathrm{d}x\Big\}\\
&- \int_{\Omega_\varepsilon}\Big[\nabla u_n\nabla\psi + V(x)u_n\psi + \lambda(I_\alpha * u_n^p)u_n^{p-2}u_n\psi\Big]\mathrm{d}x.
\end{aligned} \tag{2.27}$$

Letting $\varepsilon \to 0^+$ to the above inequality and using the fact that $|\Omega_\varepsilon| \to 0$ as $\varepsilon \to 0^+$, we have

$$\langle u_n, \psi\rangle_E + \frac{\lambda}{2p}\langle \mathbb{D}'(u_n), \psi\rangle - \int_{\mathbb{R}^3} f(x)u_n^{-\gamma}\psi \mathrm{d}x - \int_{\mathbb{R}^3} u_n^5\psi \mathrm{d}x \geq o(1),\ \forall\psi \in E.$$

This inequality also holds for $-\psi$, hence we conclude that (2.25) holds for every $\psi \in E$. $\square$

Lemma 2.10. *Assume* $0 < \|f\|_{\frac{6}{5+\gamma}} < T_1$. *Suppose* $\{u_n\} \subset \mathcal{N}_\lambda^\pm$ *satisfy* (2.12), (2.13) *and*

$$J_\lambda(u_n) \to c < c_*,\ \text{as } n \to \infty, \tag{2.28}$$

where $c\neq 0$ *and* $c_*=\frac{1}{3}S^{\frac{3}{2}}-D_*\|f\|_{\frac{6}{5+\gamma}}^{\frac{2}{\gamma+1}}$ *with* $D_*=\frac{1+\gamma}{2}\left[\frac{2S}{3(1-\gamma)}\right]^{\frac{\gamma-1}{\gamma+1}}\left[\frac{5+\gamma}{6(1-\gamma)}\right]^{\frac{2}{\gamma+1}}$, *then* $u_\lambda\not\equiv 0$ *and* $\{u_n\}$ *possesses a subsequence strongly convergent to* u_λ *in* E.

Proof. We claim that $u_\lambda\not\equiv 0$. Arguing by contradiction, $u_\lambda\equiv 0$. Then, by $u_n\in\mathcal{N}_\lambda^{\pm}\subset\mathcal{N}_\lambda$, Lemma 2.1 and (2.14), we have

$$\|u_n\|_E^2=\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x+o(1). \tag{2.29}$$

It follows from (2.29) and $J_\lambda(u_n)\to c\neq 0$ that

$$c=J_\lambda(u_n)+o(1)=\frac{1}{3}\|u_n\|_E^2+o(1). \tag{2.30}$$

If $c<0$, we get a contradiction from the last relation. If $c>0$, there exists $n_0\in\mathbb{N}$ such that $\|u_n\|_E^2\geq c$ for $n\geq n_0$. This together with (1.5) and (2.29) leads to $\lim\limits_{n\to\infty}\|u_n\|_E^2\geq S^{\frac{3}{2}}$. Then, by (2.28), (2.30) and the fact of the above relation, we obtain that

$$c<c_*=\frac{1}{3}S^{\frac{3}{2}}-D_*\|f\|_{\frac{6}{5+\gamma}}^{\frac{2}{\gamma+1}}<\frac{1}{3}S^{\frac{3}{2}}\leq\frac{1}{3}\lim_{n\to\infty}\|u_n\|_E^2=c,$$

which is a contradiction. Therefore $u_\lambda\not\equiv 0$. By Brézis-Lieb's Lemma, we have

$$\begin{aligned}&\|u_n\|_E^2=\|u_\lambda\|_E^2+\|u_n-u_\lambda\|_E^2+o(1),\\&\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x=\int_{\mathbb{R}^3}|u_\lambda|^6\mathrm{d}x+\int_{\mathbb{R}^3}|u_n-u_\lambda|^6\mathrm{d}x+o(1).\end{aligned} \tag{2.31}$$

For any $\psi\in E$, set

$$Q(u_n,\psi)=(u_n,\psi)_E+\frac{\lambda}{2p}\langle\mathbb{D}'(u_n),\psi\rangle-\int_{\mathbb{R}^3}f(x)u_n^{-\gamma}\psi\mathrm{d}x-\int_{\mathbb{R}^3}u_n^5\psi\mathrm{d}x.$$

Then,

$$\begin{aligned}J_\lambda(u_n)-\frac{1}{6}Q(u_n,u_n)=&\frac{1}{3}\|u_n\|_E^2+\lambda(\frac{1}{2p}-\frac{1}{6})\mathbb{D}(u_n)-(\frac{1}{1-\gamma}-\frac{1}{6})\int_{\mathbb{R}^3}f(x)u_n^{1-\gamma}\mathrm{d}x\\=&\frac{1}{3}\|u_n-u_\lambda\|_E^2+\frac{1}{3}\|u_\lambda\|_E^2+\lambda(\frac{1}{2p}-\frac{1}{6})\mathbb{D}(u_n)\\&-(\frac{1}{1-\gamma}-\frac{1}{6})\int_{\mathbb{R}^3}f(x)u_n^{1-\gamma}\mathrm{d}x+o(1).\end{aligned} \tag{2.32}$$

Applying (2.25) with $\psi=u_\lambda$ and using $u_n\in\mathcal{N}_\lambda^{\pm}\subset\mathcal{N}_\lambda$, (2.13), (2.14), (2.31), Lemma 2.1 lead to

$$\begin{aligned}o(1)=&-\langle u_n,u_\lambda\rangle_E-\frac{\lambda}{2p}\langle\mathbb{D}'(u_n),u_\lambda\rangle+\int_{\mathbb{R}^3}f(x)u_n^{-\gamma}u_\lambda\mathrm{d}x+\int_{\mathbb{R}^3}u_n^5u_\lambda\mathrm{d}x\\=&\|u_n\|_E^2-\langle u_n,u_\lambda\rangle_E+\lambda\mathbb{D}(u_n)-\frac{\lambda}{2p}\langle\mathbb{D}'(u_n),u_\lambda\rangle-\int_{\mathbb{R}^3}f(x)u_n^{1-\gamma}\mathrm{d}x\\&+\int_{\mathbb{R}^3}f(x)u_n^{-\gamma}u_\lambda\mathrm{d}x-\int_{\mathbb{R}^3}u_n^6\mathrm{d}x+\int_{\mathbb{R}^3}u_n^5u_\lambda\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
=&\|u_n\|_E^2 - \|u_\lambda\|_E^2 - \int_{\mathbb{R}^3} f(x)u_\lambda^{1-\gamma}\mathrm{d}x + \int_{\mathbb{R}^3} f(x)u_n^{-\gamma}u_\lambda\mathrm{d}x - \int_{\mathbb{R}^3} u_n^6\mathrm{d}x \\
&+ \int_{\mathbb{R}^3} u_\lambda^6\mathrm{d}x + o(1) \\
=&\|u_n - u_\lambda\|_E^2 - \int_{\mathbb{R}^3} f(x)u_\lambda^{1-\gamma}\mathrm{d}x + \int_{\mathbb{R}^3} f(x)u_n^{-\gamma}u_\lambda\mathrm{d}x \\
&- \int_{\mathbb{R}^3} |u_n - u_\lambda|^6\mathrm{d}x + o(1).
\end{aligned}
$$

Therefore,

$$
\lim_{n\to\infty} \|u_n - u_\lambda\|_E^2 - \int_{\mathbb{R}^3} f(x)u_\lambda^{1-\gamma}\mathrm{d}x + \lim_{n\to\infty}\int_{\mathbb{R}^3} f(x)u_n^{-\gamma}u_\lambda\mathrm{d}x = \lim_{n\to\infty}\int_{\mathbb{R}^3} |u_n - u_\lambda|^6\mathrm{d}x. \tag{2.33}
$$

By Fatou's Lemma, we can obtain

$$
\int_{\mathbb{R}^3} f(x)u_\lambda^{1-\gamma}\mathrm{d}x \le \liminf_{n\to\infty}\int_{\mathbb{R}^3} f(x)u_n^{-\gamma}u_\lambda\mathrm{d}x. \tag{2.34}
$$

We can get from (2.33) and (2.34) that

$$
\lim_{n\to\infty} \|u_n - u_\lambda\|_E^2 \le \lim_{n\to\infty}\int_{\mathbb{R}^3} |u_n - u_\lambda|^6\mathrm{d}x. \tag{2.35}
$$

Set $\lim\limits_{n\to\infty} \|u_n - u_\lambda\|_E^2 = l$, then it follows from (1.5) and (2.35) that $l \le S^{-3}l^3$, which implies that either $l = 0$ or $l \ge S^{\frac{3}{2}}$. Suppose $l \ge S^{\frac{3}{2}}$, then one can obtain from (2.28), (2.32), (2.25), (2.14), (2.2), Lemma 2.1 and Young inequalities that

$$
\begin{aligned}
c_* >&c \\
=&\frac{l}{3} + \frac{1}{3}\|u_\lambda\|_E^2 + \lambda(\frac{1}{2p} - \frac{1}{6})\mathbb{D}(u_\lambda) - (\frac{1}{1-\gamma} - \frac{1}{6})\int_{\mathbb{R}^3} f(x)u_\lambda^{1-\gamma}\mathrm{d}x \\
\ge&\frac{1}{3}S^{\frac{3}{2}} + \frac{1}{3}\|u_\lambda\|_E^2 - (\frac{1}{1-\gamma} - \frac{1}{6})\|f\|_{\frac{6}{5+\gamma}} S^{\frac{\gamma-1}{2}}\|u_\lambda\|^{1-\gamma} \\
\ge&\frac{1}{3}S^{\frac{3}{2}} - \frac{1+\gamma}{2}\left[\frac{2S}{3(1-\gamma)}\right]^{\frac{\gamma-1}{\gamma+1}}\left[\frac{5+\gamma}{6(1-\gamma)}\right]^{\frac{2}{\gamma+1}}\|f\|_{\frac{6}{5+\gamma}}^{\frac{2}{\gamma+1}} \\
=&c_*,
\end{aligned}
$$

which is a contradiction. So $l = 0$ and $u_n \to u_\lambda$ strongly in E. □

3. Existence of a first solution in $\mathcal{N}_\lambda^+$

In this section, we want to prove Theorem 1.1 by a minimization argument on $\mathcal{N}_\lambda^+$.

Proof of Theorem 1.1. Fix $0 < \|f\|_{\frac{6}{5+\gamma}} < T_0 = \min\{T_1, T_2\}$, where T_1 is defined in (2.5) and $T_2 = \dfrac{4}{(5+\gamma)S^{\frac{\gamma-1}{2}}}\left[\dfrac{S^{\frac{3}{2}}(1-\gamma)}{1+\gamma}\right]^{\frac{\gamma+1}{2}}$, then $c_* > 0$. Due to Lemma 2.2, Lemma 2.4 and Ekeland variational principle, we can obtain a minimizing sequence

$\{u_n\} \subset \mathcal{V}_\lambda^+ = \mathcal{N}_\lambda^+ \cup \{0\}$ satisfying $(2.11)^+$, $(2.12)^+$ and (2.13). According to $(2.11)^+$ and Lemma 2.6 (i), we have

$$J_\lambda(u_n) \to \tau_\lambda^+ < 0 < c_*,$$

so $\{u_n\} \subset \mathcal{N}_\lambda^+$ and applying Lemma 2.10 with $c = \tau_\lambda^+$ results in $u_\lambda \not\equiv 0$ and $u_n \to u_\lambda$ in E, up to a subsequence.

Step 1. u_λ is a solution of problem (P_λ).

One can further obtain from the above relation, $u_n \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, Lemma 2.1 and Lemma 2.7 (i) that $u_\lambda \in \mathcal{N}_\lambda$ and

$$(1+\gamma)\|u_\lambda\|_E^2 + \lambda(2p-1+\gamma)\mathbb{D}(u_\lambda) - (5+\gamma)\int_{\mathbb{R}^3} |u_\lambda|^6 \mathrm{d}x > 0.$$

Hence, $u_\lambda \in \mathcal{N}_\lambda^+$. Furthermore, passing to the limit as $n \to \infty$ in (2.25) and using Fatou's Lemma, Lemma 2.1 and (2.13) lead to

$$\int_{\mathbb{R}^3} f(x)u_\lambda^{-\gamma}\psi \mathrm{d}x \le \liminf_{n\to\infty} \int_{\mathbb{R}^3} f(x)u_n^{-\gamma}\psi \mathrm{d}x = \langle u_\lambda, \psi\rangle_E + \frac{\lambda}{2p}\langle \mathbb{D}'(u_\lambda), \psi\rangle - \int_{\mathbb{R}^3} u_\lambda^5 \psi \mathrm{d}x, \tag{3.1}$$

for any $0 \le \psi \in E$. We can repeat the arguments used in (2.26)-(2.27) to derive that (3.1) holds for any $\psi \in E$. Thus, u_λ verifies (1.4) by the arbitrariness of $\psi \in E$ in (3.1). Similar to the proof of [33, Theorem 1], we have $u_\lambda \in C^2_{loc}(\mathbb{R}^3)$. Since $u_\lambda \ge 0$, $u_\lambda \not\equiv 0$ and u_λ satisfies (1.4), the strong maximum principle implies $u_\lambda > 0$ in $\mathbb{R}^3$ and then u_λ is a solution of problem (P_λ).

Step 2. u_λ is a ground state solution of problem (P_λ).

For any $u \in \mathcal{N}_\lambda^-$, according to Lemma 2.2, there exists unique $0 < t^+(u) < t_{max} < t^-(u)$ such that $t^+(u)u \in \mathcal{N}_\lambda^+$, $t^-(u)u \in \mathcal{N}_\lambda^-$, $J_\lambda(t^+(u)u) = \inf_{0<t\le t^-(u)} J_\lambda(tu)$ and $J_\lambda(t^-(u)u) = \sup_{t\ge t_{max}} J_\lambda(tu)$. Then $t^-(u) = 1$ and there exists $\bar{t}(u) \in (t_{max}, t^-(u))$ such that $J_\lambda(t^+(u)u) < J_\lambda(\bar{t}(u)u)$. So

$$\tau_\lambda^+ \le J_\lambda(t^+(u)u) < J_\lambda(\bar{t}(u)u) \le J_\lambda(t^-(u)u) = J_\lambda(u).$$

By the arbitrariness of $u \in \mathcal{N}_\lambda^-$ and the definition of $\tau_\lambda^\pm$ and τ_λ, we have $\tau_\lambda^+ < \tau_\lambda^-$ and so $\tau_\lambda = \tau_\lambda^+$ thanks to $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ by Lemma 2.2. Therefore, $J_\lambda(u_\lambda) = \tau_\lambda^+ = \tau_\lambda$ and thus u_λ is a ground state solution of problem (P_λ).

Step 3. For any vanishing sequence $\{\lambda_n\} \subset (0,1)$, $u_{\lambda_n} \to u_0$ strongly in E where u_0 is a positive solution of problem (P_0) .

For any vanishing sequence $\{\lambda_n\} \subset (0,1)$, since $\{u_{\lambda_n}\} \subset \mathcal{N}_{\lambda_n}^+$ is a ground state solution sequence to problem (P_{λ_n}) provided by Step 2, then $J_{\lambda_n}(u_{\lambda_n}) = \tau_{\lambda_n}^+ = \tau_{\lambda_n}$ and

$$\langle u_{\lambda_n}, \psi\rangle_E + \frac{\lambda_n}{2p}\langle \mathbb{D}'(u_{\lambda_n}), \psi\rangle = \int_{\mathbb{R}^3} f(x)u_{\lambda_n}^{-\gamma}\psi \mathrm{d}x + \int_{\mathbb{R}^3} u_{\lambda_n}^5 \psi \mathrm{d}x, \tag{3.2}$$

for every $\psi \in E$ and $n \in \mathbb{N}$. By Lemma 2.3, (2.9) and (2.10), we have $\|u_{\lambda_n}\|_E < A_*$ and $\frac{1+\gamma}{3(\gamma-1)}A_*^2 \le \tau_{\lambda_n} < 0$. Thus, there exists a subsequence of $\{\lambda_n\}$, still denoted by $\{\lambda_n\}$, such that as $n \to \infty$, $\tau_{\lambda_n} \to \mu_1 \le 0$ and

$$u_{\lambda_n} \rightharpoonup u_0, \text{in } E,$$

$$\begin{aligned}&u_{\lambda_n} \to u_0, \text{ in } L^s(\mathbb{R}^3), \ s \in [2,6),\\ &u_{\lambda_n} \to u_0, \text{ a.e. in } \mathbb{R}^3,\end{aligned} \tag{3.3}$$

where u_0 is a nonnegative function in E. According to (2.10), Lemma 2.1 and weak lower semicontinuity of the norm, we can further get

$$\begin{aligned}\mu_1 =&\liminf_{n\to\infty} J_{\lambda_n}(u_{\lambda_n})\\ \le&\liminf_{n\to\infty}\left[-\frac{1+\gamma}{3(1-\gamma)}\|u_{\lambda_n}\|_E^2 + \lambda_n(3-p)\frac{1-\gamma-2p}{6p(1-\gamma)}\mathbb{D}(u_{\lambda_n})\right]\\ \le&-\frac{1+\gamma}{3(1-\gamma)}\|u_0\|_E^2\\ <&0.\end{aligned} \tag{3.4}$$

This together with $c_* > 0$ leads to $J_{\lambda_n}(u_{\lambda_n}) \to \mu_1 < 0 < c_*$. Using (3.2) and the statement in the proof of Lemma 2.10, one can similarly obtain that $u_0 \not\equiv 0$ and $u_{\lambda_n} \to u_0$ strongly in E. Then, according to $\|u_{\lambda_n}\|_E < A_*$ and $\{u_{\lambda_n}\} \subset \mathcal{N}_{\lambda_n}^+ \subset \mathcal{N}_{\lambda_n}$, we have $\|u_0\|_E \le A_*$ and $u_0 \in \mathcal{N}_0$. Passing to the lim as $n \to \infty$ in (3.2) and repeating the arguments used in Step 1, for every $\psi \in E$, we have

$$\langle u_0, \psi\rangle_E = \int_{\mathbb{R}^3} f(x)u_0^{-\gamma}\psi \mathrm{d}x + \int_{\mathbb{R}^3} u_0^5\psi \mathrm{d}x, \tag{3.5}$$

and u_0 is a positive solution of problem (P_0). Hence, $J_0(u_0) = \mu_1 \ge \tau_0$ where $\tau_0 = \inf_{u\in\mathcal{N}_0} J_0(u)$.

Step 4. u_0 is a ground state solution of problem (P_0).

In order to show u_0 is a ground state solution of problem (P_0), it is enough to prove that $J_0(u_0) = \tau_0$. Noticing that $\lambda = 0$ is allowed in Step 1 and Step 2, then problem (P_0) admits a ground state solution w_0 satisfying $0 < w_0 \in \mathcal{N}_0$ and $J_0(w_0) = \tau_0$. By Lemma 2.2, for all $n \in \mathbb{N}$, there exists $0 < t_{\lambda_n}^+ < t_{\lambda_n}^-$ such that $t_{\lambda_n}^\pm w_0 \in \mathcal{N}_{\lambda_n}^\pm \subset \mathcal{N}_{\lambda_n}$ and $J_{\lambda_n}(t_{\lambda_n}^+ w_0) = \inf_{0<t\le t_{\lambda_n}^-} J_{\lambda_n}(tw_0)$. We claim that $\{t_{\lambda_n}^-\}$ is bounded. Suppose to the contrary that there exists a subsequence of $\{t_{\lambda_n}^-\}$, still denoted by $\{t_{\lambda_n}^-\}$ such that $t_{\lambda_n}^- \to +\infty$ as $n \to \infty$. Then, by $t_{\lambda_n}^- w_0 \in \mathcal{N}_{\lambda_n}^- \subset \mathcal{N}_{\lambda_n}$ and (2.4), we have

$$\frac{1}{(t_{\lambda_n}^-)^4}\|w_0\|_E^2 + \frac{\lambda_n}{(t_{\lambda_n}^-)^{6-2p}}\mathbb{D}(w_0) = \frac{1}{(t_{\lambda_n}^-)^{5+\gamma}}\int_{\mathbb{R}^3} f(x)|w_0|^{1-\gamma}\mathrm{d}x + \int_{\mathbb{R}^3}|w_0|^6\mathrm{d}x, \tag{3.6}$$

and

$$-(6-2p)(t_{\lambda_n}^-)^{2p}\lambda_n\mathbb{D}(w_0) + (5+\gamma)(t_{\lambda_n}^-)^{1-\gamma}\int_{\mathbb{R}^3} f(x)|w_0|^{1-\gamma}\mathrm{d}x < 4(t_{\lambda_n}^-)^2\|w_0\|_E^2. \tag{3.7}$$

Moreover, $w_0 \in \mathcal{N}_0$ means

$$\|w_0\|_E^2 = \int_{\mathbb{R}^3} f(x)|w_0|^{1-\gamma}\mathrm{d}x + \int_{\mathbb{R}^3}|w_0|^6\mathrm{d}x. \tag{3.8}$$

Subtracting (3.6) with (3.8) provides

$$\left[1 - \frac{1}{(t_{\lambda_n}^-)^4}\right]\|w_0\|_E^2 - \frac{\lambda_n}{(t_{\lambda_n}^-)^{6-2p}}\mathbb{D}(w_0) = \left[1 - \frac{1}{(t_{\lambda_n}^-)^{5+\gamma}}\right]\int_{\mathbb{R}^3} f(x)|w_0|^{1-\gamma}\mathrm{d}x. \tag{3.9}$$

Passing to the limit in the above equality, we have

$$\|w_0\|_E^2 = \int_{\mathbb{R}^3} f(x)|w_0|^{1-\gamma}\mathrm{d}x,$$

a contradiction to (3.8). Therefore, $\{t_{\lambda_n}^-\}$ is bounded. Up to a subsequence, suppose that $t_{\lambda_n}^- \to t_0^-$. We claim that $t_0^- \geq 1$. Arguing by contradiction suppose that $0 < t_0^- < 1$, then it follows from (3.9) and (3.7) that

$$\Big[1 - \frac{1}{(t_0^-)^4}\Big]\|w_0\|_E^2 = \Big[1 - \frac{1}{(t_0^-)^{5+\gamma}}\Big]\int_{\mathbb{R}^3} f(x)|w_0|^{1-\gamma}\mathrm{d}x, \tag{3.10}$$

and

$$(5+\gamma)(t_0^-)^{1-\gamma}\int_{\mathbb{R}^3} f(x)|w_0|^{1-\gamma}\mathrm{d}x \leq 4(t_0^-)^2\|w_0\|_E^2. \tag{3.11}$$

Combining (3.10) with (3.11), we can deduce that

$$4(t_0^-)^{5+\gamma} - (5+\gamma)(t_0^-)^4 + 1 + \gamma \leq 0,$$

which is impossible since $4t^{5+\gamma} - (5+\gamma)t^4 + 1 + \gamma > 0$ for all $t \in (0,1)$. Therefore, $t_0^- \geq 1$. If $t_0^- > 1$, then $t_{\lambda_n}^- > 1$ for some n large enough. This together with $J_{\lambda_n}(t_{\lambda_n}^+ w_0) = \inf\limits_{0<t\leq t_{\lambda_n}^-} J_{\lambda_n}(tw_0)$ leads to $J_{\lambda_n}(w_0) \geq J_{\lambda_n}(t_{\lambda_n}^+ w_0)$ for some n large enough. If $t_0^- = 1$, then $t_{\lambda_n}^- \to 1$. For some n large enough with $t_{\lambda_n}^- \geq 1$, we have $J_{\lambda_n}(w_0) \geq J_{\lambda_n}(t_{\lambda_n}^+ w_0)$ by the similar statement above. For some n large enough with $t_{\lambda_n}^- < 1$, according to Lemma 2.2, there exists $t_{\lambda_n} \in (t_{\lambda_n}^+, t_{\lambda_n}^-)$ such that $J_{\lambda_n}(w_0) = J_{\lambda_n}(t_{\lambda_n} w_0) \geq J_{\lambda_n}(t_{\lambda_n}^+ w_0)$. Follows from above two cases, we get $J_{\lambda_n}(w_0) \geq J_{\lambda_n}(t_{\lambda_n}^+ w_0)$ for some n large enough when $t_0^- = 1$. To sum up, $J_{\lambda_n}(w_0) \geq J_{\lambda_n}(t_{\lambda_n}^+ w_0)$ for some n large enough. Hence, we can obtain from $t_{\lambda_n}^+ w_0 \in \mathcal{N}_{\lambda_n}^+$ and $\tau_{\lambda_n}^+ = \tau_{\lambda_n}$ that

$$\begin{aligned}
\tau_0 =& J_0(w_0) \\
=& J_{\lambda_n}(w_0) - \frac{\lambda_n}{2p}\mathbb{D}(w_0) \\
\geq& J_{\lambda_n}(t_{\lambda_n}^+ w_0) - \frac{\lambda_n}{2p}\mathbb{D}(w_0) \\
\geq& \tau_{\lambda_n}^+ - \frac{\lambda_n}{2p}\mathbb{D}(w_0) \\
=& \tau_{\lambda_n} - \frac{\lambda_n}{2p}\mathbb{D}(w_0),
\end{aligned}$$

for some n large enough and so

$$\limsup_{n\to+\infty} \tau_{\lambda_n} \leq \tau_0. \tag{3.12}$$

Using (3.12), one can further get

$$\tau_0 \leq J_0(u_0) = \limsup_{n\to+\infty} J_{\lambda_n}(u_{\lambda_n}) = \limsup_{n\to+\infty} \tau_{\lambda_n} \leq \tau_0.$$

This shows that $J_0(u_0) = \tau_0$ and so u_0 is a ground state solution of problem (P_0). The proof is completed. $\square$

4. Existence of a second solution in $\mathcal{N}_\lambda^-$

It is well known that S can be attained by the function

$$U_\varepsilon(x) = \frac{(3\varepsilon)^{\frac{1}{4}}}{(\varepsilon + |x|^2)^{\frac{1}{2}}},\ \varepsilon > 0,\ x \in \mathbb{R}^3, \tag{4.1}$$

and $\|U_\varepsilon\|^2 = \|U_\varepsilon\|_6^6 = S^{\frac{3}{2}}$. Let $\eta(x) \in C_0^\infty(\mathbb{R}^3)$ be a radially symmetric function such that $0 \le \eta \le 1$, $\eta|_{B_{\frac{\delta}{2}}}(0) \equiv 1$ and supp$\eta \subset B_\delta(0)$ for some $\delta > 2\delta_1$ where δ_1 is given in (f_2). Moreover, set $w_\varepsilon(x) = \eta(x)U_\varepsilon(x)$, then for $\varepsilon > 0$ small enough, we have (see [3])

$$\int_{\mathbb{R}^3} |\nabla w_\varepsilon|^2 \mathrm{d}x = K_1 + O(\varepsilon^{\frac{1}{2}}),\ \int_{\mathbb{R}^3} |w_\varepsilon|^6 \mathrm{d}x = K_2 + O(\varepsilon^{\frac{3}{2}}), \tag{4.2}$$

and

$$\int_{\mathbb{R}^3} |w_\varepsilon|^s \mathrm{d}x = \begin{cases} O(\varepsilon^{\frac{s}{4}}), & s \in [2,3), \\ O(\varepsilon^{\frac{s}{4}}|\ln\varepsilon|), & s = 3, \\ O(\varepsilon^{\frac{6-s}{4}}), & s \in (3,6), \end{cases} \tag{4.3}$$

where $K_1,\ K_2$ are positive constants and $\dfrac{K_1}{K_2^{\frac{1}{3}}} = S$. Using (4.2), we can further get

$$\frac{\int_{\mathbb{R}^3} |\nabla w_\varepsilon|^2 \mathrm{d}x}{(\int_{\mathbb{R}^3} |w_\varepsilon|^6 \mathrm{d}x)^{\frac{1}{3}}} = S + O(\varepsilon^{\frac{1}{2}}). \tag{4.4}$$

Lemma 4.1. *Assume (V_1), (V_2), (f_1) and (f_2) hold, then there exists $0 < T_{00} < T_0$ where T_0 is defined in proof of Theorem 1.1, such that for $0 < \|f\|_{\frac{6}{5+\gamma}} < T_{00}$ and $\varepsilon > 0$ small, we have*

$$\tau_\lambda^- \le \sup_{t\ge 0} J_\lambda(tw_\varepsilon) < c_*,\ \forall \lambda > 0,$$

where c_* is given in Lemma 2.10.

Proof. For $0 < \|f\|_{\frac{6}{5+\gamma}} < \dfrac{1-\gamma}{2} T_1$, by Lemma 2.2 and Lemma 2.6 (ii), there exists $t_\varepsilon > t_{max} > 0$ such that $t_\varepsilon w_\varepsilon \in \mathcal{N}_\lambda^-$ and $J_\lambda(t_\varepsilon w_\varepsilon) = \sup_{t\ge 0} J_\lambda(tw_\varepsilon) \ge \beta_0 > 0$. We can get from this and $J_\lambda(tw_\varepsilon) \to -\infty$ as $t \to +\infty$ that there exist positive constants t_{00}, t_0 independent of ε such that $t_{00} \le t_\varepsilon \le t_0$. Motivated by [10,42], let $J_\lambda(t_\varepsilon w_\varepsilon) = A(\varepsilon) + B(\varepsilon) + C(\varepsilon) - D(\varepsilon)$, where

$$A(\varepsilon) = \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3} |\nabla w_\varepsilon|^2 \mathrm{d}x - \frac{t_\varepsilon^6}{6}\int_{\mathbb{R}^3} |w_\varepsilon|^6 \mathrm{d}x,\ B(\varepsilon) = \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3} V(x)|w_\varepsilon|^2 \mathrm{d}x,$$

$$C(\varepsilon) = \lambda\frac{t_\varepsilon^{2p}}{2p}\mathbb{D}(w_\varepsilon),\ D(\varepsilon) = \frac{t_\varepsilon^{1-\gamma}}{1-\gamma}\int_{\mathbb{R}^3} f(x)|w_\varepsilon|^{1-\gamma}\mathrm{d}x.$$

For the purpose of proof, set $g_3(t) = \dfrac{t^2}{2}\displaystyle\int_{\mathbb{R}^3} |\nabla w_\varepsilon|^2 \mathrm{d}x - \dfrac{t^6}{6}\int_{\mathbb{R}^3} |w_\varepsilon|^6 \mathrm{d}x$, then one can easily get that $g_3(t)$ achieves its maximum at T_{max} with $T_{max} = \left(\dfrac{\int_{\mathbb{R}^3} |\nabla w_\varepsilon|^2 \mathrm{d}x}{\int_{\mathbb{R}^3} |w_\varepsilon|^6 \mathrm{d}x}\right)^{\frac{1}{4}}$.

Thus, it follows from (4.4) that

$$A(\varepsilon)=g_3(t_\varepsilon)\leq g_3(T_{max})=\frac{1}{3}\frac{(\int_{\mathbb{R}^3}|\nabla w_\varepsilon|^2\mathrm{d}x)^{\frac{3}{2}}}{(\int_{\mathbb{R}^3}|w_\varepsilon|^6\mathrm{d}x)^{\frac{1}{2}}}=\frac{1}{3}S^{\frac{3}{2}}+O(\varepsilon^{\frac{1}{2}}). \tag{4.5}$$

Since $t_{00}\leq t_\varepsilon\leq t_0$, one can get from $V\in C(\mathbb{R}^3)$, the definition of w_ε and (4.3) that

$$B(\varepsilon)=\frac{t_\varepsilon^2}{2}\int_{B_\delta(0)}V(x)|w_\varepsilon|^2\mathrm{d}x\leq\max_{x\in B_\delta(0)}V(x)\cdot\frac{t_0^2}{2}\int_{B_\delta(0)}|w_\varepsilon|^2\mathrm{d}x=O(\varepsilon^{\frac{1}{2}}). \tag{4.6}$$

By (2.1), (4.3) and $1+\frac{\alpha}{3}\leq p<3$, we also have

$$C(\varepsilon)\leq\lambda\frac{t_0^{2p}}{2p}d_\alpha\Big(\int_{\mathbb{R}^3}|w_\varepsilon|^{\frac{6p}{3+\alpha}}\mathrm{d}x\Big)^{\frac{3+\alpha}{3}}=\begin{cases}O(\varepsilon^{\frac{p}{2}}), & \frac{3+\alpha}{3}\leq p<\frac{3+\alpha}{2},\\ O(\varepsilon^{\frac{p}{2}}|\ln\varepsilon|^{\frac{3+\alpha}{3}}), & p=\frac{3+\alpha}{2},\\ O(\varepsilon^{\frac{3+\alpha-p}{2}}), & \frac{3+\alpha}{2}<p<3,\end{cases} \tag{4.7}$$

Similarly, by (f_2) and $\frac{3+\gamma}{2}<\beta_1<\frac{5+\gamma}{2}<3$, for any ε satisfying $0<\varepsilon\leq\delta_1^2$, we have

$$\begin{aligned}D(\varepsilon)&=\frac{t_\varepsilon^{1-\gamma}}{1-\gamma}\int_{|x|<\delta}f(x)|w_\varepsilon|^{1-\gamma}\mathrm{d}x\\&=\frac{t_\varepsilon^{1-\gamma}}{1-\gamma}\left[\int_{|x|<\delta_1}f(x)|w_\varepsilon|^{1-\gamma}\mathrm{d}x+\int_{\delta_1\leq|x|<\delta}f(x)|w_\varepsilon|^{1-\gamma}\mathrm{d}x\right]\\&\geq\frac{t_{00}^{1-\gamma}}{1-\gamma}\int_{|x|<\delta_1}\frac{\rho_1|x|^{-\beta_1}(3\varepsilon)^{\frac{1-\gamma}{4}}}{(\varepsilon+|x|^2)^{\frac{1-\gamma}{2}}}\mathrm{d}x\\&=C_3\varepsilon^{\frac{1-\gamma}{4}}\int_0^{\delta_1}\frac{r^2}{r^{\beta_1}(\varepsilon+r^2)^{\frac{1-\gamma}{2}}}\mathrm{d}r\\&=C_3\varepsilon^{\frac{\gamma+5-2\beta_1}{4}}\int_0^{\frac{\delta_1}{\sqrt{\varepsilon}}}\frac{r^2}{r^{\beta_1}(1+r^2)^{\frac{1-\gamma}{2}}}\mathrm{d}r\\&\geq C_3\varepsilon^{\frac{\gamma+5-2\beta_1}{4}}\int_0^1\frac{r^2}{2^{\frac{1-\gamma}{2}}r^{\beta_1}}\mathrm{d}r\\&=C_4\varepsilon^{\frac{\gamma+5-2\beta_1}{4}}.\end{aligned} \tag{4.8}$$

Case 1. $\frac{3+\alpha}{3}\leq p\leq\frac{3+\alpha}{2}$.

For $\frac{3+\alpha}{3}\leq p\leq\frac{3+\alpha}{2}$, using the fact that $\lim\limits_{\varepsilon\to0^+}\varepsilon^{\frac{p-1}{2}}|\ln\varepsilon|^{\frac{3+\alpha}{3}}=0$ and $\frac{p}{2}\geq\frac{3+\alpha}{6}>\frac{1}{2}$, we can obtain from (4.7) that $C(\varepsilon)=O(\varepsilon^{\frac{1}{2}})$. Combining this with (4.5), (4.6) and (4.8) leads to

$$J_\lambda(t_\varepsilon w_\varepsilon)\leq\frac{1}{3}S^{\frac{3}{2}}+O(\varepsilon^{\frac{1}{2}})-C_4\varepsilon^{\frac{\gamma+5-2\beta_1}{4}}\leq\frac{1}{3}S^{\frac{3}{2}}+C_5\varepsilon^{\frac{1}{2}}-C_4\varepsilon^{\frac{\gamma+5-2\beta_1}{4}}.$$

Set $\varepsilon = \|f\|_{\frac{6}{5+\gamma}}^{\frac{4}{1+\lambda}}$ and $T_* = \left(\frac{C_4}{C_5+D_*}\right)^{\frac{1+\gamma}{2\beta_1-3-\gamma}}$ where D_* is given in Lemma 2.10, since $\frac{3+\gamma}{2} < \beta_1 < \frac{5+\gamma}{2}$, we have

$$C_5\varepsilon^{\frac{1}{2}} - C_4\varepsilon^{\frac{\gamma+5-2\beta_1}{4}} = \|f\|_{\frac{6}{5+\gamma}}^{\frac{2}{\gamma+1}}\left(C_5 - C_4\|f\|_{\frac{6}{5+\gamma}}^{\frac{\gamma+3-2\beta_1}{1+\gamma}}\right) < -D_*\|f\|_{\frac{6}{5+\gamma}}^{\frac{2}{\gamma+1}},$$

and so

$$\sup_{t\geq 0} J_\lambda(tw_\varepsilon) = J_\lambda(t_\varepsilon w_\varepsilon) < \frac{1}{3}S^{\frac{3}{2}} - D_*\|f\|_{\frac{6}{5+\gamma}}^{\frac{2}{\gamma+1}} = c_*,$$

for all $\|f\|_{\frac{6}{5+\gamma}}$ sufficiently small with $\|f\|_{\frac{6}{5+\gamma}} < T_3 = \min\{\frac{1-\gamma}{2}T_1, T_2, T_*\}$.

Case 2. $\frac{3+\alpha}{2} < p < 3$.

When $p-\alpha \leq 2$ i.e. $\frac{3+\alpha-p}{2} \geq \frac{1}{2}$, similarly to Case 1, we can obtain $\sup_{t\geq 0} J_\lambda(tw_\varepsilon) < c_*$. Hence, we only consider the situation when $p-\alpha > 2$. It follows from $p-\alpha > 2$ and $\frac{3+\alpha}{2} < p < 3$ that $\frac{\alpha}{2} < \frac{3+\alpha-p}{2} < \frac{1}{2}$. Hence, one can get from (4.5)-(4.8) that

$$J_\lambda(t_\varepsilon w_\varepsilon) \leq \frac{1}{3}S^{\frac{3}{2}} + O(\varepsilon^{\frac{1}{2}}) + O(\varepsilon^{\frac{3+\alpha-p}{2}}) - C_4\varepsilon^{\frac{\gamma+5-2\beta_1}{4}} \leq \frac{1}{3}S^{\frac{3}{2}} + C_6\varepsilon^{\frac{\alpha}{2}} - C_4\varepsilon^{\frac{\gamma+5-2\beta_1}{4}}.$$

Set $\varepsilon = \|f\|_{\frac{6}{5+\gamma}}^{\frac{4}{\alpha(1+\gamma)}}$ and $T_{**} = \left(\frac{C_4}{C_6+D_*}\right)^{\frac{\alpha(1+\gamma)}{2\beta_1+2\alpha-5-\gamma}}$, since $\frac{5+\gamma-2\alpha}{2} < \beta_1 < \frac{5+\gamma}{2}$, we have

$$C_6\varepsilon^{\frac{\alpha}{2}} - C_4\varepsilon^{\frac{\gamma+5-2\beta_1}{4}} = \|f\|_{\frac{6}{5+\gamma}}^{\frac{2}{\gamma+1}}\left(C_6 - C_4\|f\|_{\frac{6}{5+\gamma}}^{\frac{\gamma+5-2\beta_1-2\alpha}{\alpha(1+\gamma)}}\right) < -D_*\|f\|_{\frac{6}{5+\gamma}}^{\frac{2}{\gamma+1}},$$

and so for all $\|f\|_{\frac{6}{5+\gamma}}$ sufficiently small with $\|f\|_{\frac{6}{5+\gamma}} < T_4 = \min\{\frac{1-\gamma}{2}T_1, T_2, T_{**}\}$, we have

$$\sup_{t\geq 0} J_\lambda(tw_\varepsilon) = J_\lambda(t_\varepsilon w_\varepsilon) < \frac{1}{3}S^{\frac{3}{2}} - D_*\|f\|_{\frac{6}{5+\gamma}}^{\frac{2}{\gamma+1}} = c_*.$$

To sum up, set $T_{00} = \min\{T_3, T_4\}$, then for all $\|f\|_{\frac{6}{5+\gamma}}$ sufficiently small with $\|f\|_{\frac{6}{5+\gamma}} < T_{00}$, we have

$$\tau_\lambda^- \leq J_\lambda(t_\varepsilon w_\varepsilon) = \sup_{t\geq 0} J_\lambda(tw_\varepsilon) < c_*,$$

since $t_\varepsilon w_\varepsilon \in \mathcal{N}_\lambda^-$ and this ends the proof. □

Proof of Theorem 1.2. Fix $0 < \|f\|_{\frac{6}{5+\gamma}} < T_{00}$, according to Theorem 1.1, we only need to show the existence and asymptotic behavior of another solution v_λ which is different with the first solution u_λ. Since $\mathcal{N}_\lambda^-$ is a closed set in E by Lemma 2.4, applying the Ekeland variational principle to construct a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda^-$ satisfying $(2.11)^-$, $(2.12)^-$ and (2.13) with weak limit v_λ, to not confuse with u_λ obtained in Section 2 and Section 3.

Step 1. v_λ is a solution of problem (P_λ).

We can get from $(2.11)^-$, Lemma 2.6 (ii) and Lemma 4.1 that

$$\tau_\lambda^- \geq \beta_0 > 0 \text{ and } J_\lambda(u_n) \to \tau_\lambda^- < c_*,$$

so Lemma 2.10 with $c = \tau_\lambda^-$ results in $v_\lambda \not\equiv 0$ and $u_n \to v_\lambda$ in E, up to a subsequence. Then, $J_\lambda(v_\lambda) = \tau_\lambda^-$. Moreover, $u_n \in \mathcal{N}_\lambda^- \subset \mathcal{N}_\lambda$ and $u_n \to v_\lambda$ further lead to $v_\lambda \in \mathcal{N}_\lambda$. Similarly, one can get from Lemma 2.1 and Lemma 2.7 (ii) that

$$(1+\gamma)\|v_\lambda\|_E^2 + \lambda(2p-1+\gamma)\mathbb{D}(v_\lambda) - (5+\gamma)\int_{\mathbb{R}^3} |v_\lambda|^6 \mathrm{d}x < 0,$$

therefore, $v_\lambda \in \mathcal{N}_\lambda^-$. Following the argument used for the first solution u_λ in Section 3, we see that v_λ is also a positive solution of problem (P_λ). Moreover, since $u_\lambda \in \mathcal{N}_\lambda^+$ and $v_\lambda \in \mathcal{N}_\lambda^-$, we get from Lemma 2.3 that $\|v_\lambda\|_E > \|u_\lambda\|_E$. So u_λ and v_λ are distinct.

Step 2. For any vanishing sequence $\{\lambda_n\} \subset (0,1)$, $v_{\lambda_n} \to v_0$ strongly in E where v_0 is a positive solution of problem (P_0).

For any vanishing sequence $\{\lambda_n\} \subset (0,1)$, since $\{v_{\lambda_n}\} \subset \mathcal{N}_{\lambda_n}^-$ is a positive solution sequence to problem (P_{λ_n}) provided by Step 1, then $\beta_0 \le J_{\lambda_n}(v_{\lambda_n}) = \tau_{\lambda_n}^- < c_*$, $\|v_{\lambda_n}\|_E > A^* > 0$ and

$$(v_{\lambda_n}, \psi)_E + \frac{\lambda_n}{2p}\langle \mathbb{D}'(v_{\lambda_n}), \psi\rangle = \int_{\mathbb{R}^3} f(x) v_{\lambda_n}^{-\gamma}\psi \mathrm{d}x + \int_{\mathbb{R}^3} v_{\lambda_n}^5 \psi \mathrm{d}x, \tag{4.9}$$

for every $\psi \in E$ and $n \in \mathbb{N}$. Since $v_{\lambda_n} \in \mathcal{N}_{\lambda_n}$ and $J_{\lambda_n}(v_{\lambda_n}) < c_*$, then $\{v_{\lambda_n}\}$ is bounded in E by (2.9). Thus, there exists a subsequence of $\{\lambda_n\}$, still denoted by $\{\lambda_n\}$, such that as $n \to \infty$, $\tau_{\lambda_n}^- \to \mu_2$ and

$$\begin{aligned} &v_{\lambda_n} \rightharpoonup v_0, \text{ in } E,\\ &v_{\lambda_n} \to v_0, \text{ in } L^s(\mathbb{R}^3), \ s \in [2,6),\\ &v_{\lambda_n} \to v_0, \text{ a.e. in } \mathbb{R}^3, \end{aligned} \tag{4.10}$$

where v_0 is nonnegative in E. Hence, $\mu_2 \ge \beta_0 > 0$ and $J_{\lambda_n}(v_{\lambda_n}) \to \mu_2 < c_*$. Using (4.9) and the statement in the proof of Lemma 2.10, one can similarly obtain that $v_0 \not\equiv 0$ and $v_{\lambda_n} \to v_0$ strongly in E. Then, $\|v_0\|_E \ge A^*$ follows from $\|v_{\lambda_n}\|_E > A^*$. Passing to the lim as $n \to \infty$ in (4.9) and repeating the arguments used in Step 1 in the proof of Theorem 1.1, we have that v_0 is a positive solution of problem (P_0). It follows from $\|u_0\|_E \le A_*$, $\|v_0\|_E \ge A^*$ and $A_* < A^*$ in Lemma 2.3 that $\|u_0\|_E < \|v_0\|_E$. The proof is completed. □

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References

[1] Y. Ao, *Existence of solutions for a class of nonlinear Choquard equations with critical growth*, Appl. Anal., 2021, 100(3), 465–481.

[2] T. Bartsch and Z. Wang, *Existence and multiplicity results for some superlinear elliptic problems on* $\mathbb{R}^N$, Comm. Partial Differential Equations, 1995, 20(9–10), 1725–1741.

[3] H. Brézis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., 1983, 36(4), 437–477.

[4] M. Coclite and G. Palmieri, *On a singular nonlinear Dirichlet problem*, Comm. Partial Differential Equations, 1989, 14(10), 1315–1327.

[5] A. Fiscella and P. Mishra, *The Nehari manifold for fractional Kirchhoff problems involving singular and critical terms*, Nonlinear Anal., 2019, 186, 6–32.

[6] M. Ghimenti, V. Moroz and J. Schaftingen, *Least action nodal solutions for the quadratic Choquard equation*, Proc. Amer. Math. Soc., 2017, 145, 737–747.

[7] J. Giacomoni and K. Saoudi, *Multiplicity of positive solutions for a singular and critical problem*, Nonlinear Anal., 2009, 71(9), 4060–4077.

[8] N. Hirano, C. Saccon and N. Shioji, *Existence of multiple positive solutions for singular elliptic problems with concave and convex nonlinearities*, Adv. Differential Equations, 2004, 9(1–2), 197–220.

[9] N. Hirano, C. Saccon and N. Shioji, *Brezis-Nirenberg type theorems and multiplicity of positive solutions for a singular elliptic problem*, J. Differential Equations, 2008, 245(8), 1997–2037.

[10] L. Huang, E. Rocha and J. Chen, *Positive and sign-changing solutions of a Schrödinger-Poisson system involving a critical nonlinearity*, J. Math. Anal. Appl., 2013, 408(1), 55–69.

[11] C. Lei and J. Liao, *Multiple positive solutions for Schrödinger-Poisson system involving singularity and critical exponent*, Math. Meth. Appl. Sci., 2019, 42(7), 2417–2430.

[12] C. Lei, J. Liao and C. Tang, *Multiple positive solutions for Kirchhoff type of problems with singularity and critical exponents*, J. Math. Anal. Appl., 2015, 421(1), 521–538.

[13] C. Lei, H. Suo and C. Chu, *Multiple positive solutions for a Schrödinger-Newton system with singularity and critical growth*, Electron. J. Differential Equations, 2018, 86, 1–15.

[14] F. Li, C. Gao and X. Zhu, *Existence and concentration of sign-changing solutions to Kirchhoff-type system with Hartree-type nonlinearity*, J. Math. Anal. Appl., 2017, 448(1), 60–80.

[15] G. Li, Y. Li, C. Tang and L. Yin, *Existence and concentrate behavior of ground state solutions for critical Choquard equations*, Appl. Math. Lett., 2019, 96, 101–107.

[16] G. Li and C. Tang, *Existence of a ground state solution for Choquard equation with the upper critical exponent*, Comput. Math. Appl., 2018, 76(11–12), 2635–2647.

[17] X. Li and S. Ma, *Choquard equations with critical nonlinearities*, Commun. Contemp. Math., 2020, 22(4), 1950023.

[18] X. Li, S. Ma and G. Zhang, *Existence and qualitative properties of solutions for Choquard equations with a local term*, Nonlinear Anal. Real World Appl., 2019, 45, 1–25.

[19] E. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Stud. Appl. Math., 1977, 57(2), 93–105.

[20] J. Liu, A. Hou and J. Liao, *Multiplicity of positive solutions for a class of singular elliptic equations with critical Sobolev exponent and Kirchhoff-type nonlocal term*, Electron. J. Qual. Theory Differ. Equ., 2018, 100, 1–20.

[21] D. Lü, *Existence and concentration of solutions for a nonlinear Choquard equation*, Mediterr. J. Math., 2015, 12(3), 839–850.

[22] D. Lü, *A note on Kirchhoff-type equations with Hartree-type nonlinearities*, Nonlinear Anal., 2014, 99, 35–48.

[23] C. Mercuri, V. Moroz and J. Schaftingen, *Groundstates and radial solutions to nonlinear Schrödinger-Poisson-Slater equations at the critical frequency*, Calc. Var. Partial Differ. Equ., 2016, 55, 146.

[24] V. Moroz and J. Schaftingen, *A guide to the Choquard equation*, J. Fixed Point Theory Appl., 2017, 19, 773–813.

[25] T. Mukherjee and K. Sreenadh, *Positive solutions for nonlinear Choquard equation with singular nonlinearity*, Complex Var. Elliptic Equ., 2017, 62(8), 1044–1071.

[26] S. Pekar, *Untersuchungen über Die Elektronentheorie Der Kristalle*, Akademie Verlag, Berlin, 1954.

[27] D. Ruiz amd J. Schaftingen, *Odd symmetry of least energy nodal solutions for the Choquard equation*, J. Differential Equations, 2018, 264(2), 1231–1262.

[28] J. Schaftingen and J. Xia, *Groundstates for a local nonlinear perturbation of the Choquard equations with lower critical exponent*, J. Math. Anal. Appl., 2018, 464(2), 1184–1202.

[29] J. Seok, *Limit profiles and uniqueness of ground states to the nonlinear Choquard equations*, Adv. Nonlinear Anal., 2019, 8(1), 1083–1098.

[30] J. Seok, *Nonlinear Choquard equations involving a critical local term*, Appl. Math. Lett., 2017, 63, 77–87.

[31] J. Seok, *Nonlinear Choquard equations: Doubly critical case*, Appl. Math. Lett., 2018, 76, 148–156.

[32] Y. Su and H. Chen, *Existence of nontrivial solutions for a perturbation of Choquard equation with Hardy-Littlewood-Sobolev upper critical exponent*, Electron. J. Differential Equations, 2018, 123, 1–25.

[33] Y. Sun and S. Li, *Structure of ground state solutions of singular semilinear elliptic equations*, Nonlinear Anal., 2003, 55(4), 399–417.

[34] Y. Sun and S. Wu, *An exact estimate result for a class of singular equations with critical exponents*, J. Funct. Anal., 2011, 260(5), 1257–1284.

[35] X. Wang, L. Zhao and P. Zhao, *Combined effects of singular and critical nonlinearities in elliptic problems*, Nonlinear Anal., 2013, 87, 1–10.

[36] T. Wu, *On a class of nonlocal nonlinear Schrödinger equations with potential well*, Adv. Nonlinear Anal., 2020, 9(1), 665–689.

[37] H. Yang, *Multiplicity and asymptotic behavior of positive solutions for a singular semilinear elliptic problem*, J. Differential Equations, 2003, 189(2), 487–512.

[38] S. Yu and J. Chen, *Uniqueness and asymptotical behavior of solutions to a Choquard equation with singularity*, Appl. Math. Lett., 2020, 102, 106099.

[39] S. Yu and J. Chen, *Multiple and asymptotical behavior of solutions to a Choquard equation with singularity*, J. Math. Anal. Appl., 2022, 511, 126047.

[40] S. Yu and J. Chen, *Fractional Schrödinger-Poisson system with singularity: Existence, uniqueness and asymptotic behaviour*, Glasgow Math. J., 2021, 63(1), 179–192.

[41] S. Yu and J. Chen, *Multiple positive solutions for critical elliptic problem with singularity*, Monatsh. Math., 2021, 194, 395–423.

[42] X. Zhong and C. Tang, *Ground state sign-changing solutions for a class of subcritical Choquard equations with a critical pure power nonlinearity in* $\mathbb{R}^N$, Comput. Math. Appl., 2018, 76(1), 23–34.