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HOPF AND ZERO-HOPF BIFURCATIONS FOR A CLASS OF CUBIC KOLMOGOROV SYSTEMS IN $\mathbb{R}^3$ *

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Abstract In this paper, Hopf and zero-Hopf bifurcations are investigated for a class of three-dimensional cubic Kolmogorov systems with one positive equilibrium. Firstly, by computing the singular point quantities and figuring out center conditions, we determined that the highest order of the positive equilibrium is eight as a fine focus, which yields Hopf cyclicity eight at the positive equilibrium. Secondly, by extending the normal form method, we discuss the existence of multiple periodic solutions via zero-Hopf bifurcation around the positive equilibrium. At the same time, the relevance between zero-Hopf bifurcation and Hopf bifurcation is analyzed via its special case, which are rarely studied in detail.

Keywords Three-dimensional Kolmogorov system, Hopf bifurcations, zero-Hopf bifurcations, center manifold, center problem.

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1. Introduction

Since it was proposed in 1936 [15], the Kolmogorov model has become classical and is used widely in ecology to describe the interaction between n species occupying certain same ecological habitat, which usually takes the following form

$$\frac{dx_i}{dt} = x_i f_i(x_1, x_2, \cdots, x_n), \quad i = 1, 2, \cdots, n, \tag{1.1}$$

where f_i are polynomials with respect to $x_1, x_2, \cdots, x_n$. Here, x_i represents the density of the i-th species in a biosphere, and $f_1, f_2, \cdots, f_n$ are the intrinsic growth

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rates or biotic potential of the n species, respectively. Since each species density x_i is nonnegative in reality, we only consider the behavior of the orbits in the positive quadrant $\{(x_1, x_2, \cdots, x_n)^T : x_i > 0, i = 1, 2, \cdots, n\}$. Of particular significance in applications are the existence and number of limit cycles bifurcating from positive equilibrium points, which correspond to the key dynamic behaviors of the changes in species quantities under the influences of internal and external factors. Therefore, it is natural for the topic to attract the attention of many researchers in the field of mathematical ecology [13].

For the planar Kolmogorov system, it is well known that system (1.1) does not have limit cycles if f_1 and f_2 are linear, namely it is the classical Lotka-Volterra-Gause model. When f_1 and f_2 are not linear, many results have been obtained in [5, 12, 19, 28]. For the three-dimensional Kolmogorov system, if f_1, f_2 and f_3 are linear, then system (1.1) is a quadratic Lotka-Volterra system. The relevant results of limit cycle bifurcation can be found in [26, 30] and references therein. When f_1, f_2 and f_3 are not linear, however, the works on this problem are not unusually seen, especially for the cyclicity of Hopf bifurcation, i.e., the maximal number of limit cycles which may exist in the vicinity of a Hopf singular point under proper perturbations.

In 2014, Du et al. [6] investigated one class of three-dimensional cubic Kolmogorov systems with f_1, f_2 and f_3 as quadratic polynomials, and got five small limit cycles bifurcating from a positive singular point. Recently, Gu et al. [9] proved that seven limit cycles can be generated in another class of three-dimensional cubic Kolmogorov systems. Based on the above works, we conjecture the number of limit cycles bifurcating from a single positive equilibrium point can be more than 7 for the three-dimensional cubic Kolmogorov models, and will further investigate it here.

It is well known that Hopf bifurcation is closely related to center-focus determination. For the calculation of focus values on the center manifold, there exist some available methods, such as Lyapunov-Schmidt method [11], the simple normal form method [23], the formal first integral method [7] and the displacement map method [1]. Notably, the authors of [8] presented a general method for bounding the cyclicity in the center case without any kind of reduction to center manifold. Here we will apply the method with linear recursive algorithm proposed by the authors of [27] in 2010 to directly calculate the singular point quantities on the center manifold, its some applications can be seen in [6, 9, 14, 20, 25].

For the zero-Hopf singular point with a zero eigenvalue and a pair of pure imaginary eigenvalues, under certain perturbing conditions including small change of the zero eigenvalue, a limit cycle can be generated around it, this is to say, the zero-Hopf bifurcation can occur. Recently, this problem has been getting more attention, especially in the research of many chaotic models [2, 18, 21]. The common tool for investigating this problem is the average theory, see, e.g., [4, 18, 21]. Notably, the authors of [32] applied the normal form theory to investigate the Rössler system, and showed that the method of normal forms is applicable for all types of zero-Hopf bifurcations. As for its multiple limit cycles bifurcation, there are very few results. To our knowledge, the authors of [4, 21] have discovered the multiple limit cycles by applying averaging theory of second order and first order respectively. Generally, zero-Hopf bifurcation is viewed as one degenerate type of Hopf bifurcation, yet the specific relevance between the two is rarely discussed in the literatures available for reference.

In this paper, we will investigate the multiplicity or cyclicity of Hopf and zero-

Hopf bifurcations for a class of three-dimensional cubic Kolmogorov system, i.e.,

$$\frac{dx_i}{dt} = x_i f_i(x_1, x_2, x_3), \tag{1.2}$$

where we have assumed that system (1.2) has a positive equilibrium point $E = (1, 1, 1)$, namely all $f_i(1, 1, 1) = 0,\ i = 1, 2, 3$. For the convenience of discussion of the center problem and Hopf bifurcation for the positive equilibrium E, we transform the equilibrium E to the origin by means of the transformation:

$$x_1 = x + 1, \quad x_2 = y + 1, \quad x_3 = u + 1,$$

system (1.2) can be rewritten in the following form:

$$\frac{dx}{dt} = (x + 1)\tilde{f}_1, \quad \frac{dy}{dt} = (y + 1)\tilde{f}_2, \quad \frac{du}{dt} = (u + 1)\tilde{f}_3, \tag{1.3}$$

where $\tilde{f}_i = f_i(x+1, y+1, u+1)$. Further, here we have chosen the three incomplete quadratic polynomials as follows,

$$\begin{aligned} \tilde{f}_1 &= \delta x - y + (A_{200}x^2 - A_{200}y^2 - B_{101}yu + A_{002}u^2), \\ \tilde{f}_2 &= x + \delta y + (B_{101}xu + A_{002}u^2), \\ \tilde{f}_3 &= \lambda u + (D_{200}x^2 + D_{200}y^2 + D_{101}xu + D_{011}yu + D_{002}u^2), \end{aligned} \tag{1.4}$$

with δ, λ, $A_{002}, A_{200}, B_{101}, D_{002}, D_{011}, D_{101}$ and D_{200} are nine real parameters.

The rest of this paper is organized as follows. In the next section, some preliminary methods and results are briefly introduced for the later discussion and analysis on Hopf bifurcation. In section 3, the singular point quantities of the origin corresponding to the positive equilibrium of (1.2) are calculated by deriving the recursion formula, then the center conditions of the equilibrium are determined on the center manifold. Further, it is verified that the highest order of the fine focus is eight for the positive equilibrium, which implies the Hopf cyclicity 8 at the positive equilibrium. From the all literature we know, it is the maximum number of limit cycles generated from single equilibrium point of three-dimensional systems. In section 4, by rescaling the variables and extending the normal form method, we investigate the zero-Hopf bifurcations around the positive equilibrium and verify the existence of multiple periodic solutions via zero-Hopf bifurcation. At the same time, the relevance between zero-Hopf bifurcation and Hopf bifurcation is discussed through its special case, and some related numerical illustrations are also given.

2. Preliminary results and method

In this section, we present some basic results and methods that will be used in the following sections. For the planar polynomial systems, Liu and Li [17] proposed a valid method for computing singular point quantities in complex systems in 1990, whose recent application can be found in [16]. In 2010, Wang et al. [27] generalized and developed the method to study the three-dimensional nonlinear dynamical system of the form

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \delta x - y + \displaystyle\sum_{k+j+l=2}^{\infty} A_{kjl} x^k y^j u^l = X(x,y,u), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = x + \delta y + \displaystyle\sum_{k+j+l=2}^{\infty} B_{kjl} x^k y^j u^l = Y(x,y,u), \\ \dfrac{\mathrm{d}u}{\mathrm{d}t} = -du + \displaystyle\sum_{k+j+l=2}^{\infty} d_{kjl} x^k y^j u^l = U(x,y,u), \end{cases} \tag{2.1}$$

where $x, y, u, t, d, \delta, A_{kjl}, B_{kjl}, d_{kjl} \in \mathbb{R}\,(k, j, l \in \mathbb{N})$, $d \neq 0$, and the X, Y and U are all analytic in a neighborhood of the origin. It is not difficult to find that system (1.3) is a subfamily of system (2.1).

By the transformation

$$x = \frac{z+w}{2},\ y = \frac{(w-z)\mathbf{i}}{2},\ t = -T\mathbf{i},\ \mathbf{i} = \sqrt{-1}, \tag{2.2}$$

system $(2.1)|_{\delta=0}$ can also be transformed into the following complex system

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}T} = z + \displaystyle\sum_{k+j+l=2}^{\infty} a_{kjl} z^k w^j u^l = Z(z,w,u), \\ \dfrac{\mathrm{d}w}{\mathrm{d}T} = -w - \displaystyle\sum_{k+j+l=2}^{\infty} b_{kjl} w^k z^j u^l = -W(z,w,u), \\ \dfrac{\mathrm{d}u}{\mathrm{d}T} = \mathbf{i}du + \displaystyle\sum_{k+j+l=2}^{\infty} \tilde{d}_{kjl} z^k w^j u^l = \tilde{U}(z,w,u), \end{cases} \tag{2.3}$$

where $z, w, T, a_{kjl}, b_{kjl}, \tilde{d}_{kjl} \in \mathbb{C}\,, k, j, l \in \mathbb{N}$. Moreover, the coefficients a_{kjl} and b_{kjl} of system (2.3) satisfy a conjugate relationship, namely, $b_{kjl} = \overline{a_{kjl}}$, $k, j, l \in \mathbb{N}$.

Furthermore, we can calculate the singular point quantities of the origin by the method given in Theorem 3.1 of [27], and there exists the algebraic equivalence between the m-th singular point quantity μ_m and the m-th focal value v_{2m+1} at the origin for the bifurcation equations of system (2.3) with $\delta = 0$, i.e.

$$v_{2m+1} \sim \mathbf{i}\pi\mu_m, \quad m = 1, 2, \cdots. \tag{2.4}$$

In order to prove the existence of multiple limit cycles, we introduce the following lemma.

Lemma 2.1. (see [10]) *Suppose that the focus values depend on k parameters, expressed as*

$$v_j = v_j(\epsilon_1, \epsilon_2, \cdots, \epsilon_k), \quad j = 1, 3, \cdots, 2k+1,$$

satisfying $v_j(0, 0, \cdots, 0) = 0$ for $1 \leq j \leq 2k-1$, $v_{2k+1}(0, 0, \cdots, 0) \neq 0$, and

$$\det\left[\frac{\partial(v_1, v_3, \cdots, v_{2k-1})}{\partial(\epsilon_1, \epsilon_2, \cdots, \epsilon_k)}(0, 0, \cdots, 0)\right] \neq 0, \tag{2.5}$$

then the origin of the perturbed system (2.1) has k limit cycles.

In addition, we also want to know whether the origin is center on the manifold if the first m singular point quantities vanish for system $(2.1)|_{\delta=0}$. Constructing first integrals is an effective method to determine the center conditions, and the main tool for constructing first integrals using the Darboux method is provided by the following notions and lemma (one can also see in [3, 22]).

Definition 2.1. Given a polynomial $f \in \mathbb{C}(x, y, u)$, a surface $f = 0$ is called an invariant algebraic surface of the system $(2.1)|_{\delta=0}$, if the polynomial f satisfies the equation

$$\left.\frac{df}{dt}\right|_{(2.1)} = \frac{\partial f}{\partial x}X + \frac{\partial f}{\partial y}Y + \frac{\partial f}{\partial u}U = K_f f, \tag{2.6}$$

for some polynomial $K_f \in \mathbb{C}$. The polynomial K_f is called a cofactor of f.

Definition 2.2. Let $G = \exp(g(x, y, u)/h(x, y, u)) \in \mathbb{C}(x, y, u)$ with $g, h \in \mathbb{C}(x, y, u)$, then G is an exponential factor if there exists a $K_G \in \mathbb{C}(x, y, u)$ such that

$$\left.\frac{dG}{dt}\right|_{(2.1)} = \frac{\partial G}{\partial x}X + \frac{\partial G}{\partial y}Y + \frac{\partial G}{\partial u}U = K_G G. \tag{2.7}$$

The polynomial K_G is called a cofactor of G.

Lemma 2.2. (see [3, 22]) *Suppose that system $(2.1)|_{\delta=0}$ admits p irreducible invariant algebraic curves surface $f_i = 0$ with cofactors K_i for $i = 1, 2, \cdots, p$, and q exponential factors $\exp(g_i/h_j)$ with cofactors L_j for $j = 1, 2, \cdots, q$. If there exist λ_i, η_j not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \eta_j L_j = 0, \tag{2.8}$$

then system $(2.1)|_{\delta=0}$ admits a first integral of the form

$$f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_p^{\lambda_p} (\exp(g_1/h_1))^{\eta_1} \cdots (\exp(g_q/h_q))^{\eta_q}. \tag{2.9}$$

The first integral (2.9) is called a Darboux first integral.

3. The Hopf cyclicity at the positive equilibrium

In this section, the singular point quantities of the corresponding equilibrium are computed. The necessary conditions for the equilibrium point to be a center are found by analyzing the singular point quantities, and the sufficiency of the center conditions is proved by constructing the first integral. Further, the Hopf cyclicity, namely, the maximum number of limit cycles bifurcating from the positive equilibrium is investigated.

3.1. Singular point quantities

Applying transformation (2.2), system (1.3) can become the following complex system with the same form as (2.3):

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}T} = z + Z_2 + Z_3 =: Z, \\ \dfrac{\mathrm{d}w}{\mathrm{d}T} = -w - W_2 - W_3 =: -W, \\ \dfrac{\mathrm{d}u}{\mathrm{d}T} = -\mathbf{i}\lambda u + U_2 + U_3 =: U, \end{cases} \tag{3.1}$$

where

$$\begin{aligned} &Z_2 = a_{101}zu + a_{200}z^2 + a_{020}w^2 + a_{002}u^2, \\ &Z_3 = a_{300}z^3 + a_{030}w^3 + a_{120}zw^2 + a_{102}zu^2 + a_{210}z^2w + a_{201}z^2u + a_{021}w^2u, \\ &W_2 = \overline{Z_2}, \quad W_3 = \overline{Z_3}, \\ &U_2 = d_{110}zw + d_{101}zu + d_{011}wu + d_{002}u^2, \\ &U_3 = d_{111}zwu + d_{012}wu^2 + d_{102}zu^2 + d_{003}u^3, \end{aligned}$$

with

$$a_{101} = B_{101},\ a_{200} = \frac{1-\mathbf{i}}{4} - \frac{A_{200}\mathbf{i}}{2},\ a_{020} = -\frac{1-\mathbf{i}}{4} - \frac{A_{200}\mathbf{i}}{2},\ a_{002} = A_{002}(1-\mathbf{i}),$$

$$a_{300} = a_{030} = a_{120} = a_{210} = -\frac{A_{200}\mathbf{i}}{4},\ a_{102} = -A_{002}\mathbf{i},\ a_{201} = \frac{1-\mathbf{i}}{4}B_{101},\ a_{021} = -a_{201},$$

$$b_{kjl} = \overline{a_{kjl}}, (kjl = 101, 200, 020, 002, 300, 030, 120, 102, 210, 201, 021),$$

$$d_{110} = d_{111} = -D_{200}\mathbf{i},\ d_{101} = d_{102} = -\frac{D_{011} + D_{101}\mathbf{i}}{2},\quad d_{011} = d_{012} = \frac{D_{011} - D_{101}\mathbf{i}}{2},$$

$$d_{002} = -(\lambda + D_{002})\mathbf{i},\ d_{003} = -D_{002}\mathbf{i}.$$

By using the method [27, Theorem 3.1], the recursive formulas for calculating singular point values of system (3.1) at the origin can be obtain as follows.

Lemma 3.1. *For system (3.1), the singular point values $\mu_m (m = 1, 2, \cdots)$ at the origin are determined by the following recursive formula: if $\alpha \neq \beta$ or $\alpha = \beta, \gamma \neq 0$, $c_{\alpha\beta\gamma}$ is determined by the following recursive formula:*

$$c_{\alpha\beta\gamma} = \frac{\Delta}{\beta - \alpha + \mathbf{i}\lambda\gamma} \tag{3.2}$$

where

$$\begin{aligned} \Delta = &- b_{030}(\beta+1)c[\alpha-3, \beta+1, \gamma] + (a_{300}\alpha - 2a_{300} - b_{120}\beta)c[\alpha-2, \beta, \gamma] \\ &- b_{021}(1+\beta)c[\alpha-2, 1+\beta, \gamma-1] - b_{020}(\beta+1)c[\alpha-2, \beta+1, \gamma] \\ &+ (a_{210}\alpha - a_{210} + b_{210} - b_{210}\beta + d_{111}\gamma)c[\alpha-1, \beta-1, \gamma] \\ &+ (a_{201}\alpha - a_{201} - d_{102} + d_{102}\gamma)c[\alpha-1, \beta, \gamma-1] \\ &+ (b_{200} - b_{200}\beta + d_{011}\gamma)c[\alpha, \beta-1, \gamma] + (a_{120}\alpha + 2b_{300} - b_{300}\beta)c[\alpha, \beta-2, \gamma] \\ &+ (a_{101}\alpha - d_{002} - b_{101}\beta + d_{002}\gamma)c[\alpha, \beta, \gamma-1] - b_{002}(1+\beta)c[\alpha, \beta+1, \gamma-2] \\ &+ d_{110}(1+\gamma)c[\alpha-1, \beta-1, \gamma+1] + (a_{200}\alpha - a_{200} + d_{101}\gamma)c[\alpha-1, \beta, \gamma] \end{aligned}$$

$$
\begin{aligned}
&+ a_{030}(\alpha+1)c[\alpha+1,\beta-3,\gamma] + a_{021}(\alpha+1)c[\alpha+1,\beta-2,\gamma-1]\\
&+ a_{020}(\alpha+1)c[\alpha+1,\beta-2,\gamma] + a_{002}(\alpha+1)c[\alpha+1,\beta,\gamma-2]\\
&+ (b_{201} - d_{012} - b_{201}\beta + d_{012}\gamma)c[\alpha,\beta-1,\gamma-1]\\
&+ (a_{102}\alpha - 2d_{003} - b_{102}\beta + d_{003}\gamma)c[\alpha,\beta,\gamma-2]
\end{aligned}
$$

and each $c[\alpha,\beta,\gamma]$ is namely $c_{\alpha\beta\gamma}$, and for any positive integer m, μ_m is determined by the following recursive formula:

$$
\begin{aligned}
\mu_m =& - b_{030}(1+m)c[m-3,m+1,0] + (a_{300}m - 2a_{300} - b_{120}m)c[m-2,m,0]\\
&- b_{020}(m+1)c[m-2,m+1,0] + (a_{210} - b_{210})(m-1)c[m-1,m-1,0]\\
&+ (2b_{300} + a_{120}m - b_{300}m)c[m,m-2,0] - b_{200}(m-1)c[m,m-1,0]\\
&+ a_{030}(m+1)c[m+1,m-3,0] + a_{020}(m+1)c[m+1,m-2,0]\\
&+ d_{110}c[m-1,m-1,1] + a_{200}(m-1)c[m-1,m,0],
\end{aligned}
$$

and when $\alpha<0$ or $\beta<0$ or $\gamma<0$ or $\gamma=0,\alpha=\beta$, we have let $c_{\alpha,\beta,\gamma}=0$.

Now applying the recursive formulas in Lemma 3.1 via the software Mathematica, we obtain the first two singular point quantities of system (3.1) at the origin as follows:

$$
\begin{aligned}
\mu_1 &= -\frac{\mathbf{i}A_{200}}{2},\\
\mu_2 &= -\frac{2\mathbf{i}A_{002}D_{200}^2}{\lambda^2(1+\lambda^2)}[\lambda^2 - (D_{101}+D_{011})\lambda + D_{101} - D_{011}].
\end{aligned}
\tag{3.3}
$$

To simplify the calculation, we set $\lambda=-1$ here, namely

$$\mu_2 = -\mathbf{i}A_{002}(1+2D_{101})D_{200}^2.$$

Then we do certain discussion for $\mu_2=0$ and continue to compute the following singular point quantities.

Case (i). If $A_{002}D_{200}=0$, then

$$\mu_3=\mu_4=\mu_5=\mu_6=\mu_7=\mu_8=0.$$

Case (ii). If $A_{002}D_{200}\neq 0$, from $\mu_2=0$, we have $D_{101}=-\frac{1}{2}$, then computing yields

$$\mu_3 = \frac{\mathbf{i}}{20400}A_{002}D_{200}^2 S_1, \tag{3.4}$$

letting $\mu_3=0$ yields that $S_1=0$, i.e.,

$$A_{002} = -\frac{S_2}{81600D_{200}^2}, \tag{3.5}$$

where

$$
\begin{aligned}
S_1 =& S_2 + 81600A_{002}D_{200}^2,\\
S_2 =& -1443 + 622D_{011} + 7276D_{011}^2 + 4080D_{011}^3 - 10200D_{200} + 10200B_{101}D_{200}\\
&+ 40800D_{002}D_{200} - 20400D_{011}D_{200} + 20400B_{101}D_{011}D_{200}\\
&+ 40800D_{002}D_{011}D_{200}.
\end{aligned}
$$

And we figure out

$$\mu_4 = \frac{\mathbf{i}}{44146528000000} S_2 S_3, \tag{3.6}$$

where

$$S_3 = S_n + 780 B_{101} D_{200} S_d,$$
$$S_d = -404493 + 931472 D_{011} + 6428516 D_{011}^2 + 4647120 D_{011}^3,$$

and S_n is a polynomial only with respect to D_{002}, D_{011} and D_{200}.

Next, setting $\mu_4 = 0$, we obtain $S_2 = 0$ or $S_3 = 0$. If $S_2 = 0$, from (3.5), $A_{002} = 0$ holds. This is in contradiction with the condition $A_{002} D_{200} \neq 0$, then $S_2 \neq 0$, we just consider $S_3 = 0$. By setting $S_3 = 0$ and $S_d \neq 0$, it follows that

$$B_{101} = -\frac{S_n}{780 D_{200} S_d}. \tag{3.7}$$

Further, under the above conditions we continue to compute and obtain

$$\begin{aligned} \mu_5 &= -\frac{\mathbf{i}}{2583795418624000000000 S_d^3} S_4 F_1, \\ \mu_6 &= -\frac{\mathbf{i}}{1307381773603974954762240000000000 S_d^4} S_4 F_2, \\ \mu_7 &= -\frac{\mathbf{i}}{3335423757981027399988340981760000000000000 S_d^5} S_4 F_3, \\ \mu_8 &= -\frac{\mathbf{i}}{1569986724612541205465199290460764897280000000000000000 S_d^6} S_4 F_4, \end{aligned} \tag{3.8}$$

where S_4,F_1, F_2, F_3 and F_4 are polynomials only with respect to D_{002}, D_{011} and D_{200}. In fact, S_4, F_1, F_2, F_3 and F_4 are too long to show in this paper, with terms of 22, 106, 248, 480 and 824 elements respectively, which can be found in the website:https://github.com/lujingping/KOL.git.

3.2. Center conditions

In this subsection, we investigate the center problem of system (3.1). Analyzing the singular point quantities obtained in (3.3), Cases (i) and (ii), we have the following result.

Theorem 3.2. *For system (3.1) with $\lambda = -1$, the first eight singular point quantities of the origin vanish simultaneously if and only if one of the following two conditions holds:*

$$\mathrm{K}_1 : A_{200} = A_{002} = 0, \tag{3.9}$$
$$\mathrm{K}_2 : A_{200} = D_{200} = 0. \tag{3.10}$$

Proof. From the first eight singular point quantities $\mu_1, \mu_2, \cdots, \mu_8$ in (3.3) and the cases (i) (ii), the sufficiency of the conditions in Theorem 3.2 is obvious. Then, we only need to prove the necessity of the above conditions.

Letting the first singular point quantity $\mu_1 = 0$, we obtain $A_{200} = 0$. And taking $\mu_2 = 0$ yields that $A_{002} = 0$ or $D_{200} = 0$ or $1 + 2D_{101} = 0$. For the above case (i), i.e., $A_{002} D_{200} = 0$, it can be concluded that condition K_1 or K_2 is necessary for each $\mu_i = 0$, $i = 1, 2, \cdots, 8$.

For the case (ii), when $A_{002}D_{200} \neq 0$ and $D_{101} = -\frac{1}{2}$, we note that $\mu_3 = \mu_4 = 0$ if and only if $S_1 = S_3 = 0$. Taking $\mu_5 = 0$, generates that $S_4 = 0$ or $F_1 = 0$. In fact, we compute the resultant of S_2 and S_3 with respect to B_{101}, yielding

$$\text{Resultant}[S_2, S_3, B_{101}] = -60D_{200}S_4.$$

Since $S_2 \neq 0$ in (3.5), $S_4 \neq 0$ hold necessarily. Thus $\mu_5 = 0$ if and only if $F_1 = 0$.

Next, we need to investigate whether the four equations $\mu_5 = \mu_6 = \mu_7 = \mu_8 = 0$ have common solutions, specifically, to determine whether or not the polynomials F_1, F_2, F_3 and F_4 share common zeros. For this purpose, computing the polynomial resultants of F_2, F_3, F_4 for F_1 with respect to D_{002} via Mathematica, we have

$$\begin{aligned}
\text{Resultant}[F_2, F_1, D_{002}] &= 2732274240795226681 \cdots 000000 D_{200}^{24} S_d^{12} f_{60}, \\
\text{Resultant}[F_3, F_1, D_{002}] &= -167339298139456617 \cdots 000000 D_{200}^{32} S_d^{16} f_{76}, \\
\text{Resultant}[F_4, F_1, D_{002}] &= 8691536176096270463 9 \cdots 000000 D_{200}^{40} S_d^{20} f_{92},
\end{aligned} \tag{3.11}$$

where f_{60}, f_{76} and f_{92} are all polynomials just in D_{011} and D_{200}, and the degrees of f_{60}, f_{76}, f_{92} are $60, 76, 92$ respectively. Since $D_{200}S_d \neq 0$, we just need consider whether or not the polynomials f_{60}, f_{76} and f_{92} share common zeros. Moreover, we compute the Gröbner basis of the ideal $< f_{60}, f_{76}, f_{92} >$, and we get

$$\text{GroebnerBasis}[\{f_{60}, f_{76}, f_{92}\}, \{D_{011}, D_{200}\}] = \{1\}.$$

This means that the polynomials f_{60}, f_{76} and f_{92} have no common zeros, then yielding that F_1, F_2, F_3 and F_4 have no common root. Therefore, apart from the condition K_1 or K_2, there are no other conditions such that all μ_i vanish, $i = 1, 2, \cdots, 8$. The proof of Theorem 3.2 is complete. □

However, we should note that there may exist some real values of D_{002}, D_{011} and D_{200} such that $F_1 = F_2 = F_3 = 0$, which will be discussed in the next subsection.

Furthermore, we will prove that K_1 in (3.9) and K_2 in (3.10) are two sets of center conditions of system (3.1) restricted to the center manifold. Then we give the corresponding theorem.

Theorem 3.3. *The origin of system (3.1) with $\lambda = -1$, i.e, the positive equilibrium $(1, 1, 1)$ of its corresponding real system (1.2) with $\delta = 0$ and $\lambda = -1$ is a center on the local center manifold if and only if the condition K_1 or K_2 in Theorem 3.2 holds.*

Proof. From the Theorem 3.2, the necessity is obvious. Now, we prove the sufficiency of the two conditions.

(I) If the condition K_1 holds, the system (1.3) can be rewritten as

$$\begin{cases}
\dfrac{\mathrm{d}x}{\mathrm{d}t} = -y(1+x)(1+B_{101}u), \\
\dfrac{\mathrm{d}y}{\mathrm{d}t} = x(1+y)(1+B_{101}u), \\
\dfrac{\mathrm{d}u}{\mathrm{d}t} = (1+u)(-u + D_{200}x^2 + D_{200}y^2 + D_{101}xu + D_{011}yu + D_{002}u^2).
\end{cases} \tag{3.12}$$

For system (3.12), there exists a center manifold $u = u(x, y)$, which can be expressed as the polynomial series in x and y formally as follows:

$$u = u_2(x, y) + \text{h.o.t.},$$

where u_2 is a homogeneous quadratic polynomial, h.o.t. expresses higher-order terms. Substituting $u = u(x, y)$ into the first two equations of system (3.12), it can be transformed into two-dimensional system. Thus, to prove that system (3.12) is integrable on the center manifold, we only need to find a first integral. It is easy to see that system (3.12) has two invariant surfaces

$$f_1(x, y, u) = 1 + x,$$
$$f_2(x, y, u) = 1 + y,$$

with the corresponding cofactors

$$k_1(x, y, u) = -y(1 + B_{101}u),$$
$$k_2(x, y, u) = x(1 + B_{101}u).$$

At the same time, we also find an exponential factor

$$G = e^{x+y},$$

with cofactor

$$l = (x - y)(1 + B_{101}u).$$

Now the solution of the relevant equation (2.8) in the lemma 2.2 is as follows:

$$\lambda_1 = -\eta, \quad \lambda_2 = -\eta.$$

Choosing $\eta = 1$, we obtain one first integral of system (3.12):

$$H = (1 + x)^{-1}(1 + y)^{-1}e^{x+y},$$

then its origin is a center on the local center manifold.

(II) If the condition K_2 holds, the system (1.3) can be rewritten as

$$\begin{cases} \dfrac{dx}{dt} = (1 + x)(A_{002}u^2 - B_{101}yu - y), \\ \dfrac{dy}{dt} = (1 + y)(x + B_{101}xu + A_{002}u^2), \\ \dfrac{du}{dt} = u(1 + u)(D_{101}x + D_{011}y + D_{002}u - 1). \end{cases} \tag{3.13}$$

It is not difficulty to find that system (3.13) admits an invariant algebraic surface $f(x, y, u) = u = 0$ with cofactor $k = (1 + u)(D_{101}x + D_{011}y + D_{002}u - 1)$. And we note that the surface $f(x, y, u) = u = 0$ is actually a global center manifold of system (3.13). By substituting $u = 0$ into the first and second equations of system (3.13), we can obtain the following planar system,

$$\begin{cases} \dfrac{dx}{dt} = -y(1 + x), \\ \dfrac{dy}{dt} = x(1 + y), \end{cases} \tag{3.14}$$

which has a first integral

$$H = x + y - \ln(1 + x)(1 + y).$$

This means that the origin of systems (3.14) is a center. Hence, the origin is a center for the flow of system (3.13) restricted to the center manifold. The proof of Theorem 3.3 is complete. □

3.3. Hopf bifurcation at the equilibrium E

In this section, we turn to the investigation on the maximum number of limit cycles bifurcating from the origin of system (1.3). For this purpose, we need to determine the highest order of the origin as a fine focus. From the discussion of no common zero for F_1, F_2, F_3 and F_4 in Theorem 3.2 and the center conditions given in Theorem 3.3, we still cannot determine that the upper bound of the order of fine focus at the origin of system (1.3) is eight. If and only if F_1, F_2 and F_3 can disappear at the same time, then it is true.

Next, we will figure out whether or not F_1, F_2 and F_3 share common zeros by solving the two equations $f_{60} = f_{76} = 0$ with respect to D_{011} and D_{200}, given by (3.11). Thus 164 groups of real solutions satisfying $f_{60} = f_{76} = 0$ are found, which can rigorously verified by applying Sturm's theorem of polynomial. Further, substituting them into the expression of F_1, F_2 and F_3, the real numerical solutions of D_{002} can be obtained with the aid of algebraic system Mathematica, whose existence can also be strictly verified. In this way, we get only 24 groups of real solutions satisfying the equations $F_1 = F_2 = F_3 = 0$. One of them is chosen as follows:

$$
\begin{aligned}
D_{011} &= -3.7503608886503623526497177504218126560875l3\cdots, \\
D_{200} &= 1.4045819267534722366816557146846206218692897 5\cdots, \\
D_{002} &= -0.12215312496680631191453297331446361260921 5\cdots,
\end{aligned}
\tag{3.15}
$$

at this time,

$$F_1 = F_2 = F_3 = 0, \quad F_4 = 2.17049\cdots * 10^{71} \neq 0.$$

This means that there is at least a solution such that $\mu_5 = \mu_6 = \mu_7 = 0$, but $\mu_8 \neq 0$.

On the other hand, according to the proof of theorem 3.2, we know that $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$ hold if $A_{002}D_{200}S_2S_d \neq 0$ and

$$A_{200} = 0, \quad D_{101} = -\frac{1}{2}, \quad A_{002} = -\frac{S_2}{81600D_{200}^2}, \quad B_{101} = -\frac{S_n}{780D_{200}S_d}. \tag{3.16}$$

Then under the given value conditions of (3.15), we can figure out

$$
\begin{aligned}
B_{101} &= -0.159816022767920418580982329332306655787827\cdots, \\
A_{002} &= -0.065943769804169647996394101019544501795068\cdots,
\end{aligned}
\tag{3.17}
$$

and easily verify $A_{002}D_{200}S_2S_d \neq 0$.

Thus a group of critical values is imposed as follows,

$$
\begin{aligned}
\eta &= (A_{200}, A_{002}, B_{101}, D_{101}, D_{002}, D_{011}, D_{200}) \\
&= \left(0, A_{002}^*, B_{101}^*, -\frac{1}{2}, D_{002}^*, D_{011}^*, D_{200}^*\right) \\
&=: \eta^*
\end{aligned}
\tag{3.18}
$$

where all A^*, B^*, D^* are the given parameter values in (3.15) and (3.17). Therefore, if $\eta = \eta^*$, then $\mu_1 = \cdots = \mu_7 = 0$ and $\mu_8 \neq 0$ hold necessarily, yielding the following result.

Theorem 3.4. *The highest order of the origin of system (3.1) with $\lambda = -1$, namely, the positive equilibrium $(1, 1, 1)$ of its corresponding real system (1.2) with $\delta = 0$ and $\lambda = -1$ is eight as a fine focus on the center manifold.*

According to the algebraic equivalence shown in (2.4), we can easily get the first eight focal values of the origin for system (3.1) or its conjugate real system (1.3) with $\lambda = -1$:

$$v_{2m+1} = \mathbf{i}\pi\mu_m, m = 1, 2, \cdots, 8,$$

where for each v_{2m+1}, we have set $v_{2j-1} = 0, j = 1, 2, \cdots, m$.

Furthermore, under the conditions (3.18), $v_{17}(\eta^*) \neq 0$ holds, and directly calculating the Jacobian determinant of the function group $(v_3, v_5, v_7, v_9, v_{11}, v_{13}, v_{15})$ with respect to the variable group η yields

$$\begin{aligned} J = & \left| \frac{\partial(v_3, v_5, v_7, v_9, v_{11}, v_{13}, v_{15})}{\partial(A_{200}, D_{101}, B_{101}, A_{002}, D_{002}, D_{011}, D_{200})} \right|_{\eta=\eta^*} \\ = & 70543.33212338080354505215809797213 49 \\ \neq & 0. \end{aligned} \tag{3.19}$$

By Lemma 2.1, it implies that system (3.1) can have 7 small-amplitude limit cycles bifurcating from the origin. According to the above analysis, we have the following theorem.

Theorem 3.5. *There exist seven and at most seven limit cycles bifurcating from the origin of system (3.1) with $\lambda = -1$ or the positive equilibrium $(1, 1, 1)$ of its corresponding real system (1.2) with $\delta = 0$ and $\lambda = -1$ via Hopf bifurcation restricted to a center manifold.*

Remark 1. In Theorem 3.5, since the linear parts are not involved in the perturbation of coefficients, only seven small-amplitude cycles restricted to a center manifold can appear. Just when $0 < |\delta| \ll 1$, the first two equations of the system (1.3) are perturbed in their linear parts just as system (2.1), then the multiple bifurcations of eight limit cycles from the equilibrium can occur.

4. Zero-Hopf bifurcation around the equilibrium E

Now, we consider zero-Hopf bifurcation of the positive equilibrium E. To guarantee that its eigenvalues are 0 and $\pm\mathbf{i}$, the necessary condition: $\lambda = 0$ and $\delta = 0$ should be satisfied in system (1.2). Further, we let $0 < |\lambda| \ll 1$ and $|\delta| \ll 1$ such that the system (1.3) are perturbed in their linear parts, this is also called unfolding.

4.1. Existence of multiple periodic orbits via Zero-Hopf bifurcation

Similar to the previous research on Hopf bifurcation, we will investigate the translated system (1.3) for the zero-Hopf bifurcation problem around the origin, which responds to the positive equilibrium E of system (1.2). Via the rescaling of the variables: $(x, y, u) \mapsto (\varepsilon x, \varepsilon y, \varepsilon u)$, then the corresponding perturbation form of system

(1.3) becomes as follows:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = (1+\varepsilon x)[\delta x - y + \varepsilon(A_{200}x^2 - A_{200}y^2 - B_{101}yu + A_{002}u^2)], \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = (1+\varepsilon y)[x + \delta y + \varepsilon(B_{101}xu + A_{002}u^2)], \\ \dfrac{\mathrm{d}u}{\mathrm{d}t} = (1+\varepsilon u)[\lambda u + \varepsilon(D_{200}x^2 + D_{200}y^2 + D_{101}xu + D_{011}yu + D_{002}u^2)]. \end{cases} \tag{4.1}$$

Now, we shall use the normal form theory to investigate the zero-Hopf bifurcation of system (4.1). The following conclusion can be obtained.

Theorem 4.1. *For system* (1.2)*, the zero-Hopf bifurcation can occur around the positive equilibrium E at the critical value: $\lambda = \delta = 0$. And under the perturbing condition: $0 < |\lambda| \ll 1$ and $0 < |\delta| \ll 1$, at least two limit cycle can bifurcate via setting appropriate parameter values.*

Proof. Applying the Maple program in [24,29], for system (4.1) with the unfolding added, we obtain the following normal form expressed in cylindrical coordinates [31] (for convenience, the notation u is still used in the normal form),

$$\begin{cases} \dot{u} = \lambda u + \varepsilon V_2 + \varepsilon^2 V_3 + o(\varepsilon^2), \\ \dot{r} = \delta r + \varepsilon^2 r R_2 + o(\varepsilon^2), \\ \dot{\theta} = 1 + \varepsilon B_{101}u + \varepsilon^2 E_2 + o(\varepsilon^2), \end{cases} \tag{4.2}$$

where

$$\begin{aligned} V_2 &= D_{200}R^2 + D_{002}u^2, \\ V_3 &= D_{200}R^2u + (D_{002} + A_{002}D_{011} - A_{002}D_{101})u^3, \\ R_2 &= \tfrac{1}{4}A_{200}R^2 - A_{002}(A_{200} + D_{011} - D_{101})u^2, \\ E_2 &= \tfrac{1}{12}(2A_{200} - 2A_{200}^2 - 1)R^2 + A_{002}(A_{200} - D_{011} - D_{101})u^2. \end{aligned} \tag{4.3}$$

The first two equations in the normal form (4.2) can be used for bifurcation analysis, while the third equation can be used to determine the frequency of periodic solutions.

Next, we will search for the steady-state solutions by setting $\dot{u} = \dot{r} = 0$ in (4.2).

Case (I). Considering the truncated 1-jet of the first equation and the truncated 2-jet of the second equation with respect to ε, and more by letting $\lambda = \lambda_1\varepsilon$ and $\delta = \delta_1\varepsilon^2$, then we have

$$\begin{cases} \dot{u} = \varepsilon(\lambda_1 u + V_2), \\ \dot{r} = \varepsilon^2 r(\delta_1 + R_2), \end{cases} \tag{4.4}$$

which can be called $(\varepsilon, \varepsilon^2)$-order reduced equations of system (4.2). And its all the steady-state solutions can be obtained, generally, Equations (4.4) have two groups of solutions with $r > 0$, and two unstable positive solutions, i.e., two unstable periodic orbits can be obtained easily. For example, when setting $\delta_1 = 1, \lambda_1 = \frac{1}{25}, A_{200} =$

$0, D_{200} = 1, A_{002} = 1, D_{011} = 1 + D_{101}, D_{002} = -\frac{1}{5}$, there exists such two solutions as follows

$$(u, r) = (1, \frac{2}{5}),\ \ (-1, \frac{\sqrt{6}}{5}).$$

The stability of the steady-state solutions are determined by the Jacobian of the first two equations of (4.2), evaluated at the solutions (u, r), resulting in two group of eigenvalues: $\{-\frac{4}{5}, \frac{4}{5}\}$ and $\{\frac{2\sqrt{6}}{5}, \frac{2\sqrt{6}}{5}\}$, respectively. Hence, we can determine the solutions, i.e., the two possible periodic orbits of (4.2) are unstable. Unfortunately, we can not find its stable periodic orbit of (4.2) in this case.

Case (II). Considering the truncated 2-jet of the first equation with the 1-jet part $V_2 = 0$, and the truncated 2-jet of the second equation with respect to ε, by letting $\lambda = \lambda_2 \varepsilon^2$ and still $\delta = \delta_1 \varepsilon^2$, then we have

$$\begin{cases} \dot{u} = \varepsilon^2(\lambda_2 u + V_3), \\ \dot{r} = \varepsilon^2 r(\delta_1 + R_2), \end{cases} \tag{4.5}$$

which can be called $(\varepsilon^2, \varepsilon^2)$-order reduced equations of system (4.2). And its all the steady-state solutions can also be obtained, similarly we only need the solutions with positive r. And two unstable periodic orbits and one stable periodic orbit can be obtained easily. For example, when setting $\delta_1 = 1, \lambda_2 = -2, A_{200} = -3, A_{002} = 1, D_{200} = 0, D_{002} = 0, D_{101} = 0, D_{011} = 1$, there exists such there solutions as follows

$$(u, r) = (0, \frac{2\sqrt{3}}{3}),\ (\sqrt{2}, \frac{2\sqrt{15}}{3}),\ (-\sqrt{2}, \frac{2\sqrt{15}}{3}),$$

we calculate the Jacobian of the first two equations of (4.2), evaluated at the solutions (u, r), resulting in three groups of eigenvalues:

$$\{-2,\ 2\},\ \ \{2\sqrt{30},\ \frac{2\sqrt{30}}{3}\},\ \ \{-2\sqrt{30},\ -\frac{2\sqrt{30}}{3}\},$$

respectively. Hence, we can determine the solutions, i.e., the first two possible periodic orbits of (4.2) are unstable, the third is stable.

Therefore, the proof of the theorem has been completed. □

4.2. Relevancy of Zero-Hopf and Hopf bifurcation

Here, we will probe the relevancy of Hopf and zero-Hopf bifurcation by analyzing a class of special case of system (4.1) under the condition Case (II), where the parameters are chosen as follows

$$D_{200} = 0,\ D_{002} = 0,\ D_{101} = 0,\ D_{011} = 1,\ B_{101} = 0,\ A_{002} = 1. \tag{4.6}$$

Then the corresponding system (4.1) becomes the following form

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = (1 + \varepsilon x)[\delta x - y + \varepsilon(A_{200}x^2 - A_{200}y^2 + u^2)], \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = (1 + \varepsilon y)(x + \delta y + \varepsilon u^2), \\ \dfrac{\mathrm{d}u}{\mathrm{d}t} = (1 + \varepsilon u)(\lambda u + \varepsilon y u), \end{cases} \tag{4.7}$$

namely, its corresponding system (1.3) without rescaling the variables is

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = (1+x)[\delta x - y + (A_{200}x^2 - A_{200}y^2 + u^2)], \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = (1+y)(x + \delta y + u^2), \\ \dfrac{\mathrm{d}u}{\mathrm{d}t} = (1+u)(\lambda u + yu). \end{cases} \tag{4.8}$$

When letting $\lambda = \lambda_2\varepsilon^2$ and $\delta = \delta_1\varepsilon^2$, then system (4.7) has the form of Eq.(4.5), i.e.,

$$\begin{cases} \dot{u} = \varepsilon^2 u(\lambda_2 + u^2), \\ \dot{r} = \varepsilon^2 r[\delta_1 + \dfrac{1}{4}A_{200}r^2 + (A_{200}+1)u^2], \end{cases} \tag{4.9}$$

there exists its three groups of real solutions with $r > 0$ as follows

$$(u_0, r_0) = (0,\ 2\sqrt{-\delta_1/A_{200}}),\ (u_1, r_1) = (\sqrt{-\lambda_2},\ \sqrt{-\kappa_1}),\ (u_2, r_2) = (-\sqrt{-\lambda_2}, \sqrt{-\kappa_1})$$

where $\kappa_1 = [(1+A_{200})\lambda_2 + \delta_1]/A_{200}$, and the following conditions should be satisfied

$$\lambda_2 < 0,\ \delta_1 < 0,\ A_{200} < 0,\ \kappa_1 < 0 \tag{4.10}$$

i.e., $-\delta_1 < \lambda_2 < 0$, $A_{200} < 0$ or $\lambda_2 \le -\delta_1 < 0$, $A_{200} < -\frac{\lambda_2+\delta_1}{\lambda_2}$.

Further, we evaluate the Jacobian matrix at $(u, r) = (u^*, r^*)$, then yielding

$$\begin{pmatrix} -2(1+A_{200})u^*r^* & \dfrac{3}{4}A_{200}r^{*2} - (1+A_{200})u^{*2} + \delta_1 \\ \lambda_2 + 3u^{*2} & 0 \end{pmatrix}, \tag{4.11}$$

and more the Jacobian matrix (4.11) has the following determinant and trace:

$$\mathrm{Det} = (\lambda_2 + 3u^{*2})[\frac{3}{4}A_{200}r^{*2} - (1+A_{200})u^{*2} + \delta_1], \quad \mathrm{Tr} = -2(1+A_{200})u^*r^*.$$

Then its two eigenvalues are all negative if and only if $\mathrm{Det} > 0, \mathrm{Tr} < 0$, namely the stability conditions of the periodic orbit. It is not difficult to verify that only the two solutions:(u_1, r_1) and (u_2, r_2) can correspond to the stable periodic orbits in the original three dimensional space. For the solution (u_1, r_1) under conditions (4.10), when $\frac{1}{2}(\sqrt{5}-3) < A_{200} < 0$, then $(1+A_{200}) > \sqrt{1+3A_{200}+A_{200}^2}$ holds, yielding that its Jacobian has two negative eigenvalues, i.e.,

$$\{-2\sqrt{\kappa_1\lambda_2}[(1+A_{200}) - \sqrt{\kappa_2}],\ -2\sqrt{\kappa_1\lambda_2}[(1+A_{200}) + \sqrt{\kappa_2}]\}, \tag{4.12}$$

where $\kappa_2 = 1 + 3A_{200} + A_{200}^2$. Under the conditions (4.10), for the solution (u_2, r_2), when $A_{200} < -\frac{1}{2}(\sqrt{5}+3)$, its Jacobian has two negative eigenvalues, i.e.,

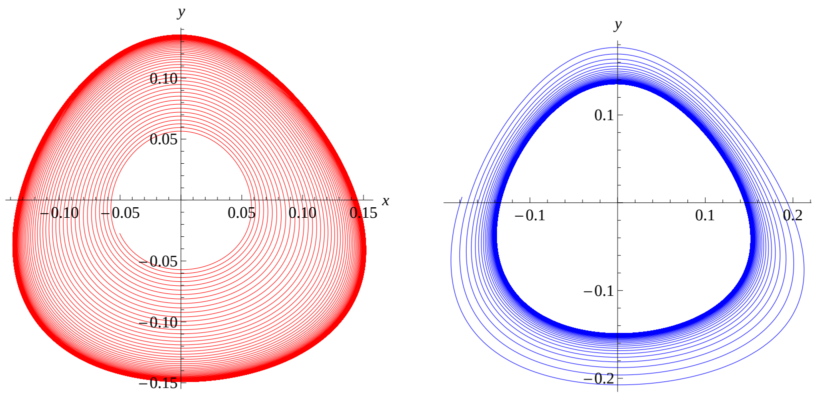
$$\{2\sqrt{\kappa_1\lambda_2}[(1+A_{200}) - \sqrt{\kappa_2}],\ 2\sqrt{\kappa_1\lambda_2}[(1+A_{200}) + \sqrt{\kappa_2}]\}. \tag{4.13}$$

While the other solution (u_0, r_0) always has positive eigenvalue.

On the one hand, from the singular quantities (3.3), we have the first two focal values for the origin of (4.8) with $\lambda < 0$:

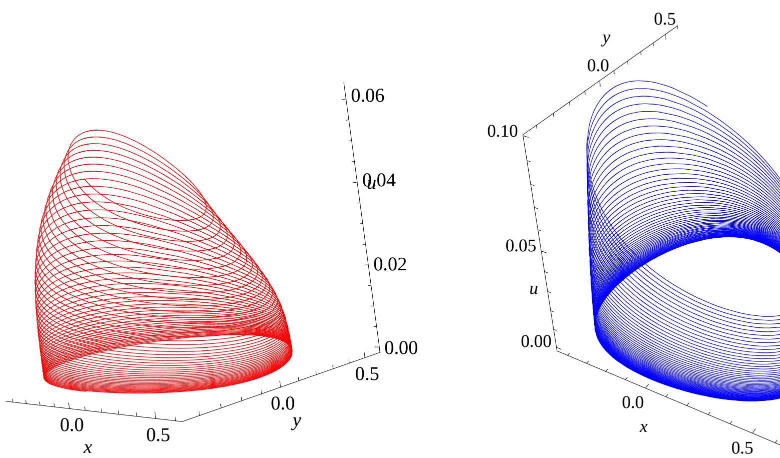
$$v_3 = \frac{\pi}{2}A_{200}, \qquad v_5 = 0, \tag{4.14}$$

where for the expression of v_5, we have already let $v_3 = 0$, i.e., $A_{200} = 0$. Obviously, when $v_3 \neq 0$, the origin of (4.8) or the equilibrium E of system (1.2) is a fine focus of order one, and if δ is disturbed sufficiently small, i.e., $0 < \delta \ll 1$, then Hopf bifurcation can occur, yielding a small amplitude limit cycle from the origin. By setting $\lambda = -1.5, \delta = 0.02$, $A_{200} = -2$, one small-amplitude cycle can appear, as shown in Figure 1. At this time, since $|\lambda|$ is not tending towards zero and relatively big, then zero-Hopf bifurcation can not occur.



(a) (b)

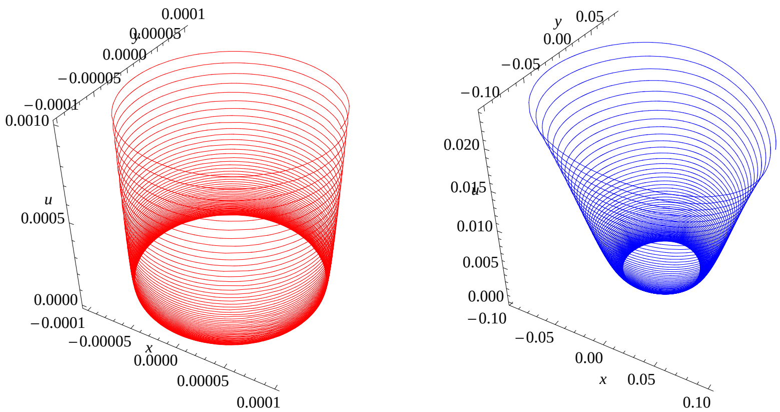
Figure 1. Projection of phase portraits on the plane x-y for system (4.8) with $\lambda = -1.5, \delta = 0.02$, $A_{200} = -2$, converging to the stable limit cycle around the origin with the initial conditions: (a) $(x_0, y_0, u_0) = (0.05, 0, 0.05)$ and (b) $(x_0, y_0, u_0) = (0.2, 0, 0.15)$.



(a) (b)

Figure 2. Simulations of system (4.8) for $\lambda = -0.01, \delta = 0.02, A_{200} = -0.3$, converging to the stable periodic orbit around the origin with the initial conditions: (a) $(x_0, y_0, u_0) = (0.05, 0, 0.08)$ and (b) $(x_0, y_0, u_0) = (0.5, 0, 0.05)$.

On the other hand, here we give a numerical example of one stable periodic orbit corresponding to the above solution (u_1, r_1) around the origin of (4.8), i.e., the equilibrium E of system (1.2) via zero-Hopf bifurcation, as shown in Figure 2.



(a) (b)

Figure 3. Simulations of system (4.8) with $\lambda = -0.01, \delta = 0,\ A_{200} = -3$, converging to the stable limit cycle around the origin with the initial conditions: (a) $(x_0, y_0, u_0) = (0.0001, 0, 0.001)$ and (b) $(x_0, y_0, u_0) = (0.1, 0, 0.02)$.

In this example, we have set $\lambda = -0.01, \delta = 0.02$, i.e., $\lambda_2 = -1, \delta_1 = 2, \varepsilon = 0.1$ and $A_{200} = -0.3$. In fact, at this time, $|\delta| \ll |A_{200}|$, and $v_3\delta < 0$, which implies that Hopf bifurcation occur around the origin, that is to say, the zero-Hopf bifurcation and Hopf bifurcation are indistinguishable around the origin.

In addition, we note that if $\delta_1 = 0$, and set $\lambda_2 < 0,\ A_{200} < 0,\ \kappa_1 < 0$, then there still exists the two groups of real solutions (u_1, r_1) and (u_2, r_2) with $r_1 > 0, r_2 > 0$. And when $A_{200} < -\frac{1}{2}(\sqrt{5}+3)$, the Jacobian at (u_2, r_2) has two negative eigenvalues with the same form as (4.13).

We give also its numerical example of one stable periodic orbit corresponding to the above solution (u_2, r_2) around the origin of (4.8), as shown in Figure 3, where by letting $\lambda = -0.01, \delta = 0$, i.e., $\lambda_2 = -1, \delta_1 = 0, \varepsilon = 0.1$ and $A_{200} = -3$. At this time, the zero-Hopf bifurcation can occur around the origin. However, since $\delta = 0$ and $|v_3|$ is relatively big, then Hopf bifurcation can not occur.

Based on the above analysis, we have the following conclusion.

Proposition 1. *For the zero-Hopf bifurcation and Hopf bifurcation of system* (4.8), *there exists certain parameter space where the two occur but cannot be distinguished, as well as the parameter spaces where only one of the two can occur.*

5. Conclusion and discussion

In this paper, we have studied Hopf bifurcation and zero-Hopf bifurcation around the positive equilibrium of a class of cubic Kolmogorov systems. Via the calculation of the singular point quantities and the determination of center conditions, the highest order fine focus is obtained, which just indicates the Hopf cyclicity 8 at the positive equilibrium as a new result. At the same time, extending the normal form theory to investigate the zero-Hopf bifurcation around the positive equilibrium, we obtain multiple periodic orbits. Further, by analyzing a class of special case of the original system, we have discussed the relevancy of zero-Hopf and Hopf bifurcation, and figured out the parameter conditions under which only one or both of the two can occur. And for the latter, there is no strict distinction between Hopf bifurcation and zero-Hopf bifurcation. We believe that there are still other interesting relevancies between the two, and further exploration is needed.

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