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SOLVABILITY OF NONLOCAL HILFER FRACTIONAL MATRIX BOUNDARY VALUE PROBLEMS WITH P-LAPLACIAN AT RESONANCE IN $\mathbb{R}^N$

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Abstract In this paper, the solvability of boundary value problems for a class of nonlinear Hilfer fractional differential equations at resonance in $\mathbb{R}^n$ is studied. In the past, research on matrix boundary value problems has consistently been conducted within the context of linear differential equations. The main contribution of this paper is the extension of linear problems to nonlinear ones. We begin by defining two Banach spaces endowed with appropriate norms and constructing suitable operators in these Banach spaces. Subsequently, by using the extension for the continuous theorem, certain sufficient conditions for the solvability of the problem are obtained. Finally, an example is provided to verify the effectiveness of our main results.

Keywords Matrix boundary value problem, p-Laplacian, continuous theorem, Hilfer fractional derivative.

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1. Introduction

This work considers the solvability of the following nonlinear Hilfer fractional matrix boundary value problems in $\mathbb{R}^n$:

$$\begin{cases} D_{0+}^{\alpha_1,\beta_1}\varphi_p(D_{0+}^{\alpha_2,\beta_2}u(t)) = f(t,u(t),D_{0+}^{\alpha_2,\beta_2}u(t)),\ 0\le t\le 1,\\ D_{0+}^{\alpha_2,\beta_2}u(0) = D_{0+}^{\gamma_2-2}u(0) = \cdots = D_{0+}^{\gamma_2-m}u(0)=\theta,\\ u(1) = A\displaystyle\int_0^1 u(t)h(t)\mathrm{d}t, \end{cases} \tag{1.1}$$

where $0<\alpha_1\le 1$, $m-1<\alpha_2\le m$, $0\le \beta_1,\beta_2\le 1$, $\gamma_2=\alpha_2+m\beta_2-\alpha_2\beta_2$, $p>1$, $\varphi_p(\vartheta)=|\vartheta|^{p-2}\vartheta$, $u=(u_i)_{n\times 1}$, $A=(a_{ii})_{n\times n}$, $a_{ii}\le 0$, $m,n\in\mathbb{N}_+$, $h(t)\ge 0$, $f\in C\big([0,1]\times\mathbb{R}^{2n},\mathbb{R}^n\big)$, θ is the zero vector in $\mathbb{R}^n$ and $D_{0+}^{\alpha,\beta}$ represents the Hilfer fractional derivative operator.

Fractional differential equations are widely used in physical and biological fields, such as elastomers, vibration and diffusion systems [1, 5–7, 9, 14, 17–19, 21, 22, 25, 28,

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30]. Fractional boundary value problems have been extensively studied, and numerous results regarding their solvability have been obtained. For example, Seal et al. [26] analyzed the convergence of solutions of fractional differential equations with integral boundary conditions by spline approximation method. In [31], Zaky discussed the existence, uniqueness and stability of solutions to nonlinear tempered fractional generalized boundary value problems. Furthermore, the method of singular spectrum collocation for obtaining the numerical solutions of these equations has been developed and analyzed. In [2], Azouzi et al. obtained the existence of solutions for generalized fractional boundary value problems by using the Mawhin continuation theorem. Moreover, Wang et al. [29] derived the existence of triple positive solutions for a class of fractional boundary value problems at resonance. Some new height functions and spectral theory are also used to solve the positive solutions. The main method used is the fixed point index theorem.

Mawhin's continuation theorem [20] is a classical method often used to study the existence of solutions for differential equations of the form $Lx = Nx$ under resonance conditions, where the operator L is an irreversible linear operator. Ge et al. [10] first generalized the result of Mawhin in [20], in which the existence theory of solutions was obtained for the non-invertible nonlinear operator L. Furthermore, Jiang [12] considered the following nonlinear problem with integral boundary conditions in one-dimensional space:

$$\begin{cases} D^{\theta}_{0+}(\varphi_p(D^{\gamma}_{0+}x))(t) + f(t, x(t), D^{\gamma-1}_{0+}x(t), D^{\gamma}_{0+}x(t)) = 0, \\ x(0) = D^{\gamma}_{0+}x(0) = 0,\ x(1) = \displaystyle\int_0^1 g(t)x(t)\mathrm{d}t, \end{cases} \tag{1.2}$$

where $p > 1$, $0 < \theta \le 1$, $1 < \gamma \le 2$, $\varphi_p(\mu) = |\mu|^{p-2}\mu$, D^{α}_{0+} denotes the Riemann-Liouville derivative operator. The author improved the results in [10] and proved the existence of the solution to the problem (1.2). Obviously, the problem (1.2) is a particular case of the problem (1.1) when $n = 1$, $m = 2$ and $\beta_1 = \beta_2 = 0$. Subsequently, Wang et al. [27] considered the solvability on the half-line at resonance for the case $n = 1$ and $\beta_1 = \beta_2 = 0$ in the problem (1.1). Baitiche et al. [3] also studied the boundary value problem similar to one of [27] by using upper and lower solution approximation. Recently, Feng et al. [8] have discussed the solvability of linear fractional boundary value problems in $\mathbb{R}^n$ without the p-Laplacian operator in the problem (1.1).

We should mention the main results obtained in [8,12,23,24], which prompts us to consider the problem (1.1). In [24], Phung et al. first researched the following second-order linear boundary value problem:

$$\begin{cases} u''(t) = g(t, u, u'),\ 0 < t < 1, \\ u'(0) = \theta,\ u(1) = Au(\xi), \end{cases} \tag{1.3}$$

where θ is a zero vector in $\mathbb{R}^n$, $0 < \xi < 1$ and A is an n-order square matrix satisfying one of the following two conditions:

$$\begin{cases} A^2 = I \ (I \text{ stands for the unit matrix}), \\ A^2 = A. \end{cases} \tag{1.4}$$

By using Mawhin's continuation theorem, the solvability conditions of the problem (1.3) were obtained. Then, Phung et al. [23] studied the following Riemann-Liouville fractional linear boundary value problem:

$$\begin{cases} D^{\mu}u(t) = g(t, u(t), D^{\mu}u(t)), \quad a.e.\ 0 < t < 1, \\ u(0) = \theta,\ D^{\mu-1}u(1) = AD^{\mu-1}u(\xi), \end{cases} \tag{1.5}$$

where $1 < \mu \leq 2$, D^{μ} is the Riemann-Liouville differential operator of order μ.

In general, the highlights of this paper can be summarized as follows.

- On the one hand, compared with the linear problems (1.3) and (1.5), the nonlinear term φ_p is introduced in the problem (1.1), which makes it more complicated to study the existence of solutions. It is worth noting that we also extend the nonlinear boundary value problem (1.3) to n-dimensional Euclidean space. (To the best of the author's knowledge, this is the first study on nonlinear boundary value problems in $\mathbb{R}^n$).
- On the other hand, the boundary condition of the problem (1.1) is presented as an integral form with a coefficient matrix, and the constraints on the coefficient matrix A have been weakened. It is no longer required the idempotent or involutory matrices in (1.4). This can be regarded as a generalization of the boundary conditions in the problem (1.3).
- In addition, the Hilfer fractional derivative in the problem (1.1) covers both Caputo and Riemann-Liouville derivatives, and can be regarded as a generalization of these two types of derivatives. Therefore, the research in this paper is not only an extension of the nonlinear boundary value problem but also provides an interesting case for the application of Hilfer fractional derivative in the field of calculus.

The rest of this paper includes the following sections. In Sect. 2, some definitions and lemmas are introduced, and two Banach spaces are constructed. In Sect. 3, we first give some preliminary results that on needed in the proof of our main theorem. Based on the extension for the continuous theorem, we then prove the existence of the solution of the problem (1.1). In Sect. 4, the main results are illustrated by an example. A conclusion is introduced in Sect. 5.

2. Preliminaries

Definition 2.1 ([10]). Suppose that Y and Z are two Banach spaces with norms of $\|\cdot\|_Y$ and $\|\cdot\|_Z$ respectively. If the continuous operator F: $domF \cap Y \to Z$ satisfies the following conditions:

(*a*) $KerF := \{u \in domF \cap Y : Fu = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$,

(*b*) $ImF := F(domF \cap Y) \subset Z$ is a closed,

where $n < \infty$, $domF$ is the domain of the operator F. Then the operator F is called quasi-linear.

Definition 2.2 ([12]). Assuming $N_\kappa : \overline{\Omega} \to Z, \kappa \in [0, 1]$ is a bounded and continuous operator, let $\Sigma_\kappa = \{x \in \overline{\Omega} : Fx = N_\kappa x\}$, $KerF = Y_1$. Suppose furthermore that at least one vector space $Z_1 \subset Z$ satisfies $dimY_1 = dimZ_1$. If there exist operators P, R and Q satisfying the following conditions for any $0 \leq \kappa \leq 1$:

(a) $KerQ = ImF$,
(b) $QNx = \theta \Leftrightarrow QN_\kappa x = \theta$,
(c) $R(\cdot, 0)$ is the zero operator, and $R(\cdot, \kappa) \mid_{\Sigma_\kappa} = (I - P) \mid_{\Sigma_\kappa}$,
(d) $F[P + R(\cdot, \kappa)] = (I - Q)N_\kappa$,

where $P : Y \to Y_1$ is a projector, $R : \overline{\Omega} \times [0, 1] \to Y_2$ is a continuous compact operator, and $Q : Z \to Z_1$ is a continuous bounded operator satisfying $Q(I - Q) = 0$, then the operator N_κ is called F-quasi-compact in $\overline{\Omega}$.

Definition 2.3 ([13]). Suppose the function $u(t)$ is defined on the interval (a, b), and $n - 1 < \mu \leq n$, $n \in N^*$. The left Riemann-Liouville fractional derivative and integral of order μ are defined as

$$D^\mu_{a^+} u(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} (I^{n-\mu}_{a^+} u)(t) \text{ and } I^\mu_{a+} u(t) = \frac{1}{\Gamma(\mu)} \int_a^t (t - \xi)^{\mu-1} u(\xi) \mathrm{d}\xi.$$

Definition 2.4 ([11]). Suppose the function $u(t)$ is defined on the interval (a, b), and $n - 1 < \mu \leq n$, $n \in N^*$, $0 \leq \delta \leq 1$. The left/right Hilfer fractional derivative of order μ and type δ is defined as

$$D^{\mu,\delta}_{a\pm} u(t) = (\pm)^n I^{\delta(n-\mu)}_{a\pm} \frac{\mathrm{d}^n}{\mathrm{d}t^n} (I^{(1-\delta)(n-\mu)}_{a\pm} u)(t).$$

Remark 2.1 ([11]). (1) The differential operator $D^{\mu,\delta}_{a\pm}$ can be equivalently expressed as $D^{\mu,\delta}_{a\pm} = I^{\delta(n-\mu)}_{a\pm} D^\gamma_{a\pm}$, $\gamma = \mu + n\delta - \mu\delta$.
(2) The Riemann-Liouville derivative is equivalent to the Hilfer derivative when $\delta = 0$, that is, $D^\mu_{a\pm} = D^{\mu,0}_{a\pm}$.
(3) The Caputo derivative is equivalent to the Hilfer derivative when $\delta = 1$, that is, ${}^C D^\mu_{a\pm} = D^{\mu,1}_{a\pm}$.

Lemma 2.1 (Theorem 2.1, [12]). *Assuming Y and Z are two Banach spaces with norms $\| \cdot \|_Y$ and $\| \cdot \|_Z$, respectively, and Ω is a bounded non-empty open subset of Y. Suppose furthermore that the operator $F : domF \cap Y \to Z$ is quasi-linear, and $N_\kappa : \overline{\Omega} \to Z$, $\kappa \in [0, 1]$ is F-quasi-compact. If*
(a) *$Fx \neq N_\kappa x$, for all $x \in domF \cap \partial\Omega$ and $\kappa \in (0, 1)$,*
(b) *$deg\{KQN, \Omega \cap KerF, 0\} \neq 0$,*

holds, where $K : ImQ \to KerF$ is a homeomorphism with $K(\theta) = \theta$, then there exists at least one solution for the abstract equation $Fx = Nx$ in $domF \cap \overline{\Omega}$.

Lemma 2.2 (Lemma 2.5, [15]). *Assume $m - 1 \leq \mu \leq m$, $m \in N^*$, Suppose furthermore that $u \in L^1(0, 1)$ and $I^{m-\mu}_{0+} u \in AC^m[0, 1]$, then*

$$I^\mu_{0+} D^\mu_{0+} u(t) = u(t) - \sum_{j=1}^m \frac{(I^{m-\mu}_{0+} u(t))^{(m-j)}\big|_{t=0}}{\Gamma(\mu - j + 1)} t^{\mu-j}.$$

Lemma 2.3 (Property 2.1, [15]). *Suppose $\mu > 0$ and $\delta > 0$, then*

$$D^\mu_{0+} t^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\mu + \delta)} t^{\mu+\delta-1}.$$

Lemma 2.4. [16] *For any x and y with $x, y \geq 0$, the following inequalities hold:*
(1) $\varphi_p(x + y) \leq 2^{p-2}(\varphi_p(x) + \varphi_p(y))$, $p \geq 2$,
(2) $\varphi_p(x + y) \leq \varphi_p(x) + \varphi_p(y)$, $1 < p \leq 2$,

where $\varphi_p(x) = |x|^{p-2}x$.

Next, we define several Banach spaces and operators. By

$$\|u\|_X = \max\{\|u\|_\infty, \|D_{0+}^{\alpha_2,\beta_2}u\|_\infty\}$$

we denote the norm of u in the space

$$X = \{u|u, D_{0+}^{\alpha_1,\beta_1}u \in C\big([0,1];\mathbb{R}^n\big)\},$$

where $\|u\|_\infty = \max_{t\in[0,1]}\max_{1\le i\le n}|u_i(t)|$. Furthermore, by $\|y\|_\infty$ we denote the norm of u in the space $Y = C\big([0,1];\mathbb{R}^n\big)$. The operators $L : domL \cap X \to Y$ and $N_\lambda : X \to Y$ are defined as follows

$$Lu(t) = D_{0+}^{\alpha_1,\beta_1}\varphi_p(D_{0+}^{\alpha_2,\beta_2}u(t)),\ t \in [0,1], \tag{2.1}$$

$$N_\lambda u(t) = \lambda f(t, u(t), D_{0+}^{\alpha_2,\beta_2}u(t)),\ \lambda \in [0,1], \tag{2.2}$$

where

$$\begin{aligned} domL =\Big\{u|u \in X, D_{0+}^{\alpha_1,\beta_1}\varphi_p(D_{0+}^{\alpha_2,\beta_2}u) \in Y, D_{0+}^{\alpha_2,\beta_2}u(0) = D_{0+}^{\gamma_2-2}u(0) = \cdots \\ = D_{0+}^{\gamma_2-m}u(0) = \theta, u(1) = A\int_0^1 u(t)h(t)\mathrm{d}t\Big\}. \end{aligned} \tag{2.3}$$

Therefore, we can write the problem (1.1) as $Lu = Nu$, $u \in domL$.

Let $T = I - A\int_0^1 h(t)t^{\gamma_2-1}\mathrm{d}t$ and T^+ be the *Moore-Penrose pseudoinverse matrix* of T. It is necessary to give the following conclusions in [4] for our subsequent research:

(a) $Im(I - T^+T) = KerT$;
(b) $ImT^+T = ImT$;
(c) $TT^+T = T$;
(d) $T^+TT^+ = T^+$.

In addition, throughout this paper, we always suppose that

$$\det\big(I - A\int_0^1 h(t)t^{\gamma_2-1}\mathrm{d}t\big) = 0.$$

3. Main results

In this section, we will prove that the problem (1.1) has at least one solution. To make the proof process clearer, six lemmas and one theorem will be given respectively.

Lemma 3.1. *Suppose the condition* $\det\big(I - A\int_0^1 h(t)t^{\gamma_2-1}\mathrm{d}t\big) = 0$ *holds, then the operator* L *defined in* (2.1) *is quasi-linear.*

Proof. It is not difficult to obtain that

$$KerL = \big\{u \in domL|u(t) = ct^{\gamma_2-1},\ c \in KerT\big\}, \tag{3.1}$$

where $T = I - A\int_0^1 h(t)t^{\gamma_2-1}\mathrm{d}t$. Now, we prove

$$ImL = \big\{y \in Y|\phi y \in ImT\big\}, \tag{3.2}$$

where $\phi : Y \to \mathbb{R}^n$ is a linear operator defined by

$$\phi y = I_{0+}^{\alpha_2}\varphi_q(I_{0+}^{\alpha_1}y(t))\big|_{t=1} - A\int_0^1 h(t)I_{0+}^{\alpha_2}\varphi_q(I_{0+}^{\alpha_1}y(t))\mathrm{d}t,\ \forall y \in Y. \tag{3.3}$$

In fact, for each $y \in ImL$, there exists a function vector $u \in domL$ such that

$$D_{0+}^{\alpha_1,\beta_1}\varphi_p(D_{0+}^{\alpha_2,\beta_2}u(t)) = y(t).$$

By Lemma 2.2 and Remark 2.1, we obtain

$$D_{0+}^{\alpha_2,\beta_2}u(t) = \varphi_q\big(I_{0+}^{\alpha_1}y(t) + c_0t^{\gamma_1-1}\big),$$

where $\gamma_1 = \alpha_1 + \beta_1 - \alpha_1\beta_1,\ q = \frac{p}{p-1}$. Since $D_{0+}^{\alpha_2,\beta_2}u(0) = D_{0+}^{\gamma_2-2}u(0) = \cdots = D_{0+}^{\gamma_2-m}u(0) = \theta$, we can get

$$u(t) = I_{0+}^{\alpha_2}\varphi_q(I_{0+}^{\alpha_1}y(t)) + c_1t^{\gamma_2-1},\ c_1 \in \mathbb{R}^n.$$

From $u(1) = A\int_0^1 h(t)u(t)\mathrm{d}t$, it can be deduced that

$$I_{0+}^{\alpha_2}\varphi_q(I_{0+}^{\alpha_1}y(t))|_{t=1} - A\int_0^1 h(t)I_{0+}^{\alpha_2}\varphi_q(I_{0+}^{\alpha_1}y(t))\mathrm{d}t + \Big(I - A\int_0^1 h(t)t^{\gamma_2-1}\mathrm{d}t\Big)c_1 = \theta. \tag{3.4}$$

Consequently,

$$ImL \subseteq \big\{y \in Y|\phi y \in ImL\big\}. \tag{3.5}$$

On the other hand, let $u(t) = I_{0+}^{\alpha_2}\varphi_q(I_{0+}^{\alpha_1}y(t)) + \xi t^{\gamma_2-1},\ \xi \in \mathbb{R}^n$, and assume that $y \in Y$ satisfies (3.4). By simple calculation, we can infer that $u(t)$ satisfies the boundary conditions of the problem (1.1) and

$$Lu(t) = D_{0+}^{\alpha_1,\beta_1}\varphi_p\big(D_{0+}^{\alpha_2,\beta_2}\big(I_{0+}^{\alpha_2}\varphi_q(I_{0+}^{\alpha_1}y(t))+\xi t^{\gamma_2-1}\big)\big) = D_{0+}^{\alpha_1,\beta_1}\varphi_p(\varphi_q I_{0+}^{\alpha_1}y(t)) = y(t).$$

Thus,

$$ImL \supseteq \big\{y \in Y|\phi y \in ImL\big\}. \tag{3.6}$$

Combining (3.5) and (3.6), we can get

$$ImL = \big\{y \in Y|\phi y \in ImL\big\}. \tag{3.7}$$

Clearly, $ImL \subset Y$ is closed. Thus, the operator L is called a quasi-linear operator. □

The operator $P : X \to KerL$ is defined as

$$(Pu)(t) = (I - T^+T)\frac{t^{\gamma_2-1}}{\Gamma(\gamma_2)}D_{0+}^{\gamma_2-1}u(0). \tag{3.8}$$

It can be derived by simple calculation that $P^2u = Pu$ and $ImP = KerL$, then $KerP \oplus KerL = X$. Hence, $P : X \to KerL$ is a projector.

The operator $Q : Y \to \mathbb{R}^n$ is defined as

$$Qy = c, \tag{3.9}$$

where c satisfies

$$\begin{aligned}&\frac{1}{\Gamma(\alpha_2)}\int_0^1(1-s)^{\alpha_2-1}\varphi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_0^s(s-\tau)^{\alpha_1-1}[y(\tau)-c]\mathrm{d}\tau\right)\mathrm{d}s+T\xi\\&-A\int_0^1\frac{h(t)}{\Gamma(\alpha_2)}\int_0^t(t-s)^{\alpha_2-1}\varphi_q\Big(\frac{1}{\Gamma(\alpha_1)}\int_0^s(s-\tau)^{\alpha_1-1}[y(\tau)-c]\mathrm{d}\tau\Big)\mathrm{d}s\mathrm{d}t=\theta.\end{aligned}\tag{3.10}$$

It can be proved that c is the unique constant vector satisfying (3.10). In fact, let

$$\begin{aligned}F(c)=&\frac{1}{\Gamma(\alpha_2)}\int_0^1(1-s)^{\alpha_2-1}\varphi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_0^s(s-\tau)^{\alpha_1-1}[y(\tau)-c]\mathrm{d}\tau\right)\mathrm{d}s+T\xi\\&-A\int_0^1\frac{h(t)}{\Gamma(\alpha_2)}\int_0^t(t-s)^{\alpha_2-1}\varphi_q\Big(\frac{1}{\Gamma(\alpha_1)}\int_0^s(s-\tau)^{\alpha_1-1}[y(\tau)-c]\mathrm{d}\tau\Big)\mathrm{d}s\mathrm{d}t\end{aligned}\tag{3.11}$$

for all $y\in Y$. Since

$$\begin{aligned}T&=I-A\int_0^1h(t)t^{\gamma_2-1}\mathrm{d}t\\&=\begin{pmatrix}1&&&\\&1&&\\&&\ddots&\\&&&1\end{pmatrix}-\begin{pmatrix}a_{11}\int_0^1h(t)t^{\gamma_2-1}\mathrm{d}t&&&\\&a_{22}\int_0^1h(t)t^{\gamma_2-1}\mathrm{d}t&&\\&&\ddots&\\&&&a_{nn}\int_0^1h(t)t^{\gamma_2-1}\mathrm{d}t\end{pmatrix}\\&=\begin{pmatrix}1-a_{11}\int_0^1h(t)t^{\gamma_2-1}\mathrm{d}t&&&\\&1-a_{22}\int_0^1h(t)t^{\gamma_2-1}\mathrm{d}t&&\\&&\ddots&\\&&&1-a_{nn}\int_0^1h(t)t^{\gamma_2-1}\mathrm{d}t\end{pmatrix},\end{aligned}$$

we have

$$T\xi=\begin{pmatrix}1-ka_{11}&&&\\&1-ka_{22}&&\\&&\ddots&\\&&&1-ka_{nn}\end{pmatrix}\begin{pmatrix}\xi_1\\\xi_2\\\vdots\\\xi_n\end{pmatrix}=\begin{pmatrix}\xi_1(1-ka_{11})\\\xi_2(1-ka_{22})\\\vdots\\\xi_n(1-ka_{nn})\end{pmatrix},\tag{3.12}$$

where $k=\int_0^1h(t)t^{\gamma_2-1}\mathrm{d}t$. Substituting (3.12) into $F(c)=\big(F_i(c)\big)_{1\times n}$ defined in (3.11), we can obtain

$$\begin{aligned}F_i(c)=&\int_0^1\frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\varphi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_0^s(s-\tau)^{\alpha_1-1}[y_i(\tau)-c_i]\mathrm{d}\tau\right)\mathrm{d}s+\xi_i(1-ka_{ii})\\&-\frac{a_{ii}}{\Gamma(\alpha_2)}\int_0^1h(t)\int_0^t(t-s)^{\alpha_2-1}\varphi_q\Big(\int_0^s\frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}[y_i(\tau)-c_i]\mathrm{d}\tau\Big)\mathrm{d}s\mathrm{d}t.\end{aligned}$$

Obviously, $F_i(c)$ is continuous and strictly decreasing in $\mathbb{R}$. Define a cone Λ in $\mathbb{R}^n$ as

$$\Lambda = \big\{(\Lambda_1, \Lambda_2, \cdots, \Lambda_n)^\top, \Lambda_i \geq 0, \Lambda_i \in \mathbb{R}, i = 1, 2, \cdots, n\big\}. \tag{3.13}$$

Take

$$b_i = \min_{t\in[0,1]} y_i(t) + m_i,\ d_i = \max_{t\in[0,1]} y_i(t) + m_i,\ i = 1, 2, \cdots n,$$

where $m_i = \frac{\xi_i(1-ka_{ii})\varphi_q(\Gamma(\alpha_1+1))\Gamma(\alpha_2+\alpha_1 q-\alpha_1+1)}{\Gamma(\alpha_1 q-\alpha_1+1)(1-a_{ii}\int_0^1 h(t)t^{\alpha_2+\alpha_1 q-\alpha_1}dt)}$, $b_i, d_i \in \mathbb{R}$. If $b_i = \min_{t\in[0,1]} y_i(t)+ m_i$, then

$$\begin{aligned}
F_i(b) \geq & 0 - \frac{m_i}{\Gamma(\alpha_2)}\int_0^1 (1-s)^{\alpha_2-1}\varphi_q\bigg(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1}\mathrm{d}\tau\bigg)\mathrm{d}s + \xi_i(1-ka_{ii}) \\
& + \frac{a_{ii}m_i}{\Gamma(\alpha_2)}\int_0^1 h(t)\int_0^t (t-s)^{\alpha_2-1}\varphi_q\bigg(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1}\mathrm{d}\tau\bigg)\mathrm{d}s\mathrm{d}t \\
= & - \frac{m_i\Gamma(\alpha_1(q-1)+1)}{\varphi_q(\Gamma(\alpha_1+1))\Gamma(\alpha_2+\alpha_1(q-1)+1)} \\
& + \frac{m_i a_{ii}\Gamma(\alpha_1(q-1)+1)\int_0^1 h(t)t^{\alpha_2+\alpha_1 q-\alpha_1}\mathrm{d}t}{\varphi_q(\Gamma(\alpha_1+1))\Gamma(\alpha_2+\alpha_1(q-1)+1)} + \xi_i(1-ka_{ii}) \\
= & - \frac{m_i\Gamma(\alpha_1(q-1)+1)\big(1-a_{ii}\int_0^1 h(t)t^{\alpha_2+\alpha_1 q-\alpha_1}\mathrm{d}t\big)}{\varphi_q(\Gamma(\alpha_1+1))\Gamma(\alpha_2+\alpha_1(q-1)+1)} + \xi_i(1-ka_{ii}) \\
= & - \xi_i(1-ka_{ii}) + \xi_i(1-ka_{ii}) \\
= & 0.
\end{aligned}$$

Similarly, if $d_i = \max_{t\in[0,1]} y_i(t) + m_i$, then $F_i(d) \leq 0$. It is not difficult to see that $F(b) \in \Lambda$ and $-F(d) \in \Lambda$, where $b = (b_1, b_2, \cdots, b_n)^\top$, $d = (d_1, d_2, \cdots, d_n)^\top$. Hence, there must be a unique c satisfying $c - b \in \Lambda$ and $d - c \in \Lambda$, such that $F(c) = \theta$. In addition, the boundedness of $Q(\Omega)$ can be deduced from the fact that space $\Omega \subset Y$ is bounded.

Remark 3.1. By the definition of Q in (3.9), it is not difficult to conclude that Q is not a projector but satisfies $Q(I - Q)y = \theta$ for all $y \in Y$.

Lemma 3.2. *The operator Q by (3.9) is continuous in Y.*

Proof. For any $g, y \in Y$, suppose $Qg = d$, $Qy = b$, where $b, d \in \mathbb{R}^n$. Since φ_q is strictly increasing, if $d_i - b_i > \max_{t\in[0,1]}\big(g_i(t) - y_i(t)\big)$, $i = 1, 2, \cdots, n$, then

$$\begin{aligned}
0 = & \frac{1}{\Gamma(\alpha_2)}\int_0^1 (1-s)^{\alpha_2-1}\varphi_q\bigg(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1}[g_i(\tau)-d_i]\mathrm{d}\tau\bigg)\mathrm{d}s + \xi_i(1-ka_{ii}) \\
& - a_{ii}\int_0^1 h(t)\frac{1}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1}\varphi_q\bigg(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1}[g_i(\tau)-d_i]\mathrm{d}\tau\bigg)\mathrm{d}s\mathrm{d}t \\
= & \int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\varphi_q\bigg(\int_0^s \frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}[(y_i(\tau)-b_i) + (g_i(\tau)-y_i(\tau)) \\
& - (d_i - b_i)]\mathrm{d}\tau\bigg)\mathrm{d}s + \xi_i(1-ka_{ii}) \\
& - a_{ii}\int_0^1 h(t)\frac{1}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1}\varphi_q\bigg(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1}[(y_i(\tau)-b_i)
\end{aligned}$$

$$
\begin{aligned}
&\quad + (g_i(\tau) - y_i(\tau)) - (d_i - b_i)]\mathrm{d}\tau\Big)\mathrm{d}s\mathrm{d}t \\
&< \frac{1}{\Gamma(\alpha_2)}\int_0^1 (1-s)^{\alpha_2-1}\varphi_q\Big(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1}[y_i(\tau)-b_i]\mathrm{d}\tau\Big)\mathrm{d}s + \xi_i(1-ka_{ii}) \\
&\quad - a_{ii}\int_0^1 \frac{h(t)}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1}\varphi_q\Big(\int_0^s \frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}[y_i(\tau)-b_i]\mathrm{d}\tau\Big)\mathrm{d}s\mathrm{d}t \\
&= 0.
\end{aligned}
$$

This is a contradiction. Conversely, if $d_i - b_i < \min_{t\in[0,1]}(g_i(t) - y_i(t))$, then

$$
\begin{aligned}
0 &= \int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\varphi_q\Big(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1}[g_i(\tau)-d_i]\mathrm{d}\tau\Big)\mathrm{d}s + \xi_i(1-ka_{ii}) \\
&\quad - a_{ii}\int_0^1 \frac{h(t)}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1}\varphi_q\Big(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1}[g_i(\tau)-d_i]\mathrm{d}\tau\Big)\mathrm{d}s\mathrm{d}t \\
&> \int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\varphi_q\Big(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1}[y_i(\tau)-b_i]\mathrm{d}\tau\Big)\mathrm{d}s + \xi_i(1-ka_{ii}) \\
&\quad - a_{ii}\int_0^1 \frac{h(t)}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1}\varphi_q\Big(\int_0^s \frac{(s-\tau)^{\alpha_1-1}}{\Gamma(\alpha_1)}[y_i(\tau)-b_i]\mathrm{d}\tau\Big)\mathrm{d}s\mathrm{d}t \\
&= 0,
\end{aligned}
$$

the contradiction appears. Consequently,

$$
\min_{t\in[0,1]}\big(g_i(t)-y_i(t)\big) \le d_i - b_i \le \max_{t\in[0,1]}\big(g_i(t)-y_i(t)\big).
$$

Then, it can be concluded that $Q : Y \to \mathbb{R}^n$ is continuous. □

Lemma 3.3. *The definition of the operator $R : X \times [0,1] \to X_2$ is*

$$
R(u,\lambda)(t) = I_{0+}^{\alpha_2}\varphi_q\big(I_{0+}^{\alpha_1}(N_\lambda u(t) - QN_\lambda u(t))\big) - T^+\phi\big(N_\lambda u(t) - QN_\lambda u(t)\big)t^{\gamma_2-1}, \tag{3.14}
$$

where ϕ is defined in (3.3), $KerL \oplus X_2 = X$. Then the operator $R : \overline{\Omega}\times[0,1] \to X_2$ is continuous and compact, where $\Omega \subset X$ is an open bounded set.

Proof. Obviously, R is continuous. Next, we show that R is compact. In fact, for any $u \in \overline{\Omega}$, by the boundedness of f on a bounded closed domain and the boundedness of Q, we obtain that there exist constants $k_1 > 0$, $k_2 > 0$ such that $\max_{(t,u)\in[0,1]\times\overline{\Omega}} |f(t,u(t),D_{0+}^{\alpha_2,\beta_2}u(t))| \le k_1$, $|Qf(t,u(t),D_{0+}^{\alpha_2,\beta_2}u(t))| \le k_2$, then

$$
\begin{aligned}
|R(u,\lambda)(t)| &= \Big|\int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\varphi_q\Big(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-r)^{\alpha_1-1}[N_\lambda u(r) - QN_\lambda u(r)]dr\Big)\mathrm{d}s \\
&\quad - T^+\phi\big(N_\lambda u(t) - QN_\lambda u(t)\big)t^{\gamma_2-1}\Big| \\
&\le \int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\varphi_q\Big(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\Big)\mathrm{d}s + \|T^+\|_*\big|\phi(N_\lambda u(t) - QN_\lambda u(t))\big| \\
&\le \frac{1}{\Gamma(\alpha_2+1)}\varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right) + \Big|I_{0+}^{\alpha_2}\varphi_q\big(I_{0+}^{\alpha_1}[N_\lambda u(t) - QN_\lambda u(t)]\big)|_{t=1}
\end{aligned}
$$

$$
\begin{aligned}
&- A\int_0^1 h(t) I_{0+}^{\alpha_2}\varphi_q\big(I_{0+}^{\alpha_1}[N_\lambda u(t)-QN_\lambda u(t)]\big)\mathrm{d}t\Big|\|T^+\|_*\\
\le&\frac{1}{\Gamma(\alpha_2+1)}\varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right)+\frac{\|T^+\|_*}{\Gamma(\alpha_2+1)}\varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right)\\
&+\frac{\|T^+\|_*\|A\|_*}{\Gamma(\alpha_2+1)}\varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right)\int_0^1 h(t)\mathrm{d}t\\
\le&\left(1+\|T^+\|_*+\|T^+\|_*\|A\|_*\int_0^1 h(t)\mathrm{d}t\right)\frac{\varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right)}{\Gamma(\alpha_2+1)}
\end{aligned}
$$

and

$$
\begin{aligned}
|D_{0+}^{\alpha_2,\beta_2}R(u,\lambda)(t)| &= \big|I_{0+}^{\beta_2(n-\alpha_2)}D_{0+}^{\gamma_2}R(u,\lambda)(t)\big|\\
&=\big|\varphi_q\big(I_{0+}^{\alpha_1}[N_\lambda u(t)-QN_\lambda u(t)]\big)\big|\\
&\le \varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right),
\end{aligned}
$$

where $\|\cdot\|_*$ stand for the max-norm of matrices, $|x|=\max\{|x_i|, i=1,2,\cdots n\}$. Therefore, R is bounded.

For any $u\in\bar{\Omega}$, $0\le\lambda\le1$ and $0\le t_1<t_2\le1$, there are

$$
\begin{aligned}
&\Big|R(u,\lambda)(t_2)-R(u,\lambda)(t_1)\Big|\\
=&\Bigg|\frac{1}{\Gamma(\alpha_2)}\int_0^{t_2}(t_2-s)^{\alpha_2-1}\varphi_q\Big(\frac{1}{\Gamma(\alpha_1)}\int_0^s(s-r)^{\alpha_1-1}\big(N_\lambda u(r)-QN_\lambda u(r)\big)\mathrm{d}r\Big)\mathrm{d}s\\
&-\frac{1}{\Gamma(\alpha_2)}\int_0^{t_1}(t_1-s)^{\alpha_2-1}\varphi_q\Big(\frac{1}{\Gamma(\alpha_1)}\int_0^s(s-r)^{\alpha_1-1}\big(N_\lambda u(r)-QN_\lambda u(r)\big)\mathrm{d}r\Big)\mathrm{d}s\\
&-T^+\Bigg[I_{0+}^{\alpha_2}\varphi_q(I_{0+}^{\alpha_1}[N_\lambda u(t)-QN_\lambda u(t)])|_{t=1}-A\int_0^1\frac{h(t_2)}{\Gamma(\alpha_2)}\int_0^{t_2}(t_2-s)^{\alpha_2-1}\\
&\times\varphi_q\big(\int_0^s\frac{(s-r)^{\alpha_1-1}}{\Gamma(\alpha_1)}[N_\lambda u(r)-QN_\lambda u(r)]\mathrm{d}r)\mathrm{d}s\mathrm{d}t_2\Bigg]t_2^{\gamma_2-1}\\
&+T^+\Bigg[I_{0+}^{\alpha_2}\varphi_q(I_{0+}^{\alpha_1}[N_\lambda u(t)-QN_\lambda u(t)])|_{t=1}-A\int_0^1\frac{h(t_1)}{\Gamma(\alpha_2)}\int_0^{t_1}(t_1-s)^{\alpha_2-1}\\
&\times\varphi_q\big(\int_0^s\frac{(s-r)^{\alpha_1-1}}{\Gamma(\alpha_1)}[N_\lambda u(r)-QN_\lambda u(r)]\mathrm{d}r)\mathrm{d}s\mathrm{d}t_1\Bigg]t_1^{\gamma_2-1}\Bigg|\\
\le&\frac{1}{\Gamma(\alpha_2)}\Big|\int_0^{t_1}[(t_2-s)^{\alpha_2-1}-(t_1-s)^{\alpha_2-1}]\\
&\times\varphi_q\big(\frac{1}{\Gamma(\alpha_1)}\int_0^s(s-r)^{\alpha_1-1}[N_\lambda u(r)-QN_\lambda u(r)]\mathrm{d}r\big)\mathrm{d}s\\
&+\int_{t_1}^{t_2}(t_2-s)^{\alpha_2-1}\varphi_q\big(\frac{1}{\Gamma(\alpha_1)}\int_0^s(s-r)^{\alpha_1-1}[N_\lambda u(r)-QN_\lambda u(r)]\mathrm{d}r)\mathrm{d}s\Big|\\
&+\frac{\|T^+\|_*}{\Gamma(\alpha_2+1)}\varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right)(t_2^{\gamma_2-1}-t_1^{\gamma_2-1})
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\|T^{+}A\|_{*}\int_0^1 h(t)\mathrm{d}t}{\Gamma(\alpha_2+1)}(t_2^{\gamma_2-1}-t_1^{\gamma_2-1})\varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right)\\
\leq&\frac{1}{\Gamma(\alpha_2+1)}\varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right)(t_2^{\alpha_2}-t_1^{\alpha_2})\\
&+\Big(\|T^{*}\|_{*}+\|T^{*}A\|_{*}\int_0^1 h(t)\mathrm{d}t\Big)\frac{1}{\Gamma(\alpha_2+1)}\varphi_q\left(\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\right)(t_2^{\gamma_2-1}-t_1^{\gamma_2-1})
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|D_{0+}^{\alpha_2,\beta_2}R(u,\lambda)(t_2)-D_{0+}^{\alpha_2,\beta_2}R(u,\lambda)(t_1)\Big|\\
=&\Big|\varphi_q\Big(\frac{1}{\Gamma(\alpha_1)}\int_0^{t_2}(t_2-s)^{\alpha_1-1}(N_\lambda u(s)-QN_\lambda u(s))\mathrm{d}s\Big)\\
&-\varphi_q\Big(\frac{1}{\Gamma(\alpha_1)}\int_0^{t_1}(t_1-s)^{\alpha_1-1}(N_\lambda u(s)-QN_\lambda u(s))\mathrm{d}s\Big)\Big|.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\Big|\frac{1}{\Gamma(\alpha_1)}\int_0^{t_2}(t_2-s)^{\alpha_1-1}(N_\lambda u(s)-QN_\lambda u(s))\mathrm{d}s\\
&-\frac{1}{\Gamma(\alpha_1)}\int_0^{t_1}(t_1-s)^{\alpha_1-1}(N_\lambda u(s)-QN_\lambda u(s))\mathrm{d}s\Big|\\
=&\frac{1}{\Gamma(\alpha_1)}\Big|\int_0^{t_1}[(t_2-s)^{\alpha_1-1}-(t_1-s)^{\alpha_1-1}](N_\lambda u(s)-QN_\lambda u(s))\mathrm{d}s\\
&+\int_{t_1}^{t_2}(t_2-s)^{\alpha_1-1}(N_\lambda u(s)-QN_\lambda u(s))\mathrm{d}s\Big|\\
\leq&\frac{k_1+k_2}{\Gamma(\alpha_1)}\Big|\int_0^{t_1}[(t_2-s)^{\alpha_1-1}-(t_1-s)^{\alpha_1-1}]\mathrm{d}s+\int_{t_1}^{t_2}(t_2-s)^{\alpha_1-1}\mathrm{d}s\Big|\\
\leq&\frac{k_1+k_2}{\Gamma(\alpha_1+1)}(t_2^{\alpha_1}-t_1^{\alpha_1}),
\end{aligned}
$$

$$
\Big|\int_0^t\frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}(N_\lambda u(s)-QN_\lambda u(s))\mathrm{d}s\Big|\leq\frac{k_1+k_2}{\Gamma(\alpha_1+1)}
$$

and $\varphi_q(\vartheta)$ is uniformly continuous on $\big[-\frac{k_1+k_2}{\Gamma(\alpha_1+1)},\frac{k_1+k_2}{\Gamma(\alpha_1+1)}\big]$. Consequently, $\{R(u,\lambda)\mid(u,\lambda)\in\overline{\Omega}\times[0,1]\}$ and $\{D_{0+}^{\alpha_2,\beta_2}R(u,\lambda)\mid(u,\lambda)\in\overline{\Omega}\times[0,1]\}$ are equicontinuous. In view of the Arzela-Ascoli Theorem, it yields that $R:\overline{\Omega}\times[0,1]\to X_2$ is compact. □

Lemma 3.4. *Suppose that Ω is a bounded, open subset of X. Then the operator N_λ defined in (2.2) is L-quasi-compact in $\bar{\Omega}$.*

Proof. It is not difficult to deduce that $dimKerL=dimImQ$, $KerQ=ImL$, $R(\cdot,0)=\theta$ and $QN_\lambda u(t)=\theta\Leftrightarrow QNu(t)=\theta$. Then (a) and (b) of Definition 2.2 hold.

For each $u \in \Sigma_\lambda = \{u \in \overline{\Omega} \mid Lu = N_\lambda u\}$, there is $N_\lambda u \in ImL = KerQ$, so $QN_\lambda u = \theta$. It follows from $N_\lambda u = Lu(t) = D_{0+}^{\alpha_1,\beta_1}\varphi_p\big(D_{0+}^{\alpha_2,\beta_2}u(t)\big)$ and $R(u,0)(t) = D_{0+}^{\alpha_2,\beta_2}R(u,0)(t) = \theta$ that

$$\begin{aligned}R(u,\lambda)(t) =& I_{0+}^{\alpha_2}\varphi_q(I_{0+}^{\alpha_1}(N_\lambda u(t) - QN_\lambda u(t)) - T^+\phi\big(N_\lambda u(t) - QN_\lambda u(t)\big)t^{\gamma_2-1}\\ =& I_{0+}^{\alpha_2}\varphi_q\big(I_{0+}^{\alpha_1}(N_\lambda u(t)\big) - T^+\phi\big(N_\lambda u(t)\big)t^{\gamma_2-1}\\ =& I_{0+}^{\alpha_2}\varphi_q\big(I_{0+}^{\alpha_1}I_{0+}^{\beta_1(1-\alpha_1)}D_{0+}^{\gamma_1}\varphi_p(D_{0+}^{\alpha_2,\beta_2}u(t)\big) - T^+\phi\big(N_\lambda u(t)\big)t^{\gamma_2-1}\\ =& u(t) - \frac{t^{\gamma_2-1}}{\Gamma(\gamma_2)}D_{0+}^{\gamma_2-1}u(0) + T^+T\frac{t^{\gamma_2-1}}{\Gamma(\gamma_2)}D_{0+}^{\gamma_2-1}u(0)\\ =& u(t) - (I - T^+T)\frac{t^{\gamma_2-1}}{\Gamma(\gamma_2)}D_{0+}^{\gamma_2-1}u(0)\\ =& (I-P)u.\end{aligned}$$

Consequently, (c) in Definition 2.2 is satisfied.

For any $u \in \overline{\Omega}$, there is

$$\begin{aligned}L[Pu + R(u,\lambda)](t) =& D_{0+}^{\alpha_1,\beta_1}\varphi_p\Big(D_{0+}^{\alpha_2,\beta_2}\big(pu(t) + R(u,\lambda)(t)\big)\Big)\\ =& D_{0+}^{\alpha_1,\beta_1}\varphi_p\Big[D_{0+}^{\alpha_2,\beta_2}(I - T^+T)\frac{t^{\gamma_2-1}}{\Gamma(\gamma_2)}D_{0+}^{\gamma_2-1}u(0)\\ &+ D_{0+}^{\alpha_2,\beta_2}\Big(I_{0+}^{\alpha_2}\varphi_q\big(I_{0+}^{\alpha_1}(N_\lambda u(t) - QN_\lambda u(t))\big)\\ &- T^+\phi\big(N_\lambda u(t) - QN_\lambda u(t)\big)t^{\gamma_2-1}\Big)\Big]\\ =& D_{0+}^{\alpha_1,\beta_1}\varphi_p\Big(D_{0+}^{\alpha_2,\beta_2}I_{0+}^{\alpha_2}\varphi_q\big(I_{0+}^{\alpha_1}(N_\lambda u(t) - QN_\lambda u(t))\big)\Big)\\ =& (I-Q)N_\lambda u(t),\end{aligned}$$

then (d) of Definition 2.2 holds. Thus, the operator N_λ is L-quasi-compact in $\overline{\Omega}$. $\square$

Next, we will give the main theorem.

Theorem 3.1. *Suppose the following conditions hold:*
(H_1) *There exists a constant* $M > 0$ *such that for every* $u \in domL$, *if* $|t^{1-\gamma_2}u(t)| > M$, $t \in [0,1]$, *then either*

$$(1)\ \langle t^{1-\gamma_2}u, Qf\rangle > 0 \quad or \quad (2)\ \langle t^{1-\gamma_2}u, Qf\rangle < 0, \quad \forall t \in [0,1]$$

holds, where $\langle\cdot,\cdot\rangle$ *is the scalar product in* $\mathbb{R}^n$.
(H_2) *There exist three non-negative functions* $a, \psi, \varsigma \in C[0,1]$ *such that*

$$\Big|f(t,\omega,\varpi)\Big| \le a(t)\varphi_p(|\omega|) + \psi(t)\varphi_p(|\varpi|) + \varsigma(t), \quad 0 \le t \le 1,$$

where $\max\{1, 2^{q-2}\}\big[\varphi_q(\|\psi\|_\infty) + \frac{2\varphi_q(\|a\|_\infty)}{\Gamma(\alpha_2+1)}\big] < \varphi_q(\Gamma(\alpha_1+1))$.
Then there exists at least one solution in X *for the problem (1.1).*

To prove Theorem 3.1, the following lemmas are first established.

Lemma 3.5. *Assume that* (H_1) *and* (H_2) *hold. Let* $\Omega_1 = \{u \in domL | Lu = N_\lambda u, \lambda \in (0,1)\}$, *then* Ω_1 *is bounded in* X.

Proof. For any $u \in \Omega_1$, we have $Lu = N_\lambda u$, $N_\lambda u \in ImL = KerQ$, then $QN_\lambda u(t) = \theta$. It is known from (H_1) that there exists $t_0 \in [0,1]$ such that $\left|t_0^{1-\gamma_2} u(t_0)\right| \leq M$. Since $Lu = N_\lambda u$, there is

$$u(t) = I_{0+}^{\alpha_2}\varphi_q(\lambda I_{0+}^{\alpha_1} Nu(t)) + \xi t^{\gamma_2 - 1}, \tag{3.15}$$

and then by (H_2), it follows that

$$\begin{aligned}
|\xi| \leq & |t_0^{1-\gamma_2} u(t)| + |t_0^{1-\gamma_2} I_{0+}^{\alpha_2}\varphi_q(I_{0+}^{\alpha_1} Nu(t))| \\
\leq & M + \frac{t_0^{1-\gamma_2}}{\Gamma(\alpha_2)}\int_0^{t_0}(t_0 - s)^{\alpha_2 - 1}\varphi_q\Big(\int_0^s \frac{(s-\varrho)^{\alpha_1 - 1}}{\Gamma(\alpha_1)}|f(\varrho, u(\varrho), D_{0+}^{\alpha_2,\beta_2}u(\varrho))|\mathrm{d}\varrho\Big)\mathrm{d}s \\
\leq & M + \frac{t_0^{1-\gamma_2}}{\Gamma(\alpha_2)}\int_0^{t_0}(t_0 - s)^{\alpha_2 - 1} \\
& \times \varphi_q\Big(\frac{\|a\|_\infty\varphi_p(\|u\|_\infty) + \|\psi\|_\infty\varphi_p(\|D_{0+}^{\alpha_2,\beta_2}u\|_\infty) + \|\varsigma\|_\infty}{\Gamma(\alpha_1 + 1)}\Big)\mathrm{d}s \\
\leq & M + \frac{\max\{1, 2^{q-2}\}\big[\varphi_q(\|a\|_\infty)\|u\|_\infty + \varphi_q(\|\psi\|_\infty)\|D_{0+}^{\alpha_2,\beta_2}u\|_\infty + \varphi_q(\|\varsigma\|_\infty)\big]}{\Gamma(\alpha_2 + 1)\varphi_q(\Gamma(\alpha_1 + 1))}.
\end{aligned}$$

Since

$$\begin{aligned}
& |D_{0+}^{\alpha_2,\beta_2}u(t)| \\
= & |D_{0+}^{\alpha_2,\beta_2} I_{0+}^{\alpha_2}\varphi_q\big(\lambda I_{0+}^{\alpha_1} Nu(t)\big) + D_{0+}^{\alpha_2,\beta_2}\xi t^{\gamma_2 - 1}| \\
\leq & \frac{\max\{1, 2^{q-2}\}\big[\varphi_q(\|a\|_\infty)\|u\|_\infty + \varphi_q(\|\varsigma\|_\infty) + \varphi_q(\|\psi\|_\infty)\|D_{0+}^{\alpha_2,\beta_2}u\|_\infty\big]}{\varphi_q(\Gamma(\alpha_1 + 1))},
\end{aligned}$$

we can get

$$\|D_{0+}^{\alpha_2,\beta_2}u\|_\infty \leq \frac{\max\{1, 2^{q-2}\}\big[\varphi_q(\|a\|_\infty)\|u\|_\infty + \varphi_q(\|\varsigma\|_\infty)\big]}{\varphi_q(\Gamma(\alpha_1 + 1)) - \max\{1, 2^{q-2}\}\varphi_q(\|\psi\|_\infty)}. \tag{3.16}$$

Therefore,

$$\begin{aligned}
|\xi| \leq & M + \frac{\max\{1, 2^{q-2}\}[\varphi_q(\|a\|_\infty)\|u\|_\infty + \varphi_q(\|\varsigma\|_\infty)]}{\Gamma(\alpha_2 + 1)\varphi_q(\Gamma(\alpha_1 + 1))} \\
& + \frac{\max\{1, 2^{q-2}\}\varphi_q(\|\psi\|_\infty)[\varphi_q(\|a\|_\infty)\|u\|_\infty + \varphi_q(\|\varsigma\|_\infty)]}{\Gamma(\alpha_2 + 1)\varphi_q(\Gamma(\alpha_1 + 1))[\varphi_q(\Gamma(\alpha_1 + 1)) - \max\{1, 2^{q-2}\}\varphi_q(\|\psi\|_\infty)]} \\
= & M + \frac{\max\{1, 2^{q-2}\}\varphi_q(\Gamma(\alpha_1 + 1))\big[\varphi_q(\|a\|_\infty)\|u\|_\infty + \varphi_q(\|\varsigma\|_\infty)\big]}{\Gamma(\alpha_2 + 1)\varphi_q(\Gamma(\alpha_1 + 1))\big[\varphi_q(\Gamma(\alpha_1 + 1)) - \max\{1, 2^{q-2}\}\varphi_q(\|\psi\|_\infty)\big]}.
\end{aligned}$$

Substituting this inequality into (3.15) to get

$$\begin{aligned}
|u(t)| \leq & |I_{0+}^{\alpha_2}\varphi_q(\lambda I_{0+}^{\alpha_1} Nu(t))| + |\xi t^{\gamma_2 - 1}| \\
\leq & \frac{\max\{2, 2^{q-1}\}\varphi_q(\Gamma(\alpha_1 + 1))\big[\varphi_q(\|a\|_\infty)\|u\|_\infty + \varphi_q(\|\varsigma\|_\infty)\big]}{\Gamma(\alpha_2 + 1)\varphi_q(\Gamma(\alpha_1 + 1))\big[\varphi_q(\Gamma(\alpha_1 + 1)) - \max\{1, 2^{q-2}\}\varphi_q(\|\psi\|_\infty)\big]} + M.
\end{aligned}$$

Then,

$$\|u\|_\infty \leq \frac{\mathcal{C}_1\varphi_q(\|\varsigma\|_\infty) + M\Gamma(\alpha_2 + 1)\big[\varphi_q(\Gamma(\alpha_1 + 1)) - \mathcal{C}_2\varphi_q(\|\psi\|_\infty)\big]}{\Gamma(\alpha_2 + 1)\big[\varphi_q(\Gamma(\alpha_1 + 1)) - \mathcal{C}_2\varphi_q(\|\psi\|_\infty)\big] - \mathcal{C}_1\varphi_q(\|a\|_\infty)}, \tag{3.17}$$

where $\mathcal{C}_1 = \max\{2, 2^{q-1}\}$, $\mathcal{C}_2 = \max\{1, 2^{q-2}\}$. Hence, together with (3.16) and (3.17), it can be deduced that Ω_1 is bounded in X. □

Lemma 3.6. *Assume that* (H_1) *holds, then* $\Omega_2 = \{u|u \in KerL, QNu = \theta\}$ *is bounded in* X.

Proof. Let $u \in \Omega_2$, we have $QNu(t) = \theta$ and $u(t) = ct^{\gamma_2-1}$, $c \in \mathbb{R}^n$. According to (H_1), there exists $t_0 \in [0,1]$ such that $\left|t_0^{1-\gamma_2}u(t_0)\right| \le M$. Thus, we get that $|c| \le M$, then Ω_2 is bounded in X. □

The following is the proof of Theorem 3.1.

Proof. Let $\Omega \supset \left(\overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \{x|x \in X, \|x\| \le M\}\right)$ be a bounded open subset of X. Lemma 3.5 implies that $Lu \ne N_\lambda u$, $u \in domL \cap \partial\Omega$, while Lemma 3.6 leads to the conclusion that $QNu \ne \theta$, $u \in KerL \cap \partial\Omega$. Let $H(u,\zeta) = \rho\zeta u + (1-\zeta)JQNu$, where $u \in KerL \cap \overline{\Omega}$, $\zeta \in [0,1]$, $J : ImQ \to KerL$ is a homeomorphism with $J\eta = \eta t^{\gamma_2-1}$, and

$$\rho = \begin{cases} 1, \text{ if } (H_1)\ (1) \text{ holds}, \\ -1, \text{ if } (H_1)\ (2) \text{ holds}. \end{cases}$$

Given any $u \in KerL \cap \partial\Omega$, there are $u(t) = \eta_0 t^{\gamma_2-1}$ and $H(u,\zeta) = \rho\zeta\eta_0 t^{\gamma_2-1} + (1-\zeta)(Qf)t^{\gamma_2-1}$.
If $\zeta = 1$, then $H(u,1) = \rho\eta_0 t^{\gamma_2-1} \ne \theta$.
If $\zeta = 0$, then $H(u,0) = (Qf)t^{\gamma_2-1} \ne \theta$.
If $0 < \zeta < 1$, suppose $H(u,\zeta) = \theta$, then $\rho\zeta\eta_0 t^{\gamma_2-1} = -(1-\zeta)(Qf)t^{\gamma_2-1}$. So there is $\eta_0 = -\frac{(1-\zeta)(Qf)}{\zeta\rho}$. It follows from (H_1) and $|\eta_0| = |t^{1-\gamma_2}u(t)| > M$ that

$$\langle \eta_0, \eta_0 \rangle = -\frac{1-\zeta}{\zeta}\frac{\langle \eta_0, Qf \rangle}{\rho} < 0.$$

This is a contradiction. Hence, $H(u,\zeta) \ne \theta$ for all $u \in KerL \cap \partial\Omega$, $\zeta \in [0,1]$. The homotopy property of degree yields the result that

$$\begin{aligned} deg(JQN|_{KerL}, \Omega \cap KerL, \theta) &= deg(H(\cdot,0), \Omega \cap KerL, \theta) \\ &= deg(H(\cdot,1), \Omega \cap KerL, \theta) \\ &= deg(\rho I, \Omega \cap KerL, \theta) \ne 0. \end{aligned}$$

Combining Lemmas 3.1-3.4 and applying Lemma 2.1, we conclude that the problem (1.1) has at least one solution in X. The proof is completed. □

4. Example

Example 4.1. We consider the following boundary value problem at resonance in $\mathbb{R}^2$:

$$\begin{cases} D_{0+}^{\frac{1}{2},\frac{1}{2}}\varphi_{\frac{5}{2}}(D_{0+}^{\frac{5}{2},\frac{1}{2}}u_1(t)) = f_1(t, u_1(t), u_2(t), D_{0+}^{\frac{5}{2},\frac{1}{2}}u_1(t), D_{0+}^{\frac{5}{2},\frac{1}{2}}u_2(t)),\ 0 \le t \le 1, \\ D_{0+}^{\frac{1}{2},\frac{1}{2}}\varphi_{\frac{5}{2}}(D_{0+}^{\frac{5}{2},\frac{1}{2}}u_2(t)) = f_2(t, u_1(t), u_2(t), D_{0+}^{\frac{5}{2},\frac{1}{2}}u_1(t), D_{0+}^{\frac{5}{2},\frac{1}{2}}u_2(t)),\ 0 \le t \le 1, \\ D_{0+}^{\frac{5}{2},\frac{1}{2}}u_1(0) = D_{0+}^{\frac{5}{2},\frac{1}{2}}u_2(0) = 0,\ D_{0+}^{\frac{3}{4}}u_1(0) = D_{0+}^{\frac{3}{4}}u_2(0) = 0, \\ u_1(1) = -2\displaystyle\int_0^1 t^{-\frac{3}{4}}u_1(t)\mathrm{d}t,\ u_2(1) = -3\displaystyle\int_0^1 t^{-\frac{3}{4}}u_2(t)\mathrm{d}t, \end{cases} \tag{4.1}$$

where $\alpha_1=\frac{1}{2}$, $\beta_1=\frac{1}{2}$, $\alpha_2=\frac{5}{2}$, $\beta_2=\frac{1}{2}$, $\gamma_2=\frac{11}{4}$, $p=\frac{5}{2}$, $h(t)=t^{-\frac{3}{4}}$, $f:[0,1]\times\mathbb{R}^4\to\mathbb{R}^2$ are defined as

$$\begin{aligned} f(t,u,z) &= \big(f_1(t,u_1,u_2,z_1,z_2), f_2(t,u_1,u_2,z_1,z_2)\big)^\top \\ &= \Big(-\frac{u_1+z_1-e^5}{20}, \frac{|u_2|+|z_2|+e^3}{40}\Big)^\top \end{aligned}$$

for any $t\in[0,1]$ and $u=(u_1,u_2)^\top, z=(z_1,z_2)^\top\in\mathbb{R}^2$.

Clearly, $A=\begin{pmatrix}-2 & 0\\ 0 & -3\end{pmatrix}$, $T=\int_0^1 h(t)t^{\gamma_2-1}\mathrm{d}t=\begin{pmatrix}2 & 0\\ 0 & \frac{5}{2}\end{pmatrix}$. Let $\xi=\begin{pmatrix}1\\ 2\end{pmatrix}$, then $T\xi=\begin{pmatrix}2\\ 5\end{pmatrix}$.

Now we prove that the conditions of Theorem 3.1 hold. Choose nonnegative integrable functions $a=\psi=\frac{1}{20}$ and $\varsigma=\frac{e^5}{20}$, then there is

$$\big|f(t,u,z)\big|\le a(t)\varphi_p(|u|)+\psi(t)\varphi_p(|z|)+\varsigma(t).$$

After some simple calculations,

$$\varphi_q(\Gamma(\alpha_1+1))-\max\{1,2^{q-2}\}\big[\varphi_q(\|\psi\|_\infty)+\frac{2}{\Gamma(\alpha_2+1)}\varphi_q(\|a\|_\infty)\big]\approx 0.6505>0$$

can be obtaind. Therefore, (H_2) is satisfied.

In order to check (H_1), let $M=3$, $c=(\|f_1\|_\infty+5.8997, \|f_2\|_\infty+12.6422)^\top$, then c satisfies (3.10). If $\big|t_0^{1-\gamma_2}u(t_0)\big|>M=3$ hold for any $t\in[0,1]$, then $\langle t^{1-\gamma_2}u, Qf\rangle=\langle t^{1-\gamma_2}u, c\rangle>0$. Hence, the condition (H_1) holds. From Theorem 3.1, it can be obtained that the problem (4.1) has at least one solution.

To intuitively illustrate the existence of solutions for the problem (4.1), we conducted numerical simulations using MATLAB. Figures 1 and 2 depict the cases for $p=2.5$ and $p=1.5$, respectively.

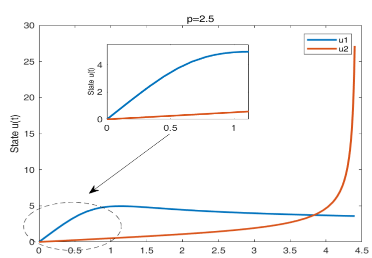

Figure 1. State $u(t)$ of the system (4.1) when $p=2.5$.

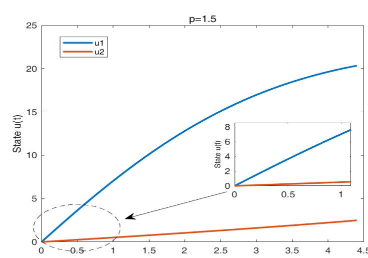

Figure 2. State $u(t)$ of the system (4.1) when $p=1.5$.

5. Conclusion

In this paper, we investigated the nonlinear Hilfer fractional boundary value problem at resonance in $\mathbb{R}^n$. By using the extension for the continuous theorem, the conclusion that the problem (1.1) has at least one solution in X was obtained. To achieve our main results, we defined two Banach spaces with specified norms and construct the appropriate operators P, Q and R within these Banach spaces. Subsequently, we proved the necessary requirements before applying Lemma 2.1. It is worth noting that the variables in the n-dimensional Euclidean space are represented as vectors or matrices, and we cannot assume a direct size relationship. The cone in (3.13) is skillfully defined, effectively resolving existing issues. Finally, we provided an example to verify the validity of our conclusion.

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References

[1] P. Aceves-Sanchez and L. Cesbron, *Fractional diffusion limit for a fractional Vlasov-Fokker-Planck equation*, SIAM J. Math. Anal., 2019, 51(1), 469–488.

[2] M. Azouzi, L. Guedda and Z. Laadjal, *Existence results of generalized proportional fractional differential equations at resonance case by the topological degree theory*, Adv. Math., 2022, 11, 613–633.

[3] Z. Baitiche, C. Derbazi and G. Wang, *Monotone iterative method for nonlinear fractional p-Laplacian differential equation in terms of ψ-Caputo fractional derivative equipped with a new class of nonlinear boundary conditions*, Math. Methods Appl. Sci., 2022, 45(2), 967–976.

[4] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, Springer-Verlag, New York, 2003.

[5] N. Cao and X. Fu, *Stationary distribution and extinction of a Lotka-Volterra model with distribute delay and nonlinear stochastic perturbations*, Chaos Solitons Fractals, 2023, 169, 113246–113254.

[6] R. Chaharpashlou and A. M. Lopes, *Hyers-ulam-rassias stability of a nonlinear stochastic fractional volterra integro-differential equation*, J. Appl. Anal. Comput., 2023, 13(5), 2799–2808.

[7] W. Ding, J. P. Hollkamp, S. Patnaik, et al., *On the fractional homogenization of one-dimensional elastic metamaterials with viscoelastic foundation*, Advanced Structured Materials, 2023, 93(1), 261–286.

[8] Y. Feng and Z. Bai, *Solvability of some nonlocal fractional boundary value problems at resonance in $\mathbb{R}^n$*, Fractal Fract., 2022, 6(1), 25–40.

[9] M. M. Freitas, A. J. A. Ramos, A. Ö. Özer, et al., *Long-time dynamics for a fractional piezoelectric system with magnetic effects and Fourier's law*, J. Differential Equations, 2021, 280, 891–927.

[10] W. Ge and J. Ren, *An extension of Mawhin's continuation theorem and its application to boundary value problems with a p-Laplacain*, Nonlinear Anal., 2004, 58(3–4), 477–488.

[11] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific Publishing Co., Inc., River Edge, NJ, 2000.

[12] W. Jiang, *Solvability of fractional differential equations with p-Laplacian at resonance*, Appl. Math. Comput., 2015, 260, 48–56.

[13] B. Jin, *Fractional Differential Equations–An Approach via Fractional Derivatives*, Springer, Cham., 2021.

[14] S. Khuri and R. Assadi, *An extended variational iteration method for fractional BVPs encountered in engineering applications*, Int. J. Numer. Method. H., 2023, 33(7), 2671–2681.

[15] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B. V., Amsterdam, 2006.

[16] J. Kuang, *Applied Inequalities*, Shandong Science and Technology Press, Jinan, 2014.

[17] T. A. M. Langlands, B. I. Henry and S. L. Wearne, *Fractional cable equation models for anomalous electrodiffusion in nerve cells: Finite domain solutions*, SIAM J. Appl. Math., 2011, 71(4), 1168–1203.

[18] Z. Li, X. Huang and Y. Liu, *Initial-boundary value problems for coupled systems of time-fractional diffusion equations*, Fract. Calc. Appl. Anal., 2023, 26(2), 533–566.

[19] P. Liu and H. Li, *Global stability of autonomous and nonautonomous hepatitis B virus models in patchy environment*, J. Appl. Anal. Comput., 2020, 10(5), 1771–1799.

[20] J. Mawhin, *Topological Degree Methods in Nonlinear Boundary Value Problems*, American Mathematical Society, Providence, RI, 1979.

[21] J. P. Ndenda, J. B. H. Njagarah and C. B. Tabi, *Fractional-order model for myxomatosis transmission dynamics: Significance of contact, vector control and culling*, SIAM J. Appl. Math., 2021, 81(2), 641–665.

[22] G. Pang, L. Lu and G. E. Karniadakis, *fPINNs: Fractional physics-informed neural networks*, SIAM J. Sci. Comput., 2019, 41(4), A2603–A2626.

[23] P. D. Phung and H. B. Minh, *Existence of solutions to fractional boundary value problems at resonance in Hilbert spaces*, Bound. Value Probl., 2017, 105, 1–20.

[24] P. D. Phung and L. X. Truong, *On the existence of a three point boundary value problem at resonance in* $\mathbb{R}^n$, J. Math. Anal. Appl., 2014, 416(2), 522–533.

[25] N. Raza, S. S. Kazmi and G. A. Basendwah, *Dynamical analysis of solitonic, quasi-periodic, bifurcation and chaotic patterns of Landau-Ginzburg-higgs model*, J. Appl. Anal. Comput., 2024, 14(1), 197–213.

[26] A. Seal and S. Natesan, *Convergence analysis of a second-order scheme for fractional differential equation with integral boundary conditions*, J. Appl. Math. Comput., 2023, 69(1), 465–489.

[27] Y. Wang, L. Liu and Y. Wu, *Extremal solutions for p-Laplacian fractional integro-differential equation with integral conditions on infinite intervals via iterative computation*, Adv. Difference Equ., 2015, 24, 1–14.

[28] P. Wang, J. Wu, X. Xu, et al., *Sharp decay estimates for Oldroyd-B model with only fractional stress tensor diffusion*, J. Funct. Anal., 2022, 282(4), 109332–109387.

[29] Y. Wang and H. Wang, *Triple positive solutions for fractional differential equation boundary value problems at resonance*, Appl. Math. Lett., 2020, 106, 376–382.

[30] Q. Xu and J. S. Hesthaven, *Discontinuous Galerkin method for fractional convection-diffusion equations*, SIAM J. Numer. Anal., 2014, 52(1), 405–423.

[31] M. A. Zaky, *Existence, uniqueness and numerical analysis of solutions of tempered fractional boundary value problems*, Appl. Numer. Math., 2019, 145, 429–457.