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IMPULSIVE CONTROL FOR A PLANT-PEST-NATURAL ENEMY MODEL WITH STAGE STRUCTURE*

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Abstract For integrated pest management (IPM), we propose a generalized stage-structured plant-pest-natural enemy system with impulsive spraying pesticide and releasing natural enemies at different fixed moment. By the stroboscopic maps, we obtain two types of periodic solutions: the plant-pest-extinction and the pest-extinction. The sufficient conditions for the global attractivity of a pest-extinction periodic solution and permanence of the system are obtained by comparison theorem and stroboscopic technique. Moreover, numerical simulations are inserted to verify the effectiveness and feasibility of the theoretical results, which show that the impulsive control plays a key role on the permanence of the system.

Keywords Impulsive control, stage-structure, global attractivity, permanence.

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1. Introduction

According to the reports of Food and Agriculture Organization of the United Nations, the warfare between human and pests has lasted for thousands of years. With the development of science and technology, to control pests many methods are available to farmers such as chemical control, biological control, remote sensing, atomic energy and so on. Among these methods, biological control and chemical control are considered as the two most effective methods to beat agricultural pests. Biological control refers to reducing the pest population by introducing other living organisms, which are often called natural enemies of the pest, or beneficial species [8]. As a matter of fact, all pests have their natural enemies. The key to successful biological control is to identify the natural enemy of the pest, and release the enemies at early stage when pest's level is still low. Chemical control means to spray pesticides on farmland or forest, which quickly destroy a significant portion of the pest population at its vulnerable stage and provide feasible method to prevent the economic

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loss. However, the sprayed pesticide assuredly bring about many ecological and environmental polluted problems, and is also recognized as a major hazard to human health and beneficial enemies. All these means that one should utilize pesticide as little as possible, i.e, achieve maximum effect at minimum levels of pesticide. How to combine biological control and chemical control is a valuable issue, which is also the primary aim of integrated pest management [14, 24].

In comparison with the growth process of biological species, the releasing natural enemies and spraying pesticides always happen in a short time or instantaneously, which should be described by impulsive perturbations [10]. In recent decades, impulsive differential equations have been extensively used as models in biology, physics, chemistry, engineering and other applied sciences [22, 23], with particular emphasis on population dynamics, see [11, 16, 18, 26–28] and references therein. Yu et al. [26] investigated an ecological model with impulsive perturbations of three species, and obtained the condition which guarantees the globally asymptotical stability of the prey and predator eradication periodic solutions. Recently, Liu et al. [18] considered the predator-prey ecosystem with general functional response and impulsive control, firstly established the sufficient conditions for the local and the global stabilities of prey eradication periodic solution. Subsequently, it follows from their analysis that the system is permanent if the impulsive perturbations are satisfied with certain conditions. In the real world, many species usually go through two distinct life stages from birth to death, immature and mature. And only mature predators can attack prey and have reproductive ability. Stage-structured systems have received great attention, see [1, 6, 12, 13, 21, 25] and references therein. In [12], Jatav and Dhar proposed a Lotka-Volterra-type plant-pest-natural enemy food chain model with stage structure by the following set of differential system

$$\begin{cases} \dfrac{dX(\tau)}{d\tau} = R_0 X(\tau)\left(1 - \dfrac{X(\tau)}{K_0}\right) - A_1 X(\tau)Y(\tau), \\ \dfrac{dY(\tau)}{d\tau} = A_1 B_1 X(\tau)Y(\tau) - A_2 Y(\tau)Z_2(\tau) - D_1 Y(\tau), \\ \dfrac{dZ_1(\tau)}{d\tau} = A_2 B_2 Y(\tau)Z_2(\tau) - (D_2 + \mu)Z_1(\tau), \\ \dfrac{dZ_2(\tau)}{d\tau} = \mu Z_1(\tau) - D_2 Z_2(\tau), \end{cases} \tag{1.1}$$

where $X(\tau)$ is the plant population, $Z_1(\tau)$, $Z_2(\tau)$ are the densities of immature and mature natural enemy populations respectively and $Y(\tau)$ is the density of pest in that region of consideration at time τ. R_0 denotes the intrinsic growth rate of the plant and K_0 is the carrying capacity. A_1 and A_2 are per capita rates of predation; B_1 and B_2 are the product of per capita rate of predation and the conversion rates; D_1 and D_2 are the death rates of pest and natural enemy, respectively. The parameter μ is the maturity rate of natural enemy and all the parameters in the model are positive constants. They studied plant-pest-natural enemy food chain model with impulsive releasing natural enemy and spraying pesticides at fixed moment. In fact, the sprayed pesticide invariably do harm to beneficial enemy [4, 7, 18, 20, 26], which was not considered in [12]. To do this, we think that spraying pesticide and releasing enemies should be applied at different moment. For simplicity, choosing the following non-dimensional variables

$$x(t) = \frac{X(\tau)}{K_0},\ y(t) = \frac{Y(\tau)}{B_1 K_0},\ z_1(t) = \frac{Z_1(\tau)}{B_1 B_2 K_0},\ z_2(t) = \frac{Z_2(\tau)}{B_1 B_2 K_0},\ t = R_0 \tau,$$

then system (1.1) can be written as the following general system

$$
\begin{cases}
\left.\begin{aligned}
\frac{dx(t)}{dt} &= x(t)(1-x(t)) - c_1x(t)y(t)\\
\frac{dy(t)}{dt} &= c_1x(t)y(t) - d_1y(t) - c_2y(t)z_2(t)\\
\frac{dz_1(t)}{dt} &= c_2y(t)z_2(t) - (d_2+m)z_1(t)\\
\frac{dz_2(t)}{dt} &= mz_1(t) - d_2z_2(t)
\end{aligned}\right\} & t \neq nT,\ t \neq (n+l-1)T,\\
\left.\begin{aligned}
x(t^+) &= x(t)\\
y(t^+) &= (1-\delta_1)y(t)\\
z_1(t^+) &= (1-\delta_2)z_1(t)\\
z_2(t^+) &= (1-\delta_3)z_2(t)
\end{aligned}\right\} & t = (n+l-1)T,\\
\left.\begin{aligned}
x(t^+) &= x(t)\\
y(t^+) &= y(t)\\
z_1(t^+) &= z_1(t) + \mu_1\\
z_2(t^+) &= z_2(t) + \mu_2
\end{aligned}\right\} & t = nT,
\end{cases}
\tag{1.2}
$$

where $c_1 = \frac{A_1B_1K_0}{R_0}$, $c_2 = \frac{B_1A_2B_2K_0}{R_0}$, $d_1 = \frac{D_1}{R_0}$, $d_2 = \frac{D_2}{R_0}$, $m = \frac{\mu}{R_0}$ and $0 < l < 1$. The constant T is the impulsive period, μ_1 and μ_2 are respectively pulse releasing amount of immature and mature natural enemies, and δ_i $(0 \leq \delta_i < 1, i = 1,2,3)$ are the death rates of the pest, immature and mature natural enemy when chemical control is used at time $t = (n+l-1)T, n \in N_+$ respectively. If $\delta_1 = \delta$, $\delta_2 = \delta_3 = 0$, $l = 1$, system (1.2) is considered by Jatav and Dhar in [12]. In this paper, we analyze the local stabilities of periodic solutions (ie. plant-pest-extinction and pest-extinction) to system (1.2), and investigate the global attractivity of a pest-extinction periodic solution and the permanence of system (1.2) with impulsive perturbations.

The rest of this paper unfolds as follows: In section 2, we obtain two types of periodic solutions: the plant-pest-extinction and the pest-extinction. The local stability and global attractivity of the periodic solutions are studied in section 3. In section 4, we investigate the permanence of system (1.2). In section 5, we give the numerical simulations and discussions. The conclusions of this paper are given in the last section.

2. Preliminaries

Let $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}^4_+ = \{x \in \mathbb{R}^4, x \geq 0\}$. Denote $f = (f_1, f_2, f_3, f_4)^T$ the map defined by the right-hand side of system (1.2). The solution of system (1.2), denoted by $X(t) = (x(t), y(t), z_1(t), z_2(t))^T$, is a piecewise continuous function $X : \mathbb{R}_+ \to \mathbb{R}^4_+$. Again, $X(t)$ is continuous on $((n-1)T, (n+l-1)T]$ and $((n+l-1)T, nT]$, $X((n+l-1)T^+) = \lim\limits_{t\to(n+l-1)T^+} X(t)$ and $X(nT^+) = \lim\limits_{t\to nT^+} X(t)$ exist. The global existence and uniqueness of solution of system (1.2) are guaranteed by the smoothness properties of f (see [2, 15]). From the biological point, we only consider system (1.2) in the following region

$$\Omega = \left\{ (x(t), y(t), z_1(t), z_2(t))^T \,|x(t) \geq 0,\ y(t) \geq 0,\ z_1(t) \geq 0,\ z_2(t) \geq 0 \right\}.$$

Let V: $\mathbb{R}_+ \times \mathbb{R}_+^4 \to \mathbb{R}_+$, then V is said to belong to class V_0 if:

(i) V is continuous in $((n-1)T, (n+l-1)T] \times \mathbb{R}_+^4$ and $((n+l-1)T, nT] \times \mathbb{R}_+^4$. For each $X \in \mathbb{R}_+^4$, $\lim\limits_{(t,u)\to(KT^-,X)} V(t,u) = V(KT, X)$ and $\lim\limits_{(t,u)\to(KT^+,X)} V(t,u) = V(KT^+, X)$ exist, where $K = n, n+l-1$.

(ii) V is locally Lipschitzian in X.

Definition 2.1. Let $V \in V_0$, then for $(t, X) \in ((n-1)T, (n+l-1)T] \times \mathbb{R}_+^4$ or $((n+l-1)T, nT] \times \mathbb{R}_+^4$, the upper right derivative of $V(t, X)$ with respect to the impulsive differential system (1.2) is defined as

$$D^+V(t,X) = \lim_{h\to 0^+} \sup \frac{1}{h}[V(t+h, X + hf(t,X)) - V(t,X)].$$

The following Lemma will be used several times throughout the paper. Let us begin with Lemma 2.1 which is well known and the readers can refer [2, 12, 15, 25] for more details.

Lemma 2.1. *Suppose $V \in V_0$ and $X(0) = X_0$. Assume that*

$$\begin{cases} D^+V(t,X) \leq g(t, V(t,X)), & t \neq nT, t \neq (n+l-1)T, \\ V(t, X(t^+)) \leq \psi_n^1(V(t,X)), & t = (n+l-1)T, \\ V(t, X(t^+)) \leq \psi_n^2(V(t,X)), & t = nT, \end{cases} \tag{2.1}$$

where $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous in $((n-1)T, (n+l-1)T] \times \mathbb{R}_+$ and $((n+l-1)T, nT] \times \mathbb{R}_+$. For $u \in \mathbb{R}_+$, $\lim\limits_{(t,v)\to(nT^+,u)} g(t,v) = g(nT^+, u)$ and $\lim\limits_{(t,v)\to((n+l-1)T^+,u)} g(t,v) = g((n+l-1)T^+, u)$ exist, and $\psi_n^i(i = 1,\ 2) : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing. Let $r(t)$ be the maximal solution of the scalar impulsive differential equation

$$\begin{cases} \dfrac{du(t)}{dt} = g(t, u(t)), & t \neq nT,\ t \neq (n+l-1)T, \\ u(t^+) = \psi_n^1(u(t)), & t = (n+l-1)T, \\ u(t^+) = \psi_n^2(u(t)), & t = nT, \\ u\left(0^+\right) = u_0 \end{cases} \tag{2.2}$$

existing on $[0, \infty)$. Then $V(0^+, X_0) \leq u_0$, implies that $V(t, X(t)) \leq r(t), t \geq 0$, where $X(t)$ is any solution of system (1.2).

For the convenience of readers, we give the following lemma and its corresponding proof.

Lemma 2.2. *There exists a constant $M > 0$ such that $x(t) \leq M, y(t) \leq M, z_1(t) \leq M, z_2(t) \leq M$ for each solution of system (1.2) with t sufficiently large enough.*

Proof. Define $V(t,X(t))$ such that

$$V(t,X(t)) = x(t) + y(t) + z_1(t) + z_2(t).$$

Then $V \in V_0$, since $\frac{dx(t)}{dt} \leq x(t)(1-x(t))$, then $x(t) \leq 1$. We calculate the right derivative of $V(t,X)$ along a solution of system (1.2). Let $0 < L < \min\{d_1, d_2\}$, then for $t \neq nT$ and $t \neq (n+l-1)T$, we obtain that

$$\begin{aligned} D^+V(t) + LV(t) &= x(t)(1-x(t)) - d_1 y(t) - d_2 z_1(t) - d_2 z_2(t) \\ &\quad + Lx(t) + Ly(t) + Lz_1(t) + Lz_2(t) \\ &= x(t) + Lx(t) - x^2(t) + (L-d_1)y(t) + (L-d_2)(z_1(t)+z_2(t)) \\ &\leq x(t) + Lx(t) - x^2(t) \\ &\leq \frac{(1+L)^2}{4}. \end{aligned}$$

When $t = (n+l-1)T$,

$$\begin{aligned} V((n+l-1)T^+) &= x((n+l-1)T) + (1-\delta_1)y((n+l-1)T) \\ &\quad + (1-\delta_2)z_1((n+l-1)T) + (1-\delta_3)z_2((n+l-1)T) \\ &\leq x((n+l-1)T) + y((n+l-1)T) + z_1((n+l-1)T) \\ &\quad + z_2((n+l-1)T) \\ &= V((n+l-1)T), \end{aligned}$$

and when $t = nT$,

$$\begin{aligned} V(nT^+) &= x(nT^+) + y(nT^+) + z_1(nT^+) + z_2(nT^+) \\ &= x(nT) + y(nT) + z_1(nT) + z_2(nT) + \mu_1 + \mu_2 \\ &= V(nT) + \mu_1 + \mu_2. \end{aligned}$$

From Lemma 2.2 [15] (page 23), we have

$$\begin{aligned} V(t) &\leq V(0)e^{-Lt} + \frac{(1+L)^2}{4L}\left(1-e^{-Lt}\right) + \frac{(\mu_1+\mu_2)e^{-L(t-T)}}{1-e^{LT}} + \frac{(\mu_1+\mu_2)e^{LT}}{e^{LT}-1} \\ &\to \frac{(1+L)^2}{4L} + \frac{(\mu_1+\mu_2)e^{LT}}{e^{LT}-1} \text{ as } t \to \infty. \end{aligned}$$

Thus $V(t,X(t))$ is ultimately bounded. Hence, by the definition of $V(t)$, there exists a constant $M := \frac{(1+L)^2}{4L} + \frac{(\mu_1+\mu_2)e^{LT}}{e^{LT}-1}$ such that $x(t) \leq M$, $y(t) \leq M$, $z_1(t) \leq M$ and $z_2(t) \leq M$ for all t large enough. This completes the proof. □

Lemma 2.3. *Let*

$$G(t) = \begin{cases} ae^{-r(t-(n-1)T)}, & t \in ((n-1)T, (n+l-1)T], \\ a(1-p)e^{-r(t-(n-1)T)}, & t \in ((n+l-1)T,\ nT] \end{cases}$$

with $a \geq 0$, $r > 0$ and $0 \leq p < 1$. Consider the following impulsive system

$$\begin{cases} \dfrac{du(t)}{dt} = b(G(t)+c) - du(t), & t \neq nT,\ t \neq (n+l-1)T, \\ u(t^+) = (1-q)u(t), & t = (n+l-1)T, \\ u(t^+) = u(t) + \mu, & t = nT, \end{cases} \tag{2.3}$$

where $b \geq 0$, $c \in \mathbb{R}$, $r > d > 0$, $0 \leq q < 1$, $0 < l < 1$ and $\mu > 0$. Then system (2.3) has a periodic solution:

$$\widetilde{u}(t) = \begin{cases} \dfrac{(ab - ab(1-p)e^{-rT} + \mu(r-d))e^{-d(t-(n-1)T)} + ab(q-p)e^{-d(t-(n-+l-2)T-rlT}}{(r-d)(1-(1-q)e^{-dT})} \\ -\dfrac{abe^{-r(t-(n-1)T}}{r-d} - \dfrac{bcqe^{-d(t-(n+l-2)T)}}{d(1-(1-q)e^{-dT})} + \dfrac{bc}{d}, \quad t \in ((n-1)T, (n+l-1)T]F, \\ \dfrac{(ab-ab(1-p)e^{-rT}+\mu(r-d))(1-q)e^{-d(t-(n-1)T)}+ab(q-p)e^{-d(t-(n+l-1)T)-rlT}}{(r-d)(1-(1-q)e^{-dT})} \\ -\dfrac{ab(1-p)e^{-r(t-(n-1)T}}{r-d} - \dfrac{bcqe^{-d(t-(n+l-1)T)}}{d(1-(1-q)e^{-dT})} + \dfrac{bc}{d}, \quad t \in ((n+l-1)T, nT], \end{cases}$$

and for any solution $u(t)$ of system (2.3), we haveu$(t) \to \widetilde{u}(t)$ as $t \to \infty$.

Proof. Now, our central task is to prove Lemma 2.3. For any $t \in ((n-1)T, (n+l-1)T]$ or $t \in ((n+l-1)T, nT]$, integrating the first equation of system (2.3) over the interval $((n-1)T, t]$ or $((n+l-1)T, t]$, we arrive at

$$u(t) = \begin{cases} \dfrac{ab(e^{-d(t-(n-1)T)} - e^{-r(t-(n-1)T)})}{r-d} + \dfrac{bc(1-e^{-d(t-(n-1)T)})}{d} \\ +u((n-1)T^+)e^{-d(t-(n-1)T)}, \quad t \in ((n-1)T, (n+l-1)T], \\ \dfrac{ab(1-p)(e^{-d(t-(n+l-1)T)-rlT} - e^{-r(t-(n-1)T)})}{r-d} + \dfrac{bc(1-e^{-d(t-(n+l-1)T)})}{d} \\ +u((n+l-1)T^+)e^{-d(t-(n+l-1)T)}, \quad t \in ((n+l-1)T, nT]. \end{cases}$$

After the successive pulse, the stroboscopic maps of system (2.3) are given by

$$\begin{aligned} u((n+l-1)T^+) = {} & \frac{ab(1-q)(e^{-dlT} - e^{-rlT})}{r-d} + \frac{bc(1-q)(1-e^{-dlT})}{d} \\ & +u((n-1)T^+)(1-q)e^{-dlT} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} u(nT^+) = {} & \frac{ab(1-p)(e^{-d(1-l)T-rlT} - e^{-rT})}{r-d} + \frac{bc(1-e^{-d(1-l)T})}{d} \\ & +u((n+l-1)T^+)e^{-d(1-l)T} + \mu. \end{aligned} \tag{2.5}$$

Substituting (2.4) into (2.5), we obtain that

$$\begin{aligned} u(nT^+) = {} & \frac{ab((1-q)e^{-dT} - (1-p)e^{-rT} + (q-p)e^{-d(1-l)T-rlT})}{r-d} \\ & + \frac{bc(1-qe^{-d(1-l)T} - (1-q)e^{-dT})}{d} + u((n-1)T^+)(1-q)e^{-dT} + \mu \\ \triangleq {} & f(u(n-1)T^+). \end{aligned} \tag{2.6}$$

It is easy to check that (2.6) has a unique positive fixed point

$$u^*((n-1)T^+) = \frac{ab((1-q)e^{-dT} - (1-p)e^{-rT} + (q-p)e^{-d(1-l)T-rlT})}{(r-d)(1-(1-q)e^{-dT})}$$

$$+\frac{bc(1-qe^{-d(1-l)T}-(1-q)e^{-dT})}{d(1-(1-q)e^{-dT})}+\frac{\mu}{1-(1-q)e^{-dT}}, \quad (2.7)$$

which satisfies $u(t) < f(u(t)) < u^*((n-1)T^+)$ if $0 < u(t) < u^*((n-1)T^+)$, and $u^*((n-1)T^+) < f(u(t)) < u(t)$ if $u(t) > u^*((n-1)T^+)$. According to the concept of [5], we can know that $u^*(nT^+)$ is globally asymptotically stable. It follows from (2.4) and (2.7) that

$$\begin{aligned}
u^*((n+l-1)T^+) &= \frac{(1-q)e^{-dlT}(ab-ab(1-p)e^{-rT}+\mu(r-d))}{(r-d)(1-(1-q)e^{-dT})} \\
&= \frac{ab(1-q)(q-p)e^{-dT-rlT}}{(r-d)(1-(1-q)e^{-dT})}-\frac{ab(1-q)e^{-rlT}}{r-d} \\
&\quad +\frac{bc(1-q)}{d}-\frac{bcq(1-q)e^{-dT}}{d(1-(1-q)e^{-dT})}. \qquad (2.8)
\end{aligned}$$

Similarly, we conclude that $u^*((n+l-1)T^+)$ is globally asymptotically stable. Therefore, the corresponding positive periodic solution of system (2.3) in the interval $((n-1)T,(n+l-1)T]\cup((n+l-1)T,nT]$ is

$$\begin{aligned}
&\widetilde{u}(t) \\
&= \begin{cases}
\dfrac{(ab-ab(1-p)e^{-rT}+\mu(r-d))e^{-d(t-(n-1)T)}+ab(q-p)e^{-d(t-(n-+l-2)T-rlT}}{(r-d)(1-(1-q)e^{-dT})} \\
-\dfrac{abe^{-r(t-(n-1)T}}{r-d}-\dfrac{bcqe^{-d(t-(n+l-2)T)}}{d(1-(1-q)e^{-dT})}+\dfrac{bc}{d}, \quad t\in((n-1)T,(n+l-1)T], \\
\dfrac{(ab-ab(1-p)e^{-rT}+\mu(r-d))(1-q)e^{-d(t-(n-1)T)}+ab(q-p)e^{-d(t-(n+l-1)T)-rlT}}{(r-d)(1-(1-q)e^{-dT})} \\
-\dfrac{ab(1-p)e^{-r(t-(n-1)T}}{r-d}-\dfrac{bcqe^{-d(t-(n+l-1)T)}}{d(1-(1-q)e^{-dT})}+\dfrac{bc}{d}, \quad t\in((n+l-1)T,\ nT],
\end{cases}
\end{aligned}$$

which is globally asymptotically stable. That is, for any solution $u(t)$ of system (2.3), we arrive at $u(t)\to\widetilde{u}(t)$ as $t\to\infty$. This completes the proof. □

In what follows, let us give two periodic solutions of system (1.2). From system (1.2), we observe that

$$\begin{cases}
\dfrac{dz_1(t)}{dt}=-(d_2+m)z_1(t), \quad t\neq nT,\ t\neq(n+l-1)T, \\
z_1(t^+)=(1-\delta_2)z_1(t), \quad t=(n+l-1)T, \\
z_1(t^+)=z_1(t)+\mu_1, \quad t=nT.
\end{cases} \qquad (2.9)$$

Thanks to Lemma 2.3, we see that system (2.9) has a globally asymptotically stable periodic solution as follows

$$\widetilde{z}_1(t)=\begin{cases}
\dfrac{\mu_1e^{-(d_2+m)(t-(n-1)T)}}{1-(1-\delta_2)e^{-(d_2+m)T}}, \quad t\in((n-1)T,(n+l-1)T], \\
\dfrac{\mu_1(1-\delta_2)e^{-(d_2+m)(t-(n-1)T)}}{1-(1-\delta_2)e^{-(d_2+m)T}}, \quad t\in((n+l-1)T,nT]
\end{cases} \qquad (2.10)$$

with

$$\widetilde{z}_1(0^+) = \widetilde{z}_1(nT^+) = \frac{\mu_1}{1-(1-\delta_2)e^{-(d_2+m)T}},$$
$$\widetilde{z}_1((n+l-1)T^+) = \frac{\mu_1(1-\delta_2)e^{-(d_2+m)lT}}{1-(1-\delta_2)e^{-(d_2+m)T}}.$$

For a solution

$$z_1(t) = \begin{cases} (1-\delta_2)^{n-1}(z_1(0^+) - \dfrac{\mu_1}{1-(1-\delta_2)e^{-(d_2+m)T}})e^{-(d_2+m)t} + \widetilde{z}_1(t), \\ t \in ((n-1)T, (n+l-1)T], \\ (1-\delta_2)^{n}(z_1(0^+) - \dfrac{\mu_1}{1-(1-\delta_2)e^{-(d_2+m)T}})e^{-(d_2+m)t} + \widetilde{z}_1(t), \\ t \in ((n+l-1)T, nT] \end{cases}$$

of system (2.9) with any initial data $z_1(0^+) \geq 0$, we have $z_1(t) \to \widetilde{z}_1(t)$, $t \to \infty$.

It follows from system (1.2) that

$$\begin{cases} \dfrac{dz_2(t)}{dt} = mz_1(t) - d_2 z_2(t), \quad t \neq nT,\ t \neq (n+l-1)T, \\ z_2(t^+) = (1-\delta_3)z_2(t), \quad t = (n+l-1)T, \\ z_2(t^+) = z_2(t) + \mu_2, \quad t = nT. \end{cases} \tag{2.11}$$

Substituting (2.10) into system (2.11), we end up with

$$\begin{cases} \dfrac{dz_2(t)}{dt} = m\widetilde{z}_1(t) - d_2 z_2(t), \quad t \neq nT,\ t \neq (n+l-1)T, \\ z_2(t^+) = (1-\delta_3)z_2(t), \quad t = (n+l-1)T, \\ z_2(t^+) = z_2(t) + \mu_2, \quad t = nT. \end{cases} \tag{2.12}$$

For system (2.12), applying Lemma 2.3 yields

$$\widetilde{z}_2(t) = \begin{cases} \dfrac{(\mu_1+\mu_2)e^{-d_2(t-(n-1)T)}}{1-(1-\delta_3)e^{-d_2T}} + \dfrac{\mu_1(\delta_3-\delta_2)e^{-d_2(t-(n-2)T)-mlT}}{(1-(1-\delta_2)e^{-(d_2+m)T})(1-(1-\delta_3)e^{-d_2T})} \\ -\dfrac{\mu_1 e^{-(d_2+m)(t-(n-1)T)}}{1-(1-\delta_2)e^{-(d_2+m)T}}, \quad t \in ((n-1)T, (n+l-1)T], \\ \dfrac{(\mu_1+\mu_2)(1-\delta_3)e^{-d_2(t-(n-1)T)}}{1-(1-\delta_3)e^{-d_2T}} \\ +\dfrac{\mu_1(\delta_3-\delta_2)e^{-d_2(t-(n-1)T)-mlT}}{(1-(1-\delta_2)e^{-(d_2+m)T})(1-(1-\delta_3)e^{-d_2T})} \\ -\dfrac{\mu_1(1-\delta_2)e^{-(d_2+m)(t-(n-1)T)}}{1-(1-\delta_2)e^{-(d_2+m)T}}, \quad t \in ((n+l-1)T,\ nT], \end{cases}$$

where

$$\widetilde{z}_2(0^+) = \widetilde{z}_2(nT^+)$$

$$
\begin{aligned}
&= \frac{\mu_1+\mu_2}{1-(1-\delta_3)e^{-d_2T}} - \frac{\mu_1}{1-(1-\delta_2)e^{-(d_2+m)T}} \\
&\quad + \frac{\mu_1(\delta_3-\delta_2)e^{-(d_2+ml)T}}{(1-(1-\delta_2)e^{-(d_2+m)T})(1-(1-\delta_3)e^{-d_2T})}, \\
\widetilde{z}_2((n+l-1)T^+) &= \frac{(\mu_1+\mu_2)(1-\delta_3)e^{-d_2lT}}{1-(1-\delta_3)e^{-d_2T}} - \frac{\mu_1(1-\delta_2)e^{-(d_2+m)lT}}{1-(1-\delta_2)e^{-(d_2+m)T}} \\
&\quad + \frac{\mu_1(\delta_3-\delta_2)e^{-(d_2+m)lT}}{(1-(1-\delta_2)e^{-(d_2+m)T})(1-(1-\delta_3)e^{-d_2T})},
\end{aligned}
$$

which is globally asymptotically stable.

Moreover, owing to pest eradication, we observe that $x(t)$ satisfies that

$$
\frac{dx(t)}{dt} = x(t)(1-x(t)).
$$

Obviously, there exists an unstable equilibrium $x=0$ and a globally asymptotically stable equilibrium $x=1$. Putting all the above solutions together, we conclude that system (1.2) has two periodic solutions: plant-pest-extinction periodic solution $(0,0,\widetilde{z}_1(t),\widetilde{z}_2(t))$ and pest-extinction periodic solution $(1,0,\widetilde{z}_1(t),\widetilde{z}_2(t))$.

3. Stability analysis

In this section, we focus on the stability of the plant-pest eradication periodic solution and the pest eradication periodic solution of system (1.2).

Theorem 3.1. *Let $(x(t),y(t),z_1(t),z_2(t))$ be any solution of system (1.2). Then*
(i) The plant-pest eradication periodic solution $(0,0,\widetilde{z}_1(t),\widetilde{z}_2(t))$ is unstable.
(ii) The pest eradication periodic solution $(1,0,\widetilde{z}_1(t),\widetilde{z}_2(t))$ of system (1.2) is locally asymptotically stable if and only if $(c_1-d_1)T-c_2(A+B+C)\le \ln\frac{1}{1-\delta_1}$ and it is unstable if and only if $(c_1-d_1)T-c_2(A+B+C)> \ln\frac{1}{1-\delta_1}$, where

$$
\begin{aligned}
A &= \frac{\mu_1\delta_2e^{-(d_2+m)lT}-\mu_1(1-(1-\delta_2)e^{-(d_2+m)T})}{(d_2+m)(1-(1-\delta_2)e^{-(d_2+m)T})}, \\
B &= \frac{(\mu_1+\mu_2)(1-(1-\delta_3)e^{-d_2T}-\delta_3e^{-d_2lT})}{d_2(1-(1-\delta_3)e^{-d_2T})}, \\
C &= \frac{\mu_1(\delta_3-\delta_2)e^{-(d_2+m)lT}(1-e^{-d_2T})}{d_2(1-(1-\delta_2)e^{-(d_2+m)T})(1-(1-\delta_3)e^{-d_2T})}.
\end{aligned}
$$

Proof. (i) To investigate the local stability of the periodic solution $(0,0,\widetilde{z}_1(t),\widetilde{z}_2(t))$, we define

$$
x(t)=\phi_1(t),\ y(t)=\phi_2(t),\ z_1(t)=\widetilde{z}_1(t)+\phi_3(t),\ z_2(t)=\widetilde{z}_2(t)+\phi_4(t), \tag{3.1}
$$

where $\phi_i(t)$, $i=1,2,3,4$ are small component amplitude perturbation of the solution $(0,0,\widetilde{z}_1(t),\widetilde{z}_2(t))$, respectively. Substituting (3.1) into system (1.2), the linearization of system (1.2) becomes

$$\left\{\begin{array}{l}\left.\begin{array}{l}\dfrac{d\phi_1(t)}{dt}=\phi_1(t)\\ \dfrac{d\phi_2(t)}{dt}=-(d_1+c_2\widetilde{z}_2(t))\phi_2(t)\\ \dfrac{d\phi_3(t)}{dt}=c_2\widetilde{z}_2(t)\phi_2(t)-(d_2+m)\phi_3(t)\\ \dfrac{d\phi_4(t)}{dt}=m\phi_3(t)-d_2\phi_4(t)\end{array}\right\}\quad t\neq nT,\ t\neq(n+l-1)T,\\ \left.\begin{array}{l}\phi_1(t^+)=\phi_1(t)\\ \phi_2(t^+)=(1-\delta_1)\phi_2(t)\\ \phi_3(t^+)=(1-\delta_2)\phi_3(t)\\ \phi_4(t^+)=(1-\delta_3)\phi_4(t)\end{array}\right\}\quad t=(n+l-1)T,\\ \left.\begin{array}{l}\phi_1(t^+)=\phi_1(t)\\ \phi_2(t^+)=\phi_2(t)\\ \phi_3(t^+)=\phi_3(t)\\ \phi_4(t^+)=\phi_4(t)\end{array}\right\}\quad t=nT.\end{array}\right. \tag{3.2}$$

Let $\Phi(t)$ be the fundamental matrix of system (3.2), then $\Phi(t)$ must satisfy

$$\frac{d\Phi(t)}{dt}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -(d_1+c_2\widetilde{z}_2(t)) & 0 & 0\\ 0 & c_2\widetilde{z}_2(t) & -(d_2+m) & 0\\ 0 & 0 & m & -d_2\end{pmatrix}\Phi(t):=K\Phi(t). \tag{3.3}$$

System (3.2) from the fifth equation to the eighth equation becomes

$$\begin{pmatrix}\phi_1((n+l-1)T^+)\\ \phi_2((n+l-1)T^+)\\ \phi_3((n+l-1)T^+)\\ \phi_4((n+l-1)T^+)\end{pmatrix}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1-\delta_1 & 0 & 0\\ 0 & 0 & 1-\delta_2 & 0\\ 0 & 0 & 0 & 1-\delta_3\end{pmatrix}\begin{pmatrix}\phi_1((n+l-1)T)\\ \phi_2((n+l-1)T)\\ \phi_3((n+l-1)T)\\ \phi_4((n+l-1)T)\end{pmatrix}.$$

System (3.2) from the ninth equation to the twelfth equation becomes

$$\begin{pmatrix}\phi_1(nT^+)\\ \phi_2(nT^+)\\ \phi_3(nT^+)\\ \phi_4(nT^+)\end{pmatrix}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\phi_1(nT)\\ \phi_2(nT)\\ \phi_3(nT)\\ \phi_4(nT)\end{pmatrix}.$$

Then the monodromy matrix of system (3.2) is given by

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\delta_1 & 0 & 0 \\ 0 & 0 & 1-\delta_2 & 0 \\ 0 & 0 & 0 & 1-\delta_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Phi(T).$$

From (3.3), we get that $\Phi(T) = \Phi(0)e^{\int_0^T K dt}$, where $\Phi(0) = I$ is the identify matrix. Hence, the monodromy matrix M has the following eigenvalues:

$$\begin{aligned} &\lambda_1 = e^T > 1, \qquad \lambda_2 = (1-\delta_1)e^{-\int_0^T (d_1 + c_2 \widetilde{z}_2(t))dt} < 1, \\ &\lambda_3 = (1-\delta_2)e^{-(d_2+m)T} < 1, \quad \lambda_4 = (1-\delta_3)e^{-d_2 T} < 1. \end{aligned}$$

According to the Floquet theory for impulsive differential equations, the plant-pest eradication periodic solution $(0, 0, \widetilde{z}_1(t), \widetilde{z}_2(t))$ of system (1.2) is unstable since $|\lambda_1| > 1$.

(ii) The local stability of the periodic solution $(1, 0, \widetilde{z}_1(t), \widetilde{z}_2(t))$ is similar to the previous case. Setting

$$x(t) = 1 + \phi_1(t),\ y(t) = \phi_2(t),\ z_1(t) = \widetilde{z}_1(t) + \phi_3(t),\ z_2(t) = \widetilde{z}_2(t) + \phi_4(t), \quad (3.4)$$

where $\phi_i(t)$, $i = 1, 2, 3, 4$ are small component amplitude perturbation of the solution $(1, 0, \widetilde{z}_1(t), \widetilde{z}_2(t))$, respectively. Plugging (3.4) into system (1.2), then the linearization of system (1.2) becomes

$$\begin{cases} \left.\begin{aligned} \frac{d\phi_1(t)}{dt} &= -\phi_1(t) - c_1\phi_2(t) \\ \frac{d\phi_2(t)}{dt} &= (c_1 - d_1 - c_2\widetilde{z}_2(t))\phi_2(t) \\ \frac{d\phi_3(t)}{dt} &= c_2\widetilde{z}_2(t)\phi_2(t) - (d_2+m)\phi_3(t) \\ \frac{d\phi_4(t)}{dt} &= m\phi_3(t) - d_2\phi_4(t) \end{aligned}\right\} & t \neq nT, t \neq (n+l-1)T, \\ \left.\begin{aligned} \phi_1(t^+) &= \phi_1(t) \\ \phi_2(t^+) &= (1-\delta_1)\phi_2(t) \\ \phi_3(t^+) &= (1-\delta_2)\phi_3(t) \\ \phi_4(t^+) &= (1-\delta_3)\phi_4(t) \end{aligned}\right\} & t = (n+l-1)T, \\ \left.\begin{aligned} \phi_1(t^+) &= \phi_1(t) \\ \phi_2(t^+) &= \phi_2(t) \\ \phi_3(t^+) &= \phi_3(t) \\ \phi_4(t^+) &= \phi_4(t) \end{aligned}\right\} & t = nT. \end{cases} \quad (3.5)$$

The fundamental matrix $\Phi(t)$ must fulfill

$$\frac{d\Phi(t)}{dt} = \begin{pmatrix} -1 & -c_1 & 0 & 0 \\ 0 & c_1 - d_1 - c_2\widetilde{z}_2(t) & 0 & 0 \\ 0 & c_2\widetilde{z}_2(t) & -(d_2+m) & 0 \\ 0 & 0 & m & -d_2 \end{pmatrix} \Phi(t).$$

It follows from system (3.5) that

$$\begin{pmatrix} \phi_1((n+l-1)T^+) \\ \phi_2((n+l-1)T^+) \\ \phi_3((n+l-1)T^+) \\ \phi_4((n+l-1)T^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\delta_1 & 0 & 0 \\ 0 & 0 & 1-\delta_2 & 0 \\ 0 & 0 & 0 & 1-\delta_3 \end{pmatrix} \begin{pmatrix} \phi_1((n+l-1)T) \\ \phi_2((n+l-1)T) \\ \phi_3((n+l-1)T) \\ \phi_4((n+l-1)T) \end{pmatrix}.$$

Similarly,

$$\begin{pmatrix} \phi_1(nT^+) \\ \phi_2(nT^+) \\ \phi_3(nT^+) \\ \phi_4(nT^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_1(nT) \\ \phi_2(nT) \\ \phi_3(nT) \\ \phi_4(nT) \end{pmatrix}.$$

We observe that the monodromy matrix of system (3.5) is

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\delta_1 & 0 & 0 \\ 0 & 0 & 1-\delta_2 & 0 \\ 0 & 0 & 0 & 1-\delta_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Phi(T), \tag{3.6}$$

which has the following eigenvalues

$$\lambda_1 = e^{-T} < 1, \quad \lambda_2 = (1-\delta_1)e^{\int_0^T (c_1 - d_1 - c_2\widetilde{z}_2(t))dt},$$
$$\lambda_3 = (1-\delta_2)e^{-(d_2+m)T} < 1, \quad \lambda_4 = (1-\delta_3)e^{-d_2 T} < 1.$$

These eigenvalues imply that the pest eradication periodic solution $(1, 0, \widetilde{z}_1(t), \widetilde{z}_2(t))$ is locally asymptotically stable if and only if $|\lambda_2| \le 1$, that is to say, $(c_1 - d_1)T - c_2(A + B + C) \le \ln \frac{1}{1-\delta_1}$. The pest eradication periodic solution $(1, 0, \widetilde{z}_1(t), \widetilde{z}_2(t))$ is unstable if and only if $|\lambda_2| > 1$, i.e., $(c_1 - d_1)T - c_2(A + B + C) > \ln \frac{1}{1-\delta_1}$. This completes the proof. □

Next, we will prove the global attractivity of the pest eradication periodic solution of system (1.2) under the condition for Theorem 3.1.

Theorem 3.2. *The pest eradication periodic solution* $(1, 0, \widetilde{z}_1(t), \widetilde{z}_2(t))$ *of system (1.2) is globally attractive if* $(c_1 - d_1)T - c_2(A + B + C) < \ln \frac{1}{1-\delta_1}$*, where* A*,* B*,* C *are the same definitions as in Theorem 3.1.*

Proof. Choose an $\varepsilon_0 > 0$ (sufficiently small) such that

$$\xi_1 = (1-\delta_1)e^{\int_{(n+l-1)T}^{(n+l)T}(c_1(1+\varepsilon_0)-d_1-c_2(\widetilde{z}_2(t)-\frac{m\varepsilon_0}{d_2}-\varepsilon_0))dt} < 1$$

as $(c_1-d_1)T - c_2(A+B+C) < \ln\frac{1}{1-\delta_1}$. Let $(x(t), y(t), z_1(t), z_2(t))$ be any solution of system (1.2). Then we get from the first equation of system (1.2) that

$$\frac{dx(t)}{dt} \le x(t)(1-x(t)),$$

which implies that $\lim\limits_{t\to+\infty}\sup x(t) = 1$. So there exists an integer $k_1 > 0$ such that $x(t) < 1+\varepsilon_0$ for all $t > k_1$. It follows from the third, seventh and eleventh equations of system (1.2) that

$$\begin{cases} \dfrac{dz_1(t)}{dt} \ge -(d_2+m)z_1(t), & t \ne nT,\ t \ne (n+l-1)T, \\ z_1(t^+) = (1-\delta_2)z_1(t), & t = (n+l-1)T, \\ z_1(t^+) = z_1(t) + \mu_1, & t = nT. \end{cases}$$

Consider the following auxiliary system

$$\begin{cases} \dfrac{du_1(t)}{dt} = -(d_2+m)u_1(t), & t \ne nT,\ t \ne (n+l-1)T, \\ u_1(t^+) = (1-\delta_2)u_1(t), & t = (n+l-1)T, \\ u_1(t^+) = u_1(t) + \mu_1, & t = nT. \end{cases} \tag{3.7}$$

With the aid of Lemma 2.3, we obtain that system (3.7) has a periodic solution

$$\widetilde{u}_1(t) = \widetilde{z}_1(t) = \begin{cases} \dfrac{\mu_1 e^{-(d_2+m)(t-(n-1)T)}}{1-(1-\delta_2)e^{-(d_2+m)T}}, & t \in ((n-1)T,\ (n+l-1)T], \\ \dfrac{\mu_1(1-\delta_2)e^{-(d_2+m)(t-(n-1)T)}}{1-(1-\delta_2)e^{-(d_2+m)T}}, & t \in ((n+l-1),\ nT], \end{cases}$$

which is globally asymptotically stable. Taking advantage of Lemma 2.1, we see that $z_1(t) \ge u_1(t)$ and $u_1(t) \to \widetilde{u}_1(t)$ as $t \to \infty$. Then there exists an integer $k_2 > \frac{k_1}{T}, t > k_2T$ such that

$$z_1(t) \ge u_1(t) > \widetilde{z}_1(t) - \varepsilon_0, \quad t \in (nT, (n+1)T],\ n > k_2. \tag{3.8}$$

Incorporating (3.8) with the fourth equation of system (1.2), we have

$$\begin{cases} \dfrac{dz_2(t)}{dt} \ge m\left(\widetilde{z}_1(t) - \varepsilon_0\right) - d_2z_2(t), & t \ne nT,\ t \ne (n+l-1)T, \\ z_2(t^+) = (1-\delta_3)z_2(t), & t = (n+l-1)T, \\ z_2(t^+) = z_2(t) + \mu_2, & t = nT. \end{cases} \tag{3.9}$$

The comparison system of system (3.9) is

$$\begin{cases} \dfrac{du_2(t)}{dt} = m\left(\widetilde{z}_1(t) - \varepsilon_0\right) - d_2u_2(t), & t \ne nT,\ t \ne (n+l-1)T, \\ u_2(t^+) = (1-\delta_3)u_2(t), & t = (n+l-1)T, \\ u_2(t^+) = u_2(t) + \mu_2, & t = nT. \end{cases} \tag{3.10}$$

Lemma 2.3 ensures that system (3.10) has a periodic solution

$$\widetilde{u}_2(t) = \begin{cases} \widetilde{z}_2(t) + \dfrac{m\varepsilon_0\delta_3 e^{-d_2(t-(n+l-2)T)}}{d_2(1-(1-\delta_3)e^{-d_2T})} - \dfrac{m\varepsilon_0}{d_2}, & t \in ((n-1)T,\ (n+l-1)T], \\ \widetilde{z}_2(t) + \dfrac{m\varepsilon_0\delta_3 e^{-d_2(t-(n+l-1)T)}}{d_2(1-(1-\delta_3)e^{-d_2T})} - \dfrac{m\varepsilon_0}{d_2}, & t \in ((n+l-1)T,\ nT], \end{cases}$$

$$> \widetilde{z}_2(t) - \frac{m\varepsilon_0}{d_2}, \quad t \in ((n-1)T,\ nT],$$

which is globally asymptotically stable. Due to Lemma 2.1, we observe that $z_2(t) \geq u_2(t)$ and $u_2(t) \to \widetilde{u}_2(t)$ as $t \to \infty$. Thus, there exists an integer $k_3 > k_2$, $t > k_3T$ such that

$$z_2(t) \geq u_2(t) > \widetilde{z}_2(t) - \frac{m\varepsilon_0}{d_2} - \varepsilon_0, \quad t \in (nT,\ (n+1)T],\ n > k_3. \tag{3.11}$$

We get from system (1.2), (3.11) and $x(t) < 1 + \varepsilon_0$ that

$$\begin{cases} \dfrac{dy(t)}{dt} \leq y(t)(c_1(1+\varepsilon_0) - d_1 - c_2(\widetilde{z}_2(t) - \dfrac{m\varepsilon_0}{d_2} - \varepsilon_0)), & t \neq (n+l-1)T, \\ y(t^+) = (1-\delta_1)y(t), & t = (n+l-1)T. \end{cases} \tag{3.12}$$

For $t \in ((n+l-1)T, (n+l)T]$ $(n > k_3 + 1)$, integrating the first equation of system (3.12) on $((n+l-1)T, t]$ yields

$$y(t) \leq y((n+l-1)T^+) e^{\int_{(n+l-1)T}^{t} (c_1(1+\varepsilon_0) - d_1 - c_2(\widetilde{z}_2(s) - \frac{m\varepsilon_0}{d_2} - \varepsilon_0))ds}.$$

After the successive pulse, we note that, due to $\xi_1 < 1$,

$$\begin{aligned} & y((n+l)T^+) \\ \leq\ & y((n+l-1)T^+)(1-\delta_1) e^{\int_{(n+l-1)T}^{(n+l)T} (c_1(1+\varepsilon_0) - d_1 - c_2(\widetilde{z}_2(t) - \frac{m\varepsilon_0}{d_2} - \varepsilon_0))dt} \\ =\ & y((n+l-1)T^+)\xi_1 \\ \leq\ & y(lT^+)\xi_1^n, \end{aligned}$$

which leads to $y((n+l)T^+) \to 0$ as $n \to \infty$. Since $0 < y(t) < y((n+l-1)T^+)\frac{1}{1-\delta_1}$ for $t \in ((n+l-1)T, (n+l)T]$ $(n > k_3+1)$, we have $y(t) \to 0$ as $n \to \infty$. Therefore, there exists an $\varepsilon_1 > 0$ (sufficiently small) such that $0 < y(t) < \varepsilon_1$ for t large enough.

Again, we get from system (1.2) that $\frac{dx(t)}{dt} \geq x(t)(1 - c_1\varepsilon_1 - x(t))$, which implies that $\lim_{t\to\infty} \inf x(t) \to 1$, i.e, $x(t) \to 1$ as $t \to \infty$. For all $n > k_3 + 1$, it follows from system (1.2) that

$$\begin{cases} \dfrac{dz_1(t)}{dt} \leq c_2\varepsilon_1 M - (d_2+m)z_1(t), & t \neq nT,\ t \neq (n+l-1)T, \\ z_1(t^+) = (1-\delta_2)z_1(t), & t = (n+l-1)T, \\ z_1(t^+) = z_1(t) + \mu_1, & t = nT. \end{cases} \tag{3.13}$$

For system (3.13), we use Lemma 2.1 and Lemma 2.3, and get

$$z_1(t) \le \begin{cases} \widetilde{z}_1(t) - \dfrac{Mc_2\varepsilon_1\delta_2 e^{-(d_2+m)(t-(n+l-2)T)}}{(d_2+m)(1-(1-\delta_2)e^{-(d_2+m)T})} + \dfrac{Mc_2\varepsilon_1}{d_2+m}, \\ t \in ((n-1)T,\ (n+l-1)T], \\ \widetilde{z}_1(t) - \dfrac{Mc_2\varepsilon_1\delta_2 e^{-(d_2+m)(t-(n+l-1)T)}}{(d_2+m)(1-(1-\delta_2)e^{-(d_2+m)T})} + \dfrac{Mc_2\varepsilon_1}{d_2+m}, \quad t \in ((n+l-1)T,\ nT] \end{cases}$$

$$< \widetilde{z}_1(t) + \frac{Mc_2\varepsilon_1}{d_2+m}, \quad t \in ((n-1)T,\ nT].$$

Therefore, we conclude that $z_1(t) \le \widetilde{z}_1(t) + \frac{c_2\varepsilon_1 M}{d_2+m}$ for t large enough. If $n > k_3+1$, then we observe that $z_2(t)$ satisfies

$$\begin{cases} \dfrac{dz_2(t)}{dt} \le m(\widetilde{z}_1(t) + \dfrac{c_2\varepsilon_1 M}{d_2+m}) - d_2 z_2(t), \quad t \ne nT,\ t \ne (n+l-1)T, \\ z_2(t^+) = (1-\delta_3)z_2(t), \quad t = (n+l-1)T, \\ z_2(t^+) = z_2(t) + \mu_2, \quad t = nT. \end{cases} \tag{3.14}$$

For system (3.14), using Lemma 2.1 and Lemma 2.3, we obtain that

$$z_2(t) \le \begin{cases} \widetilde{z}_2(t) - \dfrac{Mm\varepsilon_1 c_2\delta_3 e^{-d_2(t-(n+l-2)T)}}{d_2(d_2+m)(1-(1-\delta_3)e^{-d_2T})} + \dfrac{Mm\varepsilon_1 c_2}{d_2(d_2+m)}, \\ t \in ((n-1)T,\ (n+l-1)T], \\ \widetilde{z}_2(t) - \dfrac{Mm\varepsilon_1 c_2\delta_3 e^{-d_2(t-(n+l-1)T)}}{d_2(d_2+m)(1-(1-\delta_3)e^{-d_2T})} + \dfrac{Mm\varepsilon_1 c_2}{d_2(d_2+m)}, \\ t \in ((n+l-1)T,\ nT] \end{cases}$$

$$< \widetilde{z}_2(t) + \frac{Mm\varepsilon_1 c_2}{d_2(d_2+m)}, \quad t \in ((n-1)T,\ nT].$$

Hence, we conclude that $z_2(t) < \widetilde{z}_2(t) + \frac{Mm\varepsilon_1 c_2}{d_2(d_2+m)}$ for t large enough, which evidences that $z_1(t) \to \widetilde{z}_1(t)$ and $z_2(t) \to \widetilde{z}_2(t)$ as $t \to \infty$. This completes the proof. □

4. Permanence

In this section, the permanence of system (1.2) will be investigated. The result is stated as follows.

Theorem 4.1. *System (1.2) is permanent if* $(c_1-d_1)T - c_2(A+B+C) > \ln\frac{1}{1-\delta_1}$, *where* A, B, C *are defined as in Theorem 3.1.*

Proof. Suppose $(x(t), y(t), z_1(t), z_2(t))$ is a solution of system (1.2). From Lemma 2.2, we note that $x(t) \le M$, $y(t) \le M$, $z_1(t) \le M$ and $z_2(t) \le M$ for sufficiently large t. It follows from (1.2) that $\frac{dx(t)}{dt} \ge x(t)(1-c_1M-x(t))$, which implies that $x(t) > 1-c_1M \triangleq m_1$ for all t large enough. For sufficiently small $\varepsilon_2 > 0$, we choose $m_1 = 1-\varepsilon_2 > 0$ and also define

$$m_2 = \frac{\mu_1(1-\delta_2)e^{-(d_2+m)T}}{1-(1-\delta_2)e^{-(d_2+m)T}} - \varepsilon_2 > 0,$$

$$m_3 = \frac{m_2 m(1-\delta_3)(1-e^{-d_2T}) + d_2\mu_2(1-\delta_3)e^{-d_2T}}{d_2(1-(1-\delta_3)e^{-d_2T})} - \varepsilon_2 > 0.$$

Now, it follows from system (1.2) that

$$\begin{cases} \left.\begin{aligned} \frac{dz_1(t)}{dt} &\geq -(d_2+m)z_1(t) \\ \frac{dz_2(t)}{dt} &= mz_1(t) - d_2 z_2(t) \end{aligned}\right\} & t \neq nT,\ t \neq (n+l-1)T, \\ \left.\begin{aligned} z_1(t^+) &= (1-\delta_2)z_1(t) \\ z_2(t^+) &= (1-\delta_3)z_2(t) \end{aligned}\right\} & t = (n+l-1)T, \\ \left.\begin{aligned} z_1(t^+) &= z_1(t) + \mu_1 \\ z_2(t^+) &= z_2(t) + \mu_2 \end{aligned}\right\} & t = nT. \end{cases} \tag{4.1}$$

For system (4.1), we can easily obtain that $z_1(t) > m_2$ and $z_2(t) > m_3$ for all t large enough. Therefore, for the permanence of system (1.2), we only need to find a positive constant m_4 such that $y(t) \geq m_4$ for t large enough. The proof is divided into two steps.

Step I. Let $0 < m_5 < \frac{1}{c_1}$, $\varepsilon_3 > 0$ and $\varepsilon_4 > 0$ be small enough such that

$$\xi_2 = (1-\delta_1)e^{\int_{(n+l-1)T}^{(n+l)T}(c_1 m_1 - d_1 - c_2(\widetilde{z}_2(s) + \frac{m}{d_2}(\frac{Mc_2m_5}{d_2+m}+\varepsilon_3)+\varepsilon_4))ds} > 1$$

as $(c_1-d_1)T - c_2(A+B+C) > \ln\frac{1}{1-\delta_1}$. For sufficiently large t, we assume that $y(t) \geq m_5$ is not true. Then there exists a time $t_1 \in (0,\infty)$ such that $y(t) < m_5$ for all $t > t_1$. With the aid of this assumption, we deduce that

$$\begin{cases} \dfrac{dz_1(t)}{dt} \leq Mc_2m_5 - (d_2+m)z_1(t), \quad t \neq nT,\ t \neq (n+l-1)T, \\ z_1(t^+) = (1-\delta_2)z_1(t), \quad t = (n+l-1)T, \\ z_1(t^+) = z_1(t) + \mu_1, \quad t = nT. \end{cases}$$

Consider the following comparison system

$$\begin{cases} \dfrac{du_3(t)}{dt} = Mc_2m_5 - (d_2+m)u_3(t), \quad t \neq nT,\ t \neq (n+l-1)T, \\ u_3(t^+) = (1-\delta_2)u_3(t), \quad t = (n+l-1)T, \\ u_3(t^+) = u_3(t) + \mu_1, \quad t = nT. \end{cases} \tag{4.2}$$

Due to Lemma 2.3, then system (4.2) has a globally asymptotically stable periodic solution:

$$\widetilde{u}_3(t) = \begin{cases} \widetilde{z}_1(t) + \dfrac{Mc_2m_5}{d_2+m} - \dfrac{Mc_2m_5\delta_2 e^{-(d_2+m)(t-(n+l-2)T)}}{(d_2+m)(1-(1-\delta_2)e^{-(d_2+m)T})}, \\ t \in ((n-1)T,\ (n+l-1)T], \\ \widetilde{z}_1(t) + \dfrac{Mc_2m_5}{d_2+m} - \dfrac{Mc_2m_5\delta_2 e^{-(d_2+m)(t-(n+l-1)T)}}{(d_2+m)(1-(1-\delta_2)e^{-(d_2+m)T})}, \\ t \in ((n+l-1)T,\ nT] \end{cases}$$

$$< \widetilde{z}_1(t) + \frac{Mc_2m_5}{d_2+m}, \quad t \in ((n-1)T,\ nT].$$

According to Lemma 2.1, we have $z_1(t) \le u_3(t) < \widetilde{u}_3(t) + \varepsilon_3 < \widetilde{z}_1(t) + \frac{Mc_2m_5}{d_2+m} + \varepsilon_3$ for t large enough. From system (1.2), we arrive at

$$\begin{cases} \dfrac{dz_2(t)}{dt} \le m(\widetilde{z}_1(t) + \dfrac{Mc_2m_5}{d_2+m} + \varepsilon_3) - d_2z_2(t), \quad t \ne nT,\ t \ne (n+l-1)T, \\ z_2(t^+) = (1-\delta_3)z_2(t), \quad t = (n+l-1)T, \\ z_2(t^+) = z_2(t) + \mu_2, \quad t = nT. \end{cases} \tag{4.3}$$

Consider the comparison system of (4.3) as follows

$$\begin{cases} \dfrac{du_4(t)}{dt} = m(\widetilde{z}_1(t) + \dfrac{Mc_2m_5}{d_2+m} + \varepsilon_3) - d_2u_4(t), \quad t \ne nT,\ t \ne (n+l-1)T, \\ u_4(t^+) = (1-\delta_3)u_4(t), \quad t = (n+l-1)T, \\ u_4(t^+) = u_4(t) + \mu_2, \quad t = nT. \end{cases} \tag{4.4}$$

In the similar manner, system (4.4) also has a periodic solution

$$\begin{aligned} &\widetilde{u}_4(t) \\ &= \begin{cases} \widetilde{z}_2(t) + (\dfrac{m}{d_2} - \dfrac{m\delta_3 e^{-d_2(t-(n+l-2)T)}}{d_2(1-(1-\delta_3)e^{-d_2T})})(\dfrac{Mc_2m_5}{d_2+m} + \varepsilon_3), \\ t \in ((n-1)T,\ (n+l-1)T], \\ \widetilde{z}_2(t) + (\dfrac{m}{d_2} - \dfrac{m\delta_3 e^{-d_2(t-(n+l-1)T)}}{d_2(1-(1-\delta_3)e^{-d_2T})})(\dfrac{Mc_2m_5}{d_2+m} + \varepsilon_3), \quad t \in ((n+l-1)T,\ nT], \end{cases} \\ &< \widetilde{z}_2(t) + \frac{m}{d_2}(\frac{Mc_2m_5}{d_2+m} + \varepsilon_3), \quad t \in ((n-1)T, nT]. \end{aligned}$$

which is globally asymptotically stable. It follows that $z_2(t) \le u_4(t) < \widetilde{u}_4(t) + \varepsilon_4 < \widetilde{z}_2(t) + \frac{m}{d_2}(\frac{Mc_2m_5}{d_2+m} + \varepsilon_3) + \varepsilon_4$ for t large enough. We observe that there exists an integer $n_0 > 0$ such that

$$\begin{cases} \dfrac{dy(t)}{dt} \ge y(t)(c_1m_1 - d_1 - c_2(\widetilde{z}_2(t) + \dfrac{m}{d_2}(\dfrac{Mc_2m_5}{d_2+m} + \varepsilon_3) + \varepsilon_4)),\ t \ne (n+l-1)T, \\ y(t^+) = (1-\delta_1)y(t), \quad t = (n+l-1)T \end{cases} \tag{4.5}$$

for all $n > n_0$. For $t \in ((n+l-1)T, (n+l)T]$, integrating the first equation of system (4.5) on $((n+l-1)T, t]$, we can obtain that

$$y(t) \ge y((n+l-1)T^+)e^{\int_{(n+l-1)T}^{t}(c_1m_1 - d_1 - c_2(\widetilde{z}_2(s) + \frac{m}{d_2}(\frac{Mc_2m_5}{d_2+m} + \varepsilon_3) + \varepsilon_4))ds},$$

which leads to

$$\begin{aligned} y((n+l)T^+) &\ge y((n+l-1)T^+)(1-\delta_1) \\ &\quad \times e^{\int_{(n+l-1)T}^{(n+l)T}(c_1m_1 - d_1 - c_2(\widetilde{z}_2(s) + \frac{m}{d_2}(\frac{Mc_2m_5}{d_2+m} + \varepsilon_3) + \varepsilon_4))ds} \\ &= y((n+l-1)T^+)\xi_2. \end{aligned}$$

Then we infer that $y((n_0+l+k)T^+) \geq y((n_0+l)T^+)\xi_2^k \to \infty$ as $k \to \infty$, which is a contradiction of our assumption $y(t) < m_5$ for all $t > t_1$. Hence, there exists a time $t_2 > t_1$ such that $y(t_2) \geq m_5$.

Step II. If $y(t) \geq m_5$ for all $t \geq t_2$, then our aim will be fulfilled. Otherwise, there exists some $t > t_2$ such that $y(t) < m_5$. Let $t^* = \inf\{t | y(t) < m_5, t > t_2\}$, then there will be two cases.

Case i. Let $t^* = (n_1+l-1)T$ ($n_1 \in N_+$), then we have $y(t) \geq m_5$ for all $t \in [t_2, t^*]$ and $(1-\delta_1)m_5 \leq y(t^{*+}) = (1-\delta_1)y(t^*) < m_5$. Assume that $T_0 = n_2T + n_3T$, where $n_2 = n_2' + n_2''$, n_2', n_2'' and n_3 are positive integer and satisfy the following inequalities

$$
\begin{aligned}
&(n_2'-1)T > -\frac{1}{d_2+m}\ln\frac{\varepsilon_3}{M+\mu_1}, \quad (n_2''-1)T > -\frac{1}{d_2}\ln\frac{\varepsilon_4}{M+\mu_2},\\
&(1-\delta_1)^{n_2}e^{\eta n_2 T}\xi_2^{n_3} > (1-\delta_1)^{n_2}e^{\eta(n_2+1)T}\xi_2^{n_3} > 1
\end{aligned}
$$

with $\eta = c_1m_1 - d_1 - c_2M < 0$. Now, we claim that there exists a time $t_2' \in (t^*, t^*+T_0]$ such that $y(t_2') \geq m_5$, if it is not true, then $y(t) < m_5$ for all $t \in (t^*, t^*+T_0]$. Indeed, if system (4.2) is considered with initial data $u_3(t^{*+}) = z_1(t^{*+})$, and we get

$$
u_3(t) = \begin{cases}
(u_3(n_1T^+) - (\dfrac{Mc_2m_5}{d_2+m} + \dfrac{\mu_1}{1-(1-\delta_2)e^{-(d_2+m)T}} \\
-\dfrac{Mc_2m_5\delta_2e^{-(d_2+m)(1-l)T}}{(d_2+m)(1-(1-\delta_2)e^{-(d_2+m)T})})) \\
\times(1-\delta_2)^{n-(n_1+1)}e^{-(d_2+m)(t-n_1T)} + \widetilde{u}_3(t), \quad t \in ((n-1)T, (n+l-1)T], \\
(u_3(n_1T^+) - (\dfrac{Mc_2m_5}{d_2+m} + \dfrac{\mu_1}{1-(1-\delta_2)e^{-(d_2+m)T}} \\
-\dfrac{Mc_2m_5\delta_2e^{-(d_2+m)(1-l)T}}{(d_2+m)(1-(1-\delta_2)e^{-(d_2+m)T})})) \\
\times(1-\delta_2)^{n-1}e^{-(d_2+m)(t-n_1T)} + \widetilde{u}_3(t), \quad t \in ((n+l-1)T,\ nT],
\end{cases}
$$

where $n_1+1 \leq n \leq n_1+n_2+n_3$. Hence, we have $|u_3(t) - \widetilde{u}_3(t)| < (M+\mu_1)e^{-(d_2+m)(t-n_1T)} < \varepsilon_3$ and $z_1(t) \leq u_3(t) < \widetilde{u}_3(t) + \varepsilon_3 \leq \widetilde{z}_1(t) + \frac{Mc_2m_5}{d_2+m} + \varepsilon_3$ for all $t \in [(n_1+n_2'-1)T, t^*+T_0]$.

Now, for system (4.4) with initial data $u_4((t^*+n_2'T)^+) = z_2((t^*+n_2'T)^+)$, we conclude that $|u_4(t) - \widetilde{u}_4(t)| < (M+\mu_2)e^{-d_2(t-(n_1+n_2')T)} < \varepsilon_4$ and $z_2(t) \leq u_4(t) < \widetilde{u}_4(t) + \varepsilon_4 \leq \widetilde{z}_2(t) + \frac{m}{d_2}(\frac{Mc_2m_5}{d_2+m} + \varepsilon_3) + \varepsilon_4$ for all $t \in [(n_1+n_2'+n_2''-1)T, t^*+T_0]$, which ensures that system (4.5) holds for $[t^*+n_2T, t^*+T_0]$. Integrating the first equation of system (4.5) on $[t^*+n_2T, t^*+T_0]$, we arrive at

$$
y(t^*+T_0) \geq y(t^*+n_2T)\xi_2^{n_3}. \tag{4.6}
$$

Fourthermore, it follows from system (1.2) that

$$
\begin{cases}
\dfrac{dy(t)}{dt} \geq (c_1m_1 - d_1 - c_2M)y(t) := \eta y(t), \quad t \neq (n+l-1)T, \\
y(t^+) = (1-\delta_1)y(t), \quad t = (n+l-1)T.
\end{cases} \tag{4.7}
$$

Integrating the first equation of system (4.7) on $[t^*, t^*+n_2T]$ yields

$$y(t^*+n_2T) \geq m_5(1-\delta_1)^{n_2}e^{\eta n_2 T}. \tag{4.8}$$

Plugging (4.8) into (4.6), we deduce that

$$y(t^*+T_0) \geq m_5(1-\delta_1)^{n_2}e^{\eta n_2 T}\xi_2^{n_3} > m_5,$$

which is a contradiction of $y(t) < m_5$ for all $t \in (t^*, t^*+T_0]$. Therefore, there exists a time $t_2' \in [t^*, t^*+T_0]$ such that $y(t_2') \geq m_5$. Let $\widetilde{t} = \inf\{t|y(t) \geq m_5,\ t \geq t^*\}$, since $0 < \delta_1 < 1$, $y((n+l-1)T^+) = (1-\delta_1)y((n+l-1)T) < y((n+l-1)T)$ and $y(t) < m_5$ for $t \in (t^*, \widetilde{t})$, we have $y(\widetilde{t}) = m_5$. Suppose that $t \in (t^*, \widetilde{t})$ and $t \in (t^*+(k-1)T, t^*+kT]$ ($k \in N_+$ and $k \leq n_2+n_3$), we get from system (4.7) that

$$y(t) \geq m_5(1-\delta_1)^k e^{k\eta T} \geq m_5(1-\delta_1)^{n_2+n_3}e^{(n_2+n_3)T} \triangleq m_4.$$

For $t > \widetilde{t}$, the same arguments can be continued since $y(\widetilde{t}) \geq m_5$.

Case ii. If $t^* \neq (n_1+l-1)T$, then we have $y(t^*) = m_5$ and $y(t) \geq m_5$ for all $t \in [t_2, t^*]$. Assume that $t^* \in ((n_1'+l-1)T, (n_1'+l)T]$, $n_1' \in N_+$, we also have two subcases for $t \in (t^*, (n_1'+l)T)$.

Case a. If $y(t) < m_5$ for all $t \in (t^*, (n_1'+l)T]$, then we claim that there exists a time $t_3 \in ((n_1'+l)T, (n_1'+l)T+T_0]$ such that $y(t_3) \geq m_5$. Otherwise, integrating system (4.5) on the interval $[(n_1'+l+n_2)T, (n_1'+l+n_2+n_3)T]$, we have

$$y((n_1'+l+n_2+n_3)T) \geq y((n_1'+l+n_2)T)\xi_2^{n_3}. \tag{4.9}$$

Integrating the first equation of system (4.7) in the interval $[t^*, (n_1'+l+n_2)T]$, we obtain that

$$\begin{aligned} y((n_1'+l+n_2)T) &\geq y(t^*)(1-\delta_1)^{n_2}e^{\eta((n_1'+l+n_2)T-t^*)} \\ &\geq m_5(1-\delta_1)^{n_2}e^{\eta(n_2+1)T}. \end{aligned} \tag{4.10}$$

Putting (4.10) into (4.9), we arrive at

$$y((n_1'+l+n_2+n_3)T) \geq m_5(1-\delta_1)^{n_2}e^{\eta(1+n_2)T}\xi_2^{n_3} > m_5,$$

which is again a contradiction. Let $\bar{t} = \inf\{t|y(t) \geq m_5, t > t^*\}$, then $y(\bar{t}) = m_5$ and $y(t) < m_4$ for all $t \in (t^*, \bar{t})$. For $t \in (t^*, \bar{t})$ and $t \in ((n_1'+l-1)T+(k'-1)T, (n_1'+l-1)T+k'T]$ (k' is a positive integer and $k' \leq 1+n_2+n_3$), we obtain that

$$\begin{aligned} y(t) &\geq y(((n_1'+l-1)+k'-1)T^+)e^{\eta(t-((n_1'+l-1)+k'-1)T)} \\ &\geq y(t^*)(1-\delta_1)^{k'-1}e^{\eta(t-t^*)} \\ &\geq m_5(1-\delta_1)^{n_2+n_3}e^{(n_2+n_3+1)\eta T}. \end{aligned}$$

Letting $m_4 \triangleq m_5(1-\delta_1)^{n_2+n_3}e^{(n_2+n_3+1)\eta T}$, we have $y(t) \geq m_4$ for $t \in (t^*, \bar{t})$. For $t > \bar{t}$, the same arguments can be continued since $y(\bar{t}) \geq m_5$.

Case b. If there exists a time $t \in (t^*, (n_1'+l)T]$ such that $y(t) \geq m_5$. Let $\bar{t} = \inf\{t|y(t) \geq m_5, t > t^*\}$, then $y(t) < m_5$ for $t \in (t^*, \bar{t})$ and $y(\bar{t}) = m_5$. For $t \in (t^*, \bar{t})$, system (4.7) holds and integrating system (4.7) on (t^*, t), we have

$$y(t) \geq y(t^*)e^{\eta(t-t^*)} \geq m_5 e^{\eta T} > m_4.$$

Since $y(\bar{t}) \geq m_5$ for all $t > \bar{t}$, the same argument can be continued. Hence, we have $y(t) \geq m_4$ for all $t > t_2$. Thus in both case, we can conclude that $y(t) \geq m_4$ for all $t \geq t_2$. This completes the proof. $\square$

5. Numerical simulations

In this section, our main purpose is to numerically investigate the effects of impulsive harvest, pulse releasing amount of immature and mature natural enemies on system (1.2). For this, we will give some numerical simulations of system (1.2) with $x(0^+) = 0.5, y(0^+) = 1, z_1(0^+) = 0.8, z_2(0^+) = 4, c_1 = 1, c_2 = 0.3, d_1 = 0.01, d_2 = 0.3, m = 0.4$ and $l = 0.5$. Here all the parameter values come from [12]. If we choose $\delta_1 = \delta_2 = \delta_3 = 0, \mu_1 = \mu_2 = 0$, i.e., without impulsive control, then the plant and natural enemies decrease to zero rapidly (see Figure 1). It follows from Figure 1 that this phenomenon is harmful for the stability of ecology. If we set $\delta_1 = 0.2, \delta_2 = 0.1, \delta_3 = 0.2, \mu_1 = 2$ and $\mu_2 = 4$, we can obtain that the threshold limit for the impulsive period is $T_{\max} = 4.7364$ as $(c_1 - d_1)T - c_2(A + B + C) = \ln \frac{1}{1-\delta_1}$. Then we have $T < T_{\max}$ if $(c_1 - d_1)T - c_2(A+B+C) < \ln \frac{1}{1-\delta_1}$, and we get $T > T_{\max}$ if $(c_1 - d_1)T - c_2(A + B + C) > \ln \frac{1}{1-\delta_1}$.

It follows from Theorem 3.1 and Theorem 3.2 that the pest-extinction periodic solution $(1, 0, z_1(t), z_2(t))$ of system (1.2) is globally asymptotically stable if $T < T_{\max}$. Let $T = 4 < T_{\max}$, the pest-extinction periodic solution $(1, 0, \widetilde{z}_1(t), \widetilde{z}_2(t))$ of system (1.2) is globally attractive (see Figure 2). Further, the result of Theorem 4.1 is verified for $T > T_{\max}$, i.e., the coexistence of permanence of the system (1.2) is occurred for $T = 6.5 > T_{\max}$ (see Figure 3). The pest population will again extinct when we increase pulse releasing amounts of immature and mature natural enemies under the constant values of impulsive period, which implies that the coexistence of population transfer to the extinction of pest population (see Figure. 4). These numerical simulations and obtained results of system (1.2) show that more pulse releasing amount of the immature and mature natural enemies or shorter impulsive period is highly significant for the extinction of the pest populations.

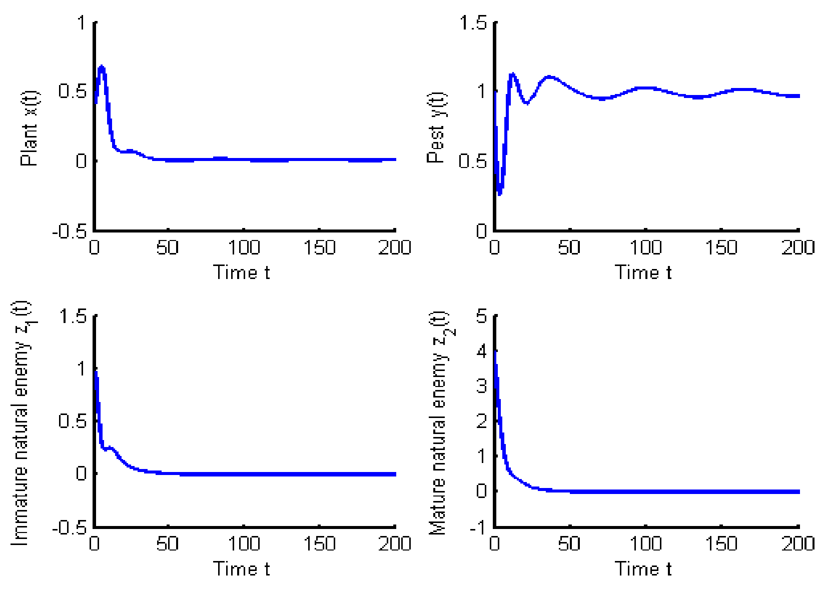

Figure 1. Dynamical behavior of system (1.2) without impulsive control.

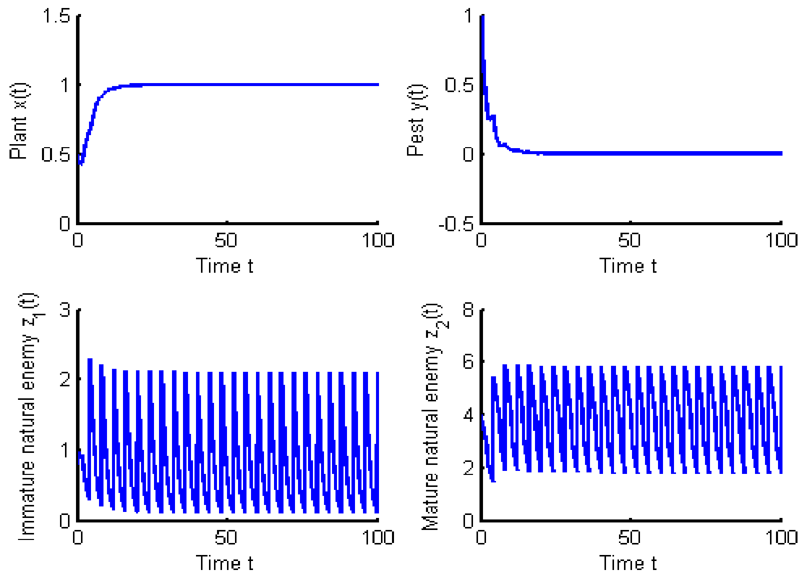

Figure 2. The pest-extinction periodic solution $(1, 0, \tilde{z}_1(t), \tilde{z}_2(t))$ of system (1.2) is globally attractive for $T = 4 < T_{\max} = 4.7364$.

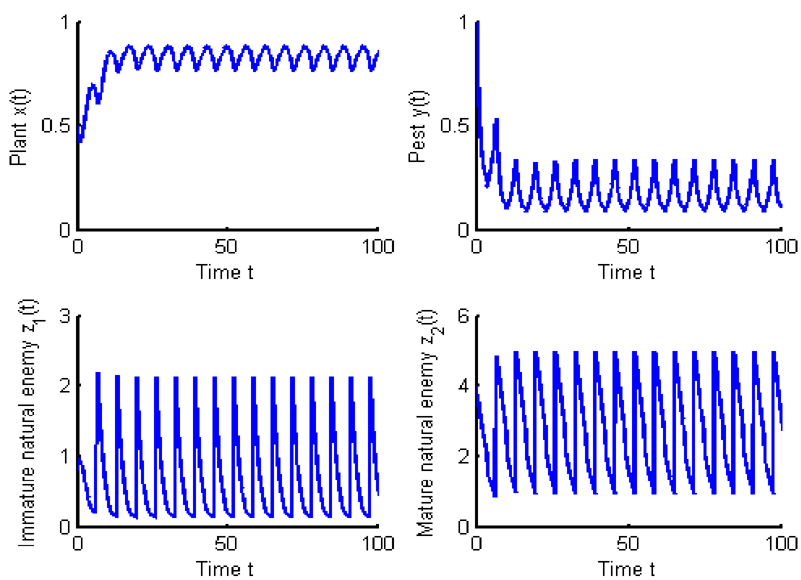

Figure 3. System (1.2) is permanent for $T = 6.5 > T_{\max} = 4.7364$.

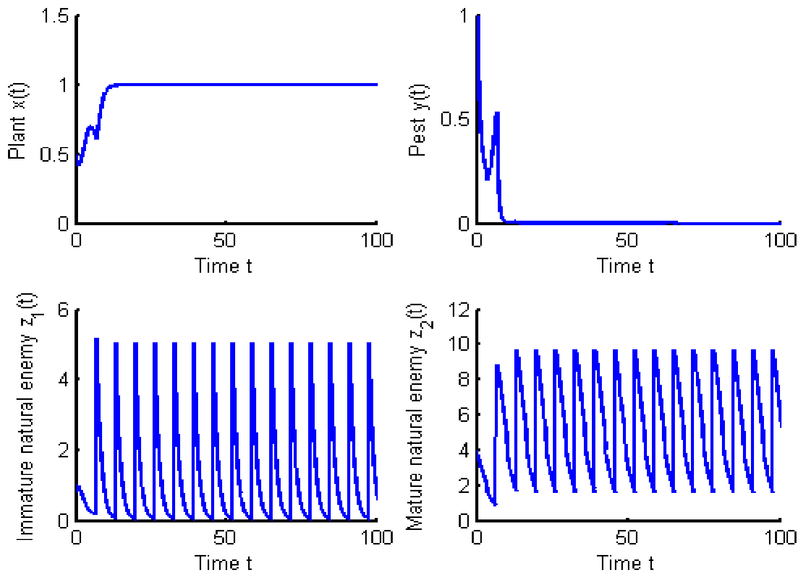

Figure 4. The coexistence of populations transfer to extinction of pests for $T = 6.5$, $\mu_1 = 5$ and $\mu_2 = 8$.

6. Conclusions and discussion

In this paper, we investigated the dynamics of a stage-structured plant-pest-natural enemy system with impulsive spraying pesticide and releasing immature and mature natural enemies at different fixed moment. It is observed both analytically and numerically that the impulsive control strategy play a significant role on the pest extinction and permanence of species.

Obviously, if $\delta_1 = \delta_2 = \delta_3 = 0$ and $\mu_1 = \mu_2 = 0$, then system (1.2) will be simplified a typical plant-pest-natural enemy model without impulsive control. As the density of pest outweigh a certain number, the plant will suffer from it. Meanwhile, the amount of natural enemies decrease sharply in a short period of time even to zero (see Figure 1). In fact, this is very worse for the stability of ecology. To maintain a stable ecology, we uses a so called impulsive control strategy, a combination of chemical and biological tactics (spraying pesticide and releasing natural enemies). By using Floquet theory for impulsive differential equations and comparison techniques, the local behavior of plant-pest-extinction periodic solution and pest-extinction periodic solution of system (1.2) are investigated in Theorem 3.1. Theorem 3.2 tells us that the pest-extinction periodic solution is globally attractive. From Theorem 4.1, we can determine the impulsive period T according to the effect of the chemical pesticides on the populations and the amount of the releasing natural enemies such that $(c_1 - d_1)T - c_2(A + B + C) > \ln \frac{1}{1-\delta_1}$. This means that the number of pest can be controlled under a safe level, so that it can do little harm to the crops as we release natural enemies and spray pesticide at different moments. As a matter of fact, the main purposes of integrated pest management are often to keep the size of the pest population under a certain economic injury level, and needn't to eradicated the pest completely. It follows from Theorem 4.1 that the densities of the pest can be controlled at a lower level and the goal of integrated pest management can be achieved by suitable pesticide input and releasing natural enemies at different moments. Hence, to combine chemical pesticide and releasing beneficial enemy are more efficacious than to use only one control method for pest control.

In this paper, we assumed that the death rate δ_1 of pests, the death rate δ_2 of immature natural enemies and the death rate δ_3 of mature natural enemies are constant. Furthermore, we assumed that the number of releasing natural enemies is also constant. In the real world, there exist lots of factors that may disturb the efficacy of pesticide sprayed and the survival ratio of natural enemies released. So the death rate δ_1 of pests, the death rate δ_2 of immature natural enemies and the death rate δ_3 of mature natural enemies should be considered as stochastic variable. The number of releasing natural enemies should also be considered as stochastic variable. If like this, our chain model will become stochastic population model and be more realistic (see [3,9,17,19,29]). We need to investigate stochastic differential equations and apply it to our model in the future.

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