Journal of Applied Analysis and Computation
Volume 15, Number 1, February 2025, 316–332

Website:http://www.jaac-online.com
DOI:10.11948/20240092

SOLVABILITY OF LANGEVIN FRACTIONAL DIFFERENTIAL EQUATION OF HIGHER-ORDER WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract This paper we concern the solvability and uniqueness of higher-order Langevin fractional differential equations subject to integral boundary conditions. We establish the existence of solutions using Krasnoselskii's fixed point theorem, while uniqueness is demonstrated through the application of the Banach fixed point theorem. The obtained results offer insights into the solution space of these complex differential equations, shedding light on their behavior and properties. To illustrate the practical implications of our findings, we provide a concrete example at the end of this paper.

Keywords Caputo fractional derivative, boundary conditions, fractional differential equation.

MSC(2010) 34A08, 34A12.

1. Introduction

In the last few decades, the branch of fractional calculus has paid more attention due to their several applications in applied mathematics and physics such as the memory of a variety of materials, signal identification and image processing, optical systems, thermal system materials and mechanical systems, control system, etc. Fractional differential equations have been of great interest recently. This is due to the intensive development of the theory of fractional calculus itself as well as its applications. Apart from diverse areas of mathematics, fractional differential equations arise in rheology, dynamical processes in self similar and porous structures,

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electrical networks, viscoelasticity, chemical physics, and many other branches of science, see [4, 5, 11, 16, 17, 19, 20, 23, 27, 31] and the references therein.

Langevin fractional differential equations represent a specialized class of stochastic differential equations that find relevance in various fields, including physics, engineering, and finance [6, 14, 28]. These equations incorporate fractional derivatives and are used to model systems characterized by random fluctuations and memory effects. The term "Langevin" is attributed to Paul Langevin, a French physicist who made pioneering contributions to the field of statistical mechanics and stochastic processes. In contrast to classical differential equations, Langevin fractional differential equations account for non-Markovian dynamics, where the future state of a system depends not only on its present state but also on its past history. This memory property is captured by fractional derivatives, which generalize the concept of differentiation to fractional orders, allowing for the modeling of long-range dependencies and irregular behaviors.

The study of Langevin fractional differential equations has gained prominence due to their ability to describe complex systems exhibiting anomalous diffusion and non-Gaussian statistical properties. Such systems can range from the motion of particles in disordered media to the pricing of financial assets subject to market fluctuations.

In recent years, researchers have made significant strides in understanding the behavior and properties of Langevin fractional differential equations, developing analytical and numerical techniques for their solution. These equations have found applications in diverse domains, including physics, biology, and engineering, making them a subject of growing interest and importance in the scientific community [22, 24, 25, 32].

This paper aims to contribute to the ongoing exploration of Langevin fractional differential equations, specifically focusing on their solvability with integral boundary conditions and addressing the fundamental issue of uniqueness. Through the utilization of well-established fixed point theorems, we provide insights into the existence and uniqueness of solutions for these equations, offering a valuable contribution to the broader field of fractional differential equations.

In the present paper, motivated by works above, we discuss the existence of solutions of the following problem with the boundary-value conditions

$$\begin{cases} {}^cD^{\alpha}\left({}^cD^{\beta}+\gamma\right)u(t)=f(t,u(t),I^{\sigma}u(t)),\, t\in[0,1], \\ u(0)=\displaystyle\int_0^{\mu}u(s)ds, \\ u(1)=\displaystyle\int_{\nu}^{1}u(s)ds, & 0<\mu<\nu<1, \\ u^{(k)}(0)=0, & k=1,2,\cdots,n-1. \end{cases} \tag{1.1}$$

with $0<\alpha\leq 1$, $n-1<\beta\leq n$, $1\leq\sigma\leq 2$, $\gamma>0$ and ${}^cD^{\alpha}(\cdot), {}^cD^{\beta}(\cdot)$ denoted respectively the Caputo fractional derivatives of order α and β, I^{σ} represents the Riemann-Liouville fractional integral of order σ, and $f:[0,1]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ is a given function.

In this paper, we extend the current body of knowledge on Langevin fractional differential equations by addressing the solvability and uniqueness of higher-order equations subject to integral boundary conditions. While previous studies have primarily concentrated on first-order equations and limited boundary conditions, our work explores more complex scenarios, providing a comprehensive understanding of the solutions and their uniqueness. In this sense, through Krasnoselskii's fixed point theorem and the Banach fixed point theorem, we investigate the existence and uniqueness of solutions for a class of Langevin fractional differential equation of higher-Order with integral boundary conditions. Therefore, the results obtained here open the doors to modeling and analyzing intricate systems exhibiting fractional dynamics.

This manuscript is organised as follows. Firstly, we give preliminaries part in which we will recall some preliminary results that we will use for the rest of the paper. The next part is devoted to study the existence and the uniqueness result.

2. Mathematical background: Preliminaries

This part is devoted to present some basic definitions and lemmas concerning the fractional calculus which will be used in our results. For more details, see [20, 23, 26, 29, 30] and the references therein.

Definition 2.1. [20, 29, 30] For a differentiable function $h : [0, +\infty) \to \mathbb{R}$, the Caputo derivative of fractional order α is defined by

$$^{c}D^{\alpha}h(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}h^{(n)}(s)ds, \quad n-1 < \alpha < n, n = [\alpha]+1,$$

where $[\alpha]$ denotes the integer part of α and Γ is the gamma function.

Definition 2.2. [20, 29, 30] The Riemann-Liouville fractional integral of order α is given by

$$I^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds,$$

where $h : \mathbb{R}^{+} \to \mathbb{R}$ is a Lebesgue measurable function, provided that the integral exists.

Lemma 2.1. [29, 30, 33] *Let $\alpha > 0$, then the fractional differential equation*

$$^{c}D^{\alpha}h(t) = 0,$$

has solutions

$$h(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n-1$, $n = [\alpha]+1$.

Lemma 2.2. [33] *Let $\alpha > 0$, then*

$$I^{\alpha}\ {}^{c}D^{\alpha}h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n-1$, $n = [\alpha]+1$.

Theorem 2.1 (**Fixed point theorem of Banach** [8]). *Let X be a Banach space and $T : X \longrightarrow X$ a contracting mapping. Then T has a unique fixed point i.e.*

$$\exists! x \in X : \ Tx = x.$$

For more details on the basic tools, we refer to [1–3, 7, 9, 10, 12, 13, 15, 18, 21].

3. Existence and uniqueness result

This section is devoted to prove the existence and uniqueness of the solution related to our problem.

As it is noted in the previous section, we remember that the fractional integer is given by

$$I^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}d\tau.$$

By applying the operator I^{α} on both sides of the first equation of the **Problem** (1.1), one has

$$\left({}^{c}D^{\beta}+\gamma\right)u(t)=I^{\alpha}f(t,u(t),I^{\sigma}u(t))+a,\ a\in\mathbb{R}. \tag{3.1}$$

Moreover, by applying again I^{β} on both sides of the Eq.(3.1), yields

$$u(t)=I^{\alpha+\beta}f(t,u(t),I^{\sigma}u(t))-\gamma I^{\beta}u(t)+\frac{a}{\Gamma(\beta+1)}t^{\beta}+\sum_{i=0}^{n-1}a_i t^i, \tag{3.2}$$

where $a_0,a_1,\cdots,a_{n-1}$ are real constants to be determined.

Now, we will use the boundary conditions to determine these constants. For this purpose, we begin by deriving both sides of the Eq.(3.2) $(n-1)-$times, so we obtain

$$u^{(n-1)}(0)=(n-1)!a_{n-1}=0.$$

Hence, based on the 4^{th} equation of our **Problem** (1.1), we get $a_{n-1}=0$.

Furthermore, by deriving the Eq.(3.2) $(n-2)-$times and using also the 4^{th} equation of our problem, we obtain

$$u^{(n-2)}(0)=(n-2)!a_{n-2}=0\Leftrightarrow a_{n-2}=0.$$

Similarly, we get $a_1=a_2=\cdots=a_{n-1}=0$.

Therefore, utilizing the second condition of our **Problem** (1.1) and the Eq.(3.2), we infer

$$u(0)=a_0=\int_0^{\mu}u(s)ds.$$

Furthermore, yields

$$\begin{aligned}u(1)&=\frac{1}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}f(s,u(s),I^{\sigma}u(s))ds\\&-\frac{\gamma}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}u(s)ds+\frac{a}{\Gamma(\beta+1)}+a_0\\&=\int_{\nu}^1 u(s)ds.\end{aligned}$$

Consequently, we get

$$a_0=\int_0^{\mu}u(s)ds \tag{3.3}$$

and

$$a=\Gamma(\beta+1)\Bigg[\int_{\nu}^1 u(s)ds+\frac{\gamma}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}u(s)ds$$

$$-\frac{1}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}f(s,u(s),I^\sigma u(s))ds-\int_0^\mu u(s)ds\Big].\qquad(3.4)$$

By substituting of Eq.(3.3) and Eq.(3.4) into the Eq.(3.2), one has

$$\begin{aligned}u(t)=&I^{\alpha+\beta}f(t,u(t),I^\sigma(t))-\gamma I^\beta u(t)+\int_0^\mu u(s)ds\\&+\Big[\int_\nu^1 u(s)ds+\frac{\gamma}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}u(s)ds\\&-\frac{1}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}f(s,u(s),I^\sigma u(s))ds-\int_0^\mu u(s)ds\Big]t^\beta.\qquad(3.5)\end{aligned}$$

Moreover, we integrate both sides of Eq.(3.5) on $[0,\mu]$, we have

$$\begin{aligned}&\int_0^\mu u(s)ds\\=&\frac{1}{\Gamma(\alpha+\beta+1)}\int_0^\mu(\mu-s)^{\alpha+\beta}f(s,u(s),I^\sigma u(s)ds-\frac{\gamma}{\Gamma(\beta+1)}\int_0^\mu(\mu-s)^\beta u(s)ds\\&+\left(\mu-\frac{\mu^{\beta+1}}{\beta+1}\right)\int_0^\mu u(s)ds+\frac{\mu^{\beta+1}}{\beta+1}\int_\nu^1 u(s)ds\\&+\frac{\mu^{\beta+1}}{\beta+1}\Big[\frac{\gamma}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}u(s)ds\\&-\frac{1}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}f(s,u(s),I^\sigma u(s))ds\Big].\end{aligned}$$

Equivalently, we obtain

$$\begin{aligned}&\left(1-\mu+\frac{\mu^{\beta+1}}{\beta+1}\right)\int_0^\mu u(s)ds-\frac{\mu^{\beta+1}}{\beta+1}\int_\nu^1 u(s)ds\\=&\frac{1}{\Gamma(\alpha+\beta+1)}\int_0^\mu(\mu-s)^{\alpha+\beta}f(s,u(s),I^\sigma u(s)ds\\&-\frac{\gamma}{\Gamma(\beta+1)}\int_0^\mu(\mu-s)^\beta u(s)ds\\&+\frac{\mu^{\beta+1}}{\beta+1}\Big[\frac{\gamma}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}u(s)ds\\&-\frac{1}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}f(s,u(s),I^\sigma u(s))ds\Big].\qquad(3.6)\end{aligned}$$

Now, let us integrate both sides of the Eq.(3.5) on $[\nu,1]$, yields

$$\begin{aligned}&\int_\nu^1 u(s)ds\\=&\frac{1}{\Gamma(\alpha+\beta+1)}\int_0^1(1-s)^{\alpha+\beta}f(s,u(s),I^\sigma u(s))ds\\&-\frac{1}{\Gamma(\alpha+\beta+1)}\int_0^\nu(\nu-s)^{\alpha+\beta}f(s,u(s),I^\sigma u(s))ds+\frac{\gamma}{\Gamma(\beta+1)}\int_0^\nu(\nu-s)^\beta u(s)ds\end{aligned}$$

$$
\begin{aligned}
&-\frac{\gamma}{\Gamma(\beta+1)}\int_0^1(1-s)^{\beta}u(s)ds+\big(1-\nu-\frac{1-\nu^{\beta+1}}{\beta+1}\big)\int_0^{\mu}u(s)ds\\
&+\frac{1-\nu^{\beta+1}}{\beta+1}\int_{\nu}^1u(s)ds+\frac{1-\nu^{\beta+1}}{\beta+1}\Big[\frac{\gamma}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}u(s)ds\\
&-\frac{1}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}f(s,u(s),I^{\sigma}u(s))ds\Big].
\end{aligned}
$$

Consequently and equivalently, we get

$$
\begin{aligned}
&-\left(1-\nu-\frac{1-\nu^{\beta+1}}{\beta+1}\right)\int_0^{\mu}u(s)ds+\left(1-\frac{1-\nu^{\beta+1}}{\beta+1}\right)\int_{\nu}^1u(s)ds\\
=&\frac{1}{\Gamma(\alpha+\beta+1)}\int_0^1(1-s)^{\alpha+\beta}f(s,u(s),I^{\sigma}u(s))ds\\
&-\frac{1}{\Gamma(\alpha+\beta+1)}\int_0^{\nu}(\nu-s)^{\alpha+\beta}f(s,u(s),I^{\sigma}u(s))ds+\frac{\gamma}{\Gamma(\beta+1)}\int_0^{\nu}(\nu-s)^{\beta}u(s)ds\\
&-\frac{\gamma}{\Gamma(\beta+1)}\int_0^1(1-s)^{\beta}u(s)ds+\frac{1-\nu^{\beta+1}}{\beta+1}\Big[\frac{\gamma}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}u(s)ds\\
&-\frac{1}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}f(s,u(s),I^{\sigma}u(s))ds\Big]. \qquad (3.7)
\end{aligned}
$$

To clarify the rest of the process, let us consider the following variables

$$
\begin{aligned}
x_1=&\int_0^{\mu}u(s)ds,\ x_2=\int_{\nu}^1u(s)ds,\\
b_1=&\frac{1}{\Gamma(\alpha+\beta+1)}\int_0^{\mu}(\mu-s)^{\alpha+\beta}f(s,u(s),I^{\sigma}u(s)ds\\
&-\frac{\gamma}{\Gamma(\beta+1)}\int_0^{\mu}(\mu-s)^{\beta}u(s)ds\\
&+\frac{\mu^{\beta+1}}{\beta+1}\Big[\frac{\gamma}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}u(s)ds\\
&-\frac{1}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}f(s,u(s),I^{\sigma}u(s))ds\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
b_2=&\frac{1}{\Gamma(\alpha+\beta+1)}\int_0^1(1-s)^{\alpha+\beta}f(s,u(s),I^{\sigma}u(s))ds\\
&-\frac{1}{\Gamma(\alpha+\beta+1)}\int_0^{\nu}(\nu-s)^{\alpha+\beta}f(s,u(s),I^{\sigma}u(s))ds\\
&+\frac{\gamma}{\Gamma(\beta+1)}\int_0^{\nu}(\nu-s)^{\beta}u(s)ds-\frac{\gamma}{\Gamma(\beta+1)}\int_0^1(1-s)^{\beta}u(s)ds\\
&+\frac{1-\nu^{\beta+1}}{\beta+1}\Big[\frac{\gamma}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}u(s)ds\\
&-\frac{1}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}f(s,u(s),I^{\sigma}u(s))ds\Big].
\end{aligned}
$$

Thus, from Eq.(3.6) and Eq.(3.7), we have the following linear system

$$Ax = b, \text{ where } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{3.8}$$

and

$$A = \begin{pmatrix} 1-\mu+\frac{\mu^{\beta+1}}{\beta+1} & -\frac{\mu^{\beta+1}}{\beta+1} \\ -(1-\nu-\frac{1-\nu^{\beta+1}}{\beta+1}) & 1-\frac{1-\nu^{\beta+1}}{\beta+1} \end{pmatrix}.$$

By a simple computation, we find that

$$\det A = \left[\frac{\nu\mu^{\beta+1} + (1-\mu)(\beta) + \nu^{\beta+1}}{\beta+1}\right] > 0.$$

Hence, using Cramer rule, and by substitution into the Eq.(3.5), we obtain the form of our solution.

Explicitly, yields

$$\begin{aligned}
\int_0^\mu u(s)ds = & \frac{1}{\nu\mu^{\beta+1} + (1-\mu)(\beta+\nu^{\beta+1})} \\
& \times\Bigg[(\beta+\nu^{\beta+1})\Bigg(\frac{1}{\Gamma(\alpha+\beta+1)}\int_0^\mu (\mu-s)^{\alpha+\beta} f(s,u(s),I^\sigma u(s)ds \\
& -\frac{\gamma}{\Gamma(\beta+1)}\int_0^\mu (\mu-s)^\beta u(s)ds + \frac{\mu^{\beta+1}}{\beta+1}\Big[\frac{\gamma}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}u(s)ds \\
& -\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1} f(s,u(s),I^\sigma u(s))ds\Big]\Bigg) \\
& +\frac{\mu^{\beta+1}}{\Gamma(\alpha+\beta+1)}\int_0^1 (1-s)^{\alpha+\beta} f(s,u(s),I^\sigma u(s))ds \\
& -\frac{\mu^{\beta+1}}{\Gamma(\alpha+\beta+1)}\int_0^\nu (\nu-s)^{\alpha+\beta} f(s,u(s),I^\sigma u(s))ds \\
& +\frac{\gamma\mu^{\beta+1}}{\Gamma(\beta+1)}\int_0^\nu (\nu-s)^\beta u(s)ds \\
& -\frac{\gamma\mu^{\beta+1}}{\Gamma(\beta+1)}\int_0^1 (1-s)^\beta u(s)ds \\
& +\frac{(1-\nu^{\beta+1})\mu^{\beta+1}}{\beta+1}\Big[\frac{\gamma}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}u(s)ds \\
& -\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1} f(s,u(s),I^\sigma u(s))ds\Big]\Bigg].
\end{aligned}$$

Let denote by $\lambda = \nu\mu^{\beta+1} + (1-\mu)(\beta+\nu^{\beta+1})$, we have

$$\begin{aligned}
\int_0^\mu u(s)ds = & \frac{\beta+\nu^{\beta+1}}{\lambda\Gamma(\alpha+\beta+1)}\int_0^\mu (\mu-s)^{\alpha+\beta} f(s,u(s),I^\sigma u(s)ds \\
& -\frac{\gamma(\beta+\nu^{\beta+1})}{\lambda\Gamma(\beta+1)}\int_0^\mu (\mu-s)^\beta u(s)ds
\end{aligned}$$

$$\begin{aligned}
&+\frac{\gamma\mu^{\beta+1}}{\lambda\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}u(s)ds\\
&-\frac{\mu^{\beta+1}}{\lambda\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}f(s,u(s),I^\sigma u(s))ds\\
&+\frac{\mu^{\beta+1}}{\lambda\Gamma(\alpha+\beta+1)}\int_0^1(1-s)^{\alpha+\beta}f(s,u(s),I^\sigma u(s))ds\\
&-\frac{\mu^{\beta+1}}{\lambda\Gamma(\alpha+\beta+1)}\int_0^\nu(\nu-s)^{\alpha+\beta}f(s,u(s),I^\sigma u(s))ds\\
&+\frac{\gamma\mu^{\beta+1}}{\lambda\Gamma(\beta+1)}\int_0^\nu(\nu-s)^{\beta}u(s)ds\\
&-\frac{\gamma\mu^{\beta+1}}{\lambda\Gamma(\beta+1)}\int_0^1(1-s)^{\beta}u(s)ds. \qquad (3.9)
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\int_\nu^1 u(s)ds\\
&=\frac{(1+\beta)(1-\mu)+\mu^{\beta+1}}{\lambda\Gamma(\alpha+\beta+1)}\int_0^1(1-s)^{\alpha+\beta}f(s,u(s),I^\sigma u(s))ds\\
&\quad-\frac{(1-\mu)(1-\nu^{\beta+1})+(1-\nu)\mu^{\beta+1}}{\lambda\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}f(s,u(s),I^\sigma u(s))ds\\
&\quad-\frac{\gamma\left[(1+\beta)(1-\mu)+\mu^{\beta+1}\right]}{\lambda\Gamma(\beta+1)}\int_0^1(1-s)^{\beta}u(s)ds\\
&\quad+\frac{\gamma\left[(1-\mu)(1-\nu^{\beta+1})+(1-\nu)\mu^{\beta+1}\right]}{\lambda\gamma(\beta)}\int_0^1(1-s)^{\beta-1}u(s)ds\\
&\quad+\frac{\gamma\left[(1+\beta)(1-\mu)+\mu^{\beta+1}\right]}{\lambda\Gamma(\beta+1)}\int_0^\nu(\nu-s)^{\beta}u(s)ds\\
&\quad-\frac{(1+\beta)(1-\mu)+\mu^{\beta+1}}{\lambda\Gamma(\alpha+\beta+1)}\int_0^\nu(\nu-s)^{\alpha+\beta}f(s,u(s),I^\sigma u(s))ds\\
&\quad+\frac{\beta(1-\nu)+\nu^{\beta+1}-\nu}{\lambda\gamma(\alpha+\beta+1)}\int_0^\mu(\mu-s)^{\alpha+\beta}f(s,u(s),I^\sigma u(s))ds\\
&\quad-\frac{\gamma\left[\beta(1-\nu)+\nu^{\beta+1}-\nu\right]}{\lambda\Gamma(\beta+1)}\int_0^\mu(\mu-s)^{\beta}u(s)ds. \qquad (3.10)
\end{aligned}$$

Now, let denote by $F(t) := f(t, u(t), I^\sigma u(t))$. So, the main result is.

Lemma 3.1. *Let u satisfy the* **Problem** (1.1)*. For $F \in C([0,1])$, our problem has a unique solution given as follows*

$$\begin{aligned}
u(t)=&I^{\alpha+\beta}f(t,u(t),I^\sigma u(t))-\gamma I^\beta u(t)+\phi(t)\Big[\gamma I^\beta u(1)+I^{\alpha+\beta+1}F(1)+\gamma I^{\beta+1}u(\nu)\\
&-I^{\alpha+\beta}F(1)-\gamma I^{\beta+1}u(1)-I^{\alpha+\beta+1}F(\nu)\Big]\\
&+\psi(t)\Big[I^{\alpha+\beta+1}F(\mu)-\gamma I^{\beta+1}u(\mu)\Big],
\end{aligned}$$

where

$$\begin{cases} \phi(t) = \dfrac{\mu^{\beta+1} + (1-\mu)(1+\beta)t^{\beta}}{\lambda}, \\ \psi(t) = \dfrac{\beta + \nu^{\beta+1} - \nu(1+\beta)t^{\beta}}{\lambda}. \end{cases}$$

Proof. By putting at mind all the previous part. Hereafter, by substitution of Eq.(3.9) and Eq.(3.10) into Eq.(3.5), one has

$$\begin{aligned}
& u(t) \\
=& I^{\alpha+\beta} f(t, u(t), I^{\sigma} u(t)) - \gamma I^{\beta} u(t) \\
& + \frac{\gamma}{\lambda\Gamma(\beta)} \left[\mu^{\beta+1} + (1-\mu)(1+\beta)t^{\beta}\right] \int_0^1 (1-s)^{\beta-1} u(s) ds \\
& - \frac{1}{\lambda\Gamma(\alpha+\beta)} \left[\mu^{\beta+1} + (1-\mu)(1+\beta)t^{\beta}\right] \int_0^1 (1-s)^{\alpha+\beta-1} f(s, u(s), I^{\sigma} u(s)) ds \\
& + \frac{1}{\lambda\Gamma(\alpha+\beta+1)} \left[\mu^{\beta+1} + (1-\mu)(1+\beta)t^{\beta}\right] \int_0^1 (1-s)^{\alpha+\beta} f(s, u(s), I^{\sigma} u(s)) ds \\
& - \frac{\gamma}{\lambda\Gamma(\beta+1)} \left[\mu^{\beta+1} + (1-\mu)(1+\beta)t^{\beta}\right] \int_0^1 (1-s)^{\beta} u(s) ds \\
& + \frac{\gamma}{\lambda\Gamma(\beta+1)} \left[\mu^{\beta+1} + (1-\mu)(1+\beta)t^{\beta}\right] \int_0^{\nu} (\nu-s)^{\beta} u(s) ds \\
& - \frac{1}{\lambda\Gamma(\alpha+\beta+1)} \left[\mu^{\beta+1} + (1-\mu)(1+\beta)t^{\beta}\right] \int_0^{\nu} (\nu-s)^{\alpha+\beta} f(s, u(s), I^{\sigma} u(s)) ds \\
& + \frac{1}{\lambda\Gamma(\alpha+\beta+1)} \left[\beta + \nu^{\beta+1} - \nu(1+\beta)t^{\beta}\right] \int_0^{\mu} (\mu-s)^{\alpha+\beta} f(s, u(s), I^{\sigma} u(s)) ds \\
& - \frac{\gamma}{\lambda\Gamma(\beta+1)} \left[\beta + \nu^{\beta+1} - \nu(1+\beta)t^{\beta}\right] \int_0^{\mu} (\mu-s)^{\beta} u(s) ds \\
=& I^{\alpha+\beta} f(t, u(t), I^{\sigma} u(t)) - \gamma I^{\beta} u(t) + \phi(t) \Bigg[\gamma I^{\beta} u(1) + I^{\alpha+\beta+1} F(1) + \gamma I^{\beta+1} u(\nu) \\
& - I^{\alpha+\beta} F(1) - \gamma I^{\beta+1} u(1) - I^{\alpha+\beta+1} F(\nu)\Bigg] + \psi(t) \Bigg[I^{\alpha+\beta+1} F(\mu) - \gamma I^{\beta+1} u(\mu)\Bigg],
\end{aligned}$$

$$\begin{cases} \phi(t) = \dfrac{\mu^{\beta+1} + (1-\mu)(1+\beta)t^{\beta}}{\lambda}, \\ \psi(t) = \dfrac{\beta + \nu^{\beta+1} - \nu(1+\beta)t^{\beta}}{\lambda}. \end{cases}$$

□

Now, in order to prove our main result by applying the **Theorem 2.1**, let consider $X = C([0,1], \mathbb{R})$ the space of all continuous functions on $[0,1]$ equipped with the norm $\|u\| = \sup_{t\in[0,1]} |u(t)|$. So, $(X, \|.\|)$ is a Banach space.

Moreover, we assume the following requirements

$(\mathcal{A}_1)$: There exist $K_f, L_f > 0$ such that for all $t \in I$, we have

$$|f(t,u,v) - f(t,u',v')| \leq K_f |u - u'| + L_f |v - v'|, \quad \forall\, u, v, u', v' \in \mathbb{R}.$$

$(\mathcal{A}_2)$: There exists a function $w \in C(I, \mathbb{R}_+)$ satisfying

$$|f(t,u,v)| \leq w(t),\ \forall t \in I,\ \forall u, v \in \mathbb{R}.$$

The existence of solutions is reformulated as follow:

Theorem 3.1. *Assume that the hypotheses* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$ *are hold. We suppose also that*

$$K := \frac{1}{\Gamma(\alpha+\beta+1)}\left[K_f + \frac{L_f}{\Gamma(\sigma+1)}\right] + \frac{\gamma}{\Gamma(\beta+1)} < 1$$

and

$$\eta_1 := \frac{\gamma\big[(\beta+\nu)\|\phi\| + \mu\|\psi\| + \beta + 1\big]}{\Gamma(\beta+2)} < 1.$$

Then the **Problem** (1.1) *has at least one solution.*

Proof. Let us consider the following closed ball $B_e = \{u \in X : \|u\| \leq e\}$ where

$$e \geq \frac{\eta_2}{1-\eta_1} \text{ with } \eta_2 := \frac{\|w\|\big[(\alpha+\beta+\nu)\|\phi\| + \mu\|\psi\| + \alpha+\beta+1\big]}{\Gamma(\alpha+\beta+2)}.$$

Now, we define the operator Θ by

$$\begin{aligned}\Theta u(t) = {} & I^{\alpha+\beta} f(t, u(t), I^{\sigma} u(t)) - \gamma I^{\beta} u(t) \\ & + \phi(t)\Big[\gamma I^{\beta} u(1) + I^{\alpha+\beta+1} F(1) + \gamma I^{\beta+1} u(\nu) \\ & - I^{\alpha+\beta} F(1) - \gamma I^{\beta+1} u(1) - I^{\alpha+\beta+1} F(\nu)\Big] \\ & + \psi(t)\Big[I^{\alpha+\beta+1} F(\mu) - \gamma I^{\beta+1} u(\mu)\Big].\end{aligned}$$

It is remarkable that Θ can be written as $\Theta u(t) = \Theta_1 u(t) + \Theta_2 u(t)$ where

$$\Theta_1 u(t) = I^{\alpha+\beta} f(t, u(t), I^{\sigma} u(t)) - \gamma I^{\beta} u(t),$$

and

$$\begin{aligned}\Theta_2 u(t) = {} & \phi(t)\Big[\gamma I^{\beta} u(1) + I^{\alpha+\beta+1} F(1) + \gamma I^{\beta+1} u(\nu) \\ & - I^{\alpha+\beta} F(1) - \gamma I^{\beta+1} u(1) - I^{\alpha+\beta+1} F(\nu)\Big] \\ & + \psi(t)\Big[I^{\alpha+\beta+1} F(\mu) - \gamma I^{\beta+1} u(\mu)\Big].\end{aligned}$$

(i) Firstly, for $u, v \in B_e$, $\Theta u + \Theta v \in B_e$.
Indeed, we have

$$\begin{aligned}& \|\Theta_1 u + \Theta_2 v\| \\ \leq {} & \sup_{t\in I}\big[I^{\alpha+\beta} f(t, u(t), I^{\sigma} u(t)) - \gamma I^{\beta} u(t)\big] \\ & + \sup_{t\in I}|\phi(t)|\Big[\gamma|I^{\beta+1} v(1) - I^{\beta} v(1)| + \gamma|I^{\beta+1} v(\nu)|\end{aligned}$$

$$+|I^{\alpha+\beta+1}F(1) - I^{\alpha+\beta}F(1)| + |I^{\alpha+\beta+1}F(\nu)|\Bigg]$$
$$+\sup_{t\in I}|\psi(t)|\Bigg[|I^{\alpha+\beta+1}F(\mu)| + \gamma|I^{\beta+1}v(\mu)|\Bigg].$$

Furthermore, we note that

$$|I^{\alpha+\beta}f(t,u(t),I^{\sigma}u(t))| \leq \frac{\|w\|}{\Gamma(\alpha+\beta+1)},\ \forall t\in I.$$

Also, we have

$$|I^{\beta}u(t)| \leq \frac{e}{\Gamma(\beta+1)},\ \forall t\in I,$$
$$|I^{\beta+1}v(1) - I^{\beta}v(1)| \leq \frac{\beta e}{\Gamma(\beta+2)},$$
$$|I^{\beta+1}v(\nu)| \leq \frac{\nu e}{\Gamma(\beta+2)},$$
$$|I^{\alpha+\beta+1}F(1) - I^{\alpha+\beta}F(1)| \leq \frac{(\alpha+\beta)\|w\|}{\Gamma(\beta+2)}$$

and

$$|I^{\alpha+\beta+1}F(\nu) \leq \frac{\nu\|w\|}{\Gamma(\alpha+\beta+2)},$$
$$|I^{\alpha+\beta+1}F(\mu) \leq \frac{\mu\|w\|}{\Gamma(\alpha+\beta+2)},$$
$$|I^{\beta+1}v(\mu) \leq \frac{\mu e}{\Gamma(\beta+2)}.$$

Consequently, we get

$$\|\Theta_1 u + \Theta_2 v\| \leq \eta_1 e + \eta_2 \leq e.$$

(ii) We show that Θ_1 is a contraction.
In fact, for $u, v \in B_e$, yields

$$\begin{aligned}
&\|\Theta_1 u - \Theta_1 v\|\\
&\leq \sup_{t\in I}\left[\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}|f(s,u(s),I^{\sigma}u(s) - f(s,v(s),I^{\sigma}v(s)|ds\right.\\
&\quad\left. + \frac{\gamma}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|u(s)-v(s)|ds\right]\\
&\leq \sup_{t\in I}\left[\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}\big(K_f|u(s)-v(s)| + L_f|I^{\sigma}u(s) - I^{\sigma}v(s)|\big)ds\right]\\
&\quad + \frac{\gamma}{\Gamma(\beta+1)}\|u-v\|\\
&\leq \left[\frac{1}{\Gamma(\alpha+\beta+1)}\left(K_f + \frac{L_f}{\Gamma(\sigma+1)}\right) + \frac{\gamma}{\Gamma(\beta+1)}\right]\|u-v\|\\
&\leq K\|u-v\|,
\end{aligned}$$

which prove that Θ_1 is a contraction.

(iii) Now, we show that Θ_2 is compact.

Note that Θ_2 is continuous and similarly to (i), for $u \in B_e$, one has

$$\|\Theta_2 u\| \leq \left[\frac{\gamma(\beta+\nu)}{\Gamma(\beta+2)} + \frac{(\alpha+\beta+\nu)\|w\|}{\Gamma(\alpha+\beta+2)}\right]\|\phi\| + \mu\left[\frac{1}{\Gamma(\beta+2)} + \frac{\|w\|}{\Gamma(\alpha+\beta+2)}\right]\|\psi\|,$$

which means that Θ is uniformly bounded on B_e.

Denote by $M_f = \sup_{(t,u,v)\in I\times B_e^2} |f(t,u,v)|$.

Hereafter, for $u \in B_e$, we have

$$\|I^\sigma u\| \leq \frac{\|u\|}{\Gamma(\sigma+1)} \leq e,$$

because $\Gamma(\sigma+1) \geq 1$ for all $\sigma \in [1,2]$.

Thus, we obtain that $I^\sigma u \in B_e$.

Moreover, for $0 < \tau_1 < \tau_2 < 1$, yields

$$\begin{aligned}
&\|\Theta_2 u(\tau_2) - \Theta_2 u(\tau_1)\\
\leq\ &\left|\gamma I^\beta u(1) + I^{\alpha+\beta+1}F(1) + \gamma I^{\beta+1}u(\nu) - I^{\alpha+\beta}F(1) - \gamma I^{\beta+1}u(1) - I^{\alpha+\beta+1}F(\nu)\right|\\
&|\phi(\tau_2) - \phi(\tau_1)| + \left|I^{\alpha+\beta+1}F(\mu) - \gamma I^{\beta+1}u(\mu)\right||\psi(\tau_2) - \psi(\tau_1)|.
\end{aligned}$$

On the other hand, we have

$$|\phi(\tau_2) - \phi(\tau_1)| \leq \frac{(1-\mu)(1+\beta)}{\lambda}|\tau_2^\beta - \tau_1^\beta|,$$

and

$$|\psi(\tau_2) - \psi(\tau_1)| \leq \frac{\nu(1+\beta)}{\lambda}|\tau_2^\beta - \tau_1^\beta|,$$

which insure that Θ_2 is equicontinuous.

Hence, it is relatively compact on B_e. So, Θ_2 is compact due to Arzela-Ascoli theorem. Finally, according to Krasnoselskii's fixed point theorem, we deduce that our **Problem** (1.1) admits at least one solution. □

Now, let denote by

$$\tilde{K} := \left(K_f + \frac{L_f}{\Gamma(\sigma+1)}\right)\frac{(\alpha+\beta+\nu)\|\phi\| + \mu\|\psi\|}{\Gamma(\alpha+\beta+2)},$$

and

$$\tilde{\eta}_2 = \frac{(\alpha+\beta+\nu)\|\phi\| + \mu\|\psi\| + \alpha+\beta+1}{\Gamma(\alpha+\beta+2)}.$$

So, we have the following result:

Theorem 3.2. *Let assume that the hypothesis* $(\mathcal{A}_1)$ *is satisfied. Suppose also that*

$$\varepsilon := K + \tilde{K} + \eta_1 < 1. \tag{3.11}$$

Then, the **Problem** (1.1) *has a unique solution.*

Proof. We consider as precedently the closed ball B_e of radius e satisfying

$$e \geq \frac{\tilde{\eta}_2 \Lambda_f}{1-\varepsilon},$$

where $\Lambda_f := \sup_{t\in I} |f(t,0,0)|$.

Let us firstly claim that the operator Θ maps B_e into itself.

For $t \in I$ and $x, y \in \mathbb{R}$, we have

$$\begin{aligned}|f(t,x,y)| &\leq |f(t,x,y) - f(t,0,0)| + |f(t,0,0)| \\ &\leq K_f|x| + L_f|y| + \Lambda_f.\end{aligned}$$

Thus, for all $(t,u) \in I \times B_e$, we obtain

$$\begin{aligned}|f(t,u(t),I^\sigma u(t))| &\leq K|u(t)| + L_f|I^\sigma u(t)| + \Lambda_f \\ &\leq K_f e + \frac{L_f}{\Gamma(\sigma+1)} e + \Lambda_f \\ &= \big(K_f + \frac{L_f}{\Gamma(\sigma+1)}\big) e + \Lambda_f.\end{aligned}$$

So, one has

$$|I^{\alpha+\beta} f(t,u(t),I^\sigma u(t))| \leq \frac{1}{\Gamma(\alpha+\beta+1)} \big[(K_f + \frac{L_f}{\Gamma(\sigma+1)})e + \Lambda_f\big].$$

For $t \in I$ and $u \in B_e$, yields that

$$\begin{aligned}|\Theta u(t)| \leq& |I^{\alpha+\beta} f(t,u(t),I^\sigma u(t))| + \gamma |I^\beta u(t)| \\ &+ \|\phi\| \Bigg[\gamma |I^{\beta+1} u(1) - I^\beta u(1)| + \gamma |I^{\beta+1} u(\nu)| \\ &+ |I^{\alpha+\beta+1} F(1) - I^{\alpha+\beta} F(1)| + \big|I^{\alpha+\beta+1} F(\nu)\big| \Bigg] \\ &+ \|\psi\| \left[|I^{\alpha+\beta+1} F(\mu)| + \gamma |I^{\beta+1} u(\mu)| \right] \\ \leq& \frac{1}{\Gamma(\alpha+\beta+1)} \Bigg[\left(K_f + \frac{L_f}{\Gamma(\sigma+1)} \right) e + \Lambda_f \Bigg] \\ &+ \frac{\gamma}{\Gamma(\beta+1)} e + \|\phi\| \Bigg[\frac{\beta\gamma e}{\Gamma(\beta+2)} + \frac{\gamma\nu e}{\Gamma(\beta+2)} \\ &+ \frac{\alpha+\beta}{\Gamma(\alpha+\beta+2)} \big[(K_f + \frac{L_f}{\Gamma(\sigma+1)}) e + \Lambda_f \big] \\ &+ \frac{\nu}{\Gamma(\alpha+\beta+2)} \big[(K_f + \frac{L_f}{\Gamma(\sigma+1)}) e + \Lambda_f \big] \\ &+ \|\psi\| \big[\frac{\mu}{\Gamma(\alpha+\beta+2)} \big[(K_f + \frac{L_f}{\Gamma(\sigma+1)}) e + \Lambda_f \big] \big] + \frac{\mu\gamma e}{\Gamma(\beta+2)} \Bigg] \\ \leq& \Bigg[\big(K_f + \frac{L_f}{\Gamma(\sigma+1)}\big) \Bigg(\frac{1}{\Gamma(\alpha+\beta+1)} + \frac{(\alpha+\beta+\nu)\|\phi\| + \mu\|\psi\|}{\Gamma(\alpha+\beta+2)} \Bigg) \\ &+ \frac{\gamma(\beta+1)}{\Gamma(\beta+2)} + \frac{\gamma\nu\|\phi\| + \gamma\mu\|\psi\| + \beta\gamma\|\phi\|}{\Gamma(\beta+2)} \Bigg] e\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\alpha+\beta+1)+(\alpha+\beta+\nu)\|\phi\|+\mu\|\psi\|}{\Gamma(\alpha+\beta+2)}\Lambda_f \\
\leq & \varepsilon e+\tilde{\eta}_2\Lambda_f \\
\leq & e.
\end{aligned}
$$

Now, let us show that Θ is a contraction.

For $u, v \in B_e$ and $t \in I$, we have

$$
\begin{aligned}
|\Theta u(t)-\Theta v(t)| &\leq |I^{\alpha+\beta} f(t, u(t), I^\sigma u(t)) - I^{\alpha+\beta} f(t, v(t), I^\sigma v(t))| \\
&\quad + \gamma |I^\beta u(t) - I^\beta v(t)| \\
&\leq \left[\frac{1}{\Gamma(\alpha+\beta+1)}\left(K_f+\frac{L_f}{\Gamma(\sigma+1)}\right)+\frac{\gamma}{\Gamma(\beta+1)}\right]\|u-v\| \\
&\leq K\|u-v\| \\
&\leq \varepsilon\|u-v\|,
\end{aligned}
$$

which prove that Θ is a contraction.

Therefore, according to the Banach fixed point theorem, we infer the result. □

Remark 3.1. Note that the assumption (3.11) leads the one given in **Theorem** (3.1).

Now, we give two examples to illustrate our obtained results.

Example 3.1. We can consider the following problem

$$
\begin{cases}
{}^cD^{\frac{1}{4}}\left({}^cD^{\frac{3}{2}}+\frac{1}{2}\right)u(t)=\dfrac{|u(t)|}{(t+3)^2}\left(\dfrac{|u(t)|}{|u(t)|+3}+3\right)+\dfrac{{}^cD^{\frac{3}{2}}u(t)}{(t+2)^2}-1, \ \ t\in[0,1], \\
u(0)=\displaystyle\int_0^{\frac{1}{4}} u(s)ds, \\
u(1)=\displaystyle\int_{\frac{1}{2}}^{1} u(s)ds, \\
u'(0)=0.
\end{cases}
$$

This problem can be abstracted into the **Problem** (1.1), where

$$
f(t,x,y)=\frac{|x|}{(t+3)^2}\left(\frac{|x|}{|x|+3}+3\right)+\frac{y}{(t+2)^2}-1.
$$

One can easily show that all assumptions of **Theorem 3.1** are satisfied with

$$
\alpha=\frac{1}{4},\ \beta=\frac{3}{2},\ n=2,\ \gamma=\nu=\frac{1}{2},\ \mu=\frac{1}{4},\ K_f=\frac{4}{9},\ L_f=\frac{1}{4}.
$$

And $K \approx 0,7693 < 1,\ \eta_1 \approx 0,8198 < 1$.

So, this problem has at least one solution.

Moreover, one note that $\tilde{K} \approx 0,5288$ and $\varepsilon \approx 2,1180 > 1$.

Consequently, one cannot ensure the uniqueness of the solution by the **Theorem 3.2**.

Example 3.2. Let the following problem,

$$\begin{cases} {}^cD^{\frac{1}{2}}\left({}^cD^{\frac{5}{2}}+\frac{1}{8}\right)u(t)=\dfrac{3+|u(t)|+|{}^cD^{\frac{3}{2}}u(t)|}{5\big(2+|u(t)+|{}^cD^{\frac{3}{2}}u(t)|\big)},\ t\in[0,1],\\ u(0)=\displaystyle\int_0^{\frac{1}{2}}u(s)ds,\\ u(1)=\displaystyle\int_{\frac{3}{4}}^{1}u(s)ds,\\ u'(0)=u''(0)=0. \end{cases}$$

The above problem can be seen as system of the **Problem** (1.1), where

$$f(t,x,y)=\frac{3+|x|+|y|}{5(2+|x|+|y|)}.$$

In this case, we have taken

$$\alpha=\frac{1}{2},\ \beta=\frac{5}{2},\ n=3,\ \gamma=\frac{1}{8},\ \sigma=\frac{3}{2},\ \mu=\frac{1}{2},\ \nu=\frac{3}{4},\ K_f=L_f=\frac{1}{5}.$$

In addition, after simple computations, we get

$$K\approx 0,0960,\ \eta_1\approx 0,0874,\ \tilde{K}\approx 0,0811.$$

Moreover, we obtain $\varepsilon\approx 0,2645<1$.

Thus, we see that, in this case, all assumptions of the **Theorem 3.2** are satisfied. So, we infer that this problem has a unique solution.

Acknowledgements. All authors' contributions to this manuscript are the same. All authors read and approved the final manuscript. We are very grateful to the anonymous reviewers for their useful comments that led to improvement of the manuscript.

Funding. The authors have not disclosed any funding.

Ethics declarations.

Conflict of interest. The author have not disclosed any competing interests.

References

[1] S. Abbas, *Existence of solutions to fractional order ordinary and delay differential equations and applications*, Electron. J. Diff. Equ., 2011, 2011(9), 1–11.

[2] S. Abbas, M. Banerjee and S. Momani, *Dynamical analysis of a fractional order modified logistic model*, Comp. Math. Appl., 2011, 62(3), 1098–1104.

[3] R. P. Agarwal, Y. Zhou and Y. He, *Existence of fractional neutral functional differential equations*, Comput. Math. Appl., 2010, 59(3), 1095–1100.

[4] M. S. Ansari, M. Malik and D. Baleanu, *Controllability of prabhakar fractional dynamical systems*, Qual. Theory Dyn. Sys., 2024, 23(2), 1–28.

[5] R. L. Bagley and P. J. Torvik, *A theoretical basis for the application of fractional calculus to viscoelasticity*, J. Rheol., 1983, 27, 201–210.

[6] Z. Baitiche, C. Derbazi and M. M. Matar, *Ulam stability for nonlinear-Langevin fractional differential equations involving two fractional orders in the ψ-Caputo sense*, Applicable Anal., 2022, 101(14), 4866–4881.

[7] M. Benchohra and F. Ouaar, *Existence results for nonlinear fractional differential equations with integral boundary conditions*, Bull. Math. Anal. Appl., 2010, 2(4), 7–15.

[8] K. Deng and H. A. Levine, *The role of critical exponents in blow-up theorems: The sequel*, J. Math. Anal. Appl., 2000, 243, 85–126.

[9] A. El Allaoui, *General Fractional Integro-Differential Equation of Order $\varrho \in (2,3]$ Involving Integral Boundary Conditions*, Commun. Math. Anal., 2023.

[10] A. M. El-Sayed and E. O. Bin-Tahar, *Positive non-decreasing solutions for a multi-term fractional-order functional differential equation with integral conditions*, Elec. J. Diff. Equ., 2011, 2011(166), 18.

[11] L. Gaul, P. Klein and S. Kemple, *Damping description involving fractional operators*, Mech. Syst. Signal Proc., 1991, 5, 81–88.

[12] R. Gorenflo, *Abel Integral Equations with Special Emphasis on Applications*, Lectures Math. Sci. vol. 13, University of Tokyo, 1996.

[13] S. B. Hadid, *Local and global existence theorems on differential equations of non-integer order*, *J. Fract. Calc.*, 1995, 7, 101–105.

[14] D. Hainaut, *A mutually exciting rough jump-diffusion for financial modelling*, Frac. Cal. Appl. Anal., 2024, 1–34.

[15] J. K. Hale and S. Verduyn, *Introduction to Functional Differential Equations*, Appl. Math. Sci., 99, Springer-Verlag, New York, 1993.

[16] H. Hassani, P. Agarwal, Z. Avazzadeh, J. A. Machado, S. Mehrabi and E. Naraghirad, *Optimal solution of a fractional epidemic model of COVID-19*, Nonlinear Studies, 2024, 31(1).

[17] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[18] R. W. Ibrahim, *Existence and uniquness of holomographyic solutions for fractinal Cauchy problem*, J. Math. Anal. Appl., 2011, 380, 232–240.

[19] T. Kanwal, A. Hussain, I. Avcı, S. Etemad, S. Rezapour and D. F. M. Torres, *Dynamics of a model of polluted lakes via fractal–fractional operators with two different numerical algorithms*, Chaos, Solitons & Fractals, 2024, 181, 114653.

[20] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Vol. 204 of North-Holland Mathematics Studies, Elsevier, Amesterdam, 2006.

[21] M. Kirane and N.-E. Tatar, *Nonexistence of solutions to a hyperbolic equation with a time fractional damping*, Z. Anal. Anwend, (J. Anal. Appl.), 2006, 25, 131–142.

[22] E. Lutz, *Fractional Langevin equation*, Phys. Rev. E, 2001, 64, 051106.

[23] I. Podlubny, *Fractional Differential Equations*, Mathematics in Science and Engineering. vol. 198, New York/London, Springer, 1999.

[24] R. Rizwan, *Existence theory and stability analysis of fractional langevin equation*, Int. J. Nonlinear Sci. Numer. Simul., 2019, 20, 833–848.

[25] R. Rizwan and A. Zada, *Nonlinear impulsive Langevin equations with mixed derivatives*, Math. Meth. Appl. Sci., 2020, 43, 427–442.

[26] S. G. Samko, A. A. Kilbas and O. I Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Amesterdam: Gordon and Breach, 1993.

[27] A. Shit and S. N. Bora, *ESR fractional model with non-zero uniform average blood velocity*, Comput. Appl. Math., 2022, 41(8), 354.

[28] C. A. Tudor, *The overdamped generalized Langevin equation with Hermite noise*, Frac. Cal. Appl. Anal., 2023, 26(3), 1082–1103.

[29] J. Vanterler da C. Sousa and E. Capelas De Oliveira, *Leibniz type rule: ψ-Hilfer fractional operator*, Commun. Nonlinear Sci. Numer. Simul., 2019, 77, 305–311.

[30] J. Vanterler da C. Sousa and E. Capelas De Oliveira, *On the ψ-Hilfer fractional derivative*, Commun. Nonlinear Sci. Numer. Simul., 2018, 60, 72–91.

[31] J. Vanterler da C. Sousa, N. N. Magnun dos Satnos, L. A. Magna and E. Capelas de Oliveira, *A new approach to the validation of an ESR fractional model*, Comput. Appl. Math., 2021, 40, 1–20.

[32] B. Zhang, R. Majeed and M. Alam, *On fractional Langevin equations with stieltjes integral conditions*, Math., 2022, 10, 3877.

[33] S. Zhang, *Positive solutions for boundary-value problems of nonlinear fractional equations*, Electron. J. Diff. Equ., 2006, 36, 12.