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TWO REGULARIZATION METHODS FOR IDENTIFYING THE UNKNOWN SOURCE OF SOBOLEV EQUATION WITH FRACTIONAL LAPLACIAN*

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Abstract In this paper, an inverse source problem for the Sobolev equation with fractional Laplacian is investigated. We prove that this kind of problem is ill-posed and apply the Quasi-boundary regularization method and fractional Landweber iterative regularization method to solve this inverse problem. Based on the result of conditional stability, the error estimates between the exact solution and the regularization solution are given under the priori and posteriori regularization parameter selection rules. Finally, three examples are given to illustrate the effectiveness and feasibility of these methods.

Keywords Sobolev equation, inverse problem, identifying source term, regularization method.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with sufficiently smooth boundary $\partial\Omega$. In this paper, we consider the following initial-boundary value problem for Sobolev equation with fractional Laplacian [14]

$$\begin{cases} u_t(x,t) - a\Delta u_t(x,t) + (-\Delta)^\beta u(x,t) = F(x), & x \in \Omega,\ t \in (0,T], \\ u(x,t) = 0, & x \in \partial\Omega,\ t \in (0,T], \\ u(x,0) = 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where $a > 0$ is the diffusion coefficient, $F(x)$ is the source function and $u(x,t)$ describes the distribution of the temperature at position x and time t. The parameter β is the fractional order of Laplacian operator with $1 \leq \beta < 2$.

Problem (1.1) is a forward problem when the function $F(x)$ is given appropriately. If the source term $F(x)$ is unknown, use the additional condition

$$u(x,T) = g(x),\ x \in \Omega, \tag{1.2}$$

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to identify the unknown source $F(x)$. This is an inverse problem. In practical applications, the input data $g(x)$ is given by measurement and the measured data $g^{\delta}(x)$ satisfies

$$\|g(\cdot) - g^{\delta}(\cdot)\| \leqslant \delta, \tag{1.3}$$

where $\|\cdot\|$ is the $L^2(\Omega)$ norm and $\delta > 0$ is the measurement error.

Fractional order Sobolev equations have been the subject of extensive research in recent years [2, 6, 8, 12, 15, 16, 26, 27]. The researches in this field have focused on various aspects, including nonlinear problems [13, 19], applications [5], and the properties of the fractional Laplacian operator [11]. The study of these equations have numerous applications in various fields, including physics, biology, and finance. The fractional Laplacian operator, an important operator that arises in fractional order Sobolev equations, has been the subject of recent research, with a focus on understanding its spectral properties, relationship to other fractional operators, and applications in various fields. The direct problem of fractional-order Sobolev equations has garnered substantial attention from a host of researchers in recent times. In [4], the authors tackled the initial-boundary value problem of Sobolev-type equation. In [1], the authors set forth foundational criteria for the approximate controllability of nonlinear impulsive delay integro-differential systems of Sobolev type, specifically within the fractional order range of $1 < q < 2$. Notably, their exploration extends to showcase the exact null controllability of identical systems under the stipulated conditions. However, research on the inverse problems of fractional-order Sobolev equations have been scarce. Therefore, this study will employ two regularization techniques for solving this equation and validating the effectiveness of the methods through corresponding numerical experiments.

The inverse problem is solved by the regularization method, such as the Tikhonov regularization method [18], the modified Tikhonov regularization method [17], the Landweber iterative regularization method [7], the fractional Landweber iterative regularization method, the Quasi-boundary regularization method, the Quasi-inverse regularization method and so on. In [21, 22], X. T. Xiong et al. used a modified Tikhonov method to solve a cauchy problem of the fractional diffusion equation. In [9, 20], T. Wei et al used the boundary element method combined with generalized Tikhonov regularization method to identify the unknown source and diffusion coefficient of fractional diffusion equation. In [7], Y.X. Gao et al. used the fractional Landweber iterative regularization method to study the inverse problem of the time-fractional Schrödinger equation. In [23, 24], F. Yang, et al. used three regularization methods(Landweber iterative regularization method, fractional Landweber iterative regularization method, Quasi-boundary regularization method) to identify the initial value of homogeneous anomalous secondary diffusion equation. In [25], J.M. Xu et al. used the modified Quasi-boundary regularization method to identify the initial value of fractional pseudo-parabolic equation.

This paper is divided into six sections. In section 2, we give the solution of the problem (1.1) and the result of conditional stability. In section 3, we use the Quasi-boundary regularization method to obtain the regularized solution of the problem (1.1). In section 4, we give the fractional Landweber iteration regularization method, Landweber iteration regularization method and their convergent estimations. Several numerical examples are given in section 5. In the final section, we give a brief conclusion.

2. The solution of the problem (1.1) and the result of conditional stability

In this section, we mainly give the uncertainty analysis, the solution of the problem and the result of conditional stability (1.1). Let λ_n and χ_n be the Dirichlet eigenvalues and eigenfunctions of $-\Delta$ on the domain Ω, satisfy [10]

$$\begin{cases} \Delta\chi_n(x) = -\lambda_n\chi_n(x), & x \in \Omega, \\ \chi_n(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$, $\lim_{n\to\infty} \lambda_n = +\infty$ and $\chi_n(x) \in H^2(\Omega) \cap H_0^1(\Omega)$, then $\{\chi_n\}_{n=1}^{\infty}$ can be normalized as the orthonormal basis in space $L^2(\Omega)$.

For any $p > 0$, we define the space

$$H^p(\Omega) = \Big\{\phi \in L^2(\Omega) \Big| \sum_{n=1}^{\infty} \lambda_n^{\beta p} |(\phi, \chi_n)|^2 < \infty\Big\}, \tag{2.2}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$, then $H^p(\Omega)$ is a Hilbert space with the norm

$$\|\phi\|_{H^p(\Omega)} := \Big(\sum_{n=1}^{\infty} \lambda_n^{\beta p} |(\phi, \chi_n)|^2\Big)^{\frac{1}{2}}. \tag{2.3}$$

The solution of problem (1.1) is obtained by using characteristic functions, variable separation method and Laplace transformation

$$u(x,t) = \sum_{n=1}^{\infty} \frac{F_n(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}t})}{\lambda_n^\beta} \chi_n(x), \tag{2.4}$$

where $F_n = (F(x), \chi_n(x))$ is the Fourier coefficient. Using $u(x,T) = g(x)$, according to (2.4), we obtain

$$g(x) = \sum_{n=1}^{\infty} \frac{F_n(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})}{\lambda_n^\beta} \chi_n(x). \tag{2.5}$$

So

$$g_n = \frac{F_n(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})}{\lambda_n^\beta}, \tag{2.6}$$

where $g_n = (g(x), \chi_n(x))$ is the Fourier coefficient. So we get the exact solution of the problem from (2.6)

$$F(x) = \sum_{n=1}^{\infty} \frac{\lambda_n^\beta g_n}{1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}} \chi_n(x). \tag{2.7}$$

Lemma 2.1. *If $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$, then the following inequality holds:*

$$\frac{C_1}{\lambda_n^\beta} \leq \frac{1}{\lambda_n^\beta}(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}), \tag{2.8}$$

where $C_1 = 1 - e^{-\frac{\lambda_1^\beta}{1+a\lambda_1}T}$.

Proof. Consider the following function

$$\Phi(z)=\frac{z^{\beta}}{1+az},\ z>0.$$

We take the derivative of the function $\Phi(z)$£¬

$$\Phi'(z)=\frac{\beta z^{\beta-1}+az^{\beta}(\beta-1)}{(1+az)^2},\ z>0.$$

So the function $\Phi(z)$ is strictly monotonically increasing, and we have

$$\frac{\lambda_1^{\beta}}{1+a\lambda_1}<\frac{\lambda_n^{\beta}}{1+a\lambda_n},\ n>1.$$

Therefore, we get

$$\frac{C_1}{\lambda_n^{\beta}}\le\frac{1}{\lambda_n^{\beta}}(1-e^{-\frac{\lambda_n^{\beta}}{1+a\lambda_n}T}). \tag{2.9}$$

□

Lemma 2.2. *For any $p>0$, $\mu>0$, $T>0$, and $0<\lambda_1\leqslant s$, the following inequality holds:*

$$A(s)=\frac{\mu s^{1-\frac{p}{2}}}{C_1+\mu s}\leqslant\begin{cases}C_2\mu^{\frac{p}{2}}, 0<p<2,\\ C_3\mu,\quad p\geqslant 2,\end{cases} \tag{2.10}$$

where $C_2=\frac{(\frac{(2-p)C_1}{P})^{1-\frac{p}{2}}}{(\frac{2-p}{p})C_1+C_1}$, $C_3=\frac{1}{C_1\lambda_1^{\frac{p}{2}}}$.

Proof. When $0<p<2$, due to $\lim_{s\to 0}A(s)=0$ and $\lim_{s\to\infty}A(s)=0$, then we obtain

$$A(s)\leqslant\sup_{s\geqslant\lambda_1}A(s)\leqslant A(s^*),$$

where s^* is the root of equation $A^{'}(s)=0$ and its value is $s^*=\frac{(2-p)C_1}{p\mu}$.

Therefore

$$A(s)\leqslant A(s^*)=\frac{\mu(\frac{(2-p)C_1}{P\mu})^{1-\frac{p}{2}}}{(\frac{2-p}{p})C_1+C_1}=:C_2\mu^{\frac{p}{2}}. \tag{2.11}$$

When $p\geqslant 2$

$$A(s)=\frac{\mu s^{1-\frac{p}{2}}}{\mu s+C_1}=\frac{\mu}{(\mu s+C_1)s^{\frac{p}{2}-1}}\leqslant\frac{\mu}{C_1\lambda_1^{1-\frac{p}{2}}}=:C_3\mu. \tag{2.12}$$

□

Lemma 2.3. *For any $p>0$, $\mu>0$, $T>0$, and $0<\lambda_1\leqslant s$, the following inequality holds:*

$$B(s)=\frac{\mu s^{\frac{2-p}{4}}}{C_1+\mu s}\leqslant\begin{cases}C_4\mu^{\frac{p+2}{4}}, 0<p<2,\\ C_5\mu,\qquad p\geqslant 2,\end{cases} \tag{2.13}$$

where $C_4:=\frac{\left(\frac{(2-p)C_1}{p+2}\right)^{\frac{2-p}{4}}}{C_1+\frac{(2-p)C_1}{p+2}}$, $C_5:=\frac{1}{\lambda_1^{\frac{p-2}{4}}C_1}$.

Proof. When $0<p<2$, due to $\lim_{s\to 0} B(s)=0$ and $\lim_{s\to\infty} B(s)=0$, then we obtain

$$B(s)\leqslant \sup_{s\geqslant\lambda_1} B(s)\leqslant B(s^*),$$

where s^* is the root of equation $B^{'}(s)=0$ and its value is $s^*=\frac{(2-p)C_1}{(p+2)\mu}$. Therefore

$$B(s)\leqslant B(s^*)=\frac{\left(\frac{(2-p)C_1}{p+2}\right)^{\frac{2-p}{4}}\mu^{\frac{2+p}{4}}}{C_1+\frac{(2-p)C_1}{p+2}}=:C_4\mu^{\frac{2+p}{4}}. \tag{2.14}$$

When $p\geqslant 2$,

$$B(s)=\frac{\mu s^{\frac{2-p}{4}}}{C_1+\mu s}\leqslant\frac{\mu}{C_1 s^{\frac{p-2}{4}}}\leqslant\frac{\mu}{\lambda_1^{\frac{p-2}{4}}C_1}=:C_5\mu. \tag{2.15}$$

□

Define operator $K: f(\cdot)\to g(\cdot)$, then problem (1.1) can be transformed into the following operator equation: $Kf(x)=g(x),\ x\in\Omega$, where K satisfies $Kf(x)=g(x)=\sum_{n=1}^{\infty}\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})F_n\chi_n(x)$. Obviously, K is a linear self-adjoint operator, and its singular values are: $\sigma_n=\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})$. Due to $g_n=F_n\cdot\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})$, thus $F_n=\sigma_n^{-1}\cdot g_n$. Therefore, we have

$$F(x)=\sum_{n=1}^{\infty}\frac{\lambda_n^\beta g_n}{1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}}\chi_n(x). \tag{2.16}$$

From(2.16), we can infer that $F(x)\to\infty$, when $n\to\infty$. Therefore this problem is ill-posed. In order to discuss the error convergence, the conditional stability of the exact solution $f(x)$ is given. Here we assume that $F(x)$ satisfies the following priori bound conditions:

$$\|F(\cdot)\|_{H^p(\Omega)}=\Big(\sum_{n=1}^{\infty}\lambda_n^{\beta p}|(F,\chi_n)|^2\Big)^{\frac{1}{2}}\leqslant E, \tag{2.17}$$

where E and p are both positive constants.

Theorem 2.1. *If $F(x)$ satisfies the priori bound condition $\|f(\cdot)\|_{H^p(\Omega)}\leqslant E$, then we obtain*

$$\|F(\cdot)\|\leqslant C_4^{\frac{1}{p+2}}E^{\frac{2}{p+2}}\|g(\cdot)\|^{\frac{p}{p+2}},\quad p>0, \tag{2.18}$$

where $C_4=\frac{1}{C_1^p}$.

Proof. According to the formula (2.7), **Lemma** 2.1, and the Hölder inequality, we have

$$\begin{aligned}&\|F(\cdot)\|^2\\ =&\|\sum_{n=1}^{\infty}\frac{\lambda_n^\beta g_n\chi_n(x)}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})}\|^2\\ =&\sum_{n=1}^{\infty}\frac{\lambda_n^{2\beta}g_n^2}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})^2}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{\lambda_n^{2\beta} g_n^{\frac{2p+4}{p+2}}}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})^2} \\
&\leq (\sum_{n=1}^{\infty} (\frac{g_n^{\frac{4}{p+2}}}{(\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}))^2})^{\frac{p+2}{2}})^{\frac{2}{p+2}} \cdot (\sum_{n=1}^{\infty} (g_n^{\frac{2p}{p+2}})^{\frac{p+2}{p}})^{\frac{p}{p+2}} \\
&= (\sum_{n=1}^{\infty} \frac{g_n^2}{(\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}))^{p+2}})^{\frac{2}{p+2}} (\sum_{n=1}^{\infty} g_n^2)^{\frac{p}{p+2}} \\
&= (\sum_{n=1}^{\infty} \frac{g_n^2}{(\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}))^2} \cdot \frac{1}{(\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}))^p})^{\frac{2}{p+2}} \cdot \|g(\cdot)\|^{\frac{2p}{p+2}} \\
&= (\sum_{n=1}^{\infty} F_n^2 \lambda_n^{\beta p} \cdot \frac{1}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})^p})^{\frac{2}{p+2}} \cdot \|g(\cdot)\|^{\frac{2p}{p+2}} \\
&\leq (\sum_{n=1}^{\infty} F_n^2 \lambda_n^{\beta p} \cdot \frac{1}{(1-e^{-\frac{\lambda_1^\beta}{1+a\lambda_1}T})^p})^{\frac{2}{p+2}} \cdot \|g(\cdot)\|^{\frac{2p}{p+2}} \\
&\leq C_4^{\frac{2}{p+2}} E^{\frac{4}{p+2}} \|g(\cdot)\|^{\frac{2p}{p+2}},
\end{aligned} \tag{2.19}
$$

where $C_4 = \frac{1}{C_1^p}$. Then we have

$$\|F(\cdot)\| \leq C_4^{\frac{1}{p+2}} E^{\frac{2}{p+2}} \|g(\cdot)\|^{\frac{p}{p+2}}.$$

Therefore, we complete the proof of **Theorem** 2.1. □

3. Quasi-boundary regularization method and its convergence estimation

In this section, we will use the Quasi-boundary regularization method to obtain the regularized solution of the problem (1.1). At the same time, we give the Hölder type error estimates between the exact solution and the regularization solution of the problem. The main idea of the Quasi-boundary regularization method is to add a penalty to the final data of the original problem to obtain an approximate solution to the original problem (1.1), i.e., $u_\mu^\delta(x,T) + \mu F_\mu^\delta(x) = g^\delta(x)$ is used to instead of $u(x,T) = g(x)$ to get the regularization solution of the problem (1.1), that is, to solve the following equation

$$
\begin{cases}
\frac{\partial^\alpha u_\mu^\delta(x,t)}{\partial t} - a\Delta \frac{\partial u_\mu^\delta(x,t)}{\partial t} + (-\Delta)^\beta u_\mu^\delta(x,t) = F_\mu^\delta(x), & x \in \Omega,\ t \in (0,T], \\
u_\mu^\delta(x,t) = 0, & x \in \partial\Omega,\ t \in (0,T], \\
u_\mu^\delta(x,0) = 0, & x \in \Omega, \\
u_\mu^\delta(x,T) + \mu F_\mu^\delta(x) = g(x), & x \in \Omega,
\end{cases} \tag{3.1}
$$

where $\mu > 0$ is the regularization parameter. Similarly, the separation of variables method and the Laplace transform can be used to obtain solution $u_\mu^\delta(x,t)$ of formula

(3.1)

$$u_\mu^\delta(x,t)=\sum_{n=1}^{\infty}\frac{(F_\mu^\delta)_n}{\lambda_n^\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}t})\chi_n(x). \tag{3.2}$$

According to $u_\mu^\delta(x,T)+\mu F_\mu^\delta(x)=g(x)$, we obtain $u_\mu^\delta(x,T)=\sum_{n=1}^{\infty}\Big(g_n^\delta(x)-\mu(F_\mu^\delta(x))_n\Big)\chi_n(x)$, $u_\mu^\delta(x,T)=\sum_{n=1}^{\infty}\frac{(F_\mu^\delta)_n}{\lambda_n^\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})\chi_n(x)$ is derived from the (3.2) formula when $t=T$. So $\Big(F_\mu^\delta(x)\Big)_n\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})=g_n^\delta(x)-\mu(F_\mu^\delta(x))_n$, thus

$$\Big(F_\mu^\delta(x)\Big)_n=\frac{\lambda_n^\beta g_n^\delta(x)}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})+\mu\lambda_n^\beta}. \tag{3.3}$$

Thus, we get the Quasi-boundary regularization solution with error and regularization solutions without error

$$F_\mu^\delta(x)=\sum_{n=1}^{\infty}\frac{\lambda_n^\beta g_n^\delta}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})+\mu\lambda_n^\beta}\chi_n(x), \tag{3.4}$$

$$F_\mu(x)=\sum_{n=1}^{\infty}\frac{\lambda_n^\beta g_n}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})+\mu\lambda_n^\beta}\chi_n(x). \tag{3.5}$$

To recovery the source item $F(x)$, we need to solve the following integral equation:

$$(KF)(x):=\int_\Omega k(x,\xi)F(\xi)d\xi=g(x),$$

where the kernel function is:

$$k(x,\xi)=\sum_{n=1}^{\infty}\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})\chi_n(x)\chi_n(\xi).$$

Next, we will give the convergent estimates between the exact and regularization solutions under the a priori regularization parameter and a posteriori regularization parameter.

3.1. The convergent error estimate with an a priori parameter choice rule

Theorem 3.1. *Assuming a priori bound* (2.17) *and a noise assumption* (1.3) *hold, then there are*

(1) *If* $0<p<2$ *and the regularization parameter* $\mu=(\frac{\delta}{E})^{\frac{2}{p+2}}$ *is selected, then there is*

$$\|F_\mu^\delta(\cdot)-F(\cdot)\|\leqslant(1+C_2)E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}; \tag{3.6}$$

(2) *If* $p\geqslant 2$ *and the regularization parameter* $\mu=(\frac{\delta}{E})^{\frac{1}{2}}$ *is selected, then there is*

$$\|F_\mu^\delta(\cdot)-F(\cdot)\|\leqslant(1+C_3)E^{\frac{1}{2}}\delta^{\frac{1}{2}}. \tag{3.7}$$

Proof. By means of a triangular inequality, we have

$$\|F_\mu^\delta(\cdot) - F(\cdot)\| \leqslant \|F_\mu^\delta(\cdot) - F_\mu(\cdot)\| + \|F_\mu(\cdot) - F(\cdot)\|. \tag{3.8}$$

Let us first give an estimate of the first term. Through (3.4), (3.5), (1.3) and **Lemma** 2.1, we obtain

$$\begin{aligned}
\|F_\mu^\delta(\cdot) - F_\mu(\cdot)\|^2 &= \Big\| \sum_{n=1}^{\infty} \frac{\lambda_n^\beta (g_n^\delta - g_n)}{(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}) + \mu\lambda_n^\beta} \chi_n(x) \Big\|^2 \\
&= \sum_{n=1}^{\infty} \Big(\frac{\lambda_n^\beta (g_n^\delta - g_n)}{(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}) + \mu\lambda_n^\beta} \Big)^2 \\
&\leqslant (\frac{\delta}{\mu})^2.
\end{aligned}$$

Then

$$\|F_\mu^\delta(\cdot) - F_\mu(\cdot)\| \leqslant \frac{\delta}{\mu}. \tag{3.9}$$

Now let us estimate the second term of equation (3.8). Using (2.13), (2.17), (3.5) and **Lemma** 2.2, we can deduce

$$\begin{aligned}
&\|F_\mu(\cdot) - F(\cdot)\|^2 \\
&= \Big\| \sum_{n=1}^{\infty} \frac{\lambda_n^\beta g_n}{(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}) + \mu\lambda_n^\beta} \chi_n(x) - \sum_{n=1}^{\infty} \frac{\lambda_n^\beta g_n}{(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})} \chi_n(x) \Big\|^2 \\
&= \Big\| \sum_{n=1}^{\infty} \Big(\frac{\lambda_n^\beta g_n}{(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}) + \mu\lambda_n^\beta} - \frac{\lambda_n^\beta g_n}{(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})} \Big) \chi_n(x) \Big\|^2 \\
&= \Big\| \sum_{n=1}^{\infty} \frac{g_n \lambda_n^{-\beta}(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}) - g_n(\lambda_n^{-\beta}(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}) + \mu)}{(\lambda_n^{-\beta}(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}) + \mu)\lambda_n^{-\beta}(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})} \chi_n(x) \Big\|^2 \\
&= \Big\| \sum_{n=1}^{\infty} \frac{g_n}{\lambda_n^{-\beta}(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}) + \mu} \cdot \frac{-\mu}{\lambda_n^{-\beta}(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})} \chi_n(x) \Big\|^2 \\
&= \Big\| \sum_{n=1}^{\infty} \frac{g_n}{\lambda_n^{-\beta}(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})} \cdot \frac{-\mu}{\lambda_n^{-\beta}(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}) + \mu} \chi_n(x) \Big\|^2 \\
&= \sum_{n=1}^{\infty} \Big(\frac{g_n}{\lambda_n^{-\beta}(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})} \Big)^2 \Big(\frac{\mu}{\lambda_n^{-\beta}(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}) + \mu} \Big)^2 \\
&= \sum_{n=1}^{\infty} \Big(\frac{g_n}{\lambda_n^{-\beta}(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})} \Big)^2 \lambda_n^{\beta p} \Big(\frac{\mu}{\lambda_n^{-\beta}(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}) + \mu} \Big)^2 \lambda_n^{-\beta p} \\
&\leqslant \sum_{n=1}^{\infty} \Big(\frac{g_n}{\lambda_n^{-\beta}(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})} \Big)^2 \lambda_n^{\beta p} \Big(\frac{\mu\lambda_n^{\beta(1-\frac{p}{2})}}{C_1 + \mu\lambda_n^\beta} \Big)^2 \\
&\leqslant E^2 \Big(\sup_{n\geqslant 1} A(n) \Big)^2,
\end{aligned}$$

where

$$A(n)=\frac{\mu\lambda_n^{\beta(1-\frac{p}{2})}}{C_1+\mu\lambda_n^{\beta}}.$$

Applying **Lemma** 2.2, we obtain

$$A(n)=\frac{\mu\lambda_n^{1-\frac{p}{2}}}{\mu\lambda_n+C_1}=\frac{\mu s^{1-\frac{p}{2}}}{\mu s+C_1}\leqslant\begin{cases}C_2\mu^{\frac{p}{2}},\ 0<p<2,\\ C_3\mu,\quad p\geqslant 2.\end{cases}\tag{3.10}$$

Therefore, we have

$$\|F_\mu(\cdot)-F(\cdot)\|\leqslant\begin{cases}C_2E\mu^{\frac{p}{2}},\ 0<p<2,\\ C_3E\mu,\quad p\geqslant 2.\end{cases}\tag{3.11}$$

Combining (3.9) with (3.11), we obtain

$$\|F_\mu^\delta(\cdot)-F(\cdot)\|\leqslant\frac{\delta}{\mu}+\begin{cases}C_2E\mu^{\frac{p}{2}},\ 0<p<2,\\ C_3E\mu,\quad p\geqslant 2.\end{cases}\tag{3.12}$$

By choosing the regularization parameters $\mu=(\frac{\delta}{E})^{\frac{2}{p+2}}(0<p<2)$ and $\mu=(\frac{\delta}{E})^{\frac{1}{2}}(p\geqslant 2)$, we have the following results.

$$\|F_\mu^\delta(\cdot)-F(\cdot)\|\leqslant\begin{cases}(1+C_2)E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}},\ 0<p<2,\\ (1+C_3)E^{\frac{1}{2}}\delta^{\frac{1}{2}},\qquad p\geqslant 2.\end{cases}\tag{3.13}$$

The proof of **Theorem** 3.1 is completed. □

3.2. The convergent error estimate with an a posteriori parameter choice rule

In this section, discrepancy principle is used to select a posteriori regularization parameter μ. The posteriori regularization parameter satisfies the following equation:

$$\|\mu(K+\mu)^{-1}(KF_\mu^\delta(\cdot)-g^\delta(\cdot))\|=\tau\delta.\tag{3.14}$$

Lemma 3.1. *Let* $\rho(\mu):=\|\mu(K+\mu)^{-1}(KF_\mu^\delta(\cdot)-g^\delta(\cdot))\|$. *If* $\|g^\delta\|>\tau\delta>0$, *we obtain*

(a) $\rho(\mu)$ *is a continuous function;*

(b) $\lim_{\mu\to 0}\rho(\mu)=0$;

(c) $\lim_{\mu\to+\infty}\rho(\mu)=\|g^\delta(\cdot)\|$;

(d) $\rho(\mu)$ *is a strictly monotone increasing function for any* $\mu\in(0,+\infty)$.

Proof. The proof of this Lemma is obtained by the expression of

$$\rho(\mu)=\Big(\sum_{n=1}^{\infty}\Big(\frac{\mu\lambda_n^\beta}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})+\mu\lambda_n^\beta}\Big)^4(g_n^\delta)^2\Big)^{\frac{1}{2}}.$$

□

The following lemmas will be used in the proof of a posteriori convergent estimate.

Theorem 3.2. *If expressions* (1.3) *and* (2.17) *hold and* μ *satisfies the regularization parameter selection rule, then*

(1) *If* $0 < p < 2$*, then the following convergent estimate is obtained*

$$\|F_\mu^\delta(\cdot) - F(\cdot)\| \leqslant (C_6(\tau+1)^{\frac{p}{p+2}} + (\frac{(C_4)^2}{\tau-1})^{\frac{2}{p+2}})E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}; \tag{3.15}$$

(2) *If* $p \geqslant 2$*, then the following convergent estimate is obtained*

$$\|F_\mu^\delta(\cdot) - F(\cdot)\| \leqslant (C_6(\tau+1)^{\frac{1}{2}} + (\frac{(C_5)^2}{\tau-1})^{\frac{1}{2}}E^{\frac{1}{2}}\delta^{\frac{1}{2}}, \tag{3.16}$$

where $C_6 = \frac{1}{C_1}$.

Proof. By means of a triangular inequality, we have

$$\|F_\mu^\delta(\cdot) - F(\cdot)\| \leqslant \|F_\mu^\delta(\cdot) - F_\mu(\cdot)\| + \|F_\mu(\cdot) - F(\cdot)\|. \tag{3.17}$$

Let us start by proving the first term of the theorem, which applies (3.9)

$$\|F_\mu^\delta(\cdot) - F_\mu(\cdot)\| \leqslant \frac{\delta}{\mu}. \tag{3.18}$$

Using formulas (3.14) and formulas (1.3), we have

$$\begin{aligned}
\tau\delta =& \Big\| \sum_{n=1}^{\infty} (\frac{\mu\lambda_n^\beta}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}) + \mu\lambda_n^\beta})^2 g_n^\delta \chi_n(x) \Big\| \\
\leqslant& \Big\| \sum_{n=1}^{\infty} (\frac{\mu\lambda_n^\beta}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}) + \mu\lambda_n^\beta})^2 (g_n^\delta - g_n)\chi_n(x) \Big\| \\
&+ \Big\| \sum_{n=1}^{\infty} (\frac{\mu\lambda_n^\beta}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}) + \mu\lambda_n^\beta})^2 g_n \chi_n(x) \Big\| \\
\leqslant& \delta + J.
\end{aligned}$$

A priori boundary condition (2.17) can be used to obtain

$$\begin{aligned}
J =& \Big\| \sum_{n=1}^{\infty} (\frac{\mu\lambda_n^\beta}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}) + \mu\lambda_n^\beta})^2 g_n \chi_n(x) \Big\| \\
=& \Big\| \sum_{n=1}^{\infty} (\frac{\mu\lambda_n^\beta}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}) + \mu\lambda_n^\beta})^2 (\frac{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})}{\lambda_n^\beta}) \frac{1}{\lambda_n^{\frac{\beta p}{2}}} \frac{\lambda_n^\beta g_n}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})} \lambda_n^{\frac{\beta p}{2}} \chi_n(x) \Big\| \\
\leqslant& E \sup_{n\geqslant 1} (B(n))^2,
\end{aligned}$$

where $B(n) = \frac{\mu\lambda_n^\beta}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}) + \mu\lambda_n^\beta} \cdot \left(\frac{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})}{\lambda_n^\beta}\right)^{\frac{1}{2}} \cdot \frac{1}{\lambda_n^{\frac{\beta p}{4}}}$.

Applying **Lemma** 2.1, we have

$$\begin{aligned}B(n) =&\frac{\mu\lambda_n^\beta}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})+\mu\lambda_n^\beta}\cdot\Big(\frac{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})}{\lambda_n^\beta}\Big)^{\frac{1}{2}}\cdot\frac{1}{\lambda_n^{\frac{\beta p}{4}}}\\ \leqslant&\frac{\mu\lambda_n^\beta}{C_1+\mu\lambda_n^\beta}\cdot(\frac{1}{\lambda_n^\beta})^{\frac{1}{2}}\cdot\frac{1}{\lambda_n^{\frac{\beta p}{4}}}=\frac{\mu}{\mu\lambda_n^\beta+C_1}\lambda_n^{\beta(\frac{2-p}{4})}.\end{aligned}$$

Applying **Lemma** 2.3, we have

$$B(n)\leqslant\begin{cases}C_4\mu^{\frac{p+2}{4}}, \ 0<p<2,\\ C_5\mu, \qquad p\geqslant 2.\end{cases} \tag{3.19}$$

So

$$(\tau-1)\delta\leqslant\begin{cases}(C_4)^2E\mu^{\frac{p+2}{2}}, \ 0<p<2,\\ (C_5)^2E\mu^2, \qquad p\geqslant 2.\end{cases} \tag{3.20}$$

Therefore,

$$\frac{1}{\mu}\leqslant\begin{cases}(\dfrac{(C_4)^2}{\tau-1})^{\frac{2}{p+2}}E^{\frac{2}{p+2}}\delta^{\frac{-2}{p+2}}, \ 0<p<2,\\ (\dfrac{(C_5)^2}{\tau-1})^{\frac{1}{2}}E^{\frac{1}{2}}\delta^{-\frac{1}{2}}, \qquad p\geqslant 2.\end{cases} \tag{3.21}$$

Substitute (3.21) to (3.18), we have

$$\|F_\mu^\delta(\cdot)-F_\mu(\cdot)\|\leqslant\frac{\delta}{\mu}\leqslant\begin{cases}(\dfrac{(C_4)^2}{\tau-1})^{\frac{2}{p+2}}E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}, \ 0<p<2,\\ (\dfrac{(C_5)^2}{\tau-1})^{\frac{1}{2}}E^{\frac{1}{2}}\delta^{\frac{1}{2}}, \qquad p\geqslant 2.\end{cases} \tag{3.22}$$

Now let us estimate the second term of formula (3.17), from (3.5), **Lemma** 2.1, **Lemma** 2.3 and the priori boundary condition of $F(x)$, we obtain

$$\begin{aligned}\|F_\mu(\cdot)-F(\cdot)\| =&\|\sum_{n=1}^{\infty}\frac{-\mu F_n\lambda_n^\beta}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})+\mu\lambda_n^\beta}\chi_n(x)\|\\ =&\|\sum_{n=1}^{\infty}(\frac{\mu(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})+\mu\lambda_n^\beta})^{\frac{p}{2}}(\frac{\mu\lambda_n^\beta}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})+\mu\lambda_n^\beta})^{1-\frac{p}{2}}\\ &\times\frac{\lambda_n^{\frac{\beta p}{2}}F_n}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})^{\frac{p}{2}}}\chi_n(x)\|\\ \leq&\|\sum_{n=1}^{\infty}(\frac{\mu\lambda_n^\beta}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})+\mu\lambda_n^\beta})^2(\frac{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})}{\lambda_n^\beta})F_n\chi_n(x)\|^{\frac{p}{p+2}}\\ &\times\|\sum_{n=1}^{\infty}\frac{\lambda_n^{\frac{\beta p}{2}}F_n}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})^{\frac{p}{2}}}\chi_n(x)\|^{\frac{2}{p+2}}\end{aligned}$$

$$\leq \|(\frac{\mu\lambda_n^\beta}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})+\mu\lambda_n^\beta})^2 g_n\chi_n\chi(x)\|^{\frac{p}{p+2}}$$
$$\times\|\sum_{n=1}^{\infty}\lambda_n^{\frac{\beta p}{2}}F_n\chi_n(x)\|^{\frac{2}{p+2}}C_1^{\frac{-p}{p+2}}$$
$$\leq(\|\sum_{n=1}^{\infty}(\frac{\mu\lambda_n^\beta}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})+\mu\lambda_n^\beta})^2(g_n-g_n^\delta)\chi_n(x)\|$$
$$+\|\sum_{n=1}^{\infty}(\frac{\mu\lambda_n^\beta}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})+\mu\lambda_n^\beta})^2 g_n^\delta\chi_n(x)\|)^{\frac{p}{p+2}}E^{\frac{2}{p+2}}C_1^{\frac{-p}{p+2}}$$
$$\leq(\frac{\tau+1}{C_1})^{\frac{p}{p+2}}E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}.$$

Therefore, we have

$$\|F_\mu(\cdot)-F(\cdot)\|\leqslant(\frac{1}{C_1})^{\frac{p}{p+2}}(\tau+1)^{\frac{p}{p+2}}E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}. \tag{3.23}$$

Combining (3.17), (3.22) and (3.23) formulas, we obtain

$$\|F_\mu^\delta(\cdot)-F(\cdot)\|\leqslant C_6(\tau+1)^{\frac{p}{p+2}}E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}+\begin{cases}(\frac{(C_4)^2}{\tau-1})^{\frac{2}{p+2}}E^{\frac{2}{p+2}}\delta^{\frac{2}{p+2}}, & 0<p<2,\\ (\frac{(C_5)^2}{\tau-1})^{\frac{1}{2}}E^{\frac{1}{2}}\delta^{\frac{1}{2}}, & p\geqslant 2,\end{cases} \tag{3.24}$$

where $C_6=\frac{1}{C_1}$, the proof of **Theorem** 3.2 is completed. □

In the Quasi-boundary regularization method, we can find the saturation effect by using the formula (3.7) and (3.16), so in the next section, we use the fractional Landweber iterative regularization method and Landweber iterative regularization method to effectively avoid this problem.

4. Fractional Landweber iteration regularization method and convergent estimations

In this section, we first give the regularization solution of the problem, then give the rules for selecting the priori regularization parameters and the posteriori regularization parameters, and discuss the Hölder type error estimation rules for exact solutions and regularization solutions under these rules. To identify the source term $F(x)$, we need to solve the following integral equation:

$$(KF)(x):=\int_\Omega k(x,\xi)F(\xi)d\xi=g(x), \tag{4.1}$$

where the kernel function is:

$$k(x,\xi)=\sum_{n=1}^{\infty}\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})\chi_n(x)\chi_n(\xi).$$

Because $k(x,\xi)=k(\xi,x)$, so K is a self-adjoint operator. According to **Theorem** 2.4 in Ref. [23], if $F\in L^2(\Omega)$, then $g\in H^2(\Omega)$. It is easy to know that $K:L^2(\Omega)\longrightarrow$

$L^2(\Omega)$ is a compact operator. Since $\chi_n(x)$ is a set of orthonormal bases in $L^2(\Omega)$, it is easy to know that

$$\sigma_n = \lambda_n^{-\beta}(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}),\ n = 1, 2, \cdots \tag{4.2}$$

is the singular value of compact operator K. Next, we use the fractional Landweber iterative regularization method to obtain the regularized solution of the problem (1.1), which is denoted as $F^{m,\delta}(x)$. Replacing $KF = g$ with $F = (I - a(K^*K)^{\frac{\gamma+1}{2}})f + a(K^*K)^{\frac{\gamma-1}{2}}K^*g$ has the following iteration format:

$$\begin{aligned} &F^{0,\delta}(x) = 0,\\ &F^{m,\delta}(x) = (I - a(K^*K)^{\frac{\gamma+1}{2}})F^{m-1,\delta}(x)\\ &\qquad\qquad + a(K^*K)^{\frac{\gamma-1}{2}}K^*g^\delta(x), \quad m = 1, 2, 3, \cdots, \end{aligned} \tag{4.3}$$

where I is a unit operator, m is the iterative step number and is also selected as the regularization parameter, a is called the relaxation factor and satisfies $0 < a < \frac{1}{\|K\|^2}$. Since K is a self-adjoint operator, we denote operator $\mathcal{R}_m : L^2(\Omega) \to L^2(\Omega)$ as follows

$$\mathcal{R}_m = a\sum_{n=0}^{m-1}(I - a(K^*K)^{\frac{\gamma+1}{2}})^n(K^*K)^{\frac{\gamma-1}{2}}K^*, 0 < \gamma \leqslant 1, \quad m = 1, 2, 3, \cdots.$$

Remark 4.1. When $\gamma = 1$, $\mathcal{R}_m$ is defined as follows:

$$\mathcal{R}_m = a\sum_{n=0}^{m-1}(I - a(K^*K))^nK^*, \quad m = 1, 2, 3, \cdots. \tag{4.4}$$

As can be seen from the literature [23], formula (4.4) is a Landweber iteration regularization operator, which is recorded as

$$\mathcal{R}_{m_1} = a\sum_{n=0}^{m_1-1}(I - a(K^*K))^nK^*, \quad m_1 = 1, 2, 3, \cdots. \tag{4.5}$$

Through calculating, we get

$$F^{m,\delta}(x) = \mathcal{R}_m g_n^\delta = a\sum_{n=0}^{m-1}(I - a(K^*K)^{\frac{\gamma+1}{2}})^n(K^*K)^{\frac{\gamma-1}{2}}K^*g_n^\delta(x). \tag{4.6}$$

Using (4.3) and the singular value σ_n of operator K, we get the fractional Landweber iteration regularization solution

$$F^{m,\delta}(x) = \sum_{n=1}^{\infty}\frac{1 - (1 - a(\lambda_n^{-\beta}(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}))^{\gamma+1})^m}{\lambda_n^{-\beta}(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})}g_n^\delta\chi_n(x), \tag{4.7}$$

where $g_n^\delta = (g^\delta(x), \chi_n(x))$. Because $\sigma_n = \lambda_n^{-\beta}(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})$ is the singular value of operator K and $0 < a < \frac{1}{\|K\|^{\gamma+1}}$, we can get $0 < a(\lambda_n^{-\beta}(1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}))^{\gamma+1} < 1$.

4.1. The convergent error estimate with an a priori parameter choice rule

Theorem 4.1. *Suppose* (1.3) *and* (2.17) *hold. The exact solution of problem* (1.1) *is formula* (2.7) *and the corresponding fractional Landweber regularization solution is given by* (4.7). *The regularization parameter is chosen by* $m = [b]$, *where* $b = \left(\frac{E}{\delta}\right)^{\frac{2(\gamma+1)}{p+2}}$. *Then we obtain the following convergent error estimate:*

$$\|F^{m,\delta}(\cdot) - F(\cdot)\| \leqslant C_7 E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, \tag{4.8}$$

where $[b]$ *denotes the largest integer less than or equal to* b *and* $C_7 := \frac{1}{a^{\gamma+1}} + C_6$ *is positive constant.*

Proof. By applying the triangular inequality, we can get

$$\|F^{m,\delta}(\cdot) - F(\cdot)\| \leqslant \|F^{m,\delta}(\cdot) - F^m(\cdot)\| + \|F^m(\cdot) - F(\cdot)\|. \tag{4.9}$$

For the first part on the right side of equation (4.9), using equation (1.3) and Bernoulli inequality, we have

$$\begin{aligned}
\|F^{m,\delta}(\cdot) - F^m(\cdot)\|^2 =& \Bigg\| \sum_{n=1}^{\infty} \frac{1-(1-a(\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}))^{\gamma+1})^m}{\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})} g_n^\delta \chi_n(x) \\
&- \sum_{n=1}^{\infty} \frac{1-(1-a(\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}))^{\gamma+1})^m}{\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})} g_n \chi_n(x) \Bigg\|^2 \\
=& \Bigg\| \sum_{n=1}^{\infty} \frac{1-(1-a(\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}))^{\gamma+1})^m}{\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})} (g_n^\delta - g_n) \chi_n(x) \Bigg\|^2 \\
=& \sum_{n=1}^{\infty} \frac{(1-(1-a(\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}))^{\gamma+1})^m)^2}{(\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}))^2} (g_n^\delta - g_n)^2 \\
=& \sum_{n=1}^{\infty} \frac{(1-(1-a(\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}))^{\gamma+1})^m)^{\frac{2}{\gamma+1}(\gamma+1)}}{(\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}))^{\frac{2}{\gamma+1}(\gamma+1)}} (g_n^\delta - g_n)^2 \\
<& \sum_{n=1}^{\infty} \frac{(1-(1-a(\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}))^{\gamma+1})^m)^{\frac{2}{\gamma+1}}}{(\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}))^{\frac{2}{\gamma+1}(\gamma+1)}} (g_n^\delta - g_n)^2 \\
\leqslant& \sum_{n=1}^{\infty} \frac{(am(\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}))^{\gamma+1})^{\frac{2}{\gamma+1}}}{(\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}))^{(\gamma+1)\frac{2}{\gamma+1}}} (g_n^\delta - g_n)^2 \\
=& a^{\frac{2}{\gamma+1}} m^{\frac{2}{\gamma+1}} \sum_{n=1}^{\infty} (g_n^\delta - g_n)^2 \\
\leqslant& a^{\frac{2}{\gamma+1}} m^{\frac{2}{\gamma+1}} \delta^2.
\end{aligned}$$

So we obtain

$$\|F^{m,\delta}(\cdot) - F^m(\cdot)\| \leqslant a^{\frac{1}{\gamma+1}} m^{\frac{1}{\gamma+1}} \delta. \tag{4.10}$$

On the other hand, using (2.17), we obtain

$$\begin{aligned}
\|F^m(\cdot)-F(\cdot)\|^2 =& \| \sum_{n=1}^{\infty} \frac{1-(1-a(\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}))^{\gamma+1})^m}{\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})} g_n\chi_n(x) \\
&- \frac{1}{\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})} g_n\chi_n(x)\|^2 \\
=& \| \sum_{n=1}^{\infty} \frac{[1-(1-a(\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}))^{\gamma+1})^m]-1}{\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})} g_n\chi_n(x)\|^2 \\
=& \| \sum_{n=1}^{\infty} \frac{-(1-a(\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}))^{\gamma+1})^m}{\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})} g_n\chi_n(x)\|^2 \\
=& \sum_{n=1}^{\infty} \frac{(1-a(\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}))^{\gamma+1})^{2m}}{(\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}))^2} g_n^2 \\
=& \sum_{n=1}^{\infty} (1-a(\lambda_n^{-\beta}(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}))^{\gamma+1})^{2m} F_n^2 (\lambda_n)^{-\beta p}(\lambda_n)^{\beta p} \\
\leq& \sum_{n=1}^{\infty} \Big(\frac{(1-a(\frac{1-e^{-\frac{\lambda_n}{\lambda_n+a\lambda_n}T}}{\lambda_n^\beta}))^{\gamma+1})^{2m}}{\lambda_n^{\beta p}} \Big) \cdot F_n^2\lambda_n^{\beta p} \\
\leq& \sup_{n\geqslant N} (D(\lambda_n))^2 \cdot \sum_{n=1}^{\infty} F_n^2 \lambda_n^{\beta p} \\
\leq& \sup_{n\geqslant N} (D(\lambda_n))^2 E^2,
\end{aligned}$$

where $C_7 = 1-e^{-\frac{T}{1+a}}$, $D(\lambda_n) := \frac{(1-a(\frac{C_7}{\lambda_n^\beta}))^{\gamma+1})^m}{\lambda_n^{\frac{\beta p}{2}}}$.

Let $G(s) := \frac{(1-a(\frac{C_7}{s}))^{\gamma+1})^m}{s^{\frac{p}{2}}}$, $s := \lambda_n^\beta$.

Suppose s_0 satisfies $G'(s_0)=0$, we have

$$s_0 = \Big(\frac{p}{aC_7^{1+\gamma}(p+2m(1+\gamma))}\Big)^{-\frac{1}{1+\gamma}}.$$

Then

$$\begin{aligned}
G(s) &\leqslant G(s_0) \\
&= \frac{(1-a(\frac{C_7}{s_0})^{\gamma+1})^m}{s_0^{\frac{p}{2}}} \\
&\leqslant s_0^{-\frac{p}{2}} \\
&= \Big(\frac{p}{aC_7^{1+\gamma}(p+2m(1+\gamma))}\Big)^{\frac{p}{2(1+\gamma)}} \\
&= \Big(\frac{p}{aC_7^{1+\gamma}}\Big)^{\frac{p}{2(1+\gamma)}} \cdot \Big(\frac{1}{p+2m\gamma+2m}\Big)^{\frac{p}{2(1+\gamma)}}
\end{aligned}$$

$$\leqslant C_8 \cdot (\frac{1}{\gamma+m})^{\frac{p}{2(1+\gamma)}},$$

where $C_8 = (\frac{p}{aC_7^{1+\gamma}})^{\frac{p}{2(1+\gamma)}}$.

Therefore, we have

$$\|F^m(\cdot) - F(\cdot)\| \leqslant D(\lambda n)E \leqslant G(S)E \leqslant C_8(\frac{1}{\gamma+m})^{\frac{p}{2(1+\gamma)}} E. \tag{4.11}$$

Combining (4.10), (4.11) and (4.12), if the regularization parameter $m = [\left(\frac{E}{\delta}\right)^{\frac{2(1+\gamma)}{p+2}}]$ is selected, then

$$\|F^{m,\delta}(\cdot) - F(\cdot)\| \leqslant a^{\frac{1}{\gamma+1}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}} + C_8 E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}} = C_9 E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, \tag{4.12}$$

where $C_9 = a^{\frac{1}{\gamma+1}} + C_8$.

The proof of **Theorem** 4.1 is completed. □

4.2. The convergent error estimate with an a posteriori parameter choice rule

In this section, we give the posteriori error estimation and the selection criteria of posteriori regularization parameters should be satisfied:

$$\|KF^{m,\delta}(\cdot) - g^\delta(\cdot)\| \leqslant \tau\delta, \tag{4.13}$$

when $m = m(\delta)$ first appears, the iteration stops, where $\|g^\delta\| \geqslant \tau\delta$.

Lemma 4.1. *Let $\rho(m) = \|KF^{m,\delta}(\cdot) - g^\delta(\cdot)\|$, then we have the following conclusions*

(a) *$\rho(m)$ is a continuous function;*

(b) $\lim_{m\to 0} \rho(m) = 0$;

(c) $\lim_{m\to +\infty} \rho(m) = \|g^\delta\|$;

(d) *$\rho(m)$ is a strictly increasing function for any $m \in (0, +\infty)$.*

Proof. The proof of this lemma is obtained by the following expression:

$$\rho(m) = \Big(\sum_{n=1}^{\infty}(1 - a(\frac{1 - e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}}{\lambda_n^\beta})^{\gamma+1})^{2m}(g_n^\delta)^2\Big)^{\frac{1}{2}}.$$

□

Remark 4.2. According to the **Lemma** 4.1, the uniqueness of m is selected by the method of formula (4.13).

Lemma 4.2. *Assume the priori condition* (2.17) *and the noise assumption* (1.3) *hold. For fixed $\tau > 1$, if we choose the regularization parameter by using Morozov's discrepancy principle* (4.13)*, then the regularization parameter $m = m(\delta)$ satisfies*

$$m \leqslant (\frac{1}{\tau-1})^{\frac{2(\gamma+1)}{p+2}} \frac{p+2}{2a\gamma C_1^{\gamma+1}} (\frac{E}{\delta})^{\frac{2(\gamma+1)}{p+2}}. \tag{4.14}$$

Proof. Due to (4.7), we obtain

$$R_m g = \sum_{n=1}^{\infty} \frac{1-\Big(1-a\Big(\frac{1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}}{\lambda_n^\beta}\Big)^{\gamma+1}\Big)^m}{\frac{1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}}{\lambda_n^\beta}} g_n\chi_n(x) \tag{4.15}$$

and

$$\|KR_m g - g\|^2 = \sum_{n=1}^{\infty}(1-a(\frac{1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}}{\lambda_n^\beta})^{\gamma+1})^{2m} g_n^2. \tag{4.16}$$

Due to $|1-a(\frac{\lambda_n^\beta}{1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}})^{\gamma+1}| < 1$, we obtain $\|KR_{m-1}-I\| \leqslant 1$. On the one hand, it is not difficult to find that m is the minimum satisfying $\|KR_m g^\delta - g^\delta\| = \|Kf^{m,\delta}-g^\delta\| \leqslant \tau\delta$. Therefore,

$$\begin{aligned}\|KR_{m-1}g-g\| &= \|KR_{m-1}g - KR_{m-1}g^\delta + KR_{m-1}g^\delta - g^\delta + g^\delta - g\| \\ &\geqslant \|KR_{m-1}g^\delta - g^\delta\| - \|(KR_{m-1}-I)(g^\delta - g)\| \\ &\geqslant \tau\delta - \|KR_{m-1}-I\|\delta \\ &\geqslant \tau\delta - \delta \\ &= (\tau-1)\delta.\end{aligned} \tag{4.17}$$

On the other hand, by using (2.17) formula, we obtain

$$\begin{aligned}&\|KR_{m-1}g-g\|^2 \\ =&\|\sum_{n=1}^{\infty}\Big(1-\Big(1-a\Big(\frac{1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}}{\lambda_n^\beta}\Big)^{\gamma+1}\Big)^{m-1}\Big)g_n\chi_n(x) - \sum_{n=1}^{\infty} g_n\chi_n(x)\|^2 \\ =&\|\sum_{n=1}^{\infty}\Big(1-a\Big(\frac{1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}}{\lambda_n^\beta}\Big)\Big)^{\gamma+1}\Big)^{m-1}\Big)g_n\chi_n(x)\|^2 \\ =&\sum_{n=1}^{\infty}\Big(\Big(1-a\Big(\frac{1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}}{\lambda_n^\beta}\Big)^{\gamma+1}\Big)^{m-1}\Big)^2\Big(\frac{1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}}{\lambda_n^\beta}\Big)^2 F_n^2\lambda_n^{\beta p}\lambda_n^{-\beta p} \\ \leqslant&\sup_{n\geqslant 1} H^2(\lambda_n)E^2,\end{aligned}$$

where $H(\lambda_n) = \Big(1-a\Big(\frac{1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}}{\lambda_n^\beta}\Big)^{\gamma+1}\Big)^{m-1}\Big)\Big(\frac{1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}}{\lambda_n^\beta}\Big)\lambda_n^{\frac{-\beta p}{2}}$.

From **Lemma** 2.1, we can obtain

$$H(\lambda_n) \leqslant \Big(1-a\Big(\frac{C_1}{\lambda_n^\beta}\Big)^{\gamma+1}\Big)^{m-1}\lambda_n^{-\beta(\frac{p}{2}+1)}.$$

Set $L(s) := \Big(1-a\Big(C_1 s\Big)^{\gamma+1}\Big)^{m-1} s^{(\frac{p}{2}+1)}$, $s := \lambda_n^{-\beta}$.

Suppose s_0 satisfies $L^{'}(s_0) = 0$, we obtain

$$s_0 = \Big(\frac{p+2}{(2(m-1)(\gamma+1)+p+2)aC_1^{\gamma+1}}\Big)^{\frac{1}{\gamma+1}}.$$

Then we have

$$\begin{aligned}L(s) \leqslant & L(s_0)\\ =&\Big(1-a\Big(C_1\Big(\frac{p+2}{(2(m-1)(\gamma+1)+p+2)aC_1^{\gamma+1}}\Big)^{\frac{1}{\gamma+1}}\Big)^{\gamma+1}\Big)^{m-1}\\ &\times\Big(\Big(\frac{p+2}{(2(m-1)(\gamma+1)+p+2)aC_1^{\gamma+1}}\Big)^{\frac{1}{\gamma+1}}\Big)^{(\frac{p}{2}+1)}\\ \leq&\Big(\Big(\frac{p+2}{(2(m-1)(\gamma+1)+p+2)aC_1^{\gamma+1}}\Big)^{\frac{1}{\gamma+1}}\Big)^{(\frac{p}{2}+1)}\\ \leq&(\frac{p+2}{2am\gamma(C_1)^{\gamma+1}})^{\frac{p+2}{2(\gamma+1)}}.\end{aligned}$$

That is

$$(\tau-1)\delta \leqslant (\frac{p+2}{2a\gamma(C_1)^{\gamma+1}})^{\frac{p+2}{2(\gamma+1)}} m^{-\frac{p+2}{2(\gamma+1)}} E. \tag{4.18}$$

From the formula (4.18), we can get

$$m \leqslant (\frac{1}{\tau-1})^{\frac{2(\gamma+1)}{p+2}} \frac{p+2}{2a\gamma(C_1)^{\gamma+1}} (\frac{E}{\delta})^{\frac{2(\gamma+1)}{p+2}}.$$

The proof of **Lemma** 4.2 is completed. □

Theorem 4.2. *The exact solution of the problem* (1.1) *is given by* (2.4)*, the fractional Landweber iterative regularization solution* $F^{m,\delta}(x)$ *is given by* (4.7)*. The regularization parameter* $m = m(\delta)$ *is obtained by the iteration stop criterion* (4.13)*, then we obtain*

$$\|F^{m,\delta}(\cdot) - F(\cdot)\| \leqslant C_{10} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, \tag{4.19}$$

where $C_{10} := ((\frac{1}{\tau-1})^{\frac{2(\gamma+1)}{p+2}} \frac{p+2}{2\gamma(C_1)^{\gamma+1}})^{\frac{1}{\gamma+1}} + (C_1)^{\frac{-p}{p+2}}(\tau+1)^{\frac{p}{p+2}}$ *is positive constant.*

Proof. Using the triangle inequality, we have

$$\|F^{m,\delta}(\cdot) - F(\cdot)\| \leqslant \|F^{m,\delta}(\cdot) - F^m(\cdot)\| + \|F^m(\cdot) - F(\cdot)\|. \tag{4.20}$$

From **Lemma** 4.2 and (4.10), we have

$$\begin{aligned}&\|F^{m,\delta}(\cdot) - F^m(\cdot)\|\\ \leqslant& a^{\frac{1}{\gamma+1}} m^{\frac{1}{\gamma+1}} \delta\\ \leqslant& a^{\frac{1}{\gamma+1}}((\frac{1}{\tau-1})^{\frac{2(\gamma+1)}{p+2}} \frac{p+2}{2a\gamma(C_1)^{\gamma+1}})^{\frac{1}{\gamma+1}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}.\end{aligned} \tag{4.21}$$

For the second term on the right side of (4.20), we have

$$\begin{aligned}K(F^m(\cdot) - F(\cdot)) =& \sum_{n=1}^{\infty} -(1-a(\frac{1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}}{\lambda_n^\beta})^{\gamma+1})^m g_n \chi_n(x)\\ =& \sum_{n=1}^{\infty} -(1-a(\frac{1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}}{\lambda_n^\beta})^{\gamma+1})^m (g_n - g_n^\delta)\chi_n(x)\\ &+ \sum_{n=1}^{\infty} -(1-a(\frac{1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}}{\lambda_n^\beta})^{\gamma+1})^m g_n^\delta \chi_n(x).\end{aligned} \tag{4.22}$$

According to (1.3) and (4.13), we can get

$$\|K(F^m(\cdot)-F(\cdot))\| \leqslant (\tau+1)\delta. \tag{4.23}$$

Due to

$$\begin{aligned}
\|F^m(\cdot)-F(\cdot)\|_{H^p(\Omega)} &= \Big(\sum_{n=1}^{\infty}(1-a(\frac{1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T}}{\lambda_n^\beta})^{\gamma+1})^{2m}\frac{\lambda_n^{2\beta}g_n^2}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})^2}\lambda_n^{\beta p}\Big)^{\frac{1}{2}}\\
&\leqslant \Big(\sum_{n=1}^{\infty}\lambda_n^{\beta p}\frac{\lambda_n^{2\beta}g_n^2}{(1-e^{-\frac{\lambda_n^\beta}{1+a\lambda_n}T})^2}\Big)^{\frac{1}{2}}\\
&= \Big(\sum_{n=1}^{\infty}\lambda_n^{\beta p}F_n^2\Big)^{\frac{1}{2}}\\
&\leqslant E.
\end{aligned}$$

Using **Theorem** 2.1, we obtain

$$\|F^m(\cdot)-F(\cdot)\| \leqslant (C_1)^{\frac{-p}{p+2}}(\tau+1)^{\frac{p}{p+2}}E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}. \tag{4.24}$$

Combining (4.20), (4.21) and (4.24), we obtain

$$\|F^{m,\delta}(\cdot)-F(\cdot)\| \leqslant C_{10}E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}, \tag{4.25}$$

where $C_{10} := ((\frac{1}{\tau-1})^{\frac{2(\gamma+1)}{p+2}}\frac{p+2}{2\gamma(C_1)^{\gamma+1}})^{\frac{1}{\gamma+1}} + (C_1)^{\frac{-p}{p+2}}(\tau+1)^{\frac{p}{p+2}}$. □

5. Numerical implementation

In this part, we use three numerical examples to prove the effectiveness and feasibility of Quasi-boundary regularization method and fractional Landweber iterative regularization method. Let $\Omega=(0,1)$, $T=1$, $a=0.5$. First of all, we obtain the final data $g(x)$ by solving the following forward problem

$$\begin{cases}
u_t(x,t)-a\Delta u_t(x,t)+(-\Delta)^\beta u(x,t)=F(x), & x\in\Omega,\ t\in(0,T],\\
u(x,t)=0, & x\in\partial\Omega,\ t\in(0,T],\\
u(x,0)=0, & x\in\Omega,
\end{cases} \tag{5.1}$$

with the given data $F(x)$. We define

$$t_z = z\tau(z=0,1,\dots,N),\quad x_j = jh(j=0,1,\cdots,M), \tag{5.2}$$

where $\tau=\frac{T}{N}$ is the step size of temporal direction and $h=\frac{1}{M}$ is the step size of spatial direction.

Next we will use the finite element difference method to carry out numerical experiments. First, the forward Euler scheme for the u_t is as follows:

$$\frac{\partial u_i^k}{\partial t} = \frac{u_i^{k+1}-u_i^k}{\tau}.$$

The Laplace operator difference scheme of one-dimensional integer order is as follows

$$\Delta(u_i^k)_t = \frac{u_{i+1}^{k+1} - 2u_i^{k+1} + u_{i-1}^{k+1} - u_{i+1}^k + 2u_i^k - u_{i-1}^k}{h^2\tau}.$$

The difference format of a one-dimensional fractional Laplace operator is as follows [28]

$$(-\Delta)^s U = C_{1,2s} B U,$$

where $C_{1,s} = \frac{4^s\Gamma(1/2+s)}{\pi^{1/2}|\Gamma(-s)|}$, B is a strictly diagonally dominant and symmetric positive definite matrix. $B \triangleq (h)_{i,p=1}^{M-1}$, and

$$h_{i,p} \triangleq \begin{cases} -\left(Z_1(i,p+1) + Z_2(i,p)\right)\dfrac{1}{i-p}, \quad 1 \le p \le i-2, \\ -\dfrac{h^{-2s}}{2-2s} - Z_2(i,i-1), \quad p = i-1, \\ -\dfrac{h^{-2s}}{2-2s} - Z_3(i,i+2), \quad p = i+1, \\ -\left(Z_3(i,p+1) + Z_4(i,p)\right)\dfrac{1}{p-i}, \quad i+2 \le p \le M, \end{cases}$$

and $h_{i,i}$ satisfies

$$h_{i,i} + \sum_{p=1,p\neq i}^{M} h_{i,p} - Y_1(i) - Y_2(i) = \begin{cases} \dfrac{h^{-2s}}{2-2s} + \dfrac{Z_4(i,M+1)}{M+1-i}, \quad i = 1, \\ \dfrac{Z_1(i,1)}{i} + \dfrac{Z_4(i,M+1)}{M+1-i}, \quad 2 \le i \le M-1, \\ \dfrac{h^{-2s}}{2-2s} + \dfrac{Z_1(i,1)}{i}, \quad i = M, \end{cases}$$

where $Z_1(i,k) = \frac{1}{h^2}\int_{x_{k-1}}^{x_k} (x_k - y)(x_i - y)^{-2s}\, dy$, $Z_2(i,k) = \frac{1}{h^2}\int_{x_{k-1}}^{x_k} (y - x_k)(x_i - y)^{-2s} dy$, $Z_3(i,k) = \frac{1}{h^2}\int_{x_{k-1}}^{x_k} (x_k - y)(y - x_i)^{-2s}\, dy$, $Z_4(i,k) = \frac{1}{h^2}\int_{x_{k-1}}^{x_k} (y - x_{k-1}) \times (y - x_i)^{-2s} dy$ and $Y_1(i) = \int_{-\infty}^{0} \frac{1}{(x_i - y)^{(1+2s)} dy}$, $Y_2(i) = \int_1^{\infty} \frac{1}{(y - x_i)^{(1+2s)}} dy$.

Through simple calculation, there are the following results

$$\begin{aligned} & Z_1(i,p+1) + Z_2(i,p) \\ & = Z_3(i,p+1) + Z_4(i,p) \\ & = \begin{cases} \dfrac{h^{-2s}}{(2s-1)(2-2s)}[2|i-p|^{2-2s} - (|i-p|-1)^{2-2s} - (|i-p|+1)^{2-2s}], \ s \neq 0.5, \\ \dfrac{1}{h}[-2|i-p|\ln(|i-p|) + (|i-p|+1)\ln(|i-p|+1) \\ +(|i-p|-1)\ln(|i-j|-1)], \ s = 0.5, \end{cases} \end{aligned}$$

$$\begin{aligned} & Z_2(i,i-1) \\ & = Z_3(i,i+2) \\ & = \begin{cases} \dfrac{h^{-2s}}{(2s-1)(2-2s)}\left(3 - 2s - 2^{2-2s}\right), \ s \neq 0.5, \\ \dfrac{1}{h}[2\ln 2 - 1], \ s = 0.5, \end{cases} \end{aligned}$$

$$Z_1(i,1)=\begin{cases}\dfrac{h^{-2s}}{(2s-1)(2-2s)}\left[i^{2-2s}-(i-1)^{2-2s}-(2-2s)i^{1-2s}\right],\ s\neq 0.5,\\ \dfrac{1}{h}\left[(1-i)\ln\left(\dfrac{i}{i-1}\right)+1\right],\ s=0.5,\end{cases}$$

$$Z_4(i,M+1)$$

$$=\begin{cases}\dfrac{h^{-2s}}{(2s-1)(2-2s)}\\ \times\left[(M+1-i)^{2-2s}-(M-i)^{2-2s}-(2-2s)(M+1-i)^{1-2s}\right],\ s\neq 0.5,\\ \dfrac{1}{h}[(i-M)]\ln\left(\dfrac{M+1-i}{M-i}\right)+1\Big],\ s=0.5,\\ \mathrm{Y}_1(i)=\dfrac{(x_i)^{-2s}}{2s},\ Y_2(i)=\dfrac{(1-x_i)^{-2s}}{2s}.\end{cases}$$

Arrange the above formulas to get the matrix B.

It is worth noting that the difference format of the fractional order Laplace operator is for the difference format of order 0-1, but when studying the error estimation, the differential format can converge to the $(3-2\beta)$ order, so in the later numerical experiments, we give numerical simulation results of order 1.1, 1.2, 1.3.

Therefore, we establish the difference format corresponding to the equation (5.1)

$$(\mathcal{A}+\mathcal{B})U^{i+1}=\mathcal{A}U^i+F,$$

where $U^i:=(u_1^i,u_2^i,u_3^i,\cdots,u_{M+1}^i)$,$i=0,1,2,\cdots,M$, $F:=(F(x_1),F(x_2),F(x_3),\cdots,F(x_{M+1}))$,

$$\mathcal{A}_{(M+1)\times(M+1)}=\begin{pmatrix}0 & & \\ & \hat{\mathcal{A}}^{-1}_{(M-1)\times(M-1)} & \\ & & 0\end{pmatrix},$$

$$\mathcal{B}_{(M+1)\times(M+1)}=\begin{pmatrix}0 & & \\ & B_{(M-1)\times(M-1)} & \\ & & 0\end{pmatrix},$$

in which

$$\hat{\mathcal{A}}_{(M-1)\times(M-1)}=\begin{pmatrix}\frac{1}{\tau}+\frac{2a}{\tau h^2} & -\frac{a}{\tau h^2} & & & \\ -\frac{a}{\tau h^2} & \frac{1}{\tau}+\frac{2a}{\tau h^2} & -\frac{a}{\tau h^2} & & \\ & -\frac{a}{\tau h^2} & \frac{1}{\tau}+\frac{2a}{\tau h^2} & \ddots & \\ & & \ddots & \ddots & -\frac{a}{\tau h^2} \\ & & & -\frac{a}{\tau h^2} & \frac{1}{\tau}+\frac{2a}{\tau h^2}\end{pmatrix}.$$

By differentiating the function, we can get the numerical solution of the function through the following iterative format,

$$\begin{cases} (\mathcal{A}+\mathcal{B})U^i = \mathcal{A}U^{i-1} + F, \ 2 \le i \le N, \\ (\mathcal{A}+\mathcal{B})U^1 = F. \end{cases}$$

On the basis of the above iterative format, the numerical solution corresponding to the function and the value of the $g(x)$ function can be obtained by using Matlab software. For the inverse problem, when applying the Quasi-boundary regularization method and fractional Landweber iterative regularization, we need to obtain a matrix K that satisfies $Kf = U^N = g^\delta$, i.e.,

$$\begin{aligned} &K^1 = (\mathcal{A}+\mathcal{B})^{-1}, \\ &K^n = (\mathcal{A}+\mathcal{B})^{-1}(K^{n-1}+I), \quad n = 2, \cdots, N, \\ &K = K^N. \end{aligned}$$

Finally, the Quasi-boundary regularization solution is obtained by the following formula:

$$F_\mu^\delta(x) = \frac{1}{K+\mu} g^\delta,$$

and the fractional Landweber iteration regularization solution is obtained by the following formula:

$$F^{m,\delta} = a \sum_{n=1}^{m-1} \left(I - a\,(K^*K)^{\frac{\gamma+1}{2}} \right)^n (K^*K)^{\frac{\gamma-1}{2}} K^* g^\delta.$$

By adding random perturbation to noise data $g(x)$, the data with errors are obtained,

$$g^\delta = g + \varepsilon \cdot \mathrm{rand}\, n(\mathrm{size}(g)),$$

where the function randn(·) produces a list of random numbers with a mean of 0 and a variance of 1. The priori regularization parameter is based on the smooth conditions of the exact solution, which is actually difficult to give in practical problem. The following examples demonstrate the validity of the Quasi-boundary regularization method and the fractional Landweber iteration regularization method based on a posteriori regularization parameter selection rule.

For the selection of parameter μ, we have given it in (3.14) with $\tau = 1.01$. For the parameter selection corresponding to the fractional order Landweber iterative regularization method, we select $\gamma = 0.1$, and the selection of the iteration step m is also given by formula (4.13). Select $M = 100$, $N = 50$. We give the following three examples.

Example 5.1. Consider the piecewise smooth function

$$F(x) = \begin{cases} 4x, & 0 \leqslant x \leqslant \dfrac{1}{2}, \\ -4(x-1), & \dfrac{1}{2} < x \leqslant 1. \end{cases}$$

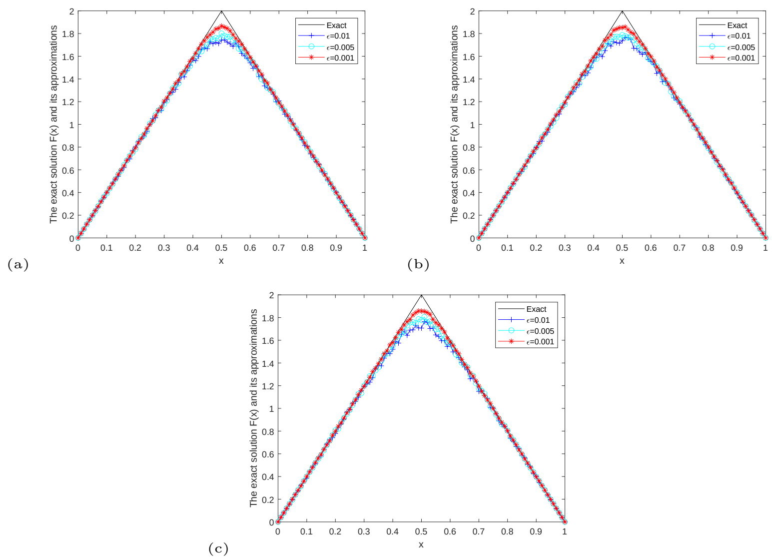

Figure 1. The comparison of the exact solution $F(x)$ and its Quasi-boundary regularization method approximation solution $F^{m,\delta}(x)$ of Example 5.1 with $\beta = 1.1(a), 1.2(b), 1.3(c)$ for $\varepsilon = 0.01, 0.005, 0.001$.

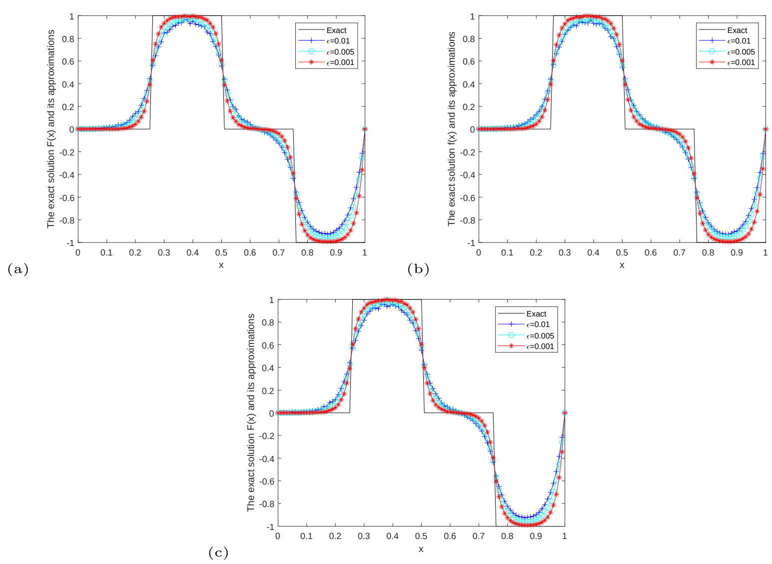

Figure 2. The comparison of the exact solution $F(x)$ and its Quasi-boundary regularization method approximation solution $F^{m,\delta}(x)$ of Example 5.2 with $\beta = 1.1(a), 1.2(b), 1.3(c)$ for $\varepsilon = 0.01, 0.005, 0.001$.

Example 5.2. Consider a non-continuous function

$$F(x) = \begin{cases} 0, & 0 \leqslant x \leqslant \dfrac{1}{4}, \\ 1, & \dfrac{1}{4} < x \leqslant \dfrac{1}{2}, \\ 0, & \dfrac{1}{2} < x \leqslant \dfrac{3}{4}, \\ -1, & \dfrac{3}{4} < x \leqslant 1. \end{cases}$$

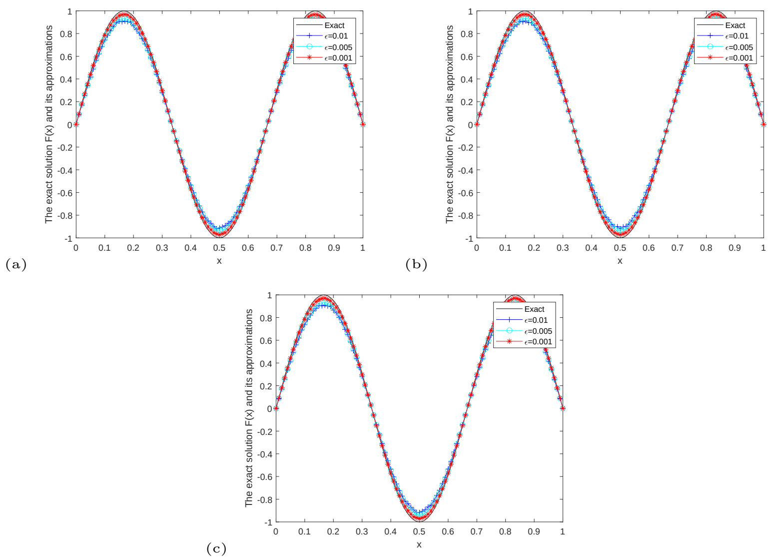

Figure 3. The comparison of the exact solution $F(x)$ and its Quasi-boundary regularization method approximation solution $F^{m,\delta}(x)$ of Example 5.3 with $\beta = 1.1, 1.2, 1.3$ for $\varepsilon = 0.01, 0.005, 0.001$.

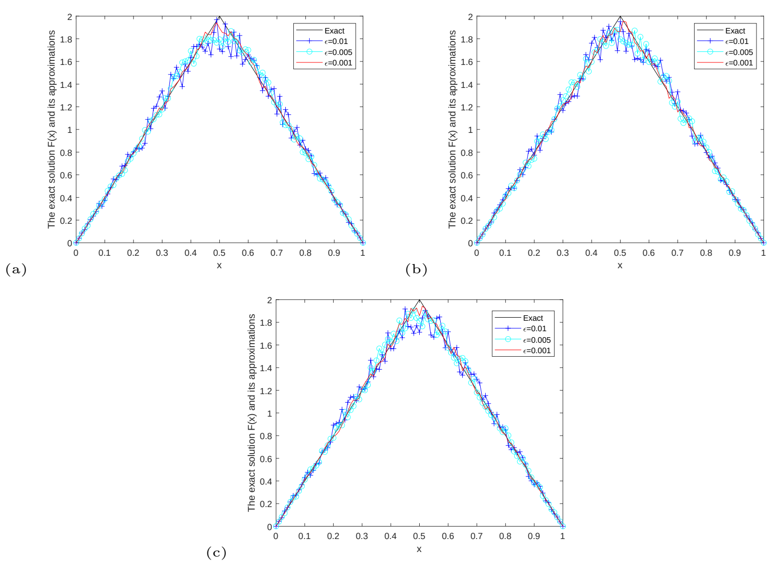

Figure 4. The comparison of the exact solution $F(x)$ and its fractional Landweber iterative regularization approximation solution $F^{m,\delta}(x)$ of Example 5.1 with $\beta = 1.1(a), 1.2(b), 1.3(c)$ for $\varepsilon = 0.01, 0.005, 0.001$.

Example 5.3. Consider the smooth function

$$F(x) = \sin(3\pi x).$$

Figures 1-3 show the error between the exact solution $F(x)$ and the Quasi-boundary regularization approximation solution $F_{\mu}^{\delta}(x)$. Figure 1 shows the exact

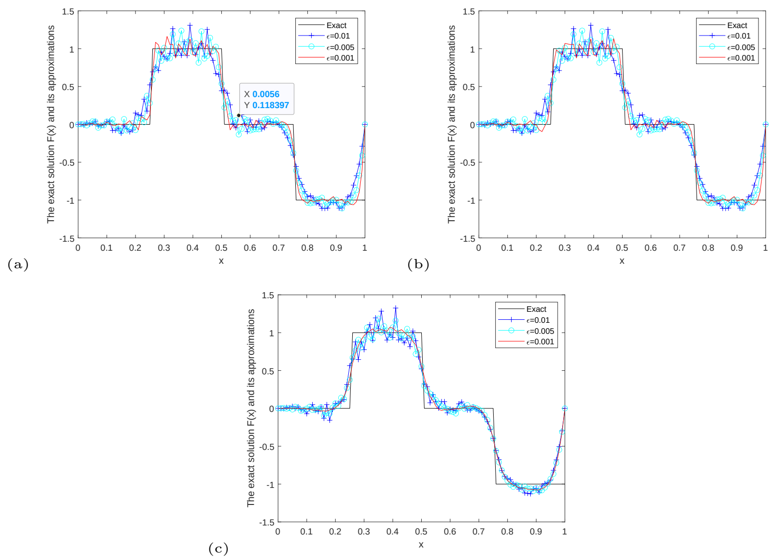

Figure 5. The comparison of the exact solution $F(x)$ and its fractional Landweber iterative regularization approximation solution $F^{m,\delta}(x)$ of Example 5.2 with $\beta = 1.1(a), 1.2(b), 1.3(c)$ for $\varepsilon = 0.01, 0.005, 0.001$.

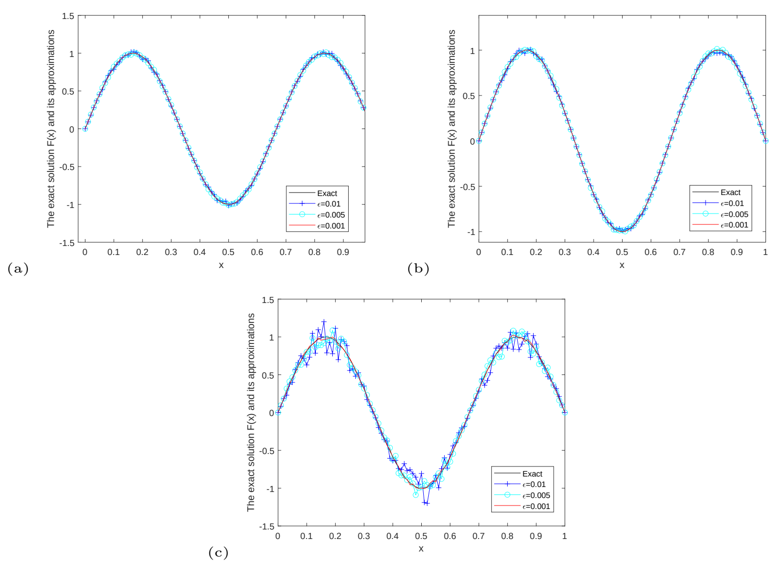

Figure 6. The comparison of the exact solution $F(x)$ and its fractional Landweber iterative regularization approximation solution $F^{m,\delta}(x)$ of Example 5.3 with $\beta = 1.1(a), 1.2(b), 1.3(c)$ for $\varepsilon = 0.01, 0.005, 0.001$.

solution $F(x)$ and the Quasi-boundary regularization approximation solution $F_\mu^\delta(x)$ of Example 5.1 for the relative error levels $\varepsilon = 0.01, 0.005, 0.001$ with various values $\beta = 1.1, 1.2, 1.3$. Figure 2 shows the exact solution $F(x)$ and the Quasi-boundary regularization approximation solution $F_\mu^\delta(x)$ of Example 5.2 for the relative error levels $\varepsilon = 0.01, 0.005, 0.001$ with various values $\beta = 1.1, 1.2, 1.3$. Figure 3 shows the exact solution $F(x)$ and the Quasi-boundary regularization approximation solution $F_\mu^\delta(x)$ of Example 5.3 for the relative error levels $\varepsilon = 0.01, 0.005, 0.001$ with various values $\beta = 1.1, 1.2, 1.3$.

Figures 4-6 show the error between the exact solution $F(x)$ and the Fractional Landweber iterative regularization approximation solution $F^{m,\delta}(x)$. Figure 4 shows the exact solution $F(x)$ and fractional Landweber iterative regularization approximation solution $F^{m,\delta}(x)$ of Example 5.1 for the relative error levels $\varepsilon = 0.01, 0.005, 0.001$ with various values $\beta = 1.1, 1.2, 1.3$. Figure 5 shows the exact solution $F(x)$ and the fractional Landweber iterative regularization approximation solution $F^{m,\delta}(x)$ of Example 5.2 for the relative error levels $\varepsilon = 0.01, 0.005, 0.001$ with various values $\beta = 1.1, 1.2, 1.3$. Figure 6 shows the exact solution $F(x)$ and the fractional Landweber iterative regularization approximation solution $F^{m,\delta}(x)$ of Example 5.3 for the relative error levels $\varepsilon = 0.01, 0.005, 0.001$ with various values $\beta = 1.1, 1.2, 1.3$.

From above six figures, we find that for the same example, the smaller ε and α, the better the fitting effect between the exact solution $F(x)$ and the regularization solutions. For different examples, the fitting results of functions with better smoothness are better than those with worse smoothness.

With the same error, fractional Landweber iterative regularization fits better than Quasi-boundary regularization method for non-smooth functions and piecewise functions, and Quasi-boundary regularization method gets better fitting results than fractional Landweber iterative regularization for smooth functions.

In addition, limited by the difference format of the fractional order Laplace operator, the smaller the order of Laplace, the better the numerical fitting effect, and the author will look for a better differential format of the fractional order Laplace operator in the next study.

6. Conclusion

The problem of inverting the source item of Sobolev equation with fractional Laplacian is studied. The regularization solutions are obtained by Quasi-boundary regularization method and fractional Landweber iteration regularization method. Based on the conditional stability result, the corresponding error estimates are obtained under the rules for selecting a priori regularization parameter and a posteriori regularization parameter. Three numerical examples are given to demonstrate the effectiveness, stability and superiority of our proposed regularization methods. Moreover, through the error estimations of the two methods, we find that the Quasi-boundary regularization method has a saturation effect, while the fractional Landweber iterative regularization method can avoid the saturation effect. In terms of numerical simulation, the Quasi-boundary method can obtain numerical results more quickly, but the corresponding results are poor, while the Landweber method needs to be continuously iterated to obtain better simulation results, and the numerical results are better than the Quasi-boundary method.

Disclosure statement

No potential conflict of interest was reported by the author.

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