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ON A PARTIALLY DEGENERATE REACTION-ADVECTION-DIFFUSION SYSTEM WITH FREE BOUNDARY CONDITIONS*

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Abstract A partially degenerate reaction-diffusion system with advection term and free boundary conditions is investigated in this paper. Firstly, we present a spreading-vanishing dichotomy for the asymptotic behavior of solutions of the system. Then, we obtain criteria for spreading and vanishing. Moreover, numerical simulation is given to illustrate the theoretical results.

Keywords Partially degenerate diffusion, free boundary problem, spreading and vanishing.

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1. Introduction

In the past decades, quite a few reaction-diffusion systems in which some but not all diffusion coefficients are zeroes called partially degenerate reaction-diffusion systems, have been introduced to give an accurate description of a wide variety of phenomena in population biology, epidemiology, and so on. Capasso and Maddalena [3] introduced an epidemic reaction-diffusion model to study the fecally-orally transmitted diseases in the European Mediterranean regions

$$\begin{cases} u_t = du_{xx} - au + cv, \\ v_t = -bv + g(u). \end{cases} \tag{1.1}$$

Zhao and Wang [43] established the existence of wavefronts and a minimal wave speed of (1.1) with monostable nonlinearity. Xu and Zhao [34] proved the existence,uniqueness and stability of traveling front of (1.1) with bistable nonlinearity. Wu [33] constructed some new entire solutions for (1.1) with bistable nonlinearity, which behaves like two increasing traveling wave solutions propagating from both sides of the x-axis and annihilating at a finite time. Hadeler and Lewis [14]

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presented the following reaction-diffusion system

$$\begin{cases} u_t = du_{xx} + f_1(u) - \gamma_1 u + \gamma_2 v, \\ v_t = \gamma_1 u - \gamma_2 v, \end{cases} \tag{1.2}$$

which describes a species population where the individuals alternate between mobile and stationary states, and only the mobile reproduce. Zhang and Zhao [38] considered the asymptotic behavior of solutions of (1.2). Further, Zhang and Li [39] established the monotonicity and uniqueness of traveling wave solutions of (1.2). Fang and Zhao [8] studied the traveling fronts and spreading speed of the following general partially degenerate reaction-diffusion system

$$\begin{cases} u_t = du_{xx} + f_1(u, v), \\ v_t = f_2(u, v). \end{cases} \tag{1.3}$$

Zhang et al. [37] considered a class of partially degenerate nonlocal diffusion systems with free boundaries. Chen et al. [4] discussed a partially degenerate epidemic model with time delay and free boundaries.

Recently, the free boundary condition has been considered in more and more ecological models to let the description of a gradual spreading process be more close to the reality. For example, we refer to [2, 7, 18, 19, 46] for single-species models. More works related to the system can be found, such as [12, 13, 30] for Lotka-Volterra competition systems, [21, 35, 40] for predator-prey systems, [27, 31, 41, 47] for epidemic models and [20, 26, 36] for other models. Wang and Cao [29] studied the spreading frontiers of (1.3) with free boundary. It is shown that a spreading-vanishing dichotomy holds, and the sharp criteria for the spreading and vanishing are obtained. Choi and Ahn [6] considered non-uniform dispersal of logistic population models with free boundaries in a spatially heterogeneous environment. They observed that the spreading-vanishing dichotomy and the asymptotic spreading speed of the moving front is uniquely determined in relation to the semi-wave speed. Ahn et al. [1] investigated the free boundary problem of a man-environment-man epidemic model: $f_1(u, v) = -au - cv$ and $f_2(u, v) = -bv + G(u)$. Kaneko et al. [16, 17] investigated the Stefan problem of nonlinear diffusion equation ($x \in \Omega(t) \subset R^N$) with positive bistable nonlinearity.

Normally, diffusion of particles in physics is random and obeys Fick's law. However, species in population dynamics or diseases in epidemiology diffuse differently owing to their initiative behaviors and activities. Some species or diseases prefer to move in one direction because of appropriate climate, food, wind direction, etc. For example, Maidana and Yang [23] studied the spread of West Nile virus (WNv) in North America. West Nile virus appeared for the first time in New York city in the summer of 1999. In the second year the wave front traveled 187km to the Northand and 1100km to the South, it spread across almost the whole America continent till 2002. Therefore, the propagation of WNv from New York city to California state is a consequence of the diffusion and advection movements of birds. Especially, bird advection becomes an important factor for lower mosquito biting rates. Recently, there are some works considering the advection. Gu et al. [9, 10] was the first one to consider the long-time behavior and the asymptotic spreading speeds of the free boundary problem with Fisher-KPP type and small advection. Gu et al. [11] further studied the long time behavior of solutions of Fisher-KPP equation with advection $\beta > 0$ and free boundaries. For a single equation with advection, there are many other works [15,24,32,44]. Besides, there are also several works devoted to the system with small advection [5,28,45]. Zhao et al. [42] considered the free boundary problem with the advection based on [1].

Inspired by the work [29], we consider the following reaction-advection-diffusion system with general reaction functions

$$\begin{cases} u_t = du_{xx} - \beta u_x + f_1(u,v), & g(t) < x < h(t), t > 0, \\ v_t = f_2(u,v), & g(t) < x < h(t), t > 0, \\ h'(t) = -\mu u_x(h(t),t), & t > 0, \\ g'(t) = -\mu u_x(g(t),t), & t > 0, \\ u(0,x) = u_0(x), v(0,x) = v_0(x), -h_0 \le x \le h_0, \\ h(0) = -g(0) = h_0, \end{cases} \tag{1.4}$$

where d, β, μ and h_0 are positive constants, $x = h(t)$ and $x = g(t)$ are moving boundaries to be determined. The initial functions $u_0(x), v_0(x) \in \Sigma(h_0)$ for some $h_0 > 0$, and

$$\begin{aligned} \Sigma(h_0) :=& \{\phi \in C^2([-h_0,h_0]) : \phi(-h_0) = \phi(h_0) = 0, \phi'(h_0) < 0, \\ & \phi'(-h_0) > 0, \phi(x) > 0 \text{ in } (-h_0,h_0)\}. \end{aligned}$$

For convenience, we denote $\Delta := f_{1u} - \frac{f_{1v}f_{2u}}{f_{2v}}$. Throughout this paper, we assume that the following holds for f_1, f_2 and β

(i) $f_i \in C^1(\mathbb{R}^2_+, \mathbb{R})(i = 1,2)$, $\frac{\partial f_1(u,v)}{\partial v} > 0$, $\frac{\partial f_2(u,v)}{\partial u} > 0$, $f_1(u,v) \le f_{1u}u + f_{1v}v$, $f_2(u,v) \le f_{2u}u + f_{2v}v$, where $f_{1u} := \frac{\partial f_1(0,0)}{\partial u}$, $f_{1v} := \frac{\partial f_1(0,0)}{\partial v}$, $f_{2u} := \frac{\partial f_2(0,0)}{\partial u}$, $f_{2v} := \frac{\partial f_2(0,0)}{\partial v} < 0$;

(ii) there exist $K_1, K_2 > 0$ such that for any $M_1 \ge K_1, M_2 \ge K_2$, there exist $\bar{M}_1 \ge M_1, \bar{M}_2 \ge M_2$ such that $f_1(\bar{M}_1, \bar{M}_2) \le 0, f_2(\bar{M}_1, \bar{M}_2) \le 0$;

(iii)
$$0 < \beta < \beta^* = \begin{cases} 2\sqrt{d(f_{1u} - \dfrac{f_{1v}f_{2u}}{f_{2v}})}, & \Delta > 0, \\ \infty, & \Delta \le 0. \end{cases}$$

2. Some basic results

Firstly, we present the existence and uniqueness of the solution by the contraction mapping theorem.

Lemma 2.1. *For any initial value $(u_0, v_0) \in \Sigma(h_0)$ and any $\alpha \in (0,1)$, there exists a positive number T such that problem (1.4) admits a unique solution $(u, v; g, h)$, which satisfies*

$$(u, v; g, h) \in C^{1+\alpha,(1+\alpha)/2}(G_T) \times C(G_T) \times (C^{1+\alpha/2}((0,T]))^2,$$

moreover,

$$\|u\|_{C^{1+\alpha,(1+\alpha)/2}(G_T)} + \|v\|_{C(G_T)} + \|g,h\|_{C^{1+\alpha/2}((0,T])} \le C,$$

where $G_T := \{(x,t) \in \mathbb{R}^2 : x \in [g(t), h(t)], t \in [0,T]\}$, *the constants* T *and* C *only depend on* $h_0, \alpha, \beta, \|u_0\|_{C^2([-h_0,h_0])}$ *and* $\|v_0\|_{C^2([-h_0,h_0])}$.

Proof. The free boundary problem is transformed into a fixed boundary problem by making a transformation. Let

$$\begin{aligned}
y &= \frac{2x - g(t) - h(t)}{h(t) - g(t)},\\
w(y,t) &= u(\frac{(h(t)-g(t))y + g(t) + h(t)}{2}),\\
z(y,t) &= v(\frac{(h(t)-g(t))y + g(t) + h(t)}{2}).
\end{aligned}$$

Then (1.4) can be transformed into the fixed boundary problem

$$\begin{cases}
w_t - d\rho^2(t)w_{yy} + (\beta\rho(t) - \xi(y,t))w_y = f_1(w,z), & -1 < y < 1, t > 0,\\
z_t - \xi(y,t)z_y = f_2(w,z), & -1 < y < 1, t > 0,\\
w(\pm 1, t) = z(\pm 1, t) = 0, & t > 0,\\
w(y,0) = u(h_0 y, 0), z(y,0) = v(h_0 y, 0), & -1 \le y \le 1,
\end{cases}$$

where

$$\rho(t) = \frac{2}{h(t)-g(t)}, \xi(y,t) = \frac{h'(t)+g'(t)}{h(t)-g(t)} + \frac{h'(t)-g'(t)}{h(t)-g(t)}y.$$

Similar to Theorem 1.1 in [22], the unique local solution is obtained by constructing a contraction map. □

Lemma 2.2. *Let* $(u, v; g, h)$ *be a solution of problem (1.4) for* $t \in [0,T]$ *for some* $T > 0$ *, then there exist constants* C_1, C_2 *independent of* T *such that*

$$\begin{aligned}
&0 < u(x,t), v(x,t) \le C_1, \ \textit{for } (x,t) \in (g(t), h(t)) \times (0,T],\\
&0 < -g'(t), h'(t) \le C_2, \ \textit{for } t \in (0,T].
\end{aligned}$$

Proof. Since the system is cooperative and $u_0, v_0 \ge 0$ are nontrivial, we have $u, v > 0$ for $(x,t) \in (g(t), h(t)) \times (0,T]$. Using the strong maximum principle, we immediately obtain $u_x(h(t),t) < 0$ and $u_x(g(t),t) > 0$. Then $h'(t) > 0, -g'(t) > 0$ in $(0,T]$.

Set $M_1 := \max\{K_1, \|u_0\|_{C^2([-h_0,h_0])}\}, M_2 := \max\{K_2, \|v_0\|_{C^2([-h_0,h_0])}\}$, then there exist $\bar{M}_1 \ge M_1$ and $\bar{M}_2 \ge M_2$ such that $f_1(\bar{M}_1, \bar{M}_2) \le 0, f_2(\bar{M}_1, \bar{M}_2) \le 0$.

Define

$$L_1 := \max_{D_T}\{u(x,t)\}, N_1 = \max\{\max_D \frac{\partial}{\partial z} f_1(z,w), \max_D \frac{\partial}{\partial w} f_2(z,w), 0\},$$

and

$$L_2 := \max_{D_T}\{v(x,t)\}, N_2 = \max\{\max_D \frac{\partial}{\partial w} f_1(z,w), \max_D \frac{\partial}{\partial z} f_2(z,w)\},$$

where $D_T = \{(x,t) \in \mathbb{R}^2 : x \in [g(t), h(t)], t \in [0,T]\}$ and $D = \{(z,w) \in \mathbb{R}^2 : 0 \leq z \leq \max\{\bar{M}_1, L_1\}, 0 \leq w \leq \max\{\bar{M}_2, L_2\}\}$.

Set

$$(U(x,t), V(x,t)) = (\bar{M}_1 - u(x,t), \bar{M}_2 - v(x,t))e^{-(N_1+N_2)t},$$

then

$$\begin{aligned} f_1(u,v) =& f_1(\bar{M}_1 - Ue^{(N_1+N_2)t}, \bar{M}_2 - Ve^{(N_1+N_2)t}) \\ =& f_1(\bar{M}_1, \bar{M}_2) - \frac{\partial}{\partial u} f_1(\xi_1(x,t), \xi_2(x,t))Ue^{(N_1+N_2)t} \\ & - \frac{\partial}{\partial v} f_1(\xi_1(x,t), \xi_2(x,t))Ve^{(N_1+N_2)t}, \end{aligned}$$

and

$$\begin{aligned} f_2(u,v) =& f_2(\bar{M}_1 - Ue^{(N_1+N_2)t}, \bar{M}_2 - Ve^{(N_1+N_2)t}) \\ =& f_2(\bar{M}_1, \bar{M}_2) - \frac{\partial}{\partial u} f_2(\gamma_1(x,t), \gamma_2(x,t))Ue^{(N_1+N_2)t} \\ & - \frac{\partial}{\partial v} f_2(\gamma_1(x,t), \gamma_2(x,t))Ve^{(N_1+N_2)t}, \end{aligned}$$

where $\xi_1(x,t), \gamma_1(x,t)$ are between $u(x,t)$ and $\bar{M}_1$, and $\xi_2(x,t), \gamma_2(x,t)$ are between $v(x,t)$ and $\bar{M}_2$. It follows that

$$\begin{cases} \quad U_t - dU_{xx} + \beta U_x & \\ \geq \frac{\partial}{\partial u} f_1(\xi_1, \xi_2)U + \frac{\partial}{\partial v} f_1(\xi_1, \xi_2)V - (N_1+N_2)U, & x \in (g(t), h(t)), t \in (0,T], \\ V_t \geq \frac{\partial}{\partial u} f_2(\gamma_1, \gamma_2)U + \frac{\partial}{\partial v} f_2(\gamma_1, \gamma_2)V - (N_1+N_2)V, & x \in (g(t), h(t)), t \in (0,T], \\ U(x,t) = \bar{M}_1 e^{-(N_1+N_2)t}, V(x,t) = \bar{M}_2 e^{-(N_1+N_2)t}, & x = h(t) \text{ or } g(t), t \in (0,T], \\ U(x,0), V(x,0) \geq 0, & x \in [-h_0, h_0]. \end{cases}$$

We now claim that $\min\{U(x,t), V(x,t)\} \geq 0$ in D_T. Otherwise, there exists $(x_0, t_0) \in D_T$ such that

$$\min\{U(x_0,t_0), V(x_0,t_0)\} = \min_{(x,t)\in D_T} \min\{U(x,t), V(x,t)\} < 0.$$

Assume that $U(x_0,t_0) = \min\{U(x_0,t_0), V(x_0,t_0)\} < 0$, then $U(x,t)$ attains its minimum at $(x_0,t_0) \in D_T$, then

$$U_t(x_0,t_0) - dU_{xx}(x_0,t_0) + \beta U_x(x_0,t_0) \leq 0.$$

On the other hand, we get

$$\begin{aligned} & \frac{\partial}{\partial u} f_1(\xi_1,\xi_2)U(x_0,t_0) + \frac{\partial}{\partial v} f_1(\xi_1,\xi_2)V(x_0,t_0) - (N_1+N_2)U(x_0,t_0) \\ \geq \ & \frac{\partial}{\partial v} f_1(\xi_1,\xi_2)V(x_0,t_0) - N_2U(x_0,t_0) \\ > \ & 0, \end{aligned}$$

which leads a contradiction. Similarly, if $V(x_0,t_0) = \min\{U(x_0,t_0), V(x_0,t_0)\} < 0$, then $V(x,t)$ attains its minimum at $(x_0,t_0) \in D_T$, which leads a contradiction again. Then $\min\{U(x,t),V(x,t)\}\geq 0$ in D_T. Hence, we obtain $0<u(x,t),v(x,t)<C_1$,where $C_1:=\min\{\bar{M}_1,\bar{M}_2\}$.

It remains to be shown that $-g'(t),h'(t)\leq C_2$ in $(0,T]$. The proof is similar to that of Lemma 2.2 in [7]. □

Since the boundedness of u,v,g and h, the global solution is guaranteed.

Theorem 2.1. *The solution of (1.4) exists and is unique for all $t\in(0,\infty)$.*

Now we give two comparison principles, which can be proven by the similar way of Lemma 2.5 in [1].

Lemma 2.3. *Assume that $T>0,\bar{g},\bar{h}\in C^1([0,T]),\bar{u}\in C(\bar{D})\cap C^{2,1}(D),\bar{v}\in C(\bar{D})\cap C^{0,1}(D)$with $D=\{(x,t)\in\mathbb{R}^2:\bar{g}(t)<x<\bar{h}(t),0<t\leq T\}$, and*

$$\begin{cases}\bar{u}_t\geq d\bar{u}_{xx}-\beta\bar{u}_x+f_1(\bar{u},\bar{v}), & \bar{g}(t)<x<\bar{h}(t),0<t\leq T,\\ \bar{v}_t\geq f_2(\bar{u},\bar{v}), & \bar{g}(t)<x<\bar{h}(t),0<t\leq T,\\ \bar{u}(\bar{g}(t),t)=\bar{u}(\bar{h}(t),t)=0, & 0<t\leq T,\\ \bar{v}(\bar{g}(t),t)=\bar{v}(\bar{h}(t),t)=0, & 0<t\leq T,\\ \bar{g}(0)\leq -h_0,\bar{g}'(t)\leq-\mu\bar{u}_x(\bar{g}(t),t), & 0<t\leq T,\\ \bar{h}(0)\geq h_0,\bar{h}'(t)\geq-\mu\bar{u}_x(\bar{h}(t),t), & 0<t\leq T,\\ \bar{u}(x,0)\geq u_0(x),\bar{v}(x,0)\geq v_0(x), & -h_0<x<h_0.\end{cases}$$

For $(u,v;g,h)$ being a solution of (1.4), then

$$g(t)\geq\bar{g}(t),h(t)\leq\bar{h}(t) \text{ for } t\in(0,T],$$
$$u(x,t)\leq\bar{u}(x,t),v(x,t)\leq\bar{v}(x,t) \text{ for } x\in[g(t),h(t)],t\in(0,T].$$

Lemma 2.4. *Assume that $T>0,\underline{g},\underline{h}\in C^1([0,T]),\underline{u}\in C(\bar{D})\cap C^{2,1}(D),\underline{v}\in C(\bar{D})\cap C^{0,1}(D)$ with $D=\{(x,t)\in\mathbb{R}^2:\underline{g}(t)<x<\underline{h}(t),0<t\leq T\}$, and*

$$\begin{cases}\underline{u}_t\leq d\underline{u}_{xx}-\beta\underline{u}_x+f_1(\underline{u},\underline{v}), & \underline{g}(t)<x<\underline{h}(t),0<t\leq T,\\ \underline{v}_t\leq f_2(\underline{u},\underline{v}), & \underline{g}(t)<x<\underline{h}(t),0<t\leq T,\\ \underline{u}(\underline{g}(t),t)=\underline{u}(\underline{h}(t),t)=0, & 0<t\leq T,\\ \underline{v}(\underline{g}(t),t)=\underline{v}(\underline{h}(t),t)=0, & 0<t\leq T,\\ \underline{g}(0)\geq -h_0,\underline{g}'(t)\geq-\mu\underline{u}_x(\underline{g}(t),t), & 0<t\leq T,\\ \underline{h}(0)\leq h_0,\underline{h}'(t)\leq-\mu\underline{u}_x(\underline{h}(t),t), & 0<t\leq T,\\ \underline{u}(x,0)\leq u_0(x),\underline{v}(x,0)\leq v_0(x), & -h_0<x<h_0.\end{cases}$$

Let $(u,v;g,h)$ be the unique solution of (1.4), then

$$g(t)\leq\underline{g}(t),h(t)\geq\underline{h}(t) \text{ for } t\in(0,T],$$
$$u(x,t)\geq\underline{u}(x,t),v(x,t)\geq\underline{v}(x,t) \text{ for } x\in[\underline{g}(t),\underline{h}(t)],t\in(0,T].$$

3. An eigenvalue problem

In this section, we discuss an eigenvalue problem and present its properties of its principal eigenvalue. Now we introduce the following eigenvalue problem

$$\begin{cases} -\lambda\phi = d\phi_{xx} - \beta\phi_x + (f_{1u} - \frac{f_{1v}f_{2u}}{f_{2v}})\phi, \ -l < x < l, \\ \phi(l) = \phi(-l) = 0. \end{cases} \tag{3.1}$$

Lemma 3.1. *Denote by $\lambda_1(l)$ the principle eigenvalue of problem (3.1) with fixed l, then*

$$\lambda_1(l) = \frac{d\pi^2}{4l^2} + \frac{\beta^2}{4d} - f_{1u} + \frac{f_{1v}f_{2u}}{f_{2v}}.$$

Proof. Let μ_1, μ_2 be the roots of

$$d\mu^2 - \beta\mu + \lambda + f_{1u} - \frac{f_{1v}f_{2u}}{f_{2v}} = 0. \tag{3.2}$$

Then $\phi(x) = Ae^{\mu_1 x} + Be^{\mu_2 x}$ is the solution of (3.1). By the boundary conditions $\phi(\pm l) = 0$, $\phi \equiv 0$ if $\beta^2 - 4d(\lambda + f_{1u} - \frac{f_{1v}f_{2u}}{f_{2v}}) \geq 0$. Then we have

$$\beta^2 - 4d(\lambda + f_{1u} - \frac{f_{1v}f_{2u}}{f_{2v}}) < 0.$$

The roots of (3.2) is

$$\mu_{1,2} = \frac{\beta \pm i\sqrt{4d(\lambda + f_{1u} - \frac{f_{1v}f_{2u}}{f_{2v}}) - \beta^2}}{2d}.$$

So

$$\begin{aligned} \phi(x) = \ & Ae^{\frac{\beta}{2d}x}[\cos\frac{\sqrt{4d(\lambda + f_{1u} - \frac{f_{1v}f_{2u}}{f_{2v}}) - \beta^2}}{2d}x \\ & + i\sin\frac{\sqrt{4d(\lambda + f_{1u} - \frac{f_{1v}f_{2u}}{f_{2v}}) - \beta^2}}{2d}x] \\ & + Be^{\frac{\beta}{2d}x}[\cos\frac{\sqrt{4d(\lambda + f_{1u} - \frac{f_{1v}f_{2u}}{f_{2v}}) - \beta^2}}{2d}x \\ & - i\sin\frac{\sqrt{4d(\lambda + f_{1u} - \frac{f_{1v}f_{2u}}{f_{2v}}) - \beta^2}}{2d}x]. \end{aligned}$$

Since $\phi(\pm l) = 0$, we obtain that

$$\frac{\sqrt{4d(\lambda + f_{1u} - \frac{f_{1v}f_{2u}}{f_{2v}}) - \beta^2}}{2d}l = \frac{\pi}{2} + k\pi(k \in \mathbb{N}).$$

Hence

$$\lambda_1(l) = \frac{d\pi^2}{4l^2} + \frac{\beta^2}{4d} - f_{1u} + \frac{f_{1v}f_{2u}}{f_{2v}}.$$

$\square$

Theorem 3.1. *The following statements of $\lambda_1(l)$ are valid:*

(i) $\lambda_1(l)$ *is continuous and strictly decreasing in* l,

$$\lim_{l\to 0}\lambda_1(l)=\infty, \lim_{l\to\infty}\lambda_1(l)=\frac{\beta^2}{4d}-f_{1u}+\frac{f_{1v}f_{2u}}{f_{2v}};$$

(ii) if $\Delta>0$ *and* $0<\beta<2\sqrt{d(f_{1u}-\frac{f_{1v}f_{2u}}{f_{2v}})}$, *there exists a threshold value* $l^*=\frac{2d\pi}{4d\Delta-\beta^2}$ *such that* $\lambda_1(l)<0$ *for* $l>l^*$, $\lambda_1(l)=0$ *for* $l=l^*$, *and* $\lambda_1(l)>0$ *for* $0<l<l^*$;

(iii) if $\Delta\le 0$, *then* $\lambda_1(l)=\frac{d\pi^2}{4l^2}+\frac{\beta^2}{4d}>0$.

4. Vanishing and spreading

In this section, we first give some sufficient conditions of vanishing and spreading, then obtain criteria for spreading and vanishing.

Lemma 4.1. *Let* $(u,v;g,h)$ *be the solution of problem (1.4). If* $h_\infty-g_\infty<\infty$, *then there exists a constant* K *such that*

$$\|u(\cdot,t)\|_{C^1([g(t),h(t)])}\le K, \forall t>1.$$

Moreover,

$$\lim_{t\to\infty}h'(t)=\lim_{t\to\infty}g'(t)=0.$$

Proof. The proof is similar to that of Proposition 3.1 in [21]. We omit the details. □

Lemma 4.2. *Let* $(u,v;g,h)$ *be the solution of problem (1.4). If* $h_\infty-g_\infty<\infty$, *then*

$$\lim_{t\to\infty}\|u(\cdot,t),v(\cdot,t)\|_{C([g(t),h(t)])}=0. \tag{4.1}$$

Proof. Since $f_1(u,v)\le f_{1u}u+f_{1v}v$, we have that

$$\begin{cases} u_t-du_{xx}+\beta u_x\ge f_{1u}u, \ g(t)<x<h(t), t>0,\\ u(g(t),t)=0, g'(t)\le-\mu u_x(g(t),t), \ t>0,\\ u(h(t),t)=0, h'(t)\ge-\mu u_x(h(t),t), \ t>0.\end{cases}$$

In view of Lemma 4.1 and Lemma 3.2 in [44], we obtain

$$\lim_{t\to\infty}\|u(\cdot,t)\|_{C([g(t),h(t)])}=0.$$

On the other hand, $v(x,t)$ satisfies

$$v_t=f_2(u,v)\le f_{2u}u+f_{2v}v, g(t)<x<h(t), t>0.$$

Hence

$$\lim_{t\to\infty}\|v(\cdot,t)\|_{C([g(t),h(t)])}=0.$$

The proof is complete. □

Lemma 4.3. *If $\Delta \le 0$, then $h_\infty - g_\infty < \infty$ and (4.1) holds.*

Proof. Direct computation gives

$$\begin{aligned}
&\frac{d}{dt}\int_{g(t)}^{h(t)}(u-\frac{f_{1v}}{f_{2v}}v)dx\\
&=\int_{g(t)}^{h(t)}[du_{xx}-\beta u_x+f_1(u,v)-\frac{f_{1v}}{f_{2v}}f_2(u,v)]dx\\
&=-\frac{d}{\mu}(h'(t)-g'(t))+\int_{g(t)}^{h(t)}[f_1(u,v)-\frac{f_{1v}}{f_{2v}}f_2(u,v)]dx.
\end{aligned}$$

Integrating from 0 to t,

$$\begin{aligned}
&\int_{g(t)}^{h(t)}(u-\frac{f_{1v}}{f_{2v}}v)dx\\
&=\int_{-h_0}^{h_0}(u_0-\frac{f_{1v}}{f_{2v}}v_0)dx+\frac{d}{\mu}[2h_0-h(t)+g(t)]\\
&\quad+\int_0^t\int_{g(t)}^{h(t)}[f_1(u,v)-\frac{f_{1v}}{f_{2v}}f_2(u,v)]dx\\
&\le\int_{-h_0}^{h_0}(u_0-\frac{f_{1v}}{f_{2v}}v_0)dx+\frac{d}{\mu}[2h_0-h(t)+g(t)]\\
&\quad+\int_0^t\int_{g(t)}^{h(t)}[f_{1u}u+f_{1v}v-\frac{f_{1v}}{f_{2v}}(f_{2u}u+f_{2v}v)]dx\\
&=\int_{-h_0}^{h_0}(u_0-\frac{f_{1v}}{f_{2v}}v_0)dx+\frac{d}{\mu}[2h_0-h(t)+g(t)]+\int_0^t\int_{g(t)}^{h(t)}[f_{1u}-\frac{f_{1v}f_{2u}}{f_{2v}}]udx.
\end{aligned}$$

In view of $\Delta \le 0$, we have

$$h(t)-g(t)\le\frac{\mu}{d}\int_{-h_0}^{h_0}(u_0-\frac{f_{1v}}{f_{2v}}v_0)dx+2h_0<\infty.$$

Thus $h_\infty - g_\infty < \infty$. Combining with Lemma 4.2, we obtain that

$$\lim_{t\to\infty}\|u(\cdot,t),v(\cdot,t)\|_{C([g(t),h(t)])}=0.$$

The proof is finished. □

Lemma 4.4. *Assume that $\Delta > 0$. If $\lambda_1(h_0) > 0$, then vanishing happens if u_0 and v_0 are sufficiently small.*

Proof. Let ϕ be the corresponding eigenfunction of $\lambda_1(h_0)$. Since $\lambda_1(h_0) > 0$, we can choose δ small enough such that

$$-\delta-\frac{\beta h_0\delta^2}{2d(2+\delta)}+\frac{3\lambda_1}{4(1+\delta)^2}>0.$$

Define

$$
\begin{aligned}
&\sigma(t) := h_0(1+\delta-\frac{\delta}{2}e^{-\delta t}), t\geq 0,\\
&w(x,t) := \epsilon e^{-\delta t}e^{\frac{\beta}{2d}(1-\frac{h_0}{\sigma(t)})x}\phi(\frac{xh_0}{\sigma(t)}), x\in[\sigma(t),\sigma(t)], t>0,\\
&z(x,t) := \frac{1}{f_{1v}(1+\delta)^2}[-\delta(2+\delta)f_{1u}-\frac{f_{1v}f_{2u}}{f_{2v}}+\frac{\lambda_1}{4}]w, x\in[-\sigma(t),\sigma(t)], t>0.
\end{aligned}
$$

Direct computations yield that

$$
\begin{aligned}
&w_t - dw_{xx} + \beta w_x - f_1(w,z)\\
\geq & w[-\delta-\frac{\beta h_0 x}{2d}\frac{\sigma'}{\sigma^2}-h_0x\frac{\sigma'}{\sigma^2}\frac{\phi'}{\phi}+(\frac{h_0}{\sigma})^2(\frac{-d\phi''}{\phi}+\frac{\beta\phi'}{\phi})-\frac{\beta^2}{4d}(1-\frac{h_0^2}{\sigma^2})]\\
&-f_{1u}w-f_{1v}z\\
\geq & w[-\delta-\frac{\beta h_0}{2d}\frac{\sigma'}{\sigma}+(\frac{h_0}{\sigma})^2\lambda_1-\frac{\beta^2}{4d}(1-\frac{h_0^2}{\sigma^2})]-f_{1u}(1-\frac{h_0^2}{\sigma^2})w-\frac{h_0^2}{\sigma^2}\frac{f_{1v}f_{2u}}{f_{2v}}w-f_{1v}z\\
\geq & w[-\delta-\frac{\beta h_0\delta^2}{2d(2+\delta)}+\frac{3\lambda_1}{4(1+\delta)^2}]\\
> & 0
\end{aligned}
$$

and

$$
\begin{aligned}
&z_t - f_2(w,z)\\
\geq & z(-\delta-\frac{\beta h_0 x}{2d}\frac{\sigma'}{\sigma^2}-h_0x\frac{\sigma'}{\sigma^2}\frac{\phi'}{\phi})-f_{2u}w-f_{2v}z\\
\geq & w[\frac{\delta}{(1+\delta)^2}(1+\frac{\beta h_0\delta}{2d(2+\delta)})(\frac{\delta(2+\delta)f_{1u}}{f_{1v}}+\frac{f_{2u}}{f_{2v}})-f_{2u}+\frac{f_{2u}}{(1+\delta)^2}-\frac{\lambda_1}{4(1+\delta)^2}\frac{f_{2v}}{f_{1v}}]\\
> & 0
\end{aligned}
$$

for $x\in(\sigma(t),\sigma(t))$ and $t>0$. Moreover,

$$
\sigma'(t)=\frac{\delta^2}{2}h_0e^{-\delta t}, w_x(t,\pm\sigma(t))=\epsilon e^{-\delta t}\phi'(\pm h_0)\frac{h_0}{\sigma}e^{\pm\frac{\beta}{2d}(\sigma-h_0)}.
$$

Set

$$
\epsilon = h_0(2+\delta)\frac{\delta^2}{4\mu}\min\{-\frac{e^{-\frac{\beta}{2d}\delta h_0}}{\phi'(h_0)},\frac{e^{\frac{\beta}{4d}\delta h_0}}{\phi'(-h_0)}\}.
$$

If u_0 and v_0 are sufficiently small such that

$$
u_0(x)\leq \epsilon\phi(\frac{x}{1+\delta/2})e^{\frac{\beta\delta x}{2d(2+\delta)}}, x\in[-h_0(1+\delta/2),h_0(1+\delta/2)]
$$

and

$$
v_0(x)\leq \frac{1}{f_{1v}(1+\delta)^2}[-\delta(2+\delta)f_{1u}-\frac{f_{1v}f_{2u}}{f_{2v}}+\frac{\lambda_1}{4}]\epsilon\phi(\frac{x}{1+\delta/2})e^{\frac{\beta\delta x}{2d(2+\delta)}}
$$

for $x \in [-h_0(1+\delta/2), h_0(1+\delta/2)]$. Then we have

$$\begin{cases} w_t \geq dw_{xx} - \beta w_x + f_1(w,z), & -\sigma(t) < x < \sigma(t), t > 0, \\ z_t \geq f_2(w,z), & -\sigma(t) < x < \sigma(t), t > 0, \\ w(-\sigma(t),t) = w(\sigma(t),t) = 0 & t \geq 0, \\ z(-\sigma(t),t) = z(\sigma(t),t) = 0 & t \geq 0, \\ -\sigma(0) \leq -h_0, -\sigma'(t) \leq -\mu w_x(-\sigma(t),t), & t > 0, \\ \sigma(0) \geq h_0, \sigma'(t) \geq -\mu w_x(\sigma(t),t), & t > 0, \\ w(x,0) \geq u_0(x), z(x,0) \geq v_0(x), & -h_0 < x < h_0. \end{cases}$$

Applying Lemma 2.3, $g(t) \geq -\sigma(t)$ and $h(t) \leq \sigma(t)$. Obviously, $h_\infty - g_\infty \leq 2h_0(1+\delta) < \infty$. Lemma 4.2 implies that $\lim_{t\to\infty} \|u(\cdot,t), v(\cdot,t)\|_{C([g(t),h(t)])} = 0$. The proof is complete. □

Lemma 4.5. *Assume that $\Delta > 0$. If $\lambda_1(h_0) < 0$, then spreading happens.*

Proof. Let ϕ be the corresponding eigenfunction of $\lambda_1(h_0)$. Define

$$\underline{u}(x,t) := \varepsilon\phi(x), \underline{v}(x,t) := -\frac{f_{2u}}{f_{2v}}\underline{u}(x,t), x \in [-h_0,h_0], t \geq 0.$$

By direct computations, we have

$$\begin{aligned} &\underline{u}_t - d\underline{u}_{xx} + \beta\underline{u}_x - f_1(\underline{u},\underline{v}) \\ =&\varepsilon(-d\phi'' + \beta\phi') - f_1(\underline{u},\underline{v}) \\ =&\varepsilon(\lambda_1\phi + f_{1u}\phi - \frac{f_{1v}f_{2u}}{f_{2v}}\phi) - \frac{\partial}{\partial u}f_1(\xi(x,t),\eta(x,t))\underline{u} - \frac{\partial}{\partial v}f_1(\xi(x,t),\eta(x,t))\underline{v} \\ =&\varepsilon\phi(\lambda_1 + f_{1u} - \frac{\partial}{\partial u}f_1(\xi(x,t),\eta(x,t)) + \frac{f_{2u}}{f_{2v}}\frac{\partial}{\partial v}f_1(\xi(x,t),\eta(x,t)) - \frac{f_{1v}f_{2u}}{f_{2v}}) \end{aligned}$$

and

$$\underline{v}_t - f_2(\underline{u},\underline{v}) = \varepsilon\phi(-\frac{\partial}{\partial u}f_2(\xi(x,t),\eta(x,t)) - \frac{\partial}{\partial v}f_2(\xi(x,t),\eta(x,t))\frac{f_{2u}}{f_{2v}})$$

for $x \in (-h_0,h_0)$ and $t > 0$, where $\xi \in (0,\underline{u})$ and $\eta \in (0,\underline{v})$. Since $\lambda_1 < 0$, we can choose ε small enough such that $u_0(x) \geq \varepsilon\phi, v_0(x) \geq -\varepsilon\phi f_{2u}/f_{2v}$ and

$$\begin{cases} \underline{u}_t \leq d\underline{u}_{xx} - \beta\underline{u}_x + f_1(\underline{u},\underline{v}), & -h_0 < x < h_0, t > 0, \\ \underline{v}_t \leq f_2(\underline{u},\underline{v}), & -h_0 < x < h_0, t > 0, \\ \underline{u}(-h_0,t) = \underline{u}(h_0,t) = 0, & t \geq 0, \\ \underline{v}(-h_0,t) = \underline{v}(h_0,t) = 0, & t \geq 0, \\ 0 \geq -\mu\underline{u}_x(-h_0,t), 0 \leq -\mu\underline{u}_x(h_0,t), & t > 0, \\ \underline{u}(x,0) \leq u_0(x), \underline{v}(x,0) \leq v_0(x), & -h_0 < x < h_0. \end{cases}$$

By applying comparison principle, $u(x,t) \geq \underline{u}(x,t)$ and $v(x,t) \geq \underline{v}(x,t)$ for $x \in [-h_0, h_0]$ and $t \geq 0$. Then

$$\lim_{t\to\infty} \|u(\cdot,t), v(\cdot,t)\|_{C([g(t),h(t)])} > 0.$$

It follows from Lemma 4.2 that $h_\infty - g_\infty = \infty$. This completes the proof. $\square$

Lemma 4.6. *Assume that* $\Delta > 0$ *and* $0 < \beta < 2\sqrt{d(f_{1u} - \frac{f_{1v}f_{2u}}{f_{2v}})}$. *If* $h_0 < l^*$, *then spreading happens if* u_0 *and* v_0 *are sufficiently large.*

Proof. By Theorem 3.1, there exists $\sqrt{T^*} > l^*$ such that $\lambda_0(\sqrt{T^*}) < 0$. Consider

$$\begin{cases} \lambda_0\phi = -d\phi'' - (\frac{1}{2} + \beta\sqrt{T^*+1})\phi', \ 0 < x < 1, \\ \phi'(0) = \phi(1) = 0. \end{cases}$$

It is well known that the first eigenvalue $\lambda_0 > 0$ and the corresponding eigenfunction $\phi > 0, \phi' < 0$ in $(0,1]$, and $\|\phi\|_{L^\infty([0,1))} = 1$. We extend ϕ to $[-1,1]$ as an even function, then we have

$$\begin{cases} \lambda_0\phi = -d\phi'' - (\frac{1}{2} + \beta\sqrt{T^*+1})sgn(x)\phi', \ -1 < x < 1, \\ \phi(-1) = \phi(1) = 0. \end{cases}$$

Now we construct a suitbale lower solution of (1.4). Define

$$\underline{h}(t) = \sqrt{t+\sigma}, 0 \leq t \leq T^*,$$

$$\underline{u}(x,t) = \begin{cases} \dfrac{\rho}{(t+\sigma)^k}\phi(\dfrac{x}{\sqrt{t+\sigma}}), -\underline{h}(t) \leq x \leq \underline{h}(t), 0 \leq t \leq T^*, \\ 0, \ |x| > \underline{h}(t), 0 \leq t \leq T^*, \end{cases}$$

and

$$\underline{v}(x,t) = \begin{cases} \dfrac{\rho}{(t+\sigma)^k}\phi(\dfrac{x}{\sqrt{t+\sigma}}), -\underline{h}(t) \leq x \leq \underline{h}(t), 0 \leq t \leq T^*, \\ 0, \ |x| > \underline{h}(t), 0 \leq t \leq T^*, \end{cases}$$

where σ, k and ρ are positive constants that are choosen later. Due to Lemma 2.2, $0 < u(x,t), v(x,t) < C_1$. Then there exists $L > 0$ such that $f_1(u,v) \geq -Lu$ and $f_2(u,v) \geq -Lv$. Now we choose

$$0 < \sigma < \min\{1, h_0^2\}, k > \lambda_0 + L(T^*+1), \rho \geq \frac{(T^*+1)^k}{2\mu\min\{\phi'(-1), -\phi'(1)\}}.$$

Direct computations yield

$$\begin{aligned} &\underline{u}_t - d\underline{u}_{xx} + \beta\underline{u}_x - f_1(\underline{u},\underline{v}) \\ \leq& -\frac{\rho}{(t+\sigma)^{k+1}}[k\phi + \frac{x}{2\sqrt{t+\sigma}}\phi' + d\phi'' - \beta\sqrt{t+\sigma}\phi' - L(t+\sigma)\phi] \\ \leq& -\frac{\rho}{(t+\sigma)^{k+1}}[k\phi + (\frac{1}{2} + \beta\sqrt{T^*+1})sgn(x)\phi' + d\phi'' - L(t+\sigma)\phi] \\ \leq& -\frac{\rho}{(t+\sigma)^{k+1}}[(\frac{1}{2} + \beta\sqrt{T^*+1})sgn(x)\phi' + d\phi'' + \lambda_0\phi] \\ =&0 \end{aligned}$$

and

$$\underline{v} - f_2(\underline{u},\underline{v}) \le -\frac{\rho}{(t+\sigma)^{k+1}}[k\phi + \frac{x}{2\sqrt{t+\sigma}}\phi' - L(t+\sigma)\phi] \le 0$$

for $x \in [-\underline{h}(t), \underline{h}(t)]$ and $t \in (0, T^*]$. On the other hand,

$$\underline{h}'(t) + \mu\underline{u}_x(\underline{h}(t),t) = \frac{1}{2\sqrt{t+\sigma}} + \frac{\mu\rho}{(t+\sigma)^{k+\frac{1}{2}}}\phi'(1) \le 0, t \in (0,T^*),$$
$$\underline{h}'(t) - \mu\underline{u}_x(\underline{h}(t),t) = \frac{1}{2\sqrt{t+\sigma}} - \frac{\mu\rho}{(t+\sigma)^{k+\frac{1}{2}}}\phi'(-1) \le 0, t \in (0,T^*).$$

If u_0 and v_0 are sufficiently large such that

$$\underline{u}(x,0) = \frac{\rho}{\sigma^k}\phi(\frac{x}{\sqrt{\sigma}}) \le u_0(x), \underline{v}(x,0) = \frac{\rho}{\sigma^k}\phi(\frac{x}{\sqrt{\sigma}}) \le v_0(x),$$

for $x \in [-\sqrt{\sigma}, \sqrt{\sigma}]$. Since $\underline{h}(0) = \sqrt{\sigma} \le h_0$, then $(\underline{u}, \underline{v}; -\underline{h}, \underline{h})$ is a lower solution of (1.4). By Lemma 2.4, we conclude that $h(t) \ge \underline{h}(t)$ and $g(t) \le -\underline{h}(t)$ in $[0, T^*]$. So $h(T^*) \ge \underline{h}(T^*) = \sqrt{T^*+\sigma} \ge \sqrt{T^*}$ and $g(T^*) \le -\sqrt{T^*}$. Then $(-l^*, l^*) \subseteq (-\sqrt{T^*}, \sqrt{T^*}) \subseteq (g(t), h(t))$ for $t \ge T^*$. Due to Lemma 4.5, we have $h_\infty - g_\infty = \infty$, that is, spreading happens. □

Theorem 4.1. *Assume that $\Delta > 0$ and $0 < \beta < 2\sqrt{d(f_{1u} - \frac{f_{1v}f_{2u}}{f_{2v}})}$. If $(u_0, v_0) = \delta(\theta, \omega)$ for θ and $\omega \in \Sigma(h_0)$, then there exists $\delta^* \ge 0$ such that spreading happens if $\delta > \delta^*$, and vanishing happens if $0 < \delta \le \delta^*$. Moreover, $\delta^* = 0$ provided $h_0 \ge l^*$, and $\delta^* > 0$ provided $h_0 < l^*$.*

Proof. This theorem follows from Lemmas 4.4 and 4.6, the detailed proof is similar to Theorem 5.2 in [46]. □

5. Numerical illustrations

In this section, we present some numerical simulations of problem (1.4). As the boundary is unknown, it is difficult to present the numerical solution of free boundary compared with the fixed boundary problem. Here, we use a similar way in [25] to deal with the problem. Firstly, we carry out the discretization by finite differences. Secondly, we use the standard implicit scheme to deal with the equation (1.4), then we get a nonlinear algebraic system with the same number of equations and unknowns. Finally, we use the Newton-Raphson method to solve this nonlinear algebraic system.

Now we fix some coefficients and initial functions. Assume that

$$d = 1, \mu = 1, \beta = 0.3; u_0(x) = a\cos(\frac{\pi x}{2h_0}), v_0(x) = b\cos(\frac{\pi x}{2h_0}),$$
$$f_1(u,v) = -a_{11}u + a_{12}v, f_2(u,v) = -a_{22}v + a_{21}u(1-u).$$

Now we consider the long time behavior of solutions (u, v).

Example 5.1. Choose $a_{11} = 2, a_{12} = a_{21} = 1, a_{22} = 1.2$, we have $\Delta \le 0$. Lemma 4.3 shows that vanishing happens if $\Delta \le 0$. The numerical solutions of problem (1.4) with $a = 0.5, b = 0.8$ and $a = 5, b = 6$ are shown in Figures 1 and 2 respectively. We observe that the free boundaries $x = h(t)$ and $x = g(t)$ increase slow, and the solution decays to zero quickly.

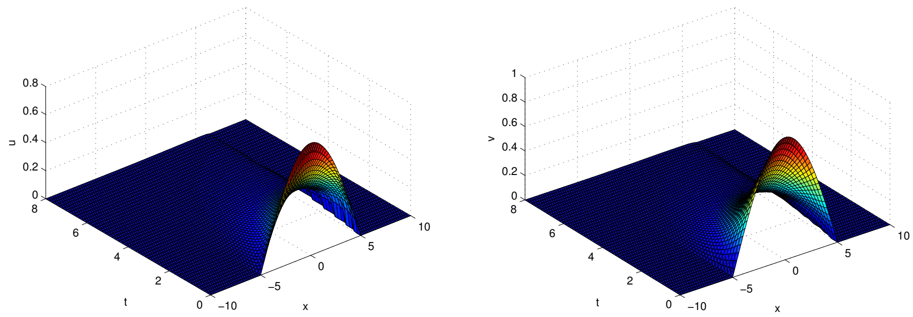

Figure 1. $a_{11} = 2, a_{12} = a_{21} = 1, a_{22} = 1.2, a = 0.5, b = 0.8, h_0 = 5.$

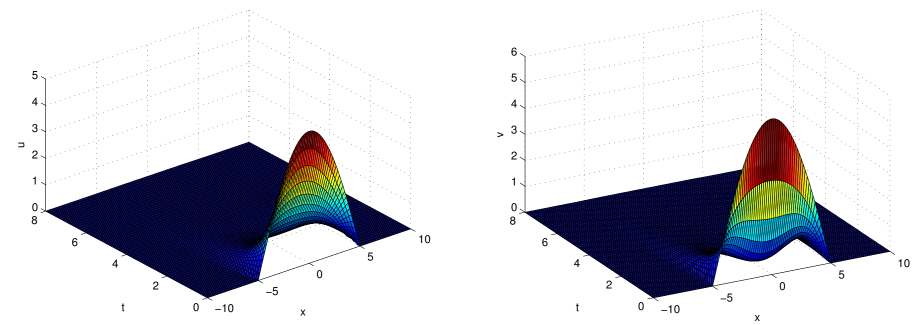

Figure 2. $a_{11} = 2, a_{12} = a_{21} = 1, a_{22} = 1.2, a = 5, b = 6, h_0 = 5.$

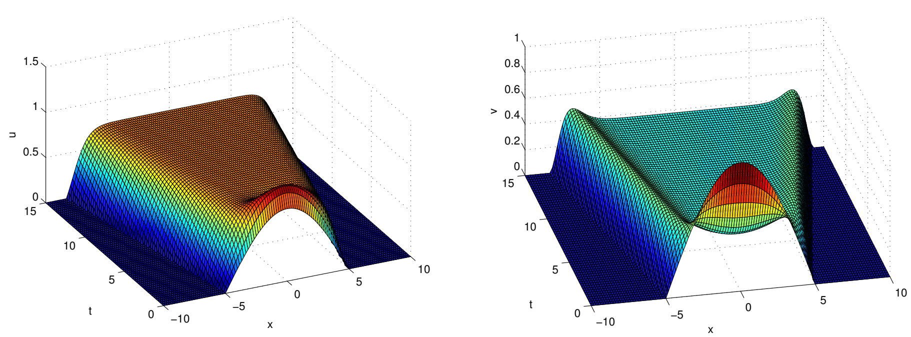

Figure 3. $a_{11} = 1, a_{12} = 2, a_{21} = 2.1, a_{22} = 1, a = 8, b = 1, h_0 = 5.$

Example 5.2. Fix $a_{11} = 1, a_{12} = 2, a_{21} = 2.1, a_{22} = 1, a = 8, b = 1, h_0 = 5$, then $\Delta > 0$ and $h_0 \geq l^*$. The simulation results are shown in Figure 3. It is easy to see that the free boundaries increase fast, and the solution stabilizes to a positive solution, which confirms to Theorem 4.1 (spreading always happens when $h_0 \geq l^*$).

Example 5.3. Let $a_{11} = 1, a_{12} = 2.5, a_{21} = 2, a_{22} = 2, h_0 = 0.8$, we have $\Delta > 0$ and $h_0 < l^*$. The numerical solutions of problem (1.4) with $a = 0.1, b = 0.2$ and $a = 1, b = 0.5$ are shown in Figures 4 and 5 respectively. It is easy to see

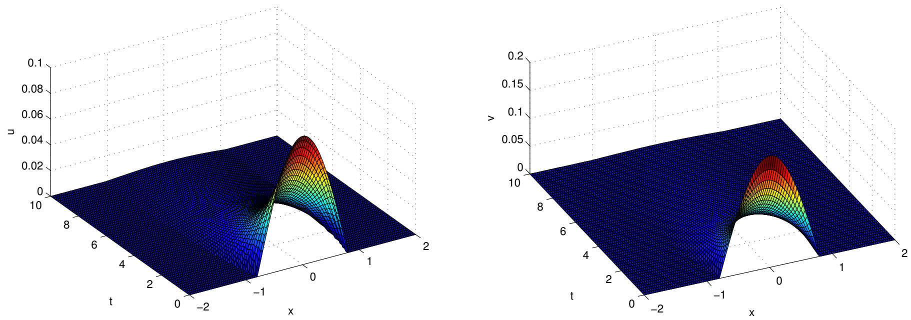

Figure 4. $a_{11} = 1, a_{12} = 2.5, a_{21} = 2, a_{22} = 2, a = 0.1, b = 0.2, h_0 = 0.8.$

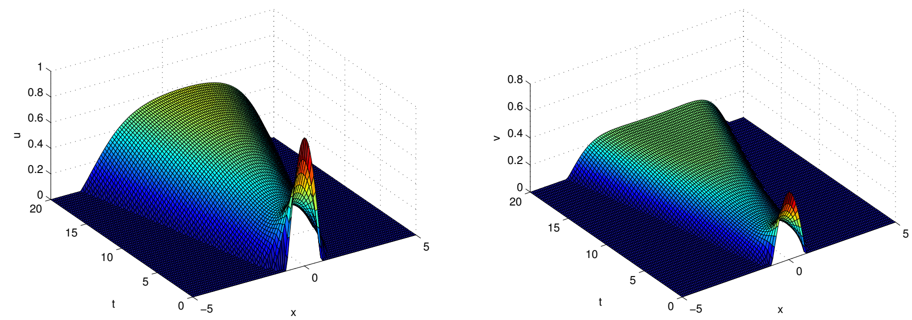

Figure 5. $a_{11} = 1, a_{12} = 2.5, a_{21} = 2, a_{22} = 2, a = 1, b = 0.5, h_0 = 0.8.$

that in Figure 4 the solution decays to zero quickly; in Figure 5 the free boundaries increase fast and the solutions stabilize to positive equilibria.They support Theorem 4.1 (If $\Delta > 0$ and $h_0 < l^*$, spreading happens if $\delta > \delta^* > 0$, vanishing happens if $0 < \delta \le \delta^*$).

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