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ON A CLASS OF $p(x,\cdot)$ -INTEGRO-DIFFERENTIAL KIRCHHOFF-TYPE PROBLEM WITH SINGULAR KERNEL

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Abstract In this paper, we consider a class of $p(x,\cdot)$-integro-differential Kirchhoff-type problem with Dirichlet boundary conditions. Considering various variational methods, we establish the existence of multiple solutions taking into account the different situations concerning the non-linearity and growth conditions.

Keywords General nonlocal integro-differential equation, variational methods, $p(x,\cdot)$-Kirchhoff type problem, generalized fractional Sobolev spaces.

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1. Introduction and statement of the problem

The study of nonlinear problems involving nonlocal integro-differential operators has seen an unusual escalation in the last decades, due mainly to their presence in many real-world applications (see, for instance, [1,10,12,27,30]). In particular, the investigation of Kirchhoff-type equations has attracted great attention in view of its physical description, which considers the effects during the vibration of elastic strings of the changes in the length of this area. More precisely, the model proposed by Kirchhoff [25] is given by the following equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.1}$$

where $\rho, L, h,$ and ρ_0 are parameters described the mass density, the length of the string, the area of cross section, and the initial tension respectively. Alternatively,

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the stationary form of equation (1.1) is expressed as follows

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)\Delta u = f(x,u), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

This formulation has garnered considerable attention since Lions's seminal work [29] by proposing a functional analysis to attack it. Following this work, on the classical case (when s is an integer and $p(x)=p$ =constant), several equations of Kirchhoff type have been widely studied. The study of Kirchhoff-type equations has expanded to include the p-Laplacian operator, which is used in equations such as

$$-M\left(\int_\Omega \frac{1}{p}|\nabla u|^p dx\right)\Delta_p u = f(x,u) \text{ in } \Omega \text{ with } u=0 \text{ on the boundary } \partial\Omega, \tag{1.3}$$

where M and f must satisfy certain conditions. Further details on these conditions can be found in references [14, 15, 20]. Furthermore, many papers prolonged the constant case by including the exponent variable, the authors showed the existence and multiplicity of weak solutions to various problems by applying different variational methods. For example, Chung in [13] used the mountain pass theorem along with Ekeland's variational principle to obtain at least two distinct nontrivial weak solutions for the problem

$$-M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} dx\right)\Delta_{p(x)} u = \lambda f(x,u) \text{ in } \Omega \text{ with } u=0 \text{ on } \partial\Omega. \tag{1.4}$$

Besides, in the case when $\lambda=1$, problem (1.4) has been investigated in reference [16] based on the direct variational approach and the theory of the variable exponent Sobolev spaces.

On the other hand, the incipience of a generalization including the fractional case was by Fiscella and Valdinocci [22], the authors have proposed a stationary Kirchhoff model in the fractional scenario arising from an interesting physical interpretation. In their correction of the initial (one-dimensional) model, they replaced the local spacial second derivative with the nonlocal operator $(-\Delta)^s$, according to this change, the tension on the string which classically has a "nonlocal" nature from the mean of the kinetic energy $\frac{|\partial_x u|^2}{2}$ on $[0,L]$, has a further nonlocal behavior provided by the H^s-norm of the function u. Moreover, the authors investigated the existence of multiple solutions for a Kirchhoff-type problem involving a nonlocal integro-differential operator, they studied the following problem

$$\begin{cases} -M\left(\int_{\mathbb{R}^{2N}} |u(x)-u(y)|^2 K(x-y)dxdy\right)\mathcal{L}_K u = \lambda f(x,u) + |u|^{2^*-2}u, & \text{in } \Omega,\\ u=0, & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.5}$$

where $2^* = \frac{2N}{N-2s}$, λ is a positive parameter, the two functions M and f are continuous and satisfying some hypothesis. In the context of the fractional Laplacian operator, reference is also made to the work by Jia et al. [23]. Besides, other authors extend the scalar case to include problems driven by the fractional p-Laplacian operator, see for instance [8, 34].

Lately, in [3,7], the authors deal with fractional variable exponent case, they generalized the Kirchhoff type problem by replacing the local $p(x)$-Laplacian operator with the nonlocal fractional $p(x,.)$-Laplacian operator. For example, in [7] based on the mountain pass theorem by Ambrosetti and Rabinowitz, direct variational techniques, and Ekeland's variational principle, the authors dealt with the existence of nontrivial weak solutions for a particular class of fractional $p(x)$-Kirchhoff-type problems with Dirichlet boundary conditions. The problem under consideration is formulated as follows

$$(\mathcal{P}_M^s)\begin{cases} \quad M\left(\displaystyle\int_Q \frac{1}{p(x,y)}\frac{|\,u(x)-u(y))^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\mathrm{d}x\ \mathrm{d}y\right)\left(-\Delta^s_{p(x)}\right)u(x)\\ =\lambda|u(x)|^{r(x)-2}u(x) \text{ in } \Omega,\\ u=0 \text{ in } \mathbb{R}^N\backslash\Omega.\end{cases}$$

This investigation explores different scenarios regarding the competition between the growth rates of functions p and r within $(\mathcal{P}_M^s)$, which is crucial in determining the eigenvalue spectrum of the problem.

More generally, this kind of problem was included also the generalized integro-differential operator $\mathcal{L}_K^{p(x,\cdot)}$ [5]. An intriguing extension of Kirchhoff-type problems has allowed numerous authors to investigate problems involving the variable order fractional Laplacian operator, providing a more accurate description of the diffusion process. For details see [28, 33, 36].

In that context, we will inspect if the generalization of the model equation introduced by Kirchhoff [25], which is an extended version of the classical D'Alambert's wave equation, can be considered in a more general functional framework. Therefore, our aim in this paper is to establish the existence and multiplicity of solutions in a smooth bounded domain Ω of $\mathbb{R}^N$ $(N \geqslant 2)$ for the following $p(x,\cdot)$-Kirchhoff-type problem

$$(\mathcal{P})\begin{cases} M\left(\sigma_K^\phi(u)\right)\mathcal{L}_K^\phi\left(u(x)\right)=f(x,u), & \text{in } \Omega,\\ u=0, & \text{on } \mathbb{R}^N\backslash\Omega,\end{cases} \tag{1.6}$$

where

- The map σ_K^ϕ is defined as

$$\sigma_K^\phi(u):=\sigma(u)=\int_Q \Phi\left(u(x)-u(y)\right)K(x,y)dxdy,$$

  and the set Q is given by $Q:=\mathbb{R}^N\times\mathbb{R}^N\backslash(\Omega^c\times\Omega^c)$.
- The operator $\mathcal{L}_K^\phi$ appointed the general nonlocal integro-differential operator introduced in [11] as follows

$$\mathcal{L}_K^\phi u(x)=p.v.\int_{\mathbb{R}^N}\phi(u(x)-u(y))K(x,y)dy, \quad \forall x\in\mathbb{R}^N,$$

  where $p.v.$ commonly refers to the principal value.
- The mapping $\phi:\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous and even function such that
  - (ϕ_1): The map Φ defined in this way

$$\begin{aligned}&\Phi:\mathbb{R}\longrightarrow\mathbb{R},\\ &t\longrightarrow\Phi(t):=\int_0^{|t|}\phi(\tau)d\tau \quad \text{is strictly convex.}\end{aligned} \tag{1.7}$$

- (ϕ_2): There exist two positive constants C_1, and C_2 such that

$$\begin{aligned} &\phi(t)t \geqslant C_1|t|^{p(x,y)} \quad \text{ and } \quad |\phi(t)| \leqslant C_2|t|^{p(x,y)-1}, \\ &\text{for all } t \in \mathbb{R} \text{ and } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N, \end{aligned} \tag{1.8}$$

  where $p(\cdot,\cdot)$ is a bounded variable exponent with suitable assumptions given in Section 2.
- (ϕ_3): For all $t \in \mathbb{R}$, we have

$$\Phi(t) \geqslant \frac{1}{p^+}\phi(t)t. \tag{1.9}$$

- The kernel $K : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0,+\infty)$ belongs to the set of measurable and symmetric functions that verify
  - (K) : There exist two positive constants k_1 and k_2 such that

$$\begin{aligned} &k_1|x-y|^{-N-sp(x,y)} \geqslant K(x,y) \geqslant k_2|x-y|^{-N-sp(x,y)}, \\ &\forall (x,y) \in \mathbb{R}^N \times \mathbb{R}^N \text{ with } x \neq y. \end{aligned} \tag{1.10}$$

- The Kirchhoff function $M : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is continuous and satisfies
  - $(\mathcal{M})$: There exist two positive constants m_1 and m_2 such that

$$m_1 t^{\alpha(x)-1} \leqslant M(t) \leqslant m_2 t^{\alpha(x)-1}, \tag{1.11}$$

    with $m_2 \geqslant m_1$, for some $\alpha \in C_+(\overline{\Omega})$ (see Section 2).
- The non-linearity $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function satisfying the following conditions
  - (f_1) : There exists a constant μ such that

$$0 < \mu F(x,t) \leq t f(x,t), \quad \text{for all} \quad (x,t) \in \overline{\Omega} \times \mathbb{R},$$

    where $F(x,t) := \int_0^t f(x,\tau)d\tau$, also μ verifies this inequality $\mu > \alpha^+ p^+$.
  - (f_2) : $\exists C > 0$, such that $\quad |f(x,t)| \leq C\left(1 + |t|^{q(x)-1}\right)$,
    with $q \in C_+(\overline{\Omega})$, and $q(x) < p_s^*(x) := \frac{N\bar{p}(x)}{N - s\bar{p}(x)}$ for all $x \in \overline{\Omega}$.
  - (f_3) : $f(x,t) = \theta\left(|t|^{\alpha^- p^+ - 1}\right) \longrightarrow 0$ as $t \longrightarrow 0$, for $x \in \overline{\Omega}$ uniformly.
  - (f_4) : $\inf_{\{x \in \Omega, |t|=1\}} F(x,t) > 0$.
  - (f_5) : There exists a positive constant ζ such that
    $f(x,t) \geqslant \zeta |t|^{\gamma(x)-1}$, $t \to 0$, where $\gamma \in C_+(\overline{\Omega})$.

Throughout this paper, it is essential to note that the assumptions $(\phi_1) - (\phi_3)$ and (K) are supposed to be verified and validated.

Now we are in the position to display our main results which are formulated in the following theorems.

Theorem 1.1. *Suppose that conditions (f_2) and $(\mathcal{M})$ hold. Moreover, if $\alpha^- p^- > q^+$. Then there exists a weak solution to problem $(\mathcal{P})$.*

Theorem 1.2. *Assume that conditions (f_1), (f_2), and $(\mathcal{M})$ hold. We suppose also that $\alpha^- p^+ < q^-$. Then problem $(\mathcal{P})$ has a non-trivial weak solution.*

Theorem 1.3. *Assume that conditions (f_1)−(f_4) and $(\mathcal{M})$ are satisfied. If f also is an even function with respect to the second variable and $\alpha(x)p^+ < q^-$ for a.e. $x \in \Omega$. Then problem $(\mathcal{P})$ has infinitely many solutions u_k satisfying $\mathcal{J}(u_k) \longrightarrow \infty$ as $k \longrightarrow \infty$.*

Theorem 1.4. *Assume that conditions (f_1), (f_2), (f_3), (f_5) and $(\mathcal{M})$ hold. If in addition, f is an even function with respect to the second variable and $\gamma^+ < \alpha(x)p^-$ for a.e. $x \in \Omega$. Then problem $(\mathcal{P})$ has a sequence of solutions $(\pm u_k)_{k=1}^{\infty}$ such that $\mathcal{J}(\pm u_k) < 0$, and $\mathcal{J}(\pm u_k) \to 0$ as $k \longrightarrow \infty$.*

The rest of this paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on the general fractional Sobolev space with variable exponent. In section 3, we are going to apply different variational methods in order to establish the existence and multiplicity of weak solutions to our problem. In section 4, we present some examples to illustrate our main results.

2. Preliminaries and functional framework

In this section, we present key facts and foundational properties relevant to the functional framework, alongside essential variational tools employed in this paper.

2.1. Variable exponent Lebesgue spaces

For a more comprehensive understanding of these spaces, we refer the reader to the works of Fan and Kovacik [21,26], as well as the insightful monograph by Diening et al. [19].

The space defined as follows

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable such that: } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\},$$

is known as the variable exponent Lebesgue space. Here, the exponent p belongs to the set

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : p(x) > 1 \quad \text{for all } x \in \overline{\Omega}\},$$

and it verifies the following property

$$1 < \inf_{x\in\overline{\Omega}} p(x) = p^- \leqslant p(x) \leqslant p^+ = \sup_{x\in\overline{\Omega}} p(x) < +\infty.$$

The generalized Lebesgue space is a reflexive, separable, and Banach space if it is equipped with the Luxemburg norm identified as

$$\|u\|_{L^{p(x)}(\Omega)} = \inf \left\{ \lambda > 0; \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} \mathrm{d}x \leqslant 1 \right\},$$

which is equivalent to the modular norm

$$\|u\|_{\rho_{p(.)}} = \inf \left\{ \lambda > 0 : \rho_{p(.)} \left(\frac{u}{\lambda} \right) \leqslant 1 \right\},$$

where the modular $\rho_{p(.)}$ is defined in this way

$$\rho_{p(x)} : L^{p(x)}(\Omega) \longrightarrow \mathbb{R},$$

$$u \longmapsto \int_\Omega |u(x)|^{p(x)}dx,$$

and it plays a crucial role in the analysis and manipulation of generalized Lebesgue spaces.
Besides, the Hölder's inequality holds for variable exponent Lebesgue spaces and it is given by the following inequality

$$\left|\int_\Omega uv \ \mathrm{d}x\right| \leqslant \left(\frac{1}{p^-}+\frac{1}{(\hat{p})^-}\right)\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{\hat{p}(x)}(\Omega)} \leqslant 2\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{\hat{p}(x)}(\Omega)}, \tag{2.1}$$

for all $u \in L^{p(x)}(\Omega)$ and $v \in L^{\hat{p}(x)}(\Omega)$, the latter space represents the conjugate space of $L^{p(x)}(\Omega)$, where its exponent is defined by the relation

$$\frac{1}{p(x)}+\frac{1}{\hat{p}(x)}=1, \text{ for all } x \in \overline{\Omega}.$$

Furthermore, the connection between the modular $\rho_{p(.)}$ and the Luxemburg norm $\|.\|_{L^{p(.)}(\Omega)}$ is summarized in the following Proposition.

Proposition 2.1. *Suppose that* $u \in L^{p(x)}(\Omega)$ *and* $\{u_n\} \subset L^{p(x)}(\Omega)$, *then we have*

i. $\|u\|_{L^{p(x)}(\Omega)} < 1(\textit{resp.} = 1, > 1) \Leftrightarrow \rho_{p(x)}(u) < 1(\textit{ resp.} = 1, > 1)$,

ii. $\|u\|_{L^{p(x)}(\Omega)} < 1 \Rightarrow \|u\|^{p+}_{L^{p(x)}(\Omega)} \leqslant \rho_{p(x)}(u) \leqslant \|u\|^{p-}_{L^{p(x)}(\Omega)}$,

iii. $\|u\|_{L^{p(x)}(\Omega)} > 1 \Rightarrow \|u\|^{p-}_{L^{p(x)}(\Omega)} \leqslant \rho_{p(x)}(u) \leqslant \|u\|^{p+}_{L^{p(x)}(\Omega)}$,

iv. $\lim_{n\to+\infty}\|u_n-u\|_{L^{p(x)}(\Omega)} = 0 \Leftrightarrow \lim_{n\to+\infty}\rho_{p(x)}(u_n-u)=0$.

2.2. The generalized fractional Sobolev spaces with variable exponents

In this subsection, our aim is to provide an overview concerning the generalized fractional Sobolev spaces with variable exponents, covering essential lemmas and properties that will prove useful later on. For more detailed information, we specifically refer to [5,6,9,18,24]. For that reason, it is important to recall the foundational findings of the fractional Sobolev spaces with variable exponents.

Henceforth, we consider $0 < s < 1$, and let $p \in C_+(\overline{Q})$ and satisfy

$$1 < p^- = \min_{(x,y)\in\overline{Q}} p(x,y) \leqslant p(x,y) \leqslant p^+ = \max_{(x,y)\in\overline{Q}} p(x,y) < +\infty, \tag{2.2}$$

along with the symmetry requirement expressed as

$$p(x,y) = p(y,x) \text{ for every } (x,y) \in \overline{Q}. \tag{2.3}$$

As well as $\overline{p}(x) = p(x,x)$ for all $x \in \Omega$. Then, the fractional Sobolev space with variable exponent is defined as follows

$$\begin{aligned} W =& W^{s,p(x,y)}(Q) \\ =& \left\{ \begin{array}{l} u : \mathbb{R}^N \longrightarrow \mathbb{R} \text{ measurable such that } u_{|\Omega} \in L^{\overline{p}(x)}(\Omega) \text{ with} \\ \displaystyle\int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}dxdy < +\infty, \text{ for some } \lambda > 0 \end{array} \right\}. \end{aligned}$$

W is a Banach space under the following norm

$$\|u\|_{W^{s,p(x,y)}(Q)} = \|u\|_{s,p(x,y)} = \|u\|_{L^{\bar{p}(x)}(\Omega)} + [u]_W,$$

with the Gagliardo seminorm is given by the next formula

$$[u]_W = [u]_{s,p(x,y)(Q)} = \inf\left\{\lambda > 0 : \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{sp(x,y)+N}}\, \mathrm{d}x\ \mathrm{d}y \leqslant 1\right\}.$$

Moreover, $\left(W, \|.\|_{W^{s,p(x,y)}(Q)}\right)$ is a reflexive and separable space.

Problem $(\mathcal{P})$ has a variational structure, therefore we need to state an appropriate variational formulation of it by introducing the suitable functional setting which is the general fractional Sobolev space with variable exponent introduced in [5] as follows

$$\begin{aligned} X &= W_K^{s,p(x,y)}(Q) \\ &= \left\{\begin{array}{l} u:\mathbb{R}^N \longrightarrow \mathbb{R} \text{ measurable such that } u_{|\Omega} \in L^{\bar{p}(x)}(\Omega) \text{ with} \\ \displaystyle\int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}} K(x,y)dxdy < +\infty, \text{ for some } \lambda > 0 \end{array}\right\}. \end{aligned}$$

The space X is endowed with the following norm

$$\|u\|_X = \|u\|_{W_K^{s,p(x,y)}(Q)} = \|u\|_{K,p(x,y)} = \|u\|_{L^{p(x)}(\Omega)} + [u]_{K,p(x,y)},$$

where the generalized Gagliardo semi-norm with variable exponent is defined by

$$[u]_{K,p(x,y)} = \inf\left\{\lambda > 0 : \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}} K(x,y)dxdy \leqslant 1\right\}. \tag{2.4}$$

$(X, \|.\|_X)$ is a separable, reflexive and Banach space [5, Corollary 1].

In the following, we consider the linear subspace of X defined as follows

$$X_0 = W_{K,0}^{s,p(x,y)}(Q) = \{u \in W_K^{s,p(x,y)}(Q) \ \text{ such that } u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}.$$

Consequently, the norm related to this functional framework is the generalized Gagliardo semi-norm with variable exponent defined in (2.4). Hence, $\left(X_0, [.]_{K,p(\cdot,\cdot)}\right)$ is a separable, reflexive and Banach space [5, Lemma 8].

A fundamental tool utilized in the analysis of spaces with variable exponents is the modular, as previously mentioned, which in this context is identified as

$$\begin{aligned} \rho^0_{K,p(x,y)} &: X_0 \longrightarrow \mathbb{R}, \\ &\quad u \longmapsto \int_Q |u(x)-u(y)|^{p(x,y)} K(x,y)dxdy. \end{aligned}$$

Similar to the results mentioned above (Proposition 2.1), the general fractional Sobolev space with variable exponent X_0 realized also the following connections between $\rho^o_{K,p(.,.)}$ and the generalized Gagliardo semi norm. As well as, to avoid confusion, we denote $\|u\|_{X_0} = [u]_{K,p(x,y)}$ for any $u \in X_0$.

Proposition 2.2. *Let $p : \overline{Q} \longrightarrow (1,+\infty)$ be a continuous variable exponent verifying conditions* (2.2) *and* (2.3). *$K : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0,+\infty)$ is a measurable and symmetric function satisfying* (1.10). *Then for any $u \in X_0$ and $(u_k) \subset X_0$, we have*

1. $1 \leqslant \|u\|_{X_0} \Rightarrow \|u\|_{X_0}^{p^-} \leqslant \rho^0_{K,p(.,.)}(u) \leqslant \|u\|_{X_0}^{p^+}$,
2. $\|u\|_{X_0} \leqslant 1 \Rightarrow \|u\|_{X_0}^{p^+} \leqslant \rho^0_{K,p(.,.)}(u) \leqslant \|u\|_{X_0}^{p^-}$,
3. $\lim_{k\to+\infty} \|u_k - u\|_{X_0} = 0 \Longleftrightarrow \lim_{k\to+\infty} \rho^0_{K,p(.,.)}(u_k - u) = 0$.

Following this, proceeding by stating a significant embedding result that forms an essential part of upcoming proofs.

Theorem 2.1 ([5, Theorem 3.1])**.** *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and $s \in (0,1)$. Let $p : \overline{Q} \longrightarrow (1,+\infty)$ be a continuous variable exponent satisfies conditions (2.2) and (2.3) with $sp^+ < N$.Consider the kernel $K : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0,+\infty)$ as a measurable and symmetric function with conditions (1.10). Furthermore, Let $q : \bar{\Omega} \longrightarrow (1,+\infty)$ be a continuous bounded variable exponent such that*

$$1 < q^- \leqslant q(x) < p_s^*(x) = \frac{N\bar{p}(x)}{N - s\bar{p}(x)}, \text{ for all } x \in \overline{\Omega}.$$

Then, the space X is continuously embedded in $L^{q(x)}(\Omega)$. Mathematically expressed by the existence of a positive constant $C = C(N,p,q,s,\Omega)$, such that for any $u \in X$, we have

$$\|u\|_{L^{q(x)}(\Omega)} \leqslant C\|u\|_X \leqslant C \max\left\{1, \tilde{k_0}\right\} \|u\|_X,$$

where $\tilde{k_0} = \max\{k_0^{-\frac{1}{p^-}}, k_0^{-\frac{1}{p^+}}\}$. Moreover, this embedding is compact.

Remark 2.1.

(1) The continuous and compact embedding results remain true if we replace X by X_0.

(2) The norms $\|.\|_{X_0}$ and $\|.\|_X$ are equivalent on X_0.

This subsection wraps up with the following lemma that outlines the basic properties of the mapping σ_K^{ϕ}.

Lemma 2.1 (Lemma 2.10, [11])**.** *Assume that conditions (ϕ_1)-(ϕ_3) hold, and the kernel is a measurable and symmetric function that verifies condition (K). Then, the following properties hold true*

(i) The functional σ_K^{ϕ} is well-defined on X_0, $\sigma_K^{\phi} \in C^1(X_0,\mathbb{R})$, and its Gâteaux derivative is given by

$$\langle {\sigma_K^{\phi}}'(u), v\rangle = \int_Q \phi(u(x)-u(y))(v(x)-v(y))K(x,y)\,dx\,dy,$$

for all $u, v \in X_0$.

(ii) The functional σ_K^{ϕ} is weakly lower semicontinuous, implies that if $u_n \rightharpoonup u$ in X_0 as $n \to +\infty$, then $\sigma_K^{\phi}(u) \leq \liminf_{n\to+\infty} \sigma_K^{\phi}(u_n)$.

(iii) The functional ${\sigma_K^{\phi}}' : X_0 \to X_0^*$ is an operator of type (S_+) on X_0, which signifies that if

$$u_n \rightharpoonup u \text{ in } X_0 \text{ and } \limsup_{n\to+\infty} \langle {\sigma_K^{\phi}}'(u_n), u_n - u\rangle \leq 0,$$

therefore $u_n \to u$ in X_0 as $n \to +\infty$.

2.3. Variational tools

In this subsection, we recall the main tools used in the investigation of the existence of weak solutions for problem $(\mathcal{P})$.

Theorem 2.2 ([32, Mountain pass theorem])**.** *Let E be a real Banach space and let $\mathcal{J} \in C^1(E, R)$ satisfies the Palais-Smale condition. Suppose that $\mathcal{J}(0) = 0$ and*

($\mathbf{R_1}$) *: There exist two positive real numbers r and δ such that $\mathcal{J}(u) \geqslant \delta$ for all $u \in E$ with $\|u\|_E = r$.*

($\mathbf{R_2}$) *: There exists $\bar{u} \in E$ with $||\bar{u}||_E > r$, such that $\mathcal{J}(\bar{u}) < 0$.*

Then, $\mathcal{J}$ possesses a critical value $c \geq \delta$. Moreover, c can be characterized as

$$c = \inf_{g\in\Gamma} \max_{z\in[0,1]} \mathcal{J}(g(z)),$$

where

$$\Gamma = \{g \in C([0, 1], E) : g(0) = 0, g(1) = \bar{u}\}.$$

Next, let E be a separable and reflexive Banach space, then we can find $e_j \in E$ and $e_j^* \in E^*$, where E^* is the dual space of E such that

$$E = \overline{\operatorname{span}\{e_j : j = 1, 2, \ldots\}}, \quad E^* = \overline{\operatorname{span}\{e_j^* : j = 1, 2, ...\}},$$

and

$$\langle e_i^*, e_j\rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

For convenience, we constitute $E_j := \operatorname{span}\{e_j\}, Y_k := \oplus_{j=1}^k E_j, Z_k := \overline{\oplus_{j=k}^\infty E_j}$.

Lemma 2.2 ([35, Lemma 4.9])**.** *Let $q \in C_+(\overline{\Omega})$ such that $q(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$. We define*

$$\beta_k = \sup\left\{\|u\|_{L^{q(x)}(\Omega)}; \|u\|_{X_0} = 1, u \in Z_k\right\}.$$

Then $\beta_k \longrightarrow 0 \quad as \quad k \longrightarrow \infty$.

Definition 2.1. Let $\mathcal{J} \in C^1(E, \mathbb{R})$, and $c \in \mathbb{R}$. The function $\mathcal{J}$ satisfies $(PS)_c^*$ (with respect to Y_n), if any sequence $\{u_{n_j}\} \subset E$ such that

$$u_{n_j} \in Y_{n_j}, \quad \mathcal{J}\left(u_{n_j}\right) \to c, \quad \mathcal{J}|'_{Y_{n_j}}\left(u_{n_j}\right) \to 0 \quad \text{in } E^* \text{ as } n_j \to \infty,$$

admits a convergent subsequence.

Theorem 2.3 ([32, Fountain theorem])**.** *Assume that E is a separable Banach space, and $\mathcal{J} \in C^1(E, \mathbb{R})$ is an even functional. Moreover, suppose that for each $k = 1, 2, \cdots$ there exist $\delta_k > r_k > 0$ such that:*

($\mathbf{A_1}$) *: $\inf_{u\in Z_k, \|u\|=r_k} \mathcal{J}(u) \to +\infty$ as $k \to +\infty$.*

($\mathbf{A_2}$) *: $\max_{u\in Y_k, \|u\|=\delta_k} \mathcal{J}(u) \leqslant 0$.*

In addition, if $\mathcal{J}$ verifies the Palais-Smale condition $(PS)_c$ for every $c > 0$. Then $\mathcal{J}$ has an unbounded sequence of critical values.

Theorem 2.4 ([32, Dual fountain theorem]). *Suppose that $\mathcal{J} \in C^1(E,\mathbb{R})$ is an even function where E is a separable Banach space, and for every $k \geqslant k_0 > 0$, there exist $\delta_k > r_k > 0$ such that:*

($\mathbf{B_1}$) *:* $\inf_{u\in Z_k,\|u\|=\delta_k} \mathcal{J}(u) \geqslant 0$.

($\mathbf{B_2}$) *:* $\max_{u\in Y_k,\|u\|=r_k}(\mathcal{J}u) < 0$.

($\mathbf{B_3}$) *:* $a_k = \inf_{u\in Z_k,\|u\|\leqslant\delta_k} \mathcal{J}(u) \longrightarrow 0$ *as* $k \longrightarrow +\infty$.

$\mathcal{J}$ verifies the Palais-Smale condition $(PS)^*_c$ for every $c \in [a_{k_0}, 0)$. Then, $\mathcal{J}$ possesses a sequence of negative critical values converging to 0.

3. Proofs of the main results

In the present section, we establish the proof of the main results outlined in Section 1 in detail.

Definition 3.1. function $u \in X_0$ is said to be weak solution of problem $(\mathcal{P})$ if for every $v \in X_0$, we have

$$M\left(\sigma^{\phi}_K(u)\right)\int_Q \phi\left(u(x)-u(y)\right)\left(v(x)-v(y)\right)K(x,y)dxdy = \int_\Omega f(x,u)vdx.$$

Associated to the problem $(\mathcal{P})$, we define the Euler-Lagrange functional $\mathcal{J}$ as follows

$$\begin{aligned}\mathcal{J} : X_0 &\longrightarrow \mathbb{R},\\ u &\longmapsto \widehat{M}\left(\sigma^{\phi}_K(u)\right) - \int_\Omega F(x,u)dx,\end{aligned}$$

where $\widehat{M}(t) = \int_0^t M(\tau)d\tau$.

Consider the functionals φ and ψ defined on X_0 as follows

$$\begin{aligned}\varphi : X_0 &\to \mathbb{R}, & \psi : X_0 &\to \mathbb{R},\\ u &\mapsto \widehat{M}(\sigma^{\phi}_k(u)), & u &\mapsto \textstyle\int_\Omega F(x,u)\,dx.\end{aligned}$$

From conditions (1.11) and (1.8), we have for any $u \in X_0$

$$\varphi(u) \leqslant \frac{m_2}{\alpha^-}\left(\frac{C_2}{p^-}\rho^0_{K,p(x,y)}(u)\right)^{\alpha(x)}.$$

Using Proposition 2.2, we acquire that φ is well defined. Furthermore, by using conditions (f_2), we get

$$\psi(u) \leqslant C\|u\|_{L^1(\Omega)} + \frac{C}{q^-}\rho_{L^{q(x)}(\Omega)}(u).$$

Since $q(x) < p^*_s(x)$ for all $x \in \Omega$, the well-posedness of ψ is obtained directly by using Proposition 2.1 and Remark 2.1.

On the other hand, $\widehat{M}$ is the primitive of the Kirchhoff function M, hence $\widehat{M} \in C^1(\mathbb{R},\mathbb{R})$, and the fact that $\Phi \in C^1(\mathbb{R},\mathbb{R})$ implies that $\sigma^{\phi}_K \in C^1(X_0,\mathbb{R})$.

Consequently, $\varphi \in C^1(X_0, \mathbb{R})$. On the other hand, f is a Carathéodory function acquires that $\psi \in C^1(X_0, \mathbb{R})$. Therefore, $\mathcal{J}$ is of Class C^1 on X_0. Moreover, its derivative is given by this formula

$$\begin{aligned}\langle \mathcal{J}'(u), v\rangle =& M\left(\sigma_K^{\phi}(u)\right)\int_Q \phi\left(u(x)-u(y)\right)\left(v(x)-v(y)\right)K(x,y)dxdy\\ &-\int_\Omega f(x,u)vdx, \quad \text{for all } (u,v) \in X_0 \times X_0.\end{aligned}$$

Consequently, weak solutions of problem $(\mathcal{P})$ correspond to the critical points of the functional $\mathcal{J}$.

3.1. Proof of Theorem 1.1

By means of the direct variational method, we establish the proof of Theorem 1.1.

Lemma 3.1. *Under the assumptions of Theorem* 1.1*, the following properties hold for the functional* $\mathcal{J}$ *on the space* X_0

(1) $\mathcal{J}$ *is bounded from below and coercive.*

(2) $\mathcal{J}$ *is weakly lower semicontinuous.*

Proof.

(1) Let $u \in X_0$ with $\|u\|_{X_0} > 1$, then we have from conditions (f_2) and $(\mathcal{M})$ that

$$\begin{aligned}\mathcal{J}(u) &= \widehat{M}\left(\sigma_K^{\phi}(u)\right) - \int_\Omega F(x,u)dx\\ &\geq \frac{m_1}{\alpha^+}\left(\sigma_K^{\phi}(u)\right)^{\alpha(x)} - C\left(\int_\Omega |u| + \frac{1}{q(x)}|u|^{q(x)}\,dx\right)\\ &\geq \frac{m_1}{\alpha^+}\left(\frac{C_1}{p^+}\rho^0_{K,p(x,y)}(u)\right)^{\alpha(x)} - C\|u\|_{L^1(\Omega)} - \frac{C}{q^-}\rho_{L^{q(x)}(\Omega)}(u).\end{aligned}$$

Since $q(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$, we obtain from the embedding results stated in Remark 2.1 the existence of two positive constants $\tilde{C}_1$ and $\tilde{C}_2$ such that

$$\|u\|_{L^1(\Omega)} \leqslant \tilde{C}_1\|u\|_{X_0} \text{ and } \|u\|_{L^{q(x)}(\Omega)} \leqslant \tilde{C}_2\|u\|_{X_0}. \tag{3.1}$$

Therefore, using Proposition 2.2 we get

$$\mathcal{J}(u) \geqslant \frac{m_1}{\alpha^+}\left(\frac{C_1}{p^+}\right)^{\alpha^-}\|u\|_{X_0}^{p^-\alpha^-} - C\tilde{C}_1\|u\|_{X_0} - \frac{C\tilde{C}_2}{q^-}\|u\|_{X_0}^{q^+}.$$

Given that $p^-\alpha^- > q^+ > 1$, then $\mathcal{J}(u) \to \infty$ as $\|u\|_{X_0} \to \infty$, as a result $\mathcal{J}$ is coercive and bounded from below.

(2) Let $\{u_n\}$ be a sequence in X_0 such that $u_n \rightharpoonup u$ in X_0 as $n \longrightarrow +\infty$. We have from Lemma 2.10 in [11] that σ_K^{ϕ} is weakly lower semicontinuous, then

$$\sigma_K^{\phi}(u) \leqslant \liminf_{n\to\infty}\sigma_K^{\phi}(u_n),$$

as $\widehat{M}$ is continuous we get that

$$\widehat{M}\left(\sigma_K^{\phi}(u)\right) \leqslant \widehat{M}\left(\liminf_{n\to\infty} \sigma_K^{\phi}(u_n)\right) = \liminf_{n\to\infty} \widehat{M}\left(\sigma_K^{\phi}(u_n)\right).$$

Moreover, the compact embedding results stated in Remark 2.1 implies that

$$u_n \longrightarrow u_0 \text{ in } L^{q(x)}(\Omega),$$
$$u_n \longrightarrow u_0 \text{ in } L^{1}(\Omega).$$

Adding the fact that $\psi \in C^1(X_0, \mathbb{R})$, we obtain

$$\lim_{n\to\infty} \psi(u_n) = \psi(u),$$

therefore ψ is weakly continuous, and consequently ψ is weakly lower semicontinuous. Finally, we conclude that $\mathcal{J}$ is weakly lower semicontinuous on X_0.

□

Conclusion. In the light of the direct variational method, the functional $\mathcal{J}$ attains its minimum on X_0, then problem $(\mathcal{P})$ admits a weak solution and Theorem 1.1 holds.

3.2. Proof of Theorem 1.2

We will employ the mountain pass theorem as our primary tool in establishing Theorem 1.2. Consequently, we state and prove the subsequent results that will come in handy later.

Lemma 3.2. *Assume that conditions* (f_1), (f_2), *and* $(\mathcal{M})$ *are satisfied, then* $\mathcal{J}$ *verifies the Palais Smale compactness condition.*

Proof. Let $\{u_n\}$ be a sequence in X_0 that verifies

$$\mathcal{J}(u_n) \to c \text{ in } \mathbb{R} \quad \text{and} \quad \mathcal{J}'(u_n) \to 0 \quad \text{as} \quad n \to +\infty \quad \text{in} \quad X_0^*. \tag{3.2}$$

From conditions (ϕ_3) and $(\mathcal{M})$, we have

$$\begin{aligned}
& c + \theta_n(1)\left\|u_n\right\|_{X_0} \\
\geq & \mathcal{J}(u_n) - \frac{1}{\mu}\left\langle \mathcal{J}'(u_n), u_n \right\rangle \\
= & \widehat{M}\left(\sigma_K^{\phi}(u_n)\right) - \int_\Omega F(x,u_n)dx + \frac{1}{\mu}\int_\Omega f(x,u_n)u_n dx \\
& - \frac{1}{\mu} M\left(\sigma_K^{\phi}(u_n)\right)\int_Q \phi\left(u_n(x)-u_n(y)\right)\left(u_n(x)-u_n(y)\right)K(x,y)dxdy, \\
\geq & \frac{m_1}{\alpha^+}\left(\sigma_K^{\phi}(u_n)\right)^{\alpha(x)} + \int_\Omega \left(\frac{1}{\mu} f(x,u_n)u_n - F(x,u_n)\right)dx \\
& - \frac{m_2 p^+}{\mu}\left(\sigma_K^{\phi}(u_n)\right)^{\alpha(x)-1}\int_Q \Phi\left(u_n(x)-u_n(y)\right)K(x,y)dxdy,
\end{aligned}$$

using condition (f_1) and Proposition 2.2, we acquire

$$c + \theta_n(1)\left\|u_n\right\|_{X_0} \geq \left(\frac{m_1}{\alpha^+} - \frac{m_2 p^+}{\mu}\right)\left(\sigma_K^{\phi}(u_n)\right)^{\alpha(x)}$$

$$\geq \left(\frac{C_1}{p^+}\right)^{\alpha^-} \left(\frac{m_1}{\alpha^+} - \frac{m_2 p^+}{\mu}\right) \|u_n\|_{X_0}^{\alpha^- p^-}.$$

In consideration of the condition that $\frac{\mu}{\alpha^+ p^+} > 1$, we get that $\{u_n\}$ is bounded in X_0. Accordingly, by the embedding result in Remark 2.1, there exists $u \in X_0$ such that for a subsequence of $\{u_n\}$, we have

$$\begin{aligned}
&u_n \rightharpoonup u \text{ in } X_0,\\
&u_n \to u \text{ a.e. in } \Omega,\\
&u_n \to u \text{ in } L^{q(\cdot)}(\Omega).
\end{aligned}$$

By the application of Hölder's inequality and condition (f_2), we have

$$\begin{aligned}
&\left|\int_\Omega f\left(x, u_n\right)\left(u_n-u\right) dx\right| \\
\leqslant & C \int_\Omega\left|u_n-u\right| dx+C \int_\Omega\left|u_n\right|^{q(x)-1}\left|u_n-u\right| dx \\
\leqslant & \tilde{C}_3\left\|u_n-u\right\|_{L^{q(x)}(\Omega)}+2 C\left\|\left|u_n\right|^{q(x)-1}\right\|_{L^{\hat{q}(x)}(\Omega)}\left\|u_n-u\right\|_{L^{q(x)}(\Omega)}.
\end{aligned}$$

It follows that

$$\int_\Omega f(x,u_n)(u_n-u)dx \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{3.3}$$

Owing to (3.2) and the weak convergence of $\{u_n\}$, we get

$$\lim_{n\to\infty} \left\langle \mathcal{J}'\left(u_n\right), u_n - u\right\rangle = 0,$$

which implies

$$\begin{aligned}
&M\left(\sigma_K^\phi(u_n)\right)\int_Q \phi\left(u_n(x)-u_n(y)\right)\left((u_n-u)(x)-(u_n-u)(y)\right)K(x,y)dxdy\\
&-\int_\Omega f(x,u_n)(u_n-u)dx \longrightarrow 0 \quad \text{as} \quad n\to\infty.
\end{aligned}$$

Combining this fact with (3.3), we obtain

$$\lim_{n\to\infty} M\left(\sigma_k^\phi(u_n)\right)\int_Q \phi\left(u_n(x)-u_n(y)\right)\left((u_n-u)(x)-(u_n-u)(y)\right)K(x,y)dxdy = 0.$$

From the boundedness of $\{u_n\}$, we can assume that

$$\int_Q \Phi\left(u(x)-u(y)\right)K(x,y)dxdy \longrightarrow \ell \geqslant 0 \quad \text{as} \quad n \longrightarrow \infty.$$

In the case when $\ell = 0$, the proof is finished. In the other case ($\ell \neq 0$), by the continuity of Kirchhoff's function, we get

$$\lim_{n\to\infty} M\left(\sigma_K^\phi(u_n)\right) = M(\ell).$$

Therefore,

$$\int_Q \phi\left(u_n(x)-u_n(y)\right)\left((u_n-u)(x)-(u_n-u)(y)\right)K(x,y)dxdy \underset{n\to\infty}{\longrightarrow} 0.$$

By the same argument, we find

$$\int_Q \phi\left(u(x)-u(y)\right)\left((u_n-u)(x)-(u_n-u)(y)\right)K(x,y)dxdy \underset{n\to\infty}{\longrightarrow} 0.$$

From Lemma 2.1, the mapping ${\sigma_K^{\phi}}'$ is of type (S^+), then $\{u_n\}$ converges strongly to u in X_0. In conclusion, $\mathcal{J}$ satisfies the Palais-Smale condition. □

Now, it remains to prove the geometrical conditions of the mountain pass theorem of Ambrosetti and Rabinowitz [2]. Therefore, we have to verify $(\mathbf{R_1})$ and $(\mathbf{R_2})$ in Theorem 2.2.

Proof of $(\mathbf{R_1})$. For any $u \in X_0$ with $\|u\|_{X_0} < 1$, we have based on conditions (f_2) and $(\mathcal{M})$ that

$$\begin{aligned}\mathcal{J}(u) &= \widehat{M}\left(\sigma_K^{\phi}(u)\right) - \int_\Omega F(x,u)\mathrm{d}x \\ &\geqslant \frac{m_1}{\alpha(x)}\left(\sigma_K^{\phi}(u)\right)^{\alpha(x)} - C\int_\Omega\left(|u| + \frac{1}{q(x)}|u|^{q(x)}\right)dx.\end{aligned} \tag{3.4}$$

Conditions (ϕ_1) and (ϕ_2), Proposition 2.2, and the embedding result (3.1) imply that

$$\begin{aligned}\mathcal{J}(u) &\geqslant \frac{m_1}{\alpha^+}\left(\frac{C_1}{p^+}\right)^{\alpha^-}\|u\|_{X_0}^{\alpha^- p^+} - C\tilde{C}_1\|u\|_{X_0} - \frac{C\tilde{C}_2}{q^-}\|u\|_{X_0}^{q^-} \\ &= \|u\|_{X_0}^{\alpha^- p^+}\left(\frac{m_1}{\alpha^+}\left(\frac{C_1}{p^+}\right)^{\alpha^-} - C\tilde{C}_1\|u\|_{X_0}^{1-\alpha^- p^+} - \frac{C\tilde{C}_2}{q^-}\|u\|_{X_0}^{q^- - \alpha^- p^+}\right).\end{aligned}$$

Next, consider the function g defined as follows

$$\begin{aligned} g &: [0,1] \longrightarrow \mathbb{R}, \\ t &\longrightarrow \frac{m_1}{\alpha^+}\left(\frac{C_1}{p^+}\right)^{\alpha^-} - C\tilde{C}_1 t^{1-\alpha^- p^+} - \frac{C\tilde{C}_2}{q^-}t^{q^- - \alpha^- p^+}.\end{aligned}$$

It is obvious to see that g is strictly positive in a neighbourhood of zero, therefore the proof of the condition $(\mathbf{R_1})$ is complete. □

Proof of $(\mathbf{R_2})$. Let us introduce the function $\delta_1 : [1,+\infty) \longrightarrow \mathbb{R}$ by

$$\delta_1(t) = t^{-\mu}F(x,t\tau) - F(x,\tau), \quad \text{for all } (x,\tau) \in \Omega\times\mathbb{R}.$$

δ_1 is differentiable, and for each $t > 1$, we have

$$\delta_1'(t) = t^{-\mu-1}\left(f(x,t\tau)t\tau - \mu F(x,t\tau)\right).$$

The (AR) condition (f_1), implies that δ_1 is an increasing function on $[1,+\infty)$, then

$$\delta_1(t) \geqslant 0 \quad \text{for all } t \in [1,+\infty), \tag{3.5}$$

which means that

$$t^{-\mu}F(x,t\tau) \geqslant F(x,\tau), \quad \text{for all } (x,\tau) \in \Omega\times\mathbb{R}. \tag{3.6}$$

Now, for any $\bar{u}_1 \in X_0$ with $\bar{u}_1 > 0$ and $t > 1$, using the previous result and condition $(\mathcal{M})$ to get

$$\begin{aligned}\mathcal{J}(t\bar{u}_1) &= \widehat{M}\left(\sigma_K^{\phi}(t\bar{u}_1)\right) - \int_{\Omega} F(x, t\bar{u}_1)\mathrm{d}x \\ &\leqslant \frac{m_2}{\alpha^-}\left(\frac{C_2}{p^-}\right)^{\alpha^+} t^{\alpha^+ p^+}\left(\rho^0_{K,p(x,y)}(\bar{u}_1)\right)^{\alpha^+} - t^{\mu}\int_{\Omega} F(x, \bar{u}_1)dx.\end{aligned}$$

Since $\mu > \alpha^+ p^+$ we get $\mathcal{J}(t\bar{u}_1) \longrightarrow -\infty$ as $t \longrightarrow +\infty$.

Consequently, for t large enough, if we take $\bar{u} = t\bar{u}_1$, then there exists $\bar{u} \in X_0$ with $\|\bar{u}\|_{X_0} > r$ such that $\mathcal{J}(\bar{u}) < 0$. □

Conclusion. As a consequence, the geometrical conditions $(\mathbf{R_1})$ and $(\mathbf{R_2})$ of mountain pass theorem are fulfilled. Moreover, from the fact that $\mathcal{J}(0) = 0$ and Lemma 3.2, we conclude that $\mathcal{J}$ has a nontrivial weak solution. Therefore Theorem 1.2 is proved.

3.3. Proof of Theorem 1.3

Now, using the fountain theorem, we show that the problem $(\mathcal{P})$ has a sequence of weak solutions with unbounded energy. To achieve this this, we will check the conditions $(\mathbf{A_1})$ and $(\mathbf{A_2})$ of Theorem 2.3.

- $(\mathbf{A_1})$: On Z_k, based on (f_2) and $(\mathcal{M})$, we obtain

$$\begin{aligned}\mathcal{J}(u) &= \widehat{M}\left(\sigma_K^{\phi}(u)\right) - \int_{\Omega} F(x, u)dx \\ &\geq \frac{m_1}{\alpha^+}\left(\sigma_K^{\phi}(u)\right)^{\alpha(x)} - C\left(\int_{\Omega}|u| + \frac{1}{q(x)}|u|^{q(x)}\,dx\right) \\ &\geq \frac{m_1}{\alpha^+}\left(\frac{C_1}{p^+}\right)^{\alpha^-}\rho^0_{K,p(x,y)}(u)^{\alpha(x)} - C\|u\|_{L^1(\Omega)} - \frac{C}{q^-}\rho_{L^{q(x)}(\Omega)}(u),\end{aligned}$$

using the embedding inequalities (3.1) stated previously and Proposition 2.2, we acquire

$$\mathcal{J}(u) \geq \frac{m_1}{\alpha^+}\left(\frac{C_1}{p^+}\right)^{\alpha^-}\|u\|^{p^-\alpha^-}_{X_0} - C\tilde{C}_1\|u\|_{X_0} - \tilde{C}_4\beta_k^{q^+}\|u\|^{q^+}_{X_0},$$

where β_k is as in Lemma 2.2, and choose $r_k = \left(\frac{\tilde{C}_4\beta_k^{q^+}q^+(p^+)^{\alpha^-}}{m_1 C_1^{\alpha^-}}\right)^{\frac{1}{\alpha^- p^- - q^+}}$. Therefore

$$\mathcal{J}(u) \geqslant m_1\left(\frac{1}{\alpha^+} - \frac{1}{q^+}\right)\left(\frac{C_1}{p^+}\right)^{\alpha^-} r_k^{\alpha^- p^-} - C\tilde{C}_1 r_k.$$

Consequently, since $\alpha^- p^- < q^+$ and from Lemma 2.2, we have that $r_k \longrightarrow +\infty$. as $k \longrightarrow +\infty$. Then, for any $u \in Z_k$ such that $\|u\|_{X_0} = r_k$, we get

$$\inf_{u\in Z_k, \|u\|_{X_0}=r_k}\mathcal{J}(u) \geqslant m_1\left(\frac{1}{\alpha^+} - \frac{1}{q^+}\right)\left(\frac{C_1}{p^+}\right)^{\alpha^-} r_k^{\alpha^- p^-} - C\tilde{C}_1 r_k \longrightarrow +\infty.$$

• $(\mathbf{A_2})$: Let δ_2 be a function defined for all $(x,t)\in\Omega\times\mathbb{R}$ as follows

$$\begin{aligned}\delta_2 : [1,+\infty) &\longrightarrow \mathbb{R},\\ \tau &\longrightarrow F(x,\tau^{-1}t)\tau^{\mu},\end{aligned}$$

δ_2 is differentiable, and for each $t\geqslant 1$, we have

$$\delta_2'(\tau)=\tau^{\mu-1}\left(\mu F(x,\tau^{-1}t)-\tau^{-1}tf(x,\tau^{-1}t)\right).$$

From condition (f_1), we infer that δ_2 is a decreasing function. Integrating this result with condition (f_4) signifies the existence of $\tilde{C}_5>0$ such that

$$F(x,t)\geqslant \tilde{C}_5|t|^{\mu},\quad \text{for any}\quad |t|\geqslant 1,\quad \text{and } x\in\Omega. \tag{3.7}$$

Condition (f_3) implies that there exists $\tilde{C}_6>0$, and $\tilde{t}>0$ such that

$$|F(x,t)|\leqslant \tilde{C}_6|t|^{\alpha^-p^+},\qquad \text{for all } x\in\Omega \qquad \text{and } 0\leqslant |t|\leqslant \tilde{t}. \tag{3.8}$$

Moreover, condition (f_2) simply means that we can find $\tilde{C}_7>0$ such that

$$|F(x,t)|\leqslant \tilde{C}_7|t|^{q^-}\leqslant \tilde{C}_7|t|^{\alpha^-p^+},\qquad \text{for all } \tilde{t}\leqslant |t|\leqslant 1. \tag{3.9}$$

We get from the previous results that

$$F(x,t)\geq \tilde{C}_5|t|^{\mu}-\left(\tilde{C}_6+\tilde{C}_7\right)|t|^{\alpha^-p^+},\ \text{for all } x\in\Omega \text{ and } t\in\mathbb{R}. \tag{3.10}$$

Therefore, combining (3.10), condition $(\mathcal{M})$ and the embedding result in Remark 2.1, we deduce

$$\begin{aligned}\mathcal{J}(u)&=\widehat{M}\left(\sigma_K^{\phi}(u)\right)-\int_{\Omega}F(x,u)dx\\ &\leqslant \frac{m_2}{\alpha^-}\left(\sigma_K^{\phi}(u)\right)^{\alpha(x)}-\tilde{C}_5\int_{\Omega}|t|^{\mu}dx+\left(\tilde{C}_6+\tilde{C}_7\right)\int_{\Omega}|t|^{\alpha^-p^+}dx\\ &\leqslant \frac{m_2}{\alpha^-}\left(\frac{C_2}{p^-}\right)^{\alpha^+}\|u\|_{X_0}^{\alpha^+p^+}-\tilde{C}_8\|u\|_{X_0}^{\mu}+\tilde{C}_9\|u\|_{X_0}^{\alpha^-p^+}.\end{aligned}$$

Since Y_k is a finite space, then all the norms are equivalents on Y_k, also $\mu>\alpha^+p^+$, which implies that condition $(\mathbf{A_2})$ holds for δ_k large enough.

Conclusion. Since $\mathcal{J}$ is an even functional, and by Lemma 3.2, $\mathcal{J}$ verifies the Palais-Smale condition. Therefore $\mathcal{J}$ satisfies the fountain Theorem assumptions, as a consequence, the proof of Theorem 1.3 is completed.

3.4. Proof of Theorem 1.4

To demonstrate the existence of infinitely many solutions with negative energy for problem $(\mathcal{P})$, we will employ the dual fountain theorem, as outlined in Theorem 2.4, as our variational tool. We begin by establishing the Palais-Smale compactness condition $(PS)_c^*$.

Lemma 3.3. *Suppose that conditions (f_1), (f_2) and $(\mathcal{M})$ hold, then $\mathcal{J}$ verifies the $(PS)_c^*$ condition.*

Proof. Let $\{u_{n_j}\}_{n_j}$ be a sequence on X_0 such that

$$u_{n_j} \in Y_{n_j}, \quad \mathcal{J}\left(u_{n_j}\right) \to c, \quad \text{and} \quad \mathcal{J}|'_{Y_{n_j}}\left(u_{n_j}\right) \to 0, \quad \text{as } n_j \to +\infty.$$

By the same argument of Lemma 3.2, we obtain the boundedness of $\{u_{n_j}\}_{n_j}$. The reflexivity of the space X_0 implies the existence of a subsequence of $\{u_{n_j}\}_{n_j}$ such that $u_{n_j} \rightharpoonup u$ in X_0.

On the other hand, we can choose $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \to u$ in X_0 (because $X_0 = \overline{\cup_{n_j} Y_{n_j}}$). Therefore,

$$\begin{aligned}
&\lim_{n_j\to\infty} \left\langle \mathcal{J}'\left(u_{n_j}\right), \left(u_{n_j} - u\right)\right\rangle \\
&= \lim_{n_j\to\infty} \left\langle \mathcal{J}'\left(u_{n_j}\right), \left(u_{n_j} - v_{n_j}\right)\right\rangle + \lim_{n_j\to\infty} \left\langle \mathcal{J}'\left(u_{n_j}\right), \left(v_{n_j} - u\right)\right\rangle \\
&= \lim_{n_j\to\infty} \left\langle \left(\mathcal{J}|_{Y_{n_j}}\right)'\left(u_{n_j}\right), \left(u_{n_j} - v_{n_j}\right)\right\rangle \\
&= 0.
\end{aligned}$$

Consequently, depending on the proof of Lemma 3.2, we get that $u_{n_j} \to u$, hence the proof is completed. □

Proof of (B₁). From conditions (f_2), and (f_3), we can find $\varepsilon > 0$ and $C_\varepsilon > 0$ such that

$$F(x,t) \leq \varepsilon |t|^{\alpha^- p^+} + C_\varepsilon |t|^{q(x)}, \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}. \tag{3.11}$$

For any $u \in Z_k$, such that $\|u\|_{X_0} = 1$, and $0 < t < 1$, we have

$$\begin{aligned}
\mathcal{J}(tu) =& \widehat{M}\left(\sigma_K^\phi(tu)\right) - \int_\Omega F(x,tu)\mathrm{d}x \\
\geqslant& \frac{m_1}{\alpha^+}\left(\sigma_K^\phi(tu)\right)^{\alpha^-} - \varepsilon \int_\Omega |tu|^{\alpha^- p^+} dx - C_\varepsilon \int_\Omega |tu|^{q(x)} dx \\
\geqslant& \frac{m_1}{\alpha^+}\left(\frac{C_1}{p^+}\right)^{\alpha^-} t^{\alpha^- p^+} (\rho^0_{K,p(x,y)})^{\alpha(x)} - \varepsilon t^{\alpha^- p^+} \int_\Omega |u|^{\alpha^- p^+} dx \\
& - t^{q^-} C_\varepsilon \int_\Omega |tu|^{q(x)} dx,
\end{aligned}$$

since $\|u\|_{X_0} = 1$, then $\rho^0_{K,p(x,y)} = 1$. In view of embedding result stated in Remark 2.1 and the definition of β_k, we obtain

$$\begin{aligned}
\mathcal{J}(tu) \geqslant& \frac{m_1}{\alpha^+}\left(\frac{C_1}{p^+}\right)^{\alpha^-} t^{\alpha^- p^+} - \varepsilon t^{\alpha^- p^+} (\tilde{C}_{10})^{\alpha^- p^+} - C_\varepsilon t^{q^-} \beta_k^{q^-} \\
\geqslant& \left(\frac{m_1}{\alpha^+}\left(\frac{C_1}{p^+}\right)^{\alpha^-} - \varepsilon (\tilde{C}_{10})^{\alpha^- p^+}\right) t^{\alpha^- p^+} - C_\varepsilon \beta_k^{q^-} t^{q^-}.
\end{aligned}$$

By choosing $\frac{m_1 (C_1)^{\alpha^-}}{2\alpha^+ (p^+)^{\alpha^-} (\tilde{C}_{10})^{\alpha^- p^+}} > \varepsilon$, we find

$$\mathcal{J}(tu) \geqslant \frac{m_1}{2\alpha^+}\left(\frac{C_1}{p^+}\right)^{\alpha^-} t^{\alpha^- p^+} - C_\varepsilon \beta_k^{q^-} t^{q^-}. \tag{3.12}$$

For k sufficiently large, we take $t = \delta_k$ small enough. Therefore, $\mathcal{J}(tu) \geqslant 0$, for $u \in Z_k$ where $\|u\|_{X_0} = 1$. Hence the condition $(\mathbf{B_1})$ of dual fountain theorem is verified. □

Proof of $(\mathbf{B_2})$. Let $u \in Y_k$ such that $\|u\|_{X_0} = 1$, and let $0 < t < \delta_k < 1$. From conditions (f_5) and $(\mathcal{M})$, we obtain by using the embedding result

$$\begin{aligned}
\mathcal{J}(tu) &= \widehat{M}\left(\sigma_K^{\phi}(tu)\right) - \int_{\Omega} F(x,tu)\mathrm{d}x \\
&\leqslant \frac{m_2}{\alpha^-}\left(\sigma_K^{\phi}(tu)\right)^{\alpha(x)} - \zeta\int_{\Omega}\frac{|tu|^{\gamma(x)}}{\gamma(x)}dx \\
&\leqslant \frac{m_2}{\alpha^-}\left(\frac{C_2}{p^-}\right)^{\alpha^+} t^{\alpha^+p^-} - \frac{\zeta}{\gamma^+}t^{\gamma^+}\int_{\Omega}|u|^{\gamma(x)}dx \\
&\leqslant \frac{m_2}{\alpha^-}\left(\frac{C_2}{p^-}\right)^{\alpha^+} t^{\alpha^+p^-} - \frac{\tau}{\gamma^+}t^{\gamma^+},
\end{aligned}$$

Since $\alpha^+p^- > \gamma^+$, we can find $0 < r_k < \delta_k$ such that $\mathcal{J}(u) < 0$, for $u \in Y_k$ with $\|u\|_{X_0} = r_k$. This shows $(\mathbf{B_2})$. □

Proof of $(\mathbf{B_3})$. We have that $Y_k \cap Z_K = \emptyset$ and for all $0 < r_k < \delta_k$. Moreover, from $(\mathbf{B_2})$, we have that $\mathcal{J}(u) < 0$, for $u \in Y_k$ with $\|u\|_{X_0} = r_k$. Then $a_k < 0$.

On the other hand, for any $0 \leqslant t \leqslant \delta_k \leqslant 1$ and $w \in Z_k$, where $\|w\|_{X_0} = 1$, we obtain by an argument similar to the one in the verification of $(\mathbf{B_1})$ that

$$\mathcal{J}(tw) \geqslant -C_\varepsilon \beta_k^{q^-} t^{q^-}.$$

Hence $0 > a_k \geqslant -C_\varepsilon \beta_k^{q^-}$. Therefore $a_k \to 0$ as $k \to \infty$. Then $(\mathbf{B_3})$ follow. □

Conclusion. In the view of the dual fountain theorem the proof of Theorem 1.4 is reached.

4. Examples

Now, we present some examples and particular cases of the function ϕ, the kernel K, and the nonlinearity f which illustrate the results of this paper. Especially, we will provide an example that verifies the conditions of Theorem 1.4. This is particularly important as the theorem contains numerous conditions on f that appear challenging to satisfy simultaneously. Our example aims to illustrate the feasibility of these conditions.

A typical cases for ϕ and the kernel K are given by

$$\phi(t) = |t|^{p(x,y)-2}t \quad \text{and} \quad K(x,y) = |x-y|^{-(N+sp(x,y))}.$$

In this case, the operator $\mathcal{L}_K^{\phi}$ reduces to the fractional $p(x,.)$-Laplacian operator $(-\Delta)^s_{p(x,.)}$ (See for instance [4,9]). Therefore

$$\sigma_K^{\phi}(u) := \sigma(u) = \int_Q \frac{1}{p(x,y)}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy.$$

We take $N = 3$ and we consider $\Omega = \left\{(x_1,x_2,x_3) \in \mathbb{R}^N :\ x_1^2 + x_2^2 + x_3^2 \leqslant 10\right\}.$

For the variable exponent $p(.,.)$, we can take

$$p(x,y) = \frac{7}{2} + \frac{1}{2}\sin\left[(x_1^2+x_2^2+x_3^2)\pi\right] + \frac{1}{2}\sin\left[(y_1^2+y_2^2+y_3^2)\pi\right].$$

It is clear that $p \in C_+(\overline{Q})$, $p^- = \frac{5}{2}$ and $p^+ = \frac{9}{2}$.

Consequently, the problem $(\mathcal{P})$ becomes

$$(\mathcal{P}_1)\begin{cases} M\left(\int_Q \frac{1}{p(x,y)}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\right)(-\Delta)^s_{p(x,.)}(u(x)) = f(x,u) & \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N\backslash\Omega. \end{cases}$$

For the non-linearity f, we can give the following example

$$f(x,t) = \begin{cases} t^{10}, & |t| > 1, \\ t^6, & |t| \le 1. \end{cases}$$

- By choosing $\mu = 11$, condition (f_1) holds.
- If we take $q(x) = 6 + \sin\left[(x_1+x_2+x_3)\pi\right]$, therefore condition (f_2) is verified.
- If we consider $\gamma(x) = \frac{11}{6} + \frac{1}{2}\sin\left[(x_1+x_2+x_3)\pi\right]$, we have that $\gamma^+ < \alpha(x)p^-$, which means that condition (f_5) is fulfilled.

Conclusion: By considering the above particular cases, for example, the assumptions of Theorem 1.4 are satisfied. So, the results obtained in Theorems 1.4 remain true for problem $(\mathcal{P}_1)$. The problem and results are all new.

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